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complexes**

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Abstract

We derive a discrete analogue of Morse-Bott theory on CW complexes and use this discrete Morse-Bott function to do some Conley theory analysis. It turns out that our discrete Morse-Bott theory is indeed a generalization of Forman's discrete Morse theory in [7].

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1 Introduction

Morse theory is a very important tool for the study of the topology of differentiable manifolds. It was first introduced by Morse in 1925, in [18]. It recovers the homology of the manifold from the critical points of a Morse function and the relations between them. The Morse inequalities are inequalities between the Betti numbers of the manifold and the numbers of critical points of fixed indices of the function. To get the homologies, one attaches a k -dimensional cell for each critical point of index k , and gluing relations between those cells then yield the homological boundary operator.

Morse-Bott theory, introduced by Bott in [2], being a generalization of Morse theory, was developed to treat the cases where instead of having critical points, one has critical submanifolds. To each critical submanifold, one associates a certain index that is determined by looking at the Hessian restricted to the normal part of the submanifold, since on the tangential part it vanishes. The Poincaré polynomial (and hence the Euler number) of the manifold is then obtained from the Morse-Bott inequalities in the sense that it is expressed in terms of the Poincaré polynomials of the critical submanifolds, taking their respective indices into account.

Conley theory on the other hand, introduced by Conley in [5], and being more dynamics related, focuses on the study of the topological invariants of a given manifold. Using the (negative) gradient flow lines generated by some (Morse) function, one obtains for each isolated invariant set its isolating neighborhood and exit set for the flow, and

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these two constitute an index pair. One obtains the Poincaré polynomial of the manifold by summing up those of the index pairs up to some correction term. In particular, the Euler number of the manifold is then obtained by summing for all isolating invariant sets the alternating sums of the dimensions of the homology groups of the index pairs.

Forman in 1998, in [7], developed a discrete version of Morse theory for CW complexes. A CW complex, introduced by J. C. Whitehead in [22], is a decomposition of a space into cells each of which is homeomorphic to an open disc. The dimension of the disc specifies the dimension of the cell. The CW construction uses a specific gluing procedure via the characteristic maps. This space is endowed with the weak topology and satisfies the condition that the closure of each cell intersects only a finite number of cells. We write $\sigma^{(k)}$ to emphasize that σ is a cell of dimension k . The topological boundary elements of a cell are called its faces. If a cell $\sigma^{(k)}$ is a face of another cell τ , we write $\sigma < \tau$ to indicate that $\dim \sigma = \dim \tau - 1$, in which case σ is called a facet of τ . We say $\sigma^{(k)} < \tau$ is a regular facet of τ if, for φ_τ , the characteristic map of τ , we have: the map $\varphi_\tau: \varphi_\tau^{-1}(\sigma^{(k)}) \rightarrow \sigma^{(k)}$ is a homeomorphism and $\overline{\varphi_\tau^{-1}(\sigma^{(k)})}$ is a closed k -ball.

We denote the cardinality of a set A by $\sharp A$.

A discrete Morse function, according to Forman, is a real-valued function defined on the set of cells such that it locally increases in dimension, except possibly in one direction. More formally, Forman's definition of a discrete Morse function f on a CW complex requires that for all cells $\sigma^{(k)}$,

$$\begin{cases} \text{for all } \tau \text{ s.t. } \sigma \text{ is an irregular face of } \tau, f(\sigma) < f(\tau); \\ Un(\sigma) := \sharp\{\tau^{(k+1)} \mid \sigma \text{ is a regular facet of } \tau \text{ and } f(\tau) \leq f(\sigma)\} \leq 1; \end{cases}$$

and

$$\begin{cases} \text{for all } \nu \text{ s.t. } \nu \text{ is an irregular face of } \sigma, f(\nu) < f(\sigma); \\ Dn(\sigma) := \sharp\{\nu^{(k-1)} \mid \nu \text{ is a regular facet of } \sigma \text{ and } f(\nu) \geq f(\sigma)\} \leq 1. \end{cases}$$

When σ is not a regular facet of some other cell, it is automatically critical; in the regular case, σ is critical if both $Dn(\sigma)$ and $Un(\sigma)$ are 0. In fact, one easily sees that at most one of them can be 1; the other then has to be 0.

A CW complex is regular if all the faces are regular. As examples one has polyhedral, cubical and simplicial complexes.

A pair $\{\sigma, \tau\}$ with $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$ is called a noncritical pair. If we draw an arrow from σ to τ whenever $\sigma < \tau$ but $f(\sigma) \geq f(\tau)$, then we get a vector field associated to this function, and each noncritical cell has precisely one arrow which is either incoming or outgoing. Therefore, for the Euler number, we only need to count the critical cells with appropriate signs according to their dimensions, since the noncritical cells cancel in pairs.

We recall that, see [9], a combinatorial vector field on a CW complex \mathbb{K} is a map $V: \mathbb{K} \rightarrow \mathbb{K} \cup \{0\}$ that satisfies: if the image of a cell is non zero, then the dimension of the image is the dimension of the cell plus one; the image cells have zero images; for each cell σ , either $V(\sigma) = 0$ or σ is a regular face of $V(\sigma)$; and the pre-image of a given

cell contains at most one element. One thinks of the function V as assigning an arrow from σ to τ whenever $V(\sigma) = \tau$. In this way, each cell can either have one incoming or outgoing arrow but not both. We write $\sigma \rightarrow \tau$ whenever there is an arrow from σ to τ .

It is an important consequence of Forman's definition that the vector field admits no closed orbits, where by a closed orbit we mean a path of the form

$$\sigma_0 \rightarrow \tau_0 > \sigma_1 \rightarrow \tau_1 > \dots \sigma_m \rightarrow \tau_m > \sigma_0.$$

Since a combinatorial vector field can also admit closed orbits, the vector field constructed from a discrete Morse function is a combinatorial vector field that has no closed orbits. Conversely, one can always construct a discrete Morse function from a combinatorial vector field that admits no closed orbits.

Forman answered the question of a discrete analogue of Conley theory for CW complexes, see [9]. He first uses a combinatorial vector field, and as isolated invariant sets he considers the rest points (which are the critical cells) and the closed orbits. The isolating neighborhoods here are the unions of all the cells in the isolated invariant sets together with the ones in their boundaries; the exit set is just the collections of cells in the isolating neighborhood that are not in the isolated invariant sets.

Our goal is to construct a discrete analogue of Morse-Bott theory. This involves generalizing discrete Morse functions to include the case where a discrete function can have larger critical collections of cells, instead of only simple critical cells. We define a discrete Morse-Bott function by requiring some conditions on specific collections of cells, where the function assumes the same value for all the cells in the collection. Using the reduced collections, excluding the noncritical pairs, we obtain some discrete Morse-Bott inequalities. That is, the Poincaré polynomial of the CW complex is expressed in terms of those of the reduced collections. The vector field originating from this discrete Morse-Bott function is such that inside each collection a cell can have possibly more than one incoming and/or outgoing arrow, but between the collections there are no closed orbits. We also do some Conley theory analysis by using the reduced collections (excluding the noncritical pairs) as the isolated invariant sets, and define their respective isolating neighborhoods and exit sets. These two constitute the index pairs. The Poincaré polynomial of the CW complex is then expressed in terms of those of the index pairs.

It should be noted that, when considering our discrete Morse-Bott function, the extracted discrete vector field is not a combinatorial vector field, thus Forman's discrete Conley theory and our approach are complementary.

2 Discrete Morse-Bott Theory

We recall, see [2], [1], that in the smooth setting, a Morse-Bott function is one that admits critical submanifolds instead of only isolated nondegenerate critical points as it is required for a Morse function (see [20], [16] and [18]). The Poincaré polynomial of the manifold is retrieved by adding those of the critical submanifolds taking their respective indices into account, up to some correction term.

As mentioned earlier, an analogue of Morse-Bott theory in the discrete case might be of importance, since extending a function defined on the set of vertices might not always result in a discrete Morse function everywhere on the cell complex. The idea is, we consider a function that is discrete Morse on a cell complex except possibly at some maximal collection of cells, where every cell in the collection has the same value. We derive the analogue of the smooth Morse-Bott inequalities, that is, inequalities involving the Poincaré polynomial (and hence the Betti numbers) of the cell complex and those of the reduced collections. First we shall need an analogue of the Morse-Bott function in the discrete setting.

Definition 2.1 (Collection). *Let f be a discrete function on some CW complex. A collection C for f is a maximal set of cells such that:*

- (i) *all the cells in C have the same function value,*
- (ii) $\bigcup_{\sigma \in C} \bar{\sigma}$ *is (path-) connected.*

We recall that a pair $\{\sigma, \tau\}$ is said to be noncritical if $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$.

Definition 2.2 (Discrete Morse-Bott function). *Let f be a function defined on a CW complex \mathbb{K} and let C^1, \dots, C^l be the collections for f . We say f is discrete Morse-Bott if, for each i and for all $\sigma^{(k)} \in C^i$,*

if σ is an irregular face of some τ , $f(\sigma) < f(\tau)$, else

$$U_n^c(\sigma) := \#\{\tau^{(k+1)} \notin C^i \mid \sigma \text{ a regular facet of } \tau, f(\tau) < f(\sigma)\} \leq 1; \quad (1)$$

if $\nu < \sigma$ is an irregular face of σ , $f(\nu) < f(\sigma)$, else

$$D_n^c(\sigma) := \#\{\nu^{(k-1)} \notin C^i \mid \nu \text{ a regular facet of } \sigma, f(\nu) > f(\sigma)\} \leq 1. \quad (2)$$

Remark 2.1. (i) If for all i , $C^i = \{\sigma_i\}$ and f is a discrete Morse-Bott function, then f is also a discrete Morse function.

- (ii) Every discrete Morse function is a discrete Morse-Bott function in which each collection has at most two elements.

Remark 2.2. From the definition above, we get that in any collection there cannot be cells σ and τ with σ an irregular face of τ . This is useful in the situation where, if a collection reduces to only two cells of adjacent dimension, then one should not be an irregular facet of the other. This is important because in discrete Morse theory, the noncritical pairs always have no contribution.

It can be shown that either $U_n^c(\sigma) = 1$ or $D_n^c(\sigma) = 1$, but not both. The argument used is the same as in the discrete Morse case, by considering a discrete Morse-Bott function for which all the collections are singletons.

Figure 1 shows an example of a discrete function that is not a discrete Morse-Bott function. Take $C = \{\nu, \nu_2, \nu_1, \sigma_1, \sigma_2, \sigma_3, \tau\}$, then the vertex ν violates (2), indeed there are w_1 and w_2 not in C , with $w_1 > \nu < w_2$ but $f(w_2) < f(\nu) > f(w_1)$.

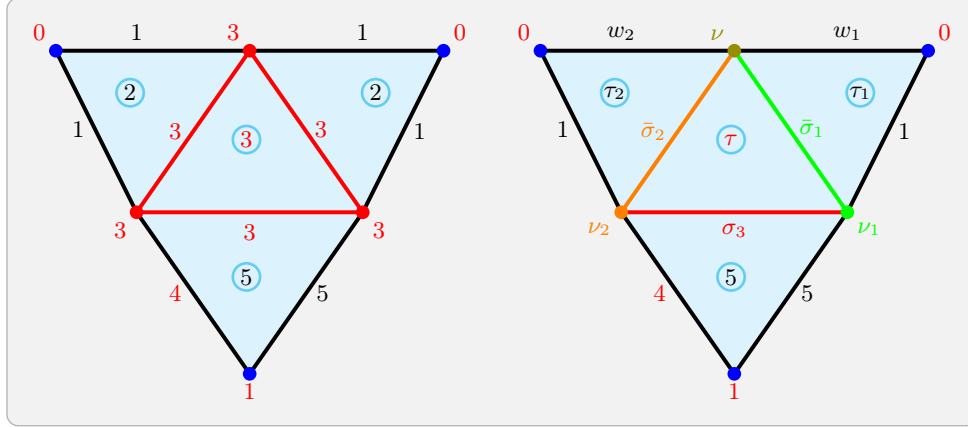


Figure 1: A discrete function that is not Morse-Bott.

Definition 2.3. Let f be a discrete Morse-Bott function and C a collection. We say that $\sigma \in C$ is **upward noncritical with respect to C** if there exists a cell $w(\sigma) \notin C$, $w(\sigma) > \sigma$ s.t. $f(\sigma) > f(w(\sigma))$.

Definition 2.4. Let f be a discrete Morse-Bott function and C a collection. We say that $\sigma \in C$ is **downward noncritical with respect to C** if there exists a cell $w(\sigma) \notin C$, $w(\sigma) < \sigma$ s.t. $f(\sigma) < f(w(\sigma))$.

Remark 2.3. If C is a singleton in the two definitions above, then we have the usual (upward and downward) noncriticalities in Forman's framework.

Observe that a cell cannot be at the same time upward and downward noncritical with respect to the same collection.

The following is the analogue of a critical submanifold.

Definition 2.5 (Reduced collection). For each collection C , we define the reduced collection C^{red} by taking out of C all the cells that are upward or downward noncritical with respect to C .

Remark 2.4. 1) If $C = \{\sigma\}$ with σ neither upward nor downward noncritical, then σ is critical in the discrete Morse sense for the function f and $C = C^{red}$.

2) If $C = \{\sigma\}$ where σ is either upward or downward noncritical, then $C^{red} = \emptyset$.

3) Observe that if C^{red} consists of only one element, then this element is not necessarily critical in the usual sense for the function f . To see this just consider a given function and a $C = \{\sigma_1, \nu, \sigma_2\}$, with $\sigma_1 > \nu < \sigma_2$ but both σ_1 and σ_2 being downward noncritical. Then $C^{red} = \{\nu\}$, but ν is not critical in the usual sense for this function since it has the same value with σ_1 and σ_2 .

The following definition helps to distinguish between a cell that we will call critical and a reduced collection that is a singleton.

Definition 2.6 (Critical cell). *A cell σ is said to be critical if*

$$\#\{\tau > \sigma | f(\tau) \leq f(\sigma)\} = 0 \quad \text{and} \quad \#\{\nu < \sigma | f(\nu) \geq f(\sigma)\} = 0.$$

The definition above tells us that if $C = C^{red} = \{\sigma\}$ then σ is critical.

The following lemma establishes the fact that the faces of upward noncritical cells w.r.t. a collection are also upward noncritical.

Lemma 2.1. *Let C be a collection for a discrete Morse-Bott function. If $\sigma \in C$ is upward noncritical with respect to C , then every face of σ in C is also upward noncritical w.r.t. C .*

In particular, if C is a subcomplex and $\sigma \in C$ is upward noncritical with respect to C , then so is $\bar{\sigma}$.

Proof. Let $\nu < \sigma$, and $\nu \in C$. The fact that σ is upward noncritical w.r.t. C implies there is an $w(\sigma) \notin C$, $w(\sigma) \in C'$ with $w(\sigma) > \sigma$, s.t. $f(\sigma) > f(w(\sigma))$. Since there are no irregular faces in a collection we get that ν is a regular face of σ , and the fact that there are no closed orbits tells us that σ is a regular face of $w(\sigma)$. We have $\nu < \sigma < w(\sigma)$, that is ν and $w(\sigma)$ are incident. Let $\tau \notin C$ be such that $w(\sigma) > \tau > \nu < \sigma$, such a τ exists by the incidence property. Then either $\tau \in C'$ or $\tau \notin C'$: if $\tau \in C'$ then $f(\tau) = f(w(\sigma))$; if $\tau \notin C'$, the definition of a discrete Morse-Bott function yields $f(w(\sigma)) > f(\tau)$. Thus, $f(\nu) = f(\sigma) > f(w(\sigma)) \geq f(\tau)$, therefore ν is also upward noncritical w.r.t. C , hence all the faces of σ in C are also upward noncritical with respect to C . \square

The most important fact about our definition of a discrete Morse-Bott function is the following.

Lemma 2.2. *If a collection C is such that $C \neq C^{red}$, then there is a one-to-one correspondence between the upward (resp. downward) noncritical cells σ w.r.t. C and the cells $w(\sigma) \notin C$, $w(\sigma) > \sigma$ with $f(\sigma) > f(w(\sigma))$ (resp. $w(\sigma) < \sigma$ with $f(\sigma) < f(w(\sigma))$).*

Proof. We already know from Lemma 2.1 that if $\sigma \in C$ is upward noncritical with respect to C , then every (regular) face of σ in C is also upward noncritical w.r.t. C . If C is a subcomplex, the interesting case would be to consider a situation where $\sigma_1, \sigma_2 \in C$ are upward noncritical w.r.t. C satisfying $\bar{\sigma}_1 \not\subseteq \bar{\sigma}_2$ and $\bar{\sigma}_2 \not\subseteq \bar{\sigma}_1$. Then, if we take a $\nu \in C$ such that $\sigma_1 > \nu < \sigma_2$, we have the existence of other cells $w(\sigma_i) \notin C$ for $i = 1, 2$ s.t. $w(\sigma_i) > \sigma_i$. Thus ν is a regular face of the σ_i 's and this also implies (by the incidence property), there exist $\tau_i \notin C$ for $i = 1, 2$ with $\nu < \tau_i < w(\sigma_i)$, and the following holds:

$$f(\nu) = f(\sigma_1) > f(w(\sigma_1)) \geq f(\tau_1) \quad \text{and} \quad f(\nu) = f(\sigma_2) > f(w(\sigma_2)) \geq f(\tau_2).$$

Thus, if $\tau_1 \neq \tau_2$, f at ν will have a value greater than that of more than one cell in the complement of C , and this will contradict the definition of a discrete Morse-Bott function. If however $\tau_1 = \tau_2$, then there is a unique $w(\nu) = \tau \notin C$ for which $f(\nu) > f(w(\nu))$.

If $\sigma \in C$ is downward noncritical, then it follows directly from the definition of f . \square

Figure 1 also shows that the result in Lemma 2.2 does not hold whenever we have a function that is not discrete Morse-Bott.

The notion of the index here however is ambiguous, but it will be shown in subsequent examples that the Euler number will always be counted with positive sign. See for instance Figure 2, Figure 3, and Figure 4; each of them involves a simplicial complex \mathbb{K} with negative, zero and positive Euler number, and also reduced collections with negative, zero and positive Euler numbers. Recall that the contribution of a cell of index k , in the computation of the Euler number of a CW complex, is given by $(-1)^k$.

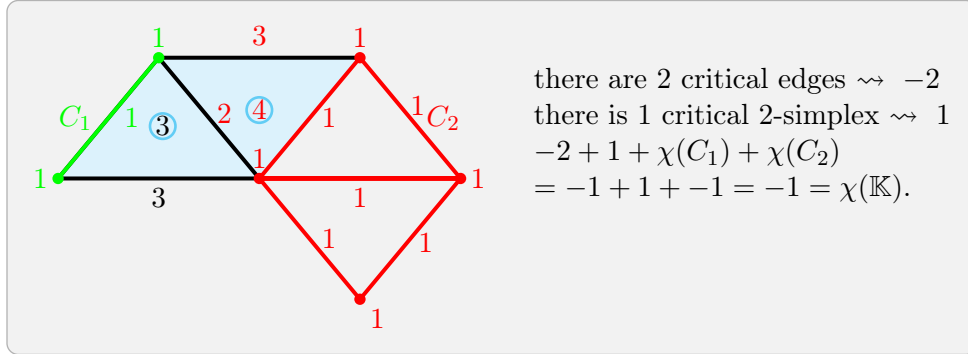


Figure 2: A discrete Morse-Bott function on \mathbb{K} with $\chi(\mathbb{K}) = -1$.

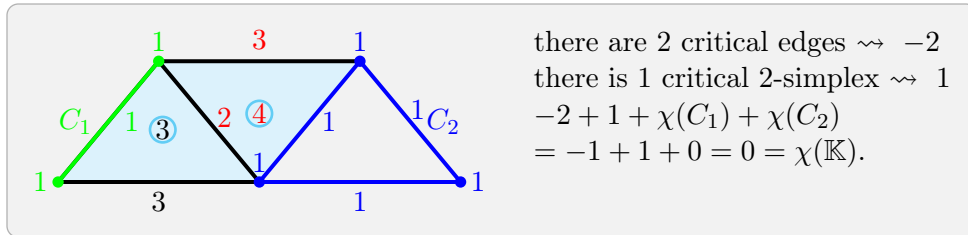


Figure 3: A discrete Morse-Bott function on \mathbb{K} with $\chi(\mathbb{K}) = 0$.

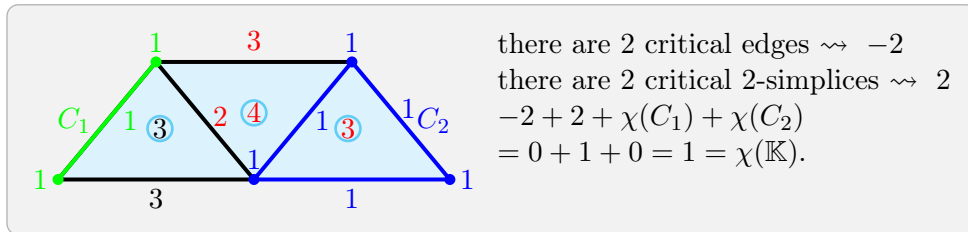


Figure 4: A discrete Morse-Bott function on \mathbb{K} with $\chi(\mathbb{K}) = 1$.

Given that the reduced collections are not necessary subcomplexes, we need to make

precise how their Betti numbers are obtained. First we prove the following:

Lemma 2.3. *If $\sigma \in \overline{C^{red}} \setminus C^{red}$ and $f(\sigma) = f(\tau)$ for $\tau \in C^{red}$, then σ cannot be downward noncritical.*

Proof. If $\sigma \in \overline{C^{red}} \setminus C^{red}$, then $\sigma \notin C^{red}$ but σ is a face of an element in C^{red} , say $\sigma < \tau$ for some $\tau \in C^{red}$.

If σ is downward noncritical, then there is a $\nu^* < \sigma$ s.t. $f(\nu^*) > f(\sigma)$ and all the other ($\nu \notin C^{red}$) $\nu < \sigma$ are such that $f(\sigma) > f(\nu)$. Note that ν^* and τ are incident. Thus there is a $\sigma^* > \tau$ s.t. $\nu^* < \sigma^* \neq \sigma$.

If $\sigma^* \in C^{red}$ we have a contradiction since $f(\nu^*) > f(\sigma) = f(\sigma^*)$ that is, $f(\nu^*)$ will be greater than $f(\sigma)$ and $f(\sigma^*)$. Thus $\sigma^* \notin C^{red}$, and $f(\sigma^*) > f(\nu^*)$ because ν^* is already upward noncritical with σ . This automatically means that $f(\sigma^*) > f(\nu^*) > f(\sigma) = f(\tau)$, which also contradicts the fact that $\tau \in C^{red}$. \square

Lemma 2.4. *For any reduced collection C^{red} , $\overline{C^{red}} \setminus C^{red}$ is a subcomplex.*

Proof. Let $\sigma \in \overline{C^{red}} \setminus C^{red}$, and $\nu < \sigma$, then by definition of $\overline{C^{red}}$, ν is also in $\overline{C^{red}}$. Thus, to show that $\nu \in \overline{C^{red}} \setminus C^{red}$, we only need to show that $\nu \notin C^{red}$.

For $\sigma \in \overline{C^{red}} \setminus C^{red}$, σ is a face of an element in C^{red} , say $\sigma < \tau$ for some $\tau \in C^{red}$. The fact that $\tau \in C^{red}$ and $\sigma < \tau$ implies that $f(\sigma) \leq f(\tau)$.

If $f(\sigma) = f(\tau)$, this means that σ is either downward or upward noncritical w.r.t. C . >From Lemma 2.3, such a σ cannot be downward noncritical. If σ is upward noncritical, then since σ cannot be downward noncritical, we have $f(\nu) \leq f(\sigma)$. If $f(\nu) = f(\sigma)$, then ν has to be in C , and by Lemma 2.1, ν also has to be upward noncritical w.r.t. C , which implies that $\nu \notin C^{red}$. If $f(\nu) < f(\sigma)$ then $\nu \notin C^{red}$ by definition of C^{red} .

If $f(\sigma) < f(\tau)$ then either $f(\nu) \leq f(\sigma) < f(\tau)$, $f(\sigma) < f(\nu) < f(\tau)$, or $f(\nu) = f(\tau) > f(\sigma)$. In any case ν cannot be in C^{red} . \square

Remark 2.5. For a given collection C , $\overline{C^{red}}$ is not always equal to C . Figure 6 illustrates this: C is the collection of the 2-cell with value 3, the two red edges, the red vertex and the two green vertices. After removing the upward noncritical vertices w.r.t. C , that is the two green vertices, and the downward noncritical 2-cell w.r.t. C , we end up with the two red edges and the red vertex. Thus, $\overline{C^{red}}$ consists of the two red edges, the red vertex and the two green vertices, which is different from C .

Let $[\tau : \sigma]$ denote the incidence number of τ and σ , which is the number of times that τ is wrapped around σ , taking the induced orientation from τ onto σ into account.

Let $C_k(C^{red}; \mathbb{Z})$ be the free \mathbb{Z} -module generated by the oriented cells of C^{red} of dimension k . One can also take $C_k(C^{red}; \mathbb{Z}_2)$.

Definition 2.7. *The boundary operator $\partial_k^{red}: C_k(C^{red}, \mathbb{Z}) \rightarrow C_{k-1}(C^{red}, \mathbb{Z})$ is given by:*

$$\partial_k^{red} \tau^{(k)} = \sum_{\sigma \in C^{red}, \sigma < \tau} [\tau : \sigma] \sigma^{(k-1)}.$$

Proposition 2.5.

$$\partial_{k-1}^{red} \circ \partial_k^{red} = 0.$$

Proof. If C^{red} is a subcomplex then it follows from the classical theory. If C^{red} is not a subcomplex, the result will follow if we show that the above boundary operator is a relative boundary operator, that is, a boundary operator for relative homology. Indeed

$$\overline{C^{red}} := \bigcup_{\sigma \in C^{red}} \bar{\sigma} \text{ is a subcomplex by definition.}$$

Let $X := \overline{C^{red}}$ and $A := \overline{C^{red}} \setminus C^{red}$ then X is a subcomplex, and A is a subcomplex of X by Lemma 2.4, and the result follows since $C^{red} = X \setminus A$. \square

Let $b_k^{red} := \dim(\ker \partial_k^{red} / \text{im } \partial_{k+1}^{red})$ then we define the Poincaré polynomial of C^{red} by

$$P_t(C^{red}) = \sum_k b_k^{red} t^k.$$

Example 2.1. Now we can change the function in Figure 1 to make it discrete Morse-Bott and this is illustrated in Figure 5. Where, C is the (subcomplex) collection of all the simplices with value 3. C has three upward noncritical vertices and one upward noncritical edge, highlighted in green. Then $C^{red} = \{\sigma_2, \sigma_3, \tau\}$. Observe that one can retrieve the Euler number of the complex just by adding that of C^{red} to those of the critical simplices. The contribution for C^{red} is -1 . There are three critical vertices and one critical edge which all together contribute for 2. Thus one has $\chi(C^{red}) + 2 = -1 + 2 = 1 = \chi(\mathbb{K})$, where $\chi(\mathbb{K})$ is the Euler number of the cell complex.

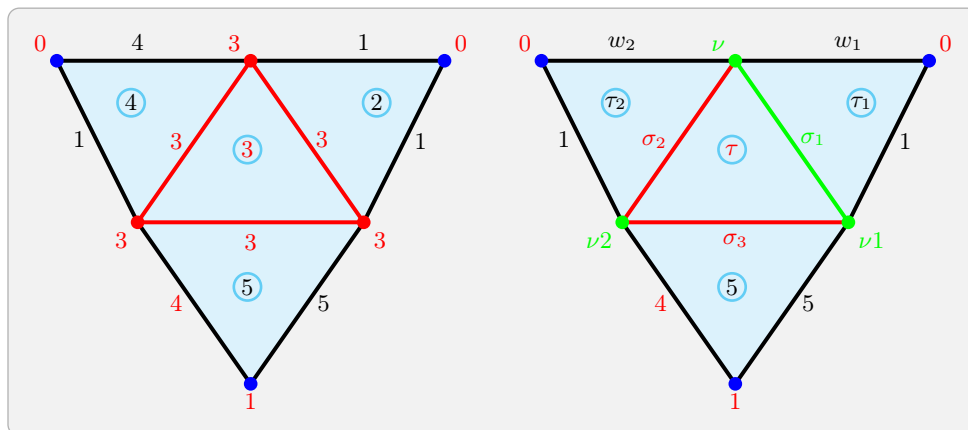


Figure 5: A discrete Morse-Bott function.

Before we state our analogue of the Morse-Bott inequalities which generalizes the idea above, we prove the following proposition.

Let $n_k := \#\{\sigma \in C^{red} \mid \dim \sigma = k\}$, then $n_k = \dim C_k(C^{red}; \mathbb{Z}) := \dim C_k^{red}$, and let $s := \dim C^{red}$.

Proposition 2.6. *Let C^{red} be a reduced collection for a discrete Morse-Bott function. Then there exists $r(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_{k=0}^s n_k t^k = P_t(C^{red}) + (1+t)r(t).$$

Proof. If $L : V \rightarrow W$ is a linear map on finitely generated R -modules, R a principal ideal domain, then

$$\dim V = \dim \ker L + \dim \operatorname{im} L.$$

$$\dim C_k^{red} = n_k = \dim \ker \partial_k^{red} + \dim \operatorname{im} \partial_k^{red},$$

and

$$b_k^{red} = \dim H_k = \dim \ker \partial_k^{red} / \operatorname{im} \partial_{k+1}^{red} = \dim \ker \partial_k^{red} - \dim \operatorname{im} \partial_{k+1}^{red}$$

for $k = 0, 1, \dots, s$, follows from the short exact sequence

$$0 \rightarrow \operatorname{im} \partial_{k+1}^{red} \hookrightarrow \ker \partial_k^{red} \rightarrow \dim H_k \rightarrow 0.$$

So,

$$\begin{aligned} \sum_{k=0}^s n_k t^k - \sum_{k=0}^s b_k^{red} t^k &= \sum_{k=0}^s (\dim \ker \partial_k^{red} + \dim \operatorname{im} \partial_k^{red}) t^k \\ &\quad - \sum_{k=0}^s (\dim \ker \partial_k^{red} - \dim \operatorname{im} \partial_{k+1}^{red}) t^k \\ &= \sum_{k=0}^s (\dim \operatorname{im} \partial_k^{red} + \dim \operatorname{im} \partial_{k+1}^{red}) t^k \\ &= \sum_{k=0}^s (n_k - \dim \ker \partial_k^{red}) t^k + \sum_{k=0}^s (n_{k+1} - \dim \ker \partial_{k+1}^{red}) t^k \\ &= (t+1) \sum_{k=1}^s (n_k - \dim \ker \partial_k^{red}) t^{k-1} \\ &\quad \text{since } n_0 = \dim \ker \partial_0^{red} \text{ and } n_{s+1} = 0 = \dim \ker \partial_{s+1}^{red}. \end{aligned}$$

The proof ends by using the fact that $\dim \ker \partial_k^{red} \leq n_k$ for all $k = 1, 2, \dots, s$. \square

Remark 2.6. In general, the contribution of each noncritical pair cancels out whenever one looks at the Euler number, thus it saves a lot of time to ignore the noncritical pairs.

The idea in Example 2.1 is the content of the following analogue of the Morse-Bott inequalities.

Theorem 2.7 (Discrete Morse-Bott inequalities). *Let f be a discrete Morse-Bott function on an n -dimensional CW complex \mathbb{K} , and let $C^{i,red}$, $i = 1, \dots, l$ be its nonempty disjoint reduced collections that are not noncritical pairs. Then there exists $R(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_{i=1}^l P_t(C^{i,red}) = P_t(\mathbb{K}) + (1+t)R(t). \quad (3)$$

The result (3) implies that, setting $t = -1$ above, the Euler number of each reduced collection should always be counted with positive sign.

If we want to have the formula

$$\sum_{i=1}^l P_t(C^{i,red}) t^{\text{ind}(C^{i,red})} = P_t(\mathbb{K}) + (1+t)R(t),$$

then we have to put $\text{ind}(C^{i,red}) = 0$ for all $i = 1, \dots, l$.

Remark 2.7. One of the reasons for defining the index as above is the fact that the Poincaré polynomial of a point in the smooth setting is given by 1 while in the discrete setting, the Poincaré polynomial of a cell of dimension k is given by t^k , as we shall see later on.

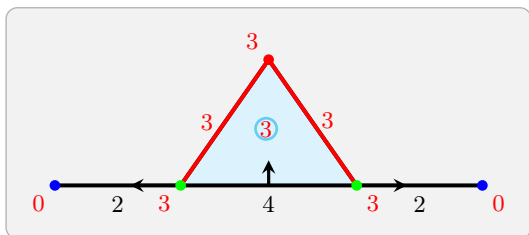


Figure 6: A discrete Morse-Bott-Conley method.

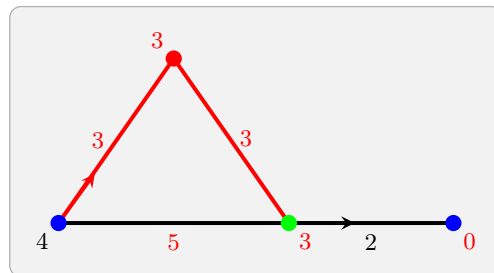


Figure 7: A reduced collection that is a noncritical pair.

Example 2.2. 1) In Figure 6, C consists of the red vertex, the two green vertices, the two red edges and the 2-simplex with value 3. Removing all the noncritical cells w.r.t. C yields that C^{red} consists of the two red edges and the red vertex. Thus, $P_t(C^{red}) = t$; in addition, there are two critical vertices, thus their overall contribution is 2. Hence the discrete Morse-Bott inequalities are satisfied since $\sum_i P_t(C^{i,red}) = t + 2$ and $P_t(\mathbb{K}) = 1$, that is, $R(t) \equiv 1$.

2) In Figure 7, C consists of the red vertex, the two red edges and the green vertex. After removing the upward or downward noncritical cells w.r.t. C , we obtain a reduced collection that is a noncritical pair, so we do not take it into consideration.

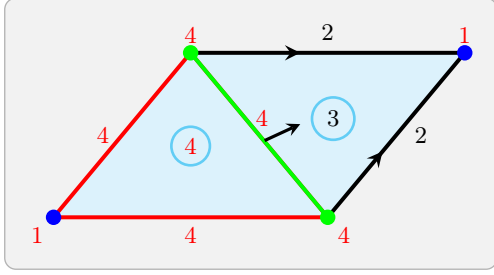


Figure 8: Example of a discrete Morse-Bott-Conley method.

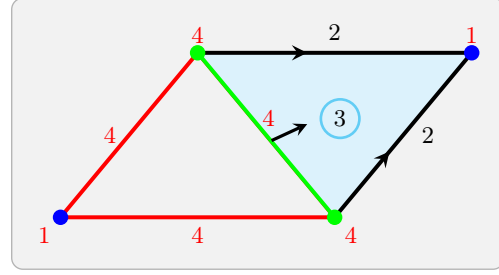


Figure 9: Another example of a discrete Morse-Bott-Conley method.

Thus we only take into account the critical edge and the critical vertex, and adding their contributions yield $t + 1$ which is the Poincaré polynomial of the complex, thus, $R(t) \equiv 0$.

- 3) In Figure 8, C is the collection of the two red edges, the green edge, the two green vertices and the 2-simplex with value 2. C^{red} consists of the two red edges and the 2-simplex. $P_t(C^{red}) = t$, to this we add the contributions for the two critical vertices. We get $\sum_i P_t(C^{i,red}) = t + 2$ and $P_t(\mathbb{K}) = 1$. Here, $R(t) \equiv 1$.
- 3) In Figure 9, C is the collection of the two red edges, the green edge and the two green vertices. C^{red} consists of the two red edges. $P_t(C^{red}) = 2t$, to this we add the contributions for the two critical vertices, and we get, $\sum_i P_t(C^{i,red}) = 2t + 2$, $P_t(\mathbb{K}) = 1 + t$. Thus, $R(t) \equiv 1$.

The following lemma is useful for the proof of Theorem 2.7.

Lemma 2.8. $\sum_i \dim \ker \partial_k^{C^{i,red}} - \dim \ker \partial_k^F \geq 0$ for each $k \geq 1$.

Proof. Suppose that the reduced collections $C^{i,red}$, $i = 1, \dots, l$ are indexed such that $f(C^{i,red}) \leq f(C^{j,red})$ for $i \leq j$. We know that if $\sigma \in \partial^F \tau$, where ∂^F denotes Forman's boundary operator, then $f(\sigma) < f(\tau)$. Using the reduced collections, if $\tau \in C^{i,red}$, and $\sigma \in \partial^F \tau$, then either $\sigma \in C^{i,red}$, in which case $f(\sigma) = f(\tau)$, or there is a $C^{j,red}$, s.t. $\sigma \in C^{j,red}$, in which case one immediately sees that $f(C^{j,red}) < f(C^{i,red})$, so $j \leq i$.

Also, we know that

$$C_k(\mathbb{K}, R) = \bigoplus_{i=1}^l C_k(C^{i,red}, R),$$

where ($R = \mathbb{Z}$ or \mathbb{Z}_2).

Let $\partial_k^i := \partial_k^{C^{i,red}}$, then $\sigma \in C_k(\mathbb{K}, R) \Rightarrow \sigma = \sigma^1 + \sigma^2 + \dots + \sigma^l$ with $\sigma^i \in C_k(C^{i,red}, R)$ where,

$$\partial^F \sigma^i = \text{Proj}_{C^{i,red}} \partial^F \sigma^i + \sum_{j=1}^{i-1} \text{Proj}_{C^{j,red}} \partial^F \sigma^i$$

$$= \partial^i \sigma^i + \sum_{j=1}^{i-1} \text{Proj}_{C^j, \text{red}} \partial^F \sigma^i, \quad \text{for } i = 1, \dots, l,$$

and Proj denotes the projection map.

If $\sigma \neq 0$ and $\sigma \in \ker \partial^F$, then $\sum_i \partial^F \sigma^i = 0$.

If $\sigma = \sigma^1$ then $\partial^F \sigma = 0 \Leftrightarrow \partial^1 \sigma^1 = 0$ and we are done.

If at least one of the σ^i 's is different from zero, since $\text{Proj}_{C^j, \text{red}} \partial^F \sigma^i = 0$ if $j > i$, we get immediately that $\partial^l \sigma^l = \text{Proj}_{C^l, \text{red}} \partial^F \sigma = 0$.

If $\sigma^l = 0$, do the same for σ^{l-1} and so on, until you get to σ^1 , which we know will have to be different from zero in order not to contradict the fact that $\sigma \neq 0$. Thus at least one of the σ^i 's must be different from zero. Hence $\sum_i \dim \ker \partial_k^i \geq \dim \ker \partial_k^F$.

Now let us suppose $\dim \ker \partial_k^F = 2$. Let $\sigma_i \neq 0$, for $i = 1, 2$ and $\sigma_1 \neq \sigma_2$. Suppose $\sigma_i \in \ker \partial_k^F$, for $i = 1, 2$ then $\sigma_1 = \sigma_1^1 + \sigma_1^2 + \dots + \sigma_1^l$ and $\sigma_2 = \sigma_2^1 + \sigma_2^2 + \dots + \sigma_2^l$. As before, we will have $\partial^l \sigma_1^l = 0$ and $\partial^l \sigma_2^l = 0$. If $\sigma_1^l = \sigma_2^l \neq 0$, then $0 \neq \bar{\sigma} := \sigma_1 + \sigma_2 = \sigma_1^1 - \sigma_2^1 + \dots + \sigma_1^{l-1} - \sigma_2^{l-1}$ satisfies $\partial^F \bar{\sigma} = 0$. From the previous step it follows that $\sum_{i=1}^{l-1} \dim \ker \partial_k^i \geq 1$. Using the fact that $\sigma_1^l \in \ker \partial_k^l$, we then get that $\sum_i^l \dim \ker \partial_k^i \geq 2$. If $\sigma_1^l = 0$ & $\sigma_2^l \neq 0$ or $\sigma_1^l \neq 0$ & $\sigma_2^l = 0$, we are done. If $\sigma_1^l = 0$ & $\sigma_2^l = 0$, then do the same for $\sigma_1 = \sigma_1^1 + \dots + \sigma_1^{l-1}$ and $\sigma_2 = \sigma_2^1 + \dots + \sigma_2^{l-1}$. Hence $\sum_i^l \dim \ker \partial_k^i \geq \dim \ker \partial_k^F$.

We assume that the statement is true for $\dim \ker \partial_k^F \leq m - 1$ and we show it is true for m . Suppose $\sigma_i \in \ker \partial_k^F$ and are all linearly independent for $i = 1, \dots, m$, where $\sigma_i = \sigma_i^1 + \dots + \sigma_i^l$.

Proceeding as before, we get $\sigma_i^l \in \ker \partial_k^l$ for $i = 1, \dots, m$. If $\sigma_i^l \neq 0$ for all i and $\sigma_i^l \neq \sigma_j^l$ for all $i \neq j$ we are done.

If $\sigma_i^l \neq 0$ for all i but $\sigma_1^l = \sigma_2^l = \dots = \sigma_m^l$, then we get $\bar{\sigma}_i := \sigma_1 - \sigma_{i+1} \in \ker \partial_k^F$ for $i = 1, \dots, m - 1$. The linear independence of the $\bar{\sigma}_i$'s follows from that of the σ_i 's, and by induction hypothesis, $\sum_{i=1}^{m-1} \dim \ker \partial_k^i \geq m - 1$. Hence, $\sum_{i=1}^m \dim \ker \partial_k^i \geq m = \dim \ker \partial_k^F$, also taking $\sigma_i^l \in \ker \partial_k^l$.

If $\sigma_i^l \neq 0$ for all i but $\sigma_1^l \neq \sigma_2^l = \dots = \sigma_m^l$, then we get $\bar{\sigma}_i := \sigma_2 - \sigma_{i+2} \in \ker \partial_k^F$ for $i = 1, \dots, m - 2$. By induction hypothesis, $\sum_{i=1}^{m-2} \dim \ker \partial_k^i \geq m - 2$. Hence, $\sum_{i=1}^m \dim \ker \partial_k^i \geq m = \dim \ker \partial_k^F$, also adding σ_1^l and σ_2^l in $\ker \partial_k^l$.

The same idea is used if there are subsets A_1, \dots, A_s of $\{1, \dots, m\}$ s.t. $\sigma_i^l = \sigma_j^l$ for all $i \neq j$, $i, j \in A_p$, but $\sigma_i^l \neq \sigma_j^l$ for $i \in A_p$ and $j \in A_q$, $p \neq q$.

Indeed: $|A_1| + \dots + |A_s| = m$, and for each A_i , we define $\bar{\sigma}_j^{A_i} := \sigma_1^{A_i} - \sigma_{j+1}^{A_i}$ for $j = 1, \dots, |A_i| - 1$. From the $\bar{\sigma}_j^{A_i}$ for $i = 1, \dots, s$ and $j = 1, \dots, |A_i| - 1$, and the induction hypothesis, we have: $\sum_{i=1}^{s-1} \dim \ker \partial_k^i \geq \sum_{i=1}^s (|A_i| - 1) = m - s$. Taking into consideration the fact that in each A_i we have $\sigma_j^l \in \ker \partial_k^l$, we get $\sum_{i=1}^l \dim \ker \partial_k^i \geq m$. If $\sigma_i^l = 0$, we do the same for σ_i^{l-1} and so on. □

We now have all the necessary tools to prove Theorem 2.7.

Proof of Theorem 2.7. Let $f_\varepsilon: \mathbb{K} \rightarrow \mathbb{R}$ given by

$$f_\varepsilon(\sigma) = f(\sigma) - \frac{\varepsilon}{\dim \sigma + 1},$$

then $f_\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$.

Let $n_k^i := \#\{\sigma \in C^{i,red} \mid \dim \sigma = k\}$ and $s^i = \dim C^{i,red}$.

Claim:

For sufficiently small ε , f_ε is discrete Morse and $\{\sigma^{(k)} \text{ critical for } f_\varepsilon\} = \bigcup_{i=1}^l \{\sigma^{(k)} \in C^{i,red}\}$, that is

$$m_k^{f_\varepsilon} = \sum_{i=1}^l n_k^i, \quad (4)$$

where $m_k^{f_\varepsilon}$ is the number of critical points of f_ε of dimension k .

Proof of the claim. Let C^i be a collection and $\sigma \in C^i$. We recall that if $\sigma < \tau$ is an irregular facet of τ , then $f(\sigma) < f(\tau)$ and the same holds for f_ε . So, it is enough to do the following at the regular facets.

$$\begin{aligned} \#\{\nu < \sigma \mid f_\varepsilon(\nu) \geq f_\varepsilon(\sigma)\} &= \#\{\nu \in C^i, \nu < \sigma \mid f_\varepsilon(\nu) \geq f_\varepsilon(\sigma)\} \\ &+ \#\{\nu \notin C^i, \nu < \sigma \mid f_\varepsilon(\nu) > f_\varepsilon(\sigma)\} \\ &=: A_1 + B_1 \end{aligned}$$

$$\begin{aligned} \#\{\tau > \sigma \mid f_\varepsilon(\tau) \leq f_\varepsilon(\sigma)\} &= \#\{\tau \in C^i, \tau > \sigma \mid f_\varepsilon(\tau) \leq f_\varepsilon(\sigma)\} \\ &+ \#\{\tau \notin C^i, \tau > \sigma \mid f_\varepsilon(\tau) < f_\varepsilon(\sigma)\} \\ &=: A_2 + B_2 \end{aligned}$$

We have, $A_1 = 0$ (resp. $A_2 = 0$) since, if $\nu < \sigma$, meaning that $\dim \nu = \dim \sigma - 1$ (resp. $\tau > \sigma$ meaning that $\dim \tau = \dim \sigma + 1$), and both are in the same collection that is $f(\sigma) = f(\nu)$ (resp. $f(\sigma) = f(\tau)$), we get that $f_\varepsilon(\sigma) > f_\varepsilon(\nu)$ (resp. $f_\varepsilon(\sigma) < f_\varepsilon(\tau)$).

For sufficiently small ε , that is, as $\varepsilon \rightarrow 0$, we have $B_1 = \#\{\nu \notin C^i, \nu < \sigma \mid f(\nu) > f(\sigma)\}$ and $B_2 = \#\{\tau \notin C^i, \tau > \sigma \mid f(\tau) < f(\sigma)\}$, so that, $B_1 \leq 1$ and $B_2 \leq 1$ follow from the discrete Morse-Bott condition for f . Hence $A_1 + B_1 \leq 1$ and $A_2 + B_2 \leq 1$ which implies that f_ε is discrete Morse.

We show that

$$\{\sigma^{(k)} \text{ critical for } f_\varepsilon\} = \bigcup_i \{\sigma^{(k)} \in C^{i,red}\}. \quad (5)$$

‘ \Rightarrow ’ Let σ be critical for f_ε , this implies that $B_1 = 0$ and $B_2 = 0$. For ε small enough, $B_1 = 0$ means that x is not downward noncritical w.r.t C^i , and $B_2 = 0$ means that σ

is not upward noncritical w.r.t C^i . Therefore σ should be in $C^{i,red}$.

' \Leftarrow ' If $\sigma \in C^{i,red}$, then σ was neither upward noncritical nor downward noncritical w.r.t. C^i for f . So for sufficiently small ε , $B_1 = 0 = B_2$ and this implies that σ is critical for f_ε . \square

Now to end the proof of the theorem: from Proposition 2.6, we have for each i ,

$$\sum_{k=0}^{s^i} n_k^i t^k = P_t(C^{i,red}) + (1+t)r_i(t), \quad (6)$$

where each $r_i(t)$ is a polynomial in t with nonnegative integer coefficients.

The function f_ε is discrete Morse on \mathbb{K} for sufficiently small ε , so from the Morse inequalities we have:

$$\begin{aligned} \sum_{k=0}^n m_k^{f_\varepsilon} t^k &= \sum_{k=0}^n b_k t^k + (1+t)r^1(t) \\ \Rightarrow \sum_{k=0}^n \left(\sum_{i=1}^l n_k^i \right) t^k &= \sum_{k=0}^n b_k t^k + (1+t)r^1(t) \quad \text{from (4)} \\ \Rightarrow \sum_{i=1}^l P_t(C^{i,red}) + (1+t)r_i(t) &= \sum_{k=0}^n b_k t^k + (1+t)r^1(t) \quad \text{from (6)} \\ \Rightarrow \sum_{i=1}^l P_t(C^{i,red}) &= P_t(\mathbb{K}) + (1+t)R(t). \end{aligned}$$

Note that

$$r^1(t) = \sum_{k=1}^n (m_k^h - \dim \ker \partial_k^F) t^{k-1} \quad \text{and} \quad r_i(t) = \sum_{k=1}^{s^i} (n_k^i - \dim \ker \partial_k^{C^{i,red}}) t^{k-1},$$

where $\partial_k^{C^{i,red}} := \partial_k^c|_{C^{i,red}}$. Thus $R(t) = \sum_{k=1}^n (\sum_i \dim \ker \partial_k^{C^{i,red}} - \dim \ker \partial_k^F) t^{k-1}$, and the result follows from Lemma 2.8. \square

After realizing that an approach of Floer's in the case of a discrete Morse function is possible one would like to ask the question if something similar can be done using a discrete Morse-Bott function.

Question: *is it possible to do some kind of Floer-related theory on a complex having a discrete Morse-Bott function?*

It turns out that, we cannot consider the whole reduced collection as one critical object. Thus the only solution is:

Solution: *we approximate the discrete Morse-Bott function on each collection to get a discrete Morse function; this is always possible as seen before, so that any upward or*

downward noncritical cell w.r.t. a given collection will not be critical after the approximation. In this way, all the cells in the reduced collections will be critical just by making the approximating function to be a trivial discrete Morse function.

The above solution shows that the only way we can have a boundary operator on some complex on which is defined our discrete Morse-Bott function is by approximating to get a discrete Morse function. We now ask: having a discrete Morse-Bott function, is it possible to develop any other theory using a more dynamics-related method to get some insight on the Euler number? This is what we will be investigating in the last part of this paper.

Now we use the discrete Morse-Bott function to derive a similar notion to smooth Conley theory, this is our version of discrete Conley theory.

3 Discrete Morse-Bott-Conley theory

We recall, see [13], [17] and [5], that Conley theory uses the idea of exit sets and isolating neighborhoods of isolated invariant sets to derive the Euler characteristic of the manifold at hand. For each isolated invariant set, the pair consisting of its isolating neighborhood and exit set is called its index pair. The Euler number of the manifold is then obtained by summing for all isolating invariant sets, the alternating sums of the dimensions of the homologies of the index pairs.

The scope of Conley theory is more general than that of Morse theory, because if we consider the discrete vector field originating from some discrete Morse function, then this combinatorial vector field does not admit any closed orbit. This makes things easy in the sense that, the only critical objects under consideration are the critical cells of the discrete Morse function, and we will show below that for a critical cell of dimension k , the homological Conley index is just given by t^k , or $(-1)^k$ for $t = -1$.

Forman, in [9], made things more interesting by considering a combinatorial vector field in general (it need not originate from some discrete Morse function and can therefore admit closed orbits). In general, a combinatorial vector field on a CW complex yields a disjoint collection C_r of rest points or closed orbits, where the rest points are exactly the critical cells. For each C_r , let $\overline{C_r}$ be the union of all the cells in C_r with the ones in their boundaries, and let $\widetilde{C_r} := \overline{C_r} \setminus C_r$. The pair $(\overline{C_r}, \widetilde{C_r})$ is taken to be an index pair for the corresponding C_r . Forman showed that with

$$m_i := \sum_{C_r} \dim H^i(\overline{C_r}, \widetilde{C_r}; \mathbb{Z}),$$

the Morse inequalities are satisfied.

Now we want to go beyond a combinatorial vector field and we do this by considering our discrete Morse-Bott function on a CW complex \mathbb{K} . The vector field originating from this function inside each collection need not be a combinatorial vector field as a cell in the collection can have more than one incoming and/or outgoing arrow, but between the collections there should not be any closed orbit.

We shall consider as isolated invariant sets the reduced collections (excluding the noncritical pairs), and their isolating neighborhood will be the subcomplex generated by the isolated invariant set. The exit set will be the part of $\partial^{top}N$ where the values of the function are smaller than on the isolated invariant set. This is the content of the following definition.

Definition 3.1. 1) I is said to be an **isolated invariant set** if it is a reduced collection that is not a noncritical pair.

2) An **isolating neighborhood** N for I is the union of all the cells in I together with all the cells in their boundaries. That is,

$$N = \bigcup_{\sigma \in I} \bar{\sigma}.$$

3) The **exit set** for the flow from I is just given by the part of N not in I where the values of f are smaller than or equal to the value on I . In other words

$$E = \{\sigma \in N \setminus I \mid f(\sigma) \leq f(\tau) \text{ for } \tau \in I\}.$$

4) We call (N, E) an **index pair** for I .

Remark 3.1. 1) The reduced collections I_i^{red} are such that there is no path (following the arrows) that moves from a cell in I_i^{red} to another cell outside of I_i^{red} , meaning that each of them is invariant. Since they constitute the building block for computing the Poincaré polynomial of the CW complex, we consider them to be isolated. Hence each collection I_i^{red} is an isolated invariant set.

2) In the definition of the exit set above, taking $I = C^{red}$, the cells in E that have the same value with the ones in I are exactly those cells that are upward noncritical w.r.t. C . Indeed, we know from Lemma 2.3 that any $\sigma \in N \setminus I$ satisfying $f(\sigma) = f(\tau)$ for $\tau \in I$ cannot be downward noncritical. Thus,

$$E = \left\{ \begin{array}{l} \sigma \in N \setminus I \text{ s.t. either } f(\sigma) < f(\tau) \text{ for } \tau \in I, \text{ or} \\ f(\sigma) = f(\tau) \text{ and } \sigma \text{ is upward noncritical w.r.t. } C \end{array} \right\}.$$

Later on, see the proof of Theorem 3.2, we will show that $E = N \setminus I$.

Definition 3.2. The **topological Conley index** of an isolated invariant set I is the homotopy type of N/E and the **homological Conley index** of I is the polynomial

$$C_t(I) := \sum_k \dim H_k(N, E; \mathbb{Z}) t^k.$$

We then have the following lemma.

Lemma 3.1. If $I = C^{red}$ with $\sharp C^{red} = 1$ of dimension k , then its homological Conley index is t^k .

Proof. Let $I = \sigma^{(k)}$, then $N = \bar{\sigma}$, and $E = \partial^{top}\bar{\sigma}$, where ∂^{top} denotes the topological boundary. Thus we obtain an index pair $(\bar{\sigma}, \partial^{top}(\bar{\sigma}))$ for σ , and the homotopy type of this index pair is just that of a k -dimensional closed disc with its boundary identified, which is that of the pair (S^k, pt) . Thus the Conley index is given by

$$C_t(\sigma^{(k)}) = \sum_i \dim H_i(\mathbb{S}^k, pt)t^i = t^k,$$

since $H_i(\mathbb{S}^k, pt) = 1$ for $i = k$ and $H_i(\mathbb{S}^k, pt) = 0$ for $i \neq k$. \square

In general, if $I = C^{red}$ is such that $C^{red} \neq C$, then there are elements that are upward noncritical w.r.t. C , and (in order not to contradict the definition of a discrete Morse-Bott function) there is a one to one correspondence between those elements and those out of C making them upward noncritical, this is the content of Lemma 2.2. Thus we have disjoint noncritical pairs $\{\sigma < \tau\}$, $\sigma \in C$ and $w(\sigma) = \tau \notin C$ with $f(\sigma) \geq f(\tau)$. We call such a σ an **exit cell**.

Remark 3.2. If $I = C^{red}$, then the exit set of I denoted E is the union of all the cells in N whose values are smaller than the value in C^{red} , together with all the exit cells of C that are contained in N .

We have the following analogue of the result in the smooth setting:

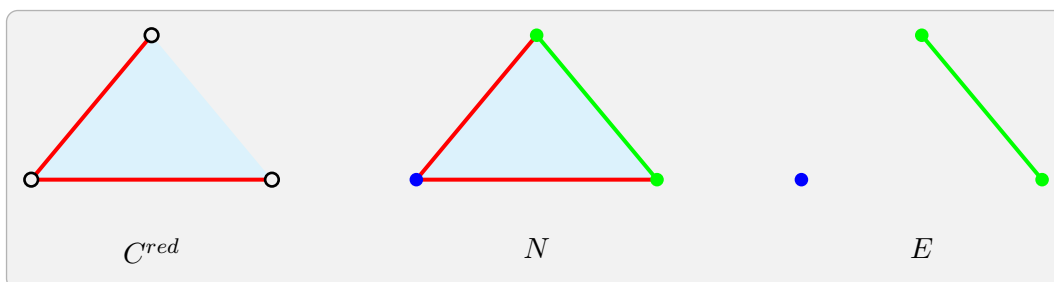


Figure 10: A reduced collection and its index pair.

Theorem 3.2. Let f be a discrete Morse-Bott function having isolated invariant sets I_1, \dots, I_l , then there exists $R(t)$, a polynomial in t with nonnegative integer coefficients, such that

$$P_t(\mathbb{K}) + (1+t)R(t) = \sum_{j=1}^l C_t(I_j). \quad (7)$$

Recall that $P_t(N, E) = P_t(N/E) - 1$.

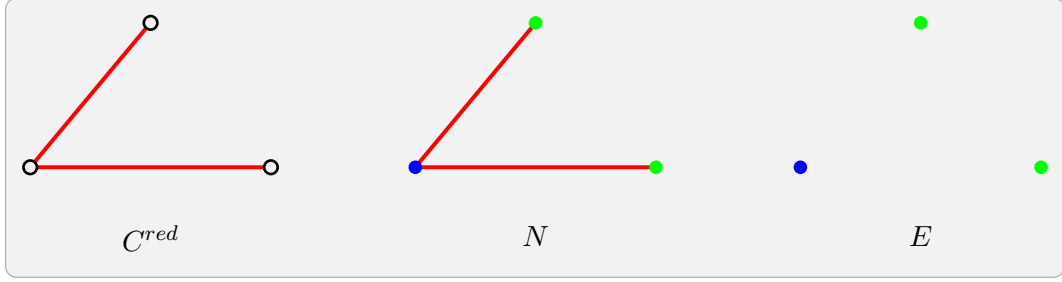


Figure 11: Another reduced collection and its index pair.

- Example 3.1.** 1) For the discrete Morse-Bott function in Figure 8, we get the respective index pair in Figure 10. Thus $P_t(N, E) = t$. Observe that this can be achieved geometrically by performing N/E and using the fact that $P_t(N, E) = P_t(N/E) - 1$. Hence, $P_t(\mathbb{K}) = 1$ and $\sum_j C_t(I_j) = t + 2$, this implies that $R(t) \equiv 1$.
- 2) Using Figure 9, the respective index pair for the discrete Morse-Bott function is given by Figure 11. In this case, we get $P_t(N, E) = 2t$. Thus $P_t(\mathbb{K}) = 1 + t$ and $\sum_j C_t(I_j) = 2t + 2$, this implies that $R(t) \equiv 1$.

Proof of Theorem 3.2. • We have shown above, see Lemma 3.1, that whenever I is a singleton, a cell of dimension k ,

$$C_t(I) = t^k.$$

- If $I = C^{red}$ where $\#C^{red} > 1$, we only need to show that $C^{red} = N \setminus E$ (which holds for all C^{red}).

The fact that $C^{red} \subseteq N \setminus E$ follows from the definitions of those sets.

We show that $N \setminus E \subseteq C^{red}$.

Let $\sigma \in N \setminus E$, then σ is either in C^{red} or it is a face of an element in C^{red} . If $\sigma \in C^{red}$, we are done. If there is a $\tau \in C^{red}$ s.t. $\sigma < \tau$, then from the definition of C^{red} and the fact that σ is not in E , we must have $f(\sigma) = f(\tau)$ and σ is not upward noncritical. Lemma 2.3 also establishes the fact that σ cannot be downward noncritical. Thus, $f(\sigma) = f(\tau)$ and σ is neither upward nor downward noncritical, this implies σ is in C^{red} .

Thus $N := N(I)$ is a subcomplex by definition, and $E := E(I) = N \setminus I := \overline{C^{red}} \setminus C^{red}$ is a subcomplex by Lemma 2.4. Hence, the relative homology is well defined. Thus,

$$C_t(I) = \sum_k \dim H_k(N(I), E(I); \mathbb{Z}) t^k = P_t(I),$$

and the second equality follows from Theorem 2.7 since $C^{red} = N \setminus E$ and both $N := N(I_j)$ and $E := E(I_j)$ are subcomplexes, indeed

$$\sum_{j=1}^l C_t(I_j) = \sum_{j=1}^l \sum_k \dim H_k(N(I_j), E(I_j); \mathbb{Z}) t^k$$

$$= \sum_{j=1}^l P_t(I_j) = P_t(\mathbb{K}) + (1+t)R(t).$$

□

A consequence of the proof above is the following:

Lemma 3.3. *If (N, E) is an index pair for $C^{i,red}$, then*

$$\chi(C^{i,red}) = \chi(N, E) = \chi(N) - \chi(E).$$

Remark 3.3. The lemma above is not true in general if we consider the Poincaré polynomials instead of the Euler numbers.

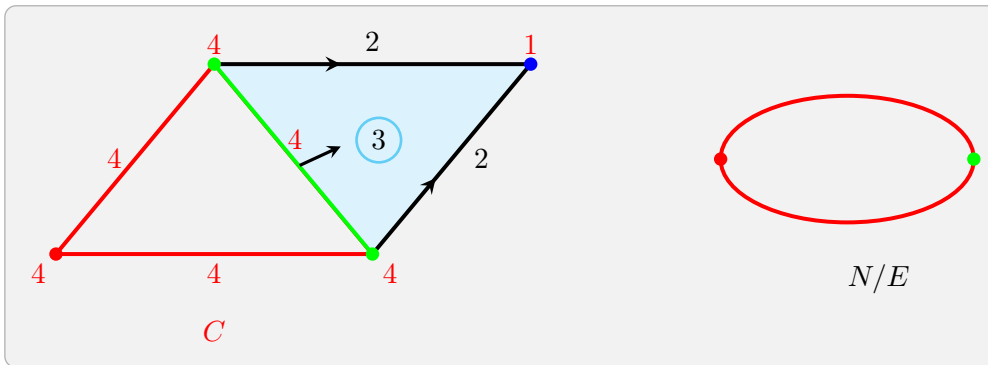


Figure 12: The corresponding quotient space to an index pair.

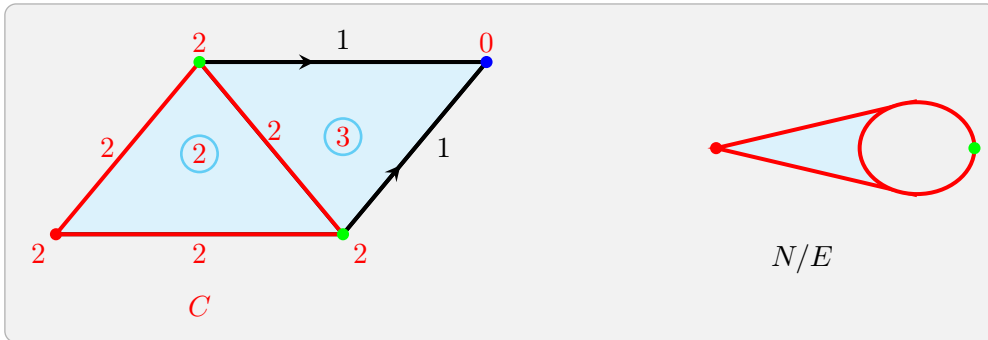


Figure 13: The quotient space corresponding to an index pair.

Example 3.2. 1) Using Figure 12, C^{red} consists of the two red edges and the red vertex, N is the subcomplex generated by the two red edges, so that E is the collection of the two green vertices. $P_t(N, E) = P_t(N/E) - 1 = t$. The critical vertex contributes 1. Thus $\sum_j C_t(I_j) = t + 1 = P_t(\mathbb{K})$ that is $R(t) \equiv 0$.

- 2) In Figure 13, C^{red} takes all the elements of C except the two green vertices. $N = C$ and E is the two green vertices. $P_t(N, E) = t$, there is one critical 2-simplex whose contribution is t^2 , and one critical vertex whose contribution is 1. Hence $\sum_j C_t(I_j) = t^2 + t + 1$, but $P_t(\mathbb{K}) = 1$ this implies that $R(t) = t$.

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References

- [1] A. Banyaga, D. Hurtubise, Lectures on Morse homology, Kluwer Texts in Mathematical Sciences vol 29, Kluwer Academic Publishers, Dordrecht, 2004.
- [2] R. Bott, Nondegenerate critical manifolds, Ann of Math (2) 60, 248–261. MR 16, 1954.
- [3] R. Bott, Lectures on Morse theory, old and new, Bull. Amer. Math. Soc. (N.S.) 7, no. 2, 331–358, 1982.
- [4] G. E. Bredon, Topology and Geometry, Graduate Texts in Mathematics 139, Springer-Verlag, 1997.
- [5] C. Conley, Isolated invariant sets and the Morse index, CBMS Regional Conference Ser. in Math. 38, Amer. Math. Soc., 1978.
- [6] T. Dieck, Algebraic Topology, EMS Textbooks in Mathematics, Eur. Math. Soc. (EMS) Zürich, 2008.
- [7] R. Forman, Morse theory for cell complexes, Adv.Math. 134, 90–145, 1998.
- [8] R. Forman, Witten-Morse theory for cell complexes, Topology 37, 945–979, 1998.
- [9] R. Forman, Combinatorial vector fields and dynamical systems, Mathematische Zeitschrift 228(4), 629–681, 1998.
- [10] R. Fritsch, R. A. Piccinini, Cellular structures in topology, Cambridge University Press, Cambridge, 1990.
- [11] R. Geoghegan, Topological Methods in Group Theory, Grad. texts in Math. 243, Springer-Verlag, New York, 2008.
- [12] P. J. Hilton, An introduction to homotopy theory, Cambridge: University Press, 1953.
- [13] J. Jost, Dynamical systems, Universitext, Springer-Verlag, Berlin, 2005.

- [14] D. Kozlov, *Combinatorial Algebraic Topology*, Springer-Verlag, New York, 2008.
- [15] A. T. Lundell, S. Weingram, *The Topology of CW Complexes*, Van Nostrand, 1969.
- [16] J. Milnor, *Morse theory*, *Annals of Math. Studies* 51, Princeton Univ. Press, 1963.
- [17] K. Mischaikow, *Conley index theory: a brief introduction*, *Banach Center Publications* 47, 1999.
- [18] M. Morse, *Relations between the critical points of a real function of n independent variables*, *Trans. Amer. Math. Soc* 27, 1925.
- [19] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Reading, MA, 1984.
- [20] M. Schwarz, *Morse homology*, Birkhäuser, 1993.
- [21] J. Vick, *Homology theory*, Academic Press, Inc., New York, 1973.
- [22] J. C. H. Whitehead, *Combinatorial homotopy I*, *Bull. Amer. Math. Soc.* 55, 213–245, 1949.