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by

Xue-Na Zhu, Ming Li, and Shao-Ming Fei

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Xue-Na Zhu[1], Ming Li[2], and Shao-Ming Fei[3,4]
[1]*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*
[2]*College of the Science, China University of Petroleum, Qingdao 266580, China*
[3]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[4]*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

We present a lower bound of concurrence for four-partite systems in terms of the concurrence for $M\,(2\leq M\leq 3)$ part quantum systems and give an analytical lower bound for $2\otimes 2\otimes 2\otimes 2$ mixed quantum sates. It is shown that these lower bounds are able to improve the existing bounds and detect entanglement better. Furthermore, our approach can be generalized to multipartite quantum systems.

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I. INTRODUCTION

Quantifying entanglement is a basic and long standing problem in quantum information theory [1]. Concurrence [2–6] is one of the well-accepted entanglement measures [7–12]. Different from the entanglement of formation which is defined for bipartite systems, concurrence can be generalized to arbitrary multipartite systems. Nevertheless, calculation of concurrence is a formidable task for higher-dimensional cases. For arbitrary S-dimensional bipartite quantum states, Ref. [13] provided an analytical lower bound of concurrence by decomposing the joint Hilbert space into many $s\ (2\leq s\leq S-1)$-dimensional subspaces, which may be used to improve all the known lower bounds of concurrence. For arbitrary qubit systems, Ref. [14] provided analytical lower bounds of concurrence in terms of the monogamy inequality of concurrence for qubit systems. For arbitrary N-partite S-dimensional quantum states, Ref. [15] provided an analytical lower bound of concurrence in terms of the concurrence for N-partite $s\ (2\leq s\leq S-1)$-dimensional quantum systems. More generally, for arbitrary N-partite arbitrary dimensional quantum states, Ref. [17] provided an analytical lower bound of concurrence in terms of the concurrence for two part quantum systems. A natural problem is whether the arbitrary dimensional N-partite quantum states can be dealt with M-partite $(2\leq M\leq N-1)$ quantum systems.

In this paper we provide the lower bound of concurrence for 4-partite quantum states in terms of tripartite and bipartite quantum systems. The generalized lower bound of concurrence can be generalized to the multipartite case.

II. LOWER BOUND OF CONCURRENCE FOR FOUR-PARTITE QUANTUM SYSTEMS

To investigate multi-entanglement we first introduce the M-partite concurrence in N-partite systems. For a pure N-partite quantum state $|\psi\rangle\in H_1\otimes H_2\otimes\cdots\otimes H_N$, $dimH_i=d_i$, $i=1,...,N$, the concurrence of $|\psi\rangle$ is defined by [6, 16]

$$C_N(|\psi\rangle)=2^{1-\frac{N}{2}}\sqrt{(2^N-2)-\sum_\alpha Tr(\rho_\alpha^2)},\tag{1}$$

where $\rho_\alpha=Tr_{\bar{\alpha}}(|\psi\rangle\langle\psi|)$, $\alpha\subseteq\{1,2,...,N\}$, $\bar{\alpha}$ is the compliment of α, ρ_α labels all the different reduced density matrices of $|\psi\rangle\langle\psi|$. We list all the 2^N-2 reduced matrices in the following way: $\{\rho_1,\rho_2,...,\rho_N,\rho_{12},\rho_{13},...,\rho_{1N},\rho_{23},...,\rho_{12...N-1},...,\rho_{23...N}\}$, by noticing that $Tr(\rho_\alpha^2)=Tr(\rho_{\bar{\alpha}}^2)$ for any pure states. For a mixed multipartite quantum state, $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, the corresponding concurrence is given by the convex roof extension,

$$C_N(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_iC_N(|\psi_i\rangle),\tag{2}$$

where the minimum is taken over all possible pure state decompositions $\{p_i,|\psi_i\rangle\}$ of ρ. This multipartite concurrence can be used to detect and classify various genuine multipartite entanglements. It has been shown in [17] that a multipartite quantum state is genuinely multipartite entangled if the multipartite concurrence is larger than certain quantities given by the number and the dimension of the subsystems.

For the N-partite quantum state $|\psi\rangle\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, we denote the general M-partite decomposition of $|\psi\rangle$, $\{i^1\},...,\{i^{M_1}\},\{k_1^1,k_2^1\},...,\{k_1^{M_2},k_2^{M_2}\},...,\{q_1^{M_j},...,q_j^{M_j}\}$, where all the subspaces in one bracket $\{\}$ are taken to be one part. Hence, $\{i^1, ..., i^{M_1}, k_1^1, k_2^1, ..., k_1^{M_2}, k_2^{M_2}, ..., q_1^{M_j}, ..., q_j^{M_j}\} = \{1, 2, ..., N\}$ and $\sum_{k=1}^j M_k = M$, $\sum_{k=1}^j kM_k = N$. The concurrence of the state $|\psi\rangle$ under such M-partite partition is given by

$$C_M(|\psi\rangle) = 2^{1-\frac{M}{2}} \sqrt{(2^M - 2) - \sum_{\beta} Tr(\rho_\beta^2)}, \tag{3}$$

where $\beta \in \{\{i^1\}, ..., \{i^{M_1}\}, ..., \{q_1^{M_j}, ..., q_j^{M_j}\}\}$. Take $N = 4$ and $M = 3$ as an example, one has $M_1 = 2$ and $M_2 = 1$. There are six different tripartite decompositions: 1|2|34, 1|23|4, 1|24|3, 12|3|4, 13|2|4 and 14|2|3. For convenience, we denote $C_{i|jk|l}(\rho) = C_3(\rho_{i|jk|l})$ and $C_{ij|kl}(\rho) = C_2(\rho_{ij|kl})$.

Theorem 1: For any mixed quantum state, $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$, the concurrence is bounded by

$$\begin{aligned} C_4^2(\rho) \ \geq\ & \frac{1}{12}\big(2C_{1|2|34}^2(\rho) + 2C_{1|3|24}^2(\rho) \\ +\ & 2C_{1|4|23}^2(\rho) + 2C_{12|3|4}^2(\rho) \\ +\ & 2C_{13|2|4}^2(\rho) + 2C_{14|2|3}^2(\rho) \\ +\ & C_{12|34}^2(\rho) + C_{13|24}^2 + C_{14|23}^2(\rho)\big). \end{aligned} \tag{4}$$

[Proof:] We start the proof with a pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$. According to the definition (1), one has that

$$\begin{aligned} C_4(|\psi\rangle) &= \frac{1}{2}\sqrt{14 - \sum_{\alpha} Tr(\rho_\alpha^2)} \\ &= \frac{1}{2}\sqrt{\sum_{\alpha} (1 - Tr(\rho_\alpha^2))}, \end{aligned} \tag{5}$$

where ρ_α labels all the different reduced density matrices of $|\psi\rangle\langle\psi|$. On the other hand, we have

$$\begin{aligned} C_{i|jk|l}^2(|\psi\rangle) \ =\ & \frac{1}{2}\big[(1 - Tr(\rho_i^2)) + (1 - Tr(\rho_{jk}^2)) \\ +\ & (1 - Tr(\rho_l^2)) + (1 - Tr(\rho_{ijk}^2)) \\ +\ & (1 - Tr(\rho_{il}^2)) + (1 - Tr(\rho_{jkl}^2))\big], \end{aligned} \tag{6}$$

and

$$C_{ij|kl}^2(|\psi\rangle) = (1 - Tr(\rho_{ij}^2)) + (1 - Tr(\rho_{kl}^2)). \tag{7}$$

From (5), (6) and (7) , we have

$$\begin{aligned} C_4^2(|\psi\rangle) \ =\ & \frac{1}{12}\big(2C_{1|2|34}^2(|\psi\rangle) + 2C_{1|3|24}^2(|\psi\rangle) \\ +\ & 2C_{1|4|23}^2(|\psi\rangle) + 2C_{12|3|4}^2(|\psi\rangle) \\ +\ & 2C_{13|2|4}^2(|\psi\rangle) + 2C_{14|2|3}^2(|\psi\rangle) \\ +\ & C_{12|34}^2(|\psi\rangle) + C_{13|24}^2(|\psi\rangle) + C_{14|23}^2(|\psi\rangle)\big). \end{aligned} \tag{8}$$

Let $\rho = \sum_i p_i |\psi\rangle_i\langle\psi|$ be the optimal pure state decomposition of (2). We have

$$\begin{aligned} C_4(\rho) &= \sum_i p_i C_4(|\psi\rangle_i) \\ &= \sum_i p_i \sqrt{\frac{1}{6}C_{1|2|34}^2(|\psi\rangle_i) + ... + \frac{1}{6}C_{14|2|3}^2(|\psi\rangle_i) + \frac{1}{12}C_{12|34}^2(|\psi\rangle_i) + ... + \frac{1}{12}C_{14|24}^2(|\psi\rangle_i)} \\ &\geq \sqrt{(\sum_i p_i \frac{1}{\sqrt{6}} C_{1|2|34}(|\psi\rangle_i))^2 + ... + (\sum_i p_i \frac{1}{\sqrt{12}} C_{14|23}(|\psi\rangle_i))^2} \\ &\geq \frac{1}{\sqrt{12}}\big(2C_{1|2|34}^2(\rho) + 2C_{1|3|24}^2(\rho) + 2C_{1|4|23}^2(\rho) + 2C_{12|3|4}^2(\rho) + 2C_{13|2|4}^2(\rho) + 2C_{14|2|3}^2(\rho) \\ &\quad + C_{12|34}^2(\rho) + C_{13|24}^2(\rho) + C_{14|23}^2(\rho)\big)^{\frac{1}{2}}, \end{aligned} \tag{9}$$

where the Cauchy-Schwarz inequality $(\sum_j(\sum_i y_{ij})^2)^{\frac{1}{2}} \leq \sum_i(\sum_j y_{ij}^2)^{\frac{1}{2}}$ has been used in the second inequality. $\square$

For a mixed quantum state $\rho \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$, a lower bound of $C_4^2(\rho)$ has been derived based on bipartite partitions in Ref. [17]. By using the following relation [17],

$$C^2_{i|j|kl}(\rho) \geq \frac{1}{2}\left(C^2_{i|jkl}(\rho) + C^2_{ij|kl}(\rho) + C^2_{ikl|j}(\rho)\right), \tag{10}$$

from (4) we have

$$\begin{aligned} C_4^2(\rho) \ &\geq \ \frac{1}{12}\big(2C^2_{1|2|34}(\rho) + 2C^2_{1|3|24}(\rho) + 2C^2_{1|4|23}(\rho) + 2C^2_{12|3|4}(\rho) + 2C^2_{13|2|4}(\rho) + 2C^2_{14|2|3}(\rho) \\ &+ \ C^2_{12|34}(\rho) + C^2_{13|24} + C^2_{14|23}(\rho)\big) \\ &\geq \ \frac{1}{4}\big(C^2_{1|234}(\rho) + C^2_{2|134}(\rho) + C^2_{3|124}(\rho) + C^2_{4|123}(\rho) \\ &+ \ C^2_{12|34}(\rho) + C^2_{13|24}(\rho) + C^2_{14|23}(\rho)\big) = \Delta, \end{aligned} \tag{11}$$

where Δ is the lower bound obtained in [17]. Hence, our bound (4) is better than the lower bound in [17] for four-partite quantum mixed states.

III. ANALYTICAL LOWER BOUND FOR $2 \otimes 2 \otimes 2 \otimes 2$ MIXED STATES

Let H_A, H_B and H_C be $2, 2$ and 4-dimensional Hilbert spaces associated with the systems A, B and C, respectively. A pure state $|\varphi\rangle \in H_A \otimes H_B \otimes H_C$ has the form

$$|\varphi\rangle = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{3} a_{ijk}|ijk\rangle, \tag{12}$$

where $a_{ijk} \in \mathbb{C}$, $\sum_{ijk}|a_{ijk}|^2 = 1$, $\{|ijk\rangle\}$ is the basis of $H_A \otimes H_B \otimes H_C$. The concurrence of $|\varphi\rangle$ can be equivalently written as [6],

$$C_3(|\varphi\rangle) = \sqrt{\frac{1}{2}\sum(|a_{ijk}a_{pqm} - a_{ijm}a_{pqk}|^2 + |a_{ijk}a_{pqm} - a_{iqk}a_{pjm}|^2 + |a_{ijk}a_{pqm} - a_{pjk}a_{iqm}|^2)}. \tag{13}$$

To evaluate $C_3(\rho)$, we project $2 \otimes 2 \otimes 4$ dimensional states to $2 \otimes 2 \otimes 2$ sub-states. For a given $2 \otimes 2 \otimes 4$ pure state, we define its "$2 \otimes 2 \otimes 2$" pure state $|\varphi\rangle_{2\otimes2\otimes2} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k\in\{k_1,k_2\}} a_{ijk}|ijk\rangle$, where $\{k_1, k_2\} \in \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}\}$. In fact, for any $2 \otimes 2 \otimes 4$ pure state $|\varphi\rangle$, there are 6 different $2 \otimes 2 \otimes 2$ substates with respect to $|\varphi\rangle$. Without causing confusion, in the following we simply use $|\varphi\rangle_{2\otimes2\otimes2}$ to denote one of such states, as these substates will always be considered together. The concurrence $C(|\varphi\rangle_{2\otimes2\otimes2})$ is similarly given by Eq. (13), with the subindices i and j, associated with the systems A and B respectively, running from 0 to 1, and with the subindex k associated with the system C taking values k_1 and k_2

Correspondingly, for a mixed state ρ, we define its $2\otimes2\otimes2$ mixed (unnormalized) substates $\rho_{2\otimes2\otimes2}$. The concurrence of $\rho_{2\otimes2\otimes2}$ is defined by $C(\rho_{2\otimes2\otimes2}) = \min\sum_i p_i C(|\phi_i\rangle)$, minimized over all possible $2\otimes2\otimes2$ pure-state decompositions of $\rho_{2\otimes2\otimes2} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, with$\sum_i p_i = Tr(\rho_{2\otimes2\otimes2})$. The $2 \otimes 2 \otimes 2$ submatrices $\rho_{2\otimes2\otimes2}$ have the following form,

$$\rho_{2\otimes2\otimes2} = \begin{pmatrix} \rho_{00k_1,00k_1} & \rho_{00k_1,00k_2} & \rho_{00k_1,01k_1} & \rho_{00k_1,01k_2} & \rho_{00k_1,10k_1} & \rho_{00k_1,10k_2} & \rho_{00k_1,11k_1} & \rho_{00k_1,11k_2} \\ \rho_{00k_2,00k_1} & \rho_{00k_2,00k_2} & \rho_{00k_2,01k_1} & \rho_{00k_2,01k_2} & \rho_{00k_2,10k_1} & \rho_{00k_2,10k_2} & \rho_{00k_2,11k_1} & \rho_{00k_2,11k_2} \\ \rho_{01k_1,01k_1} & \rho_{01k_1,00k_2} & \rho_{01k_1,01k_1} & \rho_{01k_1,01k_2} & \rho_{01k_1,10k_1} & \rho_{01k_1,10k_2} & \rho_{01k_1,11k_1} & \rho_{01k_1,11k_2} \\ \rho_{01k_2,00k_1} & \rho_{01k_2,00k_2} & \rho_{01k_2,01k_1} & \rho_{01k_2,01k_2} & \rho_{01k_2,10k_1} & \rho_{01k_2,10k_2} & \rho_{01k_2,11k_1} & \rho_{01k_2,11k_2} \\ \rho_{10k_1,00k_1} & \rho_{10k_1,00k_2} & \rho_{10k_1,01k_1} & \rho_{10k_1,01k_2} & \rho_{10k_1,10k_1} & \rho_{10k_1,10k_2} & \rho_{10k_1,11k_1} & \rho_{10k_1,11k_2} \\ \rho_{10k_2,00k_1} & \rho_{10k_2,00k_2} & \rho_{10k_2,01k_1} & \rho_{10k_2,01k_2} & \rho_{10k_2,10k_1} & \rho_{10k_2,10k_2} & \rho_{10k_2,11k_1} & \rho_{10k_2,11k_2} \\ \rho_{11k_1,00k_1} & \rho_{11k_1,00k_2} & \rho_{11k_1,01k_1} & \rho_{11k_1,01k_2} & \rho_{11k_1,10k_1} & \rho_{11k_1,10k_2} & \rho_{11k_1,11k_1} & \rho_{11k_1,11k_2} \\ \rho_{11k_2,00k_1} & \rho_{11k_2,00k_2} & \rho_{11k_2,01k_1} & \rho_{11k_2,01k_2} & \rho_{11k_2,10k_1} & \rho_{11k_2,10k_2} & \rho_{11k_2,11k_1} & \rho_{11k_2,11k_2} \end{pmatrix}, \tag{14}$$

where $0 \leq k_1 < k_2 \leq 3$ associated to the space H_C.

Theorem 2: For any $2 \otimes 2 \otimes 4$ tripartite mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C_3^2(\rho) \geq \frac{1}{3} \sum C_3^2(\rho_{2\otimes 2\otimes 2}), \tag{15}$$

where $\sum$ stands for summing over all possible $2 \otimes 2 \otimes 2$ mixed sub-states $\rho_{2\otimes 2\otimes 2}$.

[Proof]. From the expression of concurrence (13), it is straightforward to prove that the concurrence of pure state $|\varphi\rangle$ and the concurrence of $|\varphi\rangle_{2\otimes 2\otimes 2}$ with respect to $|\varphi\rangle$ have the following relation,

$$C_3^2(|\varphi\rangle) \geq \sum \frac{1}{3} C_3^2(|\varphi\rangle_{2\otimes 2\otimes 2}). \tag{16}$$

Therefore for mixed state $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$, we have

$$\begin{aligned}
C_3(\rho) &= \min \sum_i p_i C_3(|\varphi_i\rangle) \\
&\geq \min \sqrt{\frac{1}{3}} \sum_i p_i \left(\sum C_3^2(|\varphi_i\rangle_{2\otimes 2\otimes 2})\right)^{\frac{1}{2}} \\
&\geq \min \sqrt{\frac{1}{3}} \left[\sum \left(\sum_i p_i C_3(|\varphi_i\rangle_{2\otimes 2\otimes 2})\right)^2\right]^{\frac{1}{2}} \\
&\geq \sqrt{\frac{1}{3}} \left[\sum \left(\min \sum_i p_i C_3(|\varphi_i\rangle_{2\otimes 2\otimes 2})\right)^2\right]^{\frac{1}{2}} \\
&= \sqrt{\frac{1}{3}} \left[\sum C_3^2(\rho_{2\otimes 2\otimes 2})\right]^{\frac{1}{2}},
\end{aligned}$$

where the relation $(\sum_j (\sum_i x_{ij})^2)^{\frac{1}{2}} \leq \sum_i (\sum_j x_{ij}^2)^{\frac{1}{2}}$ has been used in the second inequality, the first three minimizations run over all possible pure state decompositions of the mixed state ρ, while the last minimization runs over all $2 \otimes 2 \otimes 2$ pure state decompositions of $\rho_{2\otimes 2\otimes 2}$ associated with ρ. □

According to Theorem 1 and Theorem 2, we have the following Corollary 1:

Corollary 1: For any $2 \otimes 2 \otimes 2 \otimes 2$ mixed quantum state ρ, the concurrence $C_4(\rho)$ satisfies

$$C_4^2(\rho) \geq \frac{1}{12} \left(\sum \frac{2}{3} C_3^2(\rho_{2\otimes 2\otimes 2}) + \sum_{2\leq j\leq 4} C^2_{1j|\{1,2,3,4\}\setminus\{1,j\}}(\rho)\right), \tag{17}$$

where $\sum$ stands for summing over all possible $2 \otimes 2 \otimes 2$ mixed sub-states $\rho_{2\otimes 2\otimes 2}$ of $\rho_{i|j|kl}$, $1 \leq i < j \leq 4$, $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$).

For any four-qubit mixed quantum state ρ, Ref. [14] provided analytical lower bounds of concurrence in terms of the monogamy inequality of concurrence:

$$C_4^2(\rho) \geq \frac{1}{2} \sum_{i=1}^{3} \sum_{j>i}^{4} (T_i + T_j) C_{ij}^2(\rho),$$

where T_i $(i = 1, 2, 3, 4)$ are given in Ref. [14] and the difference of a constant factor $\frac{1}{2}$ defining the concurrence for four qubit pure states has already been taken into account. The bounds given in Corollary 1 can be used to improve the bounds of concurrence presented in [14]. Let us consider the following example.

Example: We consider the quantum state $\rho = \frac{1-t}{16} I_{16} + t|\psi\rangle\langle\psi|$, where $|\psi\rangle = (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)/2$, I_{16} is the 16×16 identity matrix.

From our Theorem 1, we need to compute the lower bounds of $C_{i|j|kl}(\rho)$ and $C_{ij|kl}(\rho)$. For convenience, we denote

$$Z_1 = \frac{(5t-1)^2}{128},$$

$$Z_2 = \frac{1}{128}(1-9t)^2(1+t)^2,$$

$$Z_3 = -\frac{1}{256}(1+t)^2\left[5(-51+4\sqrt{17})t^2+(26+4\sqrt{17})t-3\right],$$

and

$$Z_4 = \frac{3(1+t)^4}{128}\left(\sqrt{\frac{1+7t}{4(t+1)}}-3\sqrt{\frac{1-t}{4(t+1)}}\right)^2.$$

For $C(\rho_{i|j|kl})$, we use Theorem 2 and the lower bound of [15] of $2\otimes 2\otimes 2$ mixed states. We obtain the lower bound Z of $\sum_{1\leq i<j\leq 4,\{k,l\}=\{1,2,3,4\}/\{i,j\}} C^2_{i|j|kl}(\rho)$:

$$Z = \begin{cases} 2Z_2, & t\in(\frac{1}{9},0.2],\\ 32Z_1+2Z_2, & t\in(0.2,0.308051],\\ 32Z_1+Z_2+Z_3, & t\in(0.308051,1]. \end{cases}$$

For $C(\rho_{ij|kl})$, we use the lower bound of [13]. We have

$$\sum_{1<j\leq 4,\{k,l\}=\{1,2,3,4\}\backslash\{1,j\}} C^2_{1j|kl}(\rho)\geq Z_4$$

with $t\in(0.5,1]$.

Then the lower bound of $C_4^2(\rho)$ can be obtained, see Fig. 1.

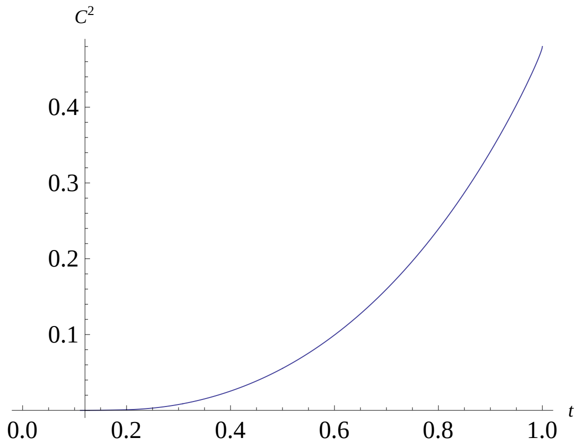

Fig. 1: Lower bound of $C_4^2(\rho)$ for $0\leq t\leq 1$.

From Fig. 1, we see that the lower bound can detect entanglement of ρ for $t>\frac{1}{9}$. From Fig. 2, we see that the our result is better than the lower bound from [17] and [14] for $t\in(\frac{1}{9},0.4)$, where the difference of a constant factor $\frac{1}{2}$ in defining the concurrence for four qubit pure states has already been taken into account.

By generalizing our results to arbitrary dimensional n-partite systems, we have

Corollary 2: For any N-partite arbitrary dimensional mixed state $\rho\in H_1\otimes H_2\otimes\cdots\otimes H_N$,

$$C_N^2(\rho)\geq\sum_{m=2}^{M}q_m\sum_i C^2_{i_1^1 i_2^1\ldots i_{k_1}^1|i_1^2 i_2^2\ldots i_{k_2}^2|\ldots|i_1^m i_2^m\ldots i_{k_m}^m}(\rho), \tag{18}$$

where $2\leq M\leq N-1$, $\sum_{i=1}^m k_m=N$, $\{i_1^1,...,i_{k_1}^1,...,i_1^m i_2^m...i_{k_m}^m\}=\{1,2,...,n\}$ and q_m is a fixed number depending on $k_1,k_2,\cdots,k_m$.

Corollary 2 says that the lower bound of the concurrence of an N-partite quantum state can be expressed by the concurrences of its $2,3,...,N-1$-partite substates.

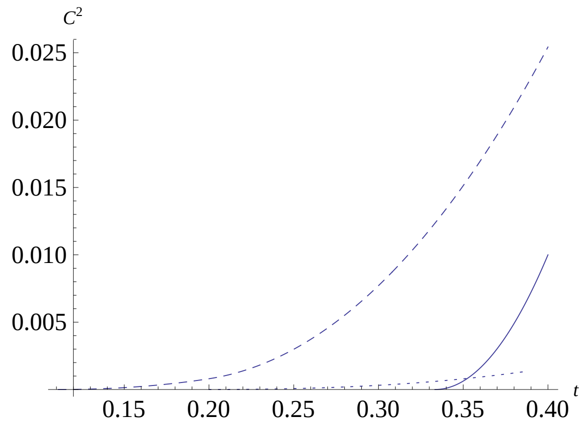

Fig. 2: Dashed line for the bound (4), dotted line for that from Ref. [17], solid line for that from Ref. [14].

IV. CONCLUSIONS AND REMARKS

In summary, we have proposed a new approach in constructing hierarchy of lower bounds of concurrence for mixed multipartite quantum states in terms of the less part decomposed quantum systems. Besides, our approach can be generalized to N part systems to obtain the lower bound of the concurrence for $M\,(2 \leq M \leq N-1)$ part systems.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki Rev. Mod. Phys. **81**, 865(2009).
[2] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[3] X. F. Qi, T. Gao, and F. L. Yan, Quantum Inf Process **16**, 23 (2017).
[4] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).
[5] P. Rungta, V. Bužek, C. M. Caves, M. Hillery and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).
[6] S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclass. Opt. 3, 223 (2001).
[7] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. **92**, 167902 (2004).
[8] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. **95**, 040504 (2005).
[9] H. P. Breuer, J. Phys. A: Math. Gen. **39**, 11847 (2006).
[10] H. P. Breuer, Phys. Rev. Lett. **97**, 080501 (2006).
[11] J. I. de Vicente, Phys. Rev. A **75**, 052320 (2007).
[12] C. J. Zhang, Y. S. Zhang, S. Zhang, and G. C. Guo, Phys. Rev. A **76**, 012334 (2007).
[13] M. J. Zhao, X. N. Zhu and S. M. Fei, Phys. Rev. A **84**, 062322 (2011).
[14] X. N. Zhu and S.M. Fei, Quantum Inf Process (2014) 13:815-823.
[15] X. N. Zhu, M. J. Zhao, and S. M. Fei, Phys. Rev. A **86** , 022307 (2012).
[16] A.R.R. Carvalho, F. Mintert, A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).
[17] M. Li, S. M. Fei, X. Q. Li-Jost and H. Fan, Phys. Rev. A **92**, 062338 (2015).