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by

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The wave-particle duality demonstrates a competition relation between wave and particle behavior for a particle going through an interferometer. This duality can be formulated as an inequality, which upper bounds the sum of interference visibility and path information. However, if the particle is entangled with a quantum memory, then the bound may decrease. Here, we find the duality relation between coherence and path information for a particle going through a multipath interferometer in the presence of a quantum memory, offering an upper bound on the duality relation which is directly connected with the amount of entanglement between the particle and the quantum memory.

I. INTRODUCTION

Quantum coherence, defined as the degree of superposition in a given reference basis, can be used to characterize the quantumness in a single system, and plays an important role in a variety of applications, ranging from metrology [1] to thermodynamics [2, 3]. Recently, the development of a resource theory of coherence has attracted much attention [4–10]. One of the main advantages that a resource theory offers is the lucid quantitative and operational description at ones disposal. In order to quantify the amount of coherence, two coherence measures have been proposed, namely, l_1 norm of coherence and relative entropy of coherence [4]. As coherence can be used to characterize the wave behavior of a particle, here we investigate the duality relation between coherence and path information for a particle going through a multipath interferometer with the access to a quantum memory, where the coherence is quantified by l_1 norm and relative entropy of coherence.

The wave-particle duality illustrates that a particle can exhibit both wave and particle behavior when it goes through an interferometer. A number of quantitative formulations for this duality have been proposed [11–20]. One well-known duality relation for a two-path interferometer is given in [13, 14] as follows,

$$D^2 + V^2 \leq 1, \tag{1}$$

where particle behavior is quantified by the path information (or path distinguishability) D and wave behavior is quantified by the interference visibility V. This tradeoff relation shows that the path information D will give a limitation on the interference visibility V and vice versa. Besides, the connection between wave-particle duality and Heisenberg' uncertainty principle has been investigated [21–24], where Heisenberg' uncertainty principle demonstrates that the complementary observables cannot be measured precisely at the same time. The equivalence between these two concepts in certain formulation, where the particle and wave behavior are captured by the measurements on the so-called particle and wave observables, illustrates the significance of wave-particle duality in the operational tasks [24].

The wave-particle duality for multipath interferometers is first investigated in [25] in terms of the density matrix of the particle represented in the path basis. In a similar scenario, the duality relations between path coherence and path information have been proposed in terms of the coherence measures in the resource theory of coherence [26–28]. For a given reference basis $\{\,|\,i\rangle\}_{i=1}^{N}$, l_1 norm of coherence is defined as $C_{l_1}(\rho)=\sum_{i\neq j}|\rho_{ij}|$ with $\rho_{ij}=\langle i|\rho|\,j\rangle$ and relative entropy of coherence is defined as $C_r(\rho)=S(\rho^{(\mathrm{d})})-S(\rho)$ where $S(\rho)=-\operatorname{Tr}[\rho\log\rho]$ is the von Neumann entropy and $\rho^{(\mathrm{d})}=\sum_i\rho_{ii}|i\rangle\langle i|$ is the diagonal part of ρ [4]. The wave behavior of the particle is quantified by these two coherence measures. To detect path information, detectors are used to interact with the particle and the path information is quantified by the discrimination of detector states [26, 27]. It has been proved that the sum of path coherence and the optimal success probability of discriminating detector states (by unambiguous state discrimination) is less than or equal to one [26]. Later an improved duality relation between coherence and path information in N-path interferometer is proved, with the path information quantified by the ambiguous (or minimal error) state discrimination,

$$(P_s-\frac{1}{N})^2+X^2\leq(1-\frac{1}{N})^2, \tag{2}$$

where P_s is the optimal success probability to discriminate detector states by ambiguous state discrimination and X is the normalized l_1 norm coherence of the particle defined by $X=\frac{1}{N}C_{l_1}(\rho)$ [27]. The quantity P_s-1/N quantifies the advantage to discriminate detector states by prior knowledge compared with random guessing [27].

Equation (2) bounds the duality between coherence and path information without quantum memory. However, this bound may decrease if the particle is entangled with a quantum memory. Here, we show a duality relation between coherence and path information for a particle A going through an N-path interferometer in the presence of a quantum memory B (see Fig.1), which provides an upper bound on the sum of the coherence and path information that depends on the amount of entanglement between the particle A and the quantum memo-

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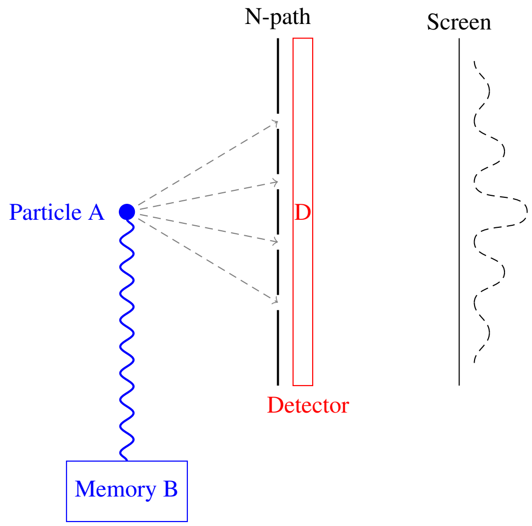

FIG. 1. Schematic diagram of an N-path interferometer. The particle A goes through an N-path interferometer while A is entangled with a quantum memory B. The detector D is used to detect which path the particle A goes through.

ry B,

$$\left(P_s^A-\tfrac{1}{N}\right)^2+X_A^2 \;\leq\; \left(1-\tfrac{1}{N}\right)^2 + \tfrac{2(N-1)}{N^2}\left(\mathrm{Tr}\left[\rho_A^2\right]-\mathrm{Tr}\left[\rho_{AB}^2\right]\right). \tag{3}$$

The extra term $\mathrm{Tr}\left[\rho_A^2\right]-\mathrm{Tr}\left[\rho_{AB}^2\right]$ on the right-hand side of (3) quantifies the amount of entanglement between particle A and the memory B, as

$$\mathrm{Tr}\left[\rho_A^2\right]-\mathrm{Tr}\left[\rho_{AB}^2\right]<0 \tag{4}$$

only if ρ_{AB} is entangled [29]. Inequality (4) provides a powerful tool in experimental detection of entanglement [30, 31]. Moreover, for $N=2$, the relation (3) becomes an equality.

Let us begin with a pure bipartite state $|\psi\rangle_{AB}$ between particle A and quantum memory B. Note that we can always consider a pure bipartite state between particle A and a lager quantum memory B′ even if the initial bipartite state between A and B is not pure.

In an N-path interferometer, if the orthonormal basis states $\{\,|\,i\rangle\,\}_{i=1}^N$ correspond to the N possible slits or paths, then the state of the particle A after crossing the slit can be described in terms of $\{\,|\,i\rangle\,\}_{i=1}^N$, and thus the bipartite pure state $|\psi\rangle_{AB}$ can be represented as $|\psi\rangle_{AB}=\sum_{i=1}^N\sum_{j=1}^{d_B}a_{ij}\,|i\rangle_A\,|j\rangle_B$, with $a_{ij}\in\mathbb{C}$, $\sum_{i,j}|a_{ij}|^2=1$ and d_B being the dimension of quantum memory B. Denote $p_i=\sum_j|a_{ij}|^2$ and $|u_i\rangle_B=\sum_j a_{ij}\,|j\rangle_B/\sqrt{p_i}$. $|\psi\rangle_{AB}$ can also be written as

$$|\psi\rangle_{AB}=\sum_{i=1}^N\sqrt{p_i}\,|i\rangle_A\,|u_i\rangle_B\,, \tag{5}$$

where p_i is the probability to take the ith path and $|u_i\rangle_B$ is the normalized pure state on memory B for any $i\in\{1,\ldots,N\}$. Here $|u_i\rangle_B$ are not necessary orthogonal. In order to detect particle A, another quantum system called detector interacts with the particle A inside the interferometer, and the interaction is described as the controlled unitary $U(|i\rangle_A\,|\phi_0\rangle_D)=|i\rangle_A\,|\phi_i\rangle_D$ with $|\phi_0\rangle_D$ being the initial state of the detector. After the interaction, the state of the whole system becomes

$$|\Psi\rangle_{ABD}=\sum_{i=1}^N\sqrt{p_i}\,|i\rangle_A\,|u_i\rangle_B\,|\phi_i\rangle_D\,. \tag{6}$$

Thus the reduced density matrix of the combined system A,B is

$$\begin{aligned}\rho_{AB}&=\mathrm{Tr}_D\left[|\Psi\rangle\langle\Psi|_{ABD}\right]\\&=\sum_{i,j=1}^N\sqrt{p_ip_j}\langle\phi_j|\phi_i\rangle\,|i\rangle\,\langle j|_A\otimes|u_i\rangle\,\langle u_j|_B\,,\end{aligned} \tag{7}$$

and the reduced density matrix of the particle A is

$$\begin{aligned}\rho_A&=\mathrm{Tr}_{BD}\left[|\Psi\rangle\langle\Psi|_{ABD}\right]\\&=\sum_{i,j=1}^N\sqrt{p_ip_j}\langle\phi_j|\phi_i\rangle\langle u_j|u_i\rangle\,|i\rangle\,\langle j|_A\,.\end{aligned} \tag{8}$$

Hence the normalized l_1 norm of coherence measure X_A is

$$X_A=\frac{1}{N}C_{l_1}(\rho_A)=\frac{1}{N}\sum_{\substack{i,j=1\\i\neq j}}^N\sqrt{p_ip_j}|\langle\phi_j|\phi_i\rangle||\langle u_j|u_i\rangle|. \tag{9}$$

Besides, the path information is stored in the detector states and the density matrix of the detector can be expressed as

$$\rho_D=\mathrm{Tr}_{AB}\left[|\Psi\rangle\langle\Psi|_{ABD}\right]=\sum_{i=1}^N p_i|\phi_i\rangle\langle\phi_i|_D. \tag{10}$$

In order to obtain the path information, we need perform state discrimination on the set of quantum states $\{\,|\,\phi_i\rangle_D\,\}_{i=1}^N$ with the probability $\{\,p_i\,\}_{i=1}^N$. For a positive operator valued measure (POVM) $\{\Pi_i\}_{i=1}^N$ with $\Pi_i\geq 0$ and $\sum_i\Pi_i=\mathbb{I}$, the success probability to discriminate the states $\{\,|\,\phi_i\rangle_D\,\}$ is $\sum_{i=1}^N p_i\,\mathrm{Tr}\left[\Pi_i|\phi_i\rangle\langle\phi_i|\right]$. Thus the optimal success probability among all POVMs is

$$P_s^A=\max_{\{\Pi_i\}}\sum_{i=1}^N p_i\,\mathrm{Tr}\left[\Pi_i|\phi_i\rangle\langle\phi_i|\right]. \tag{11}$$

For ambiguous state discrimination, an upper bound for the optimal success probability P_s^A is given by [27],

$$P_s^A\leq\frac{1}{N}+\frac{1}{2N}\sum_{i,j=1}^N\left\|T_{ij}\right\|_1, \tag{12}$$

where the operators $\{T_{ij}\}$ are given by $T_{ij}=p_i|\phi_i\rangle\langle\phi_i|-p_j|\phi_j\rangle\langle\phi_j|$ and trace norm of T_{ij} for $i\neq j$ is given by

$$\left\|T_{ij}\right\|_1=2\sqrt{\left(\frac{p_i+p_j}{2}\right)^2-p_ip_j|\langle\phi_i|\phi_j\rangle|^2}.$$

Here, the quantum memory B is not involved in the discrimination of the detector states, as the quantum memory may be far away from the interferometer and one may not be able to perform a joint measurement on both the memory B and the detector D. Note that, if the joint measurement on B and D is allowed, then the memory B would be able to increase the optimal success probability to discriminate states and the relation (3) will reduce to (2) following the method in [27].

The bound (12) gives an upper bound for the sum of path information $\left(P_s^A-1/N\right)^2$ and the square of the normalized l_1 norm of coherence X_A^2 for particle A,

$$\left(P_s^A-\frac{1}{N}\right)^2+X_A^2\leq\frac{1}{N^2}\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{1}{2}\left\|T_{ij}\right\|_1\right)^2+\frac{1}{N^2}\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\sqrt{p_ip_j}|\langle\phi_i|\phi_j\rangle||\langle u_i|u_j\rangle|\right)^2. \quad (13)$$

Due to the Schwarz inequality, we can get some upper bounds for the two terms on the right-hand side of (13), where the trace norm of T_{ij} is upper bounded as follows,

$$\begin{aligned}&\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{1}{2}\left\|T_{ij}\right\|_1\right)^2\\&=\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\sqrt{\left(\frac{p_i+p_j}{2}\right)^2-p_ip_j|\langle\phi_i|\phi_j\rangle|^2}\right)^2\\&\leq\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{p_i+p_j}{2}\right)\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{p_i+p_j}{2}-2\frac{p_ip_j}{p_i+p_j}|\langle\phi_i|\phi_j\rangle|^2\right)\\&=(N-1)\left[(N-1)-\sum_{\substack{i,j=1\\i\neq j}}^{N}2\frac{p_ip_j}{p_i+p_j}|\langle\phi_i|\phi_j\rangle|^2\right], \quad (14)\end{aligned}$$

and the l_1 norm of coherence is upper bounded as

$$\begin{aligned}&\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\sqrt{p_ip_j}|\langle\phi_i|\phi_j\rangle||\langle u_i|u_j\rangle|\right)^2\\&\leq\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{p_i+p_j}{2}\right)\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}2\frac{p_ip_j}{p_i+p_j}|\langle\phi_i|\phi_j\rangle|^2|\langle u_i|u_j\rangle|^2\right)\\&=(N-1)\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}2\frac{p_ip_j}{p_i+p_j}|\langle\phi_i|\phi_j\rangle|^2|\langle u_i|u_j\rangle|^2\right), \quad (15)\end{aligned}$$

where the relation $\sum_{i\neq j=1}^{N}\frac{p_i+p_j}{2}=N-1$ has been taken into account. Substituting (14) and (15) into (13), we have

$$\begin{aligned}&\left(P_s^A-\frac{1}{N}\right)^2+X_A^2\\&\leq\left(1-\frac{1}{N}\right)^2-2\frac{N-1}{N^2}\sum_{\substack{i,j=1\\i\neq j}}^{N}\frac{p_ip_j}{p_i+p_j}|\langle\phi_i|\phi_j\rangle|^2\left(1-|\langle u_i|u_j\rangle|^2\right)\\&\leq\left(1-\frac{1}{N}\right)^2-2\frac{N-1}{N^2}\sum_{\substack{i,j=1\\i\neq j}}^{N}p_ip_j|\langle\phi_i|\phi_j\rangle|^2\left(1-|\langle u_i|u_j\rangle|^2\right)\\&=\left(1-\frac{1}{N}\right)^2-2\frac{N-1}{N^2}\left(\mathrm{Tr}\left[\rho_D^2\right]-\mathrm{Tr}\left[\rho_A^2\right]\right), \quad (16)\end{aligned}$$

where second inequality comes from the fact that $p_i+p_j\leq1$ and $|\langle u_i|u_j\rangle|\leq1$ for any $i\neq j$ and the last equality comes from the fact that $\mathrm{Tr}\left[\rho_D^2\right]$ and $\mathrm{Tr}\left[\rho_A^2\right]$ can be expressed as

$$\mathrm{Tr}\left[\rho_A^2\right]=\sum_{\substack{i,j=1\\i\neq j}}^{N}p_ip_j|\langle\phi_i|\phi_j\rangle|^2|\langle u_i|u_j\rangle|^2+\sum_{i=1}^{N}p_i^2, \quad (17)$$

and

$$\mathrm{Tr}\left[\rho_D^2\right]=\sum_{\substack{i,j=1\\i\neq j}}^{N}p_ip_j|\langle\phi_i|\phi_j\rangle|^2+\sum_{i=1}^{N}p_i^2. \quad (18)$$

Besides, since ρ_{AB} and ρ_D are the reduced states of the pure state $|\Psi\rangle_{ABD}$, the purity of ρ_{AB} and ρ_D are equal, $\mathrm{Tr}\left[\rho_{AB}^2\right]=\mathrm{Tr}\left[\rho_D^2\right]$. Therefore, we obtain the duality relation (3) in the presence of a quantum memory. In view of the equations (17) and (18), we find that $\mathrm{Tr}\left[\rho_A^2\right]\leq\mathrm{Tr}\left[\rho_D^2\right]$ which means $\mathrm{Tr}\left[\rho_A^2\right]\leq\mathrm{Tr}\left[\rho_{AB}^2\right]$. Thus the right-hand side of (3) is less than or equal to $(1-1/N)^2$. Furthermore, if $\mathrm{Tr}\left[\rho_A^2\right]<\mathrm{Tr}\left[\rho_{AB}^2\right]$, then ρ_{AB} is entangled and the right-hand side of (3) is strictly less than $(1-1/N)^2$. Note that if the initial bipartite state $|\psi\rangle_{AB}$ is separable, $|\psi\rangle_{AB}=|\psi'\rangle_A|u\rangle_B$ where $|u_i\rangle_B$ in (5) is equal to $|u\rangle_B$ up to a phase, then $\mathrm{Tr}\left[\rho_A^2\right]=\mathrm{Tr}\left[\rho_{AB}^2\right]$ and the relation (3) reduces to (2).

For $N=2$, the equality in (12) holds [32], that is, the optimal success probability is given by

$$P_s^A=\frac{1}{2}+\sqrt{\frac{1}{4}-p_1p_2|\langle\phi_1|\phi_2\rangle|^2},$$

and the normalized l_1 norm coherence can be written as

$$X_A=\sqrt{p_1p_2}|\langle\phi_1|\phi_2\rangle||\langle u_1|u_2\rangle|.$$

In this case we have

$$\begin{aligned}\left(P_s^A-\frac{1}{2}\right)^2+X_A^2&=\frac{1}{4}-p_1p_2|\langle\phi_1|\phi_2\rangle|^2(1-|\langle u_1|u_2\rangle|^2)\\&=\frac{1}{4}+\frac{1}{2}\left(\mathrm{Tr}\left[\rho_A^2\right]-\mathrm{Tr}\left[\rho_{AB}^2\right]\right). \quad (19)\end{aligned}$$

That is, the equality in duality relation (3) holds for two-path interferometer.

If the particle A has no quantum memory and the initial state is $|\tilde{\Psi}\rangle_A=\sum_{i=1}^{N}\sqrt{p_i}|i\rangle_A$, then after the interaction with the detector, the reduced state $\tilde{\rho}_A = \sum_{i,j}\sqrt{p_i p_j}\langle\phi_j|\phi_i\rangle\,|i\rangle\langle j|_A$ and $\tilde{\rho}_D$ is just given by the equation (10). According to [27], the coherence $\tilde{X}_A$ and the path information $\tilde{P}_s^A$ satisfy the relation (2). Compared with the case that A has a quantum memory, the difference between the bounds of (2) and (3) comes from the loss of coherence of particle in the presence of entanglement, where the amount of coherence-loss can be quantified as

$$\begin{aligned}\tfrac{1}{N^2}\left(\mathrm{Tr}\left[\rho_{AB}^2\right]-\mathrm{Tr}\left[\rho_A^2\right]\right) &\le \Delta X_A^2\\ &\le \tfrac{2(N-1)^2}{N}\left(\mathrm{Tr}\left[\rho_{AB}^2\right]-\mathrm{Tr}\left[\rho_A^2\right]\right),\end{aligned} \tag{20}$$

where $\Delta X_A^2 = \tilde{X}_A^2 - X_A^2$ is the amount of coherence-loss, ρ_{AB} and ρ_A are given by (7) and (8) respectively. The first inequality comes from the fact that

$$\begin{aligned}&C_{l_1}^2(\tilde{\rho}_A)-C_{l_1}^2(\rho_A)\\ &\ge \sum_{\substack{i,j=1\\ i\ne j}}^{N} p_i p_j|\langle\phi_i|\phi_j\rangle|^2(1-|\langle u_i|u_j\rangle|^2)\\ &= \mathrm{Tr}\left[\rho_{AB}^2\right]-\mathrm{Tr}\left[\rho_A^2\right],\end{aligned}$$

and the second inequality is due to that

$$\begin{aligned}&C_{l_1}^2(\tilde{\rho}_A)-C_{l_1}^2(\rho_A)\\ &=(C_{l_1}(\tilde{\rho}_A)+C_{l_1}(\rho_A))(C_{l_1}(\tilde{\rho}_A)-C_{l_1}(\rho_A))\\ &\le 2(N-1)(\sum_{\substack{i,j=1\\ i\ne j}}^{N}\sqrt{p_i p_j}|\langle\phi_i|\phi_j\rangle|(1-|\langle u_i|u_j\rangle|))\\ &\le 2(N-1)(\sum_{\substack{i,j=1\\ i\ne j}}^{N}\sqrt{p_i p_j}|\langle\phi_i|\phi_j\rangle|(\sqrt{1-|\langle u_i|u_j\rangle|^2}))\\ &\le 2(N-1)N(N-1)(\sum_{\substack{i,j=1\\ i\ne j}}^{N}p_i p_j|\langle\phi_i|\phi_j\rangle|^2(1-|\langle u_i|u_j\rangle|^2)\\ &\le 2(N-1)^2N(\mathrm{Tr}\left[\rho_{AB}^2\right]-\mathrm{Tr}\left[\rho_A^2\right]),\end{aligned}$$

where the third line comes from the fact that $C_{l_1}(\tilde{\rho}_A), C_{l_1}(\rho_A) \le N-1$. Hence, the loss of coherence in particle A depends on the entanglement between A and B. This provides an interpretation for the decrease of duality bound in (3): in the presence of quantum memory B, part of the coherence in particle A is encoded in the entanglement between A and B, which leads to the loss of coherence in the particle A and the decrease of the bound for duality relation between coherence and path information.

The duality relation (3) also provides a tighter bound on duality relation (2) for mixed states without quantum memory. Suppose a particle A goes through an N-path interferometer while the initial state of particle A is a mixed state ρ_A^0. The orthonormal basis states $\{\,|\,i\rangle\}_{i=1}^N$ correspond to the N possible paths. Then there exists another quantum system B such that the bipartite state between A and B can be expressed as in (5). Thus the initial density matrix of particle A is

$$\rho_A^0 = \sum_{i,j=1}^{N}\sqrt{p_i p_j}\langle u_j|u_i\rangle\,|i\rangle\langle j|_A\,.$$

After the interaction with detector, the bipartite state between A and D is given by

$$\begin{aligned}\rho_{AD} &= U\left(\rho_A^0\otimes|\phi_0\rangle\langle\phi_0|_D\right)U^\dagger\\ &=\sum_{i,j=1}^{N}\sqrt{p_i p_j}\langle u_j|u_i\rangle\,|i\rangle\langle j|_A\otimes|\phi_i\rangle\langle\phi_j|_D\,,\end{aligned}$$

where $U(|i\rangle_A|\phi_0\rangle_D) = |i\rangle_A|\phi_i\rangle_D$. Then the reduced density matrix of particle A and detector D are given by (8) and (10), respectively. Therefore, according to (16), we get the duality relation for mixed state ρ_A without quantum memory,

$$\begin{aligned}&\left(P_s^A-\frac{1}{N}\right)^2+X_A^2\\ &\le\left(1-\frac{1}{N}\right)^2+2\frac{N-1}{N^2}\left(\mathrm{Tr}\left[\rho_A^2\right]-\mathrm{Tr}\left[\rho_D^2\right]\right).\end{aligned} \tag{21}$$

Since $\mathrm{Tr}\left[\rho_A^2\right] \le \mathrm{Tr}\left[\rho_D^2\right]$, the right-hand side of (21) is less than $(1-1/N)^2$, which provides a tighter bound than that of relation (2) for mixed states. If ρ_A^0 is pure, $\mathrm{Tr}\left[\rho_A^2\right]$ and $\mathrm{Tr}\left[\rho_D^2\right]$ are equal to ρ_A and ρ_D, which are the reduced states of the pure state $\rho_{AD} = U\left(\rho_A^0\otimes|\phi_0\rangle\langle\phi_0|_D\right)U^\dagger$. Also, the relation (21) becomes an equality for N=2.

Now, let us recall an entropic version of duality relation between path coherence and the path information without quantum memory, that is, the duality relation between relative entropy coherence of the particle A and the mutual information between detector states and measurement outcomes [27],

$$I(D:M)+C_r(\rho_A)\le H(\{\,p_i\,\}), \tag{22}$$

where $H(\{\,p_i\,\}) = -\sum_i p_i\log p_i$ is the Shannon entropy, $C_r(\rho_A)$ is the relative entropy coherence of particle A, and D,M are two random variables corresponding to the detector states and the measurement outcomes of a POVM $\mathscr{M} = \{\,\Pi_i\,\}_{i=1}^N$ respectively, where the joint distribution for D,M is $p_{ij} = p(D=i, M=j) = p_i\,\mathrm{Tr}\left[|\phi_i\rangle\langle\phi_i|\Pi_j\right]$ [27]. Note that the path information is quantified by the mutual information $I(D:M)$ defined by

$$I(D:M) = H(D)+H(M)-H(D,M), \tag{23}$$

where $H(D) = H(\{\,p_i\,\})$ and $H(M) = H(\{\,q_j\,\})$ with $q_j = \sum_i p_{ij}$.

In the following, we show an entropic duality relation between coherence and path information in the presence of a quantum memory B,

$$I(D:M)+C_r(\rho_A)\le H(\{\,p_i\,\})+S(B|A), \tag{24}$$

where $I(D:M)$ is the mutual information between detector states and measurement outcomes of a POVM $\mathscr{M} = \{\,\Pi_i\,\}_{i=1}^N$ as defined in [27] and the conditional entropy $S(B|A) = S(\rho_{AB}) - S(\rho_A)$. The extra term $S(B|A)$ on the right-hand side of (24) quantifies the amount of entanglement between particle A and the memory B, as $S(B|A) < 0$ indicates the entanglement of ρ_{AB} [33].

Due to the presence of the quantum memory B, equation (6) is the state of the whole system after particle A interacts with the detector. Equations (8) and (10) are the reduced density matrix of particle ρ_A and ρ_D respectively. The relative entropy of coherence for ρ_A is given by

$$C_r(\rho_A) = S(\rho_A^{(\mathrm{d})}) - S(\rho_A) = H(\{p_i\}) - S(\rho_A). \quad (25)$$

In view of the Holevo bound, the path information $I(D:M)$ is upper bounded as

$$I(D:M) \leq S(\rho_D) - \sum_{i=1}^{N} p_i S(|\phi_i\rangle\langle\phi_i|) = S(\rho_D),$$

where the von Neumann entropy for pure state is zero. Thus,

$$I(D:M) + C_r(\rho_A) \leq H(\{p_i\}) - S(\rho_A) + S(\rho_D). \quad (26)$$

Since ρ_{AB} and ρ_D are the reduced states of the pure state $|\Psi\rangle_{ABD}$, the von Neumann entropy of ρ_{AB} and ρ_D are equal, $S(\rho_D) = S(\rho_{AB})$. Therefore, we obtain the duality relation (24) for the case of the presence of a quantum memory. If $S(B|A) < 0$, ρ_{AB} is entangled and the right-hand side of (24) is strictly less than $H(\{p_i\})$. That is, it provides a tighter bound than (22) in this case. Also, if the initial bipartite state $|\psi\rangle_{AB}$ between A and B is separable, $|\psi\rangle_{AB} = |\psi'\rangle_A |u\rangle_B$, where $|u_i\rangle$ in (5) is equal to $|u\rangle$ up to a phase, then the whole state of A,B and D after the interaction between particle A and D is of the form, $|\Psi\rangle_{ABD} = U(|\psi'\rangle_A |\phi_0\rangle_D) \otimes |u\rangle_B$. Thus $S(\rho_A) = S(\rho_{AB})$ and the duality relation (24) reduces to the relation (22). Besides, as the accessible information is defined as $Acc(D) = \max_{POVM\,\mathscr{M}} H(D:M)$ and the duality relation holds for any POVM on detector state ρ_D, we obtain the following relation between the accessible information and relative entropy of coherence in the presence of quantum memory,

$$Acc(D) + C_r(\rho_A) \leq H(\{p_i\}) + S(B|A). \quad (27)$$

For $N = 2$, the von Neumman entropies of ρ_A and ρ_{AB} can be analytically calculated,

$$S(\rho_{AB}) = -\left(\frac{1}{2}+\lambda_1\right)\log\left(\frac{1}{2}+\lambda_1\right) - \left(\frac{1}{2}-\lambda_1\right)\log\left(\frac{1}{2}-\lambda_1\right),$$
$$S(\rho_A) = -\left(\frac{1}{2}+\lambda_2\right)\log\left(\frac{1}{2}+\lambda_2\right) - \left(\frac{1}{2}-\lambda_2\right)\log\left(\frac{1}{2}-\lambda_2\right),$$

where

$$\lambda_1 = \left(\frac{p_1-p_2}{2}\right)^2 + p_1p_2|\langle\phi_1|\phi_2\rangle|^2,$$
$$\lambda_2 = \left(\frac{p_1-p_2}{2}\right)^2 + p_1p_2|\langle\phi_1|\phi_2\rangle|^2|\langle u_1|u_2\rangle|^2.$$

As $\lambda_2 \leq \lambda_1$, we have $S(\rho_A) \geq S(\rho_{AB})$ or $S(B|A) \leq 0$. Hence the right-hand side of (24) is less than $H(\{p_i\})$ for two-path interferometer.

Similar to the case of l_1 norm measure, the duality relation (24) with quantum memory also gives a tighter bound for mixed states without quantum memory. The duality relation for mixed states without quantum memory is described by the equation (26), with the reduced density matrices of particle A and detector D given by (8) and (10), respectively. It is easy to see that $S(\rho_A) \geq S(\rho_D)$ for $N = 2$.

As another interesting scenario, we may also consider two entangled particles A and B, such that A goes through an N-path interferometer and B goes through another. Then coherence and path information of A and B both satisfy the relation (3). We have

$$\left(P_s^A - \frac{1}{N}\right)^2 + \left(P_s^B - \frac{1}{N}\right)^2 + X_A^2 + X_B^2$$
$$\leq 2\left(1-\frac{1}{N}\right)^2 + \frac{2(N-1)}{N^2}\left(\mathrm{Tr}\left[\rho_A^2\right] + \mathrm{Tr}\left[\rho_B^2\right] - 2\,\mathrm{Tr}\left[\rho_{AB}^2\right]\right),$$

where ρ_A, ρ_B and ρ_{AB} are the reduced states of A, B and the system AB after the interaction with the individual detectors. The relation (28) becomes an inequality for $N = 2$. Note that in this scenario, the relation between the single-particle visibility and the mutual information of the two particles has also been investigated [34].

In conclusion, we have obtained two duality relations between path information and coherence for a particle going through a multipath interferometer in the presence of a quantum memory, for both coherence quantifier l_1 norm of coherence and relative entropy of coherence. We have shown that the entanglement between the particle and the quantum memory will lower down the upper bounds of these duality relations, due to the decrease of coherence in the presence of entanglement. Moreover, our bonds for wave-particle duality relations with quantum memory also provide the corresponding bonds for particles in mixed initial states without quantum memory. These results provide a new insight into the wave-particle duality and reveal the role of quantum entanglement in the wave-particle duality.

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