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Oset Hypersurfaces and Persistent Homology of Algebraic Varieties

by

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Abstract

In this paper, we study the true persistent homology of algebraic varieties. We prove the algebraicity of two quantities central to the computation of persistent homology. Moreover, we connect persistent homology and algebraic optimization (Euclidean Distance Degree). Namely, we express the degree corresponding to the distance variable of the offset hypersurface in terms of the Euclidean Distance degree of the starting variety, obtaining a new way to compute these degrees. Finally, we describe the non-properness locus of the offset construction and use this to describe the set of points that are topologically interesting (the medial axis and center points of the bounded components of the complement of the variety) and relevant to the computation of persistent homology.

1. Introduction

Experimental research is based on collecting and analyzing data. It is very important to understand the background mathematical model that defines a given phenomenon. One of the possibilities is that the data is driven by a geometric model, say an algebraic variety or a manifold. In this case, we would like to "learn the geometric object" from the data (for more details see [**?**]). For example, we would like to understand the topological features of the underlying model. A classical way to do this is by *persistent homology* ([**?**, **?**, **?**]), which studies the homology of the set of points within a range of distances from the data set, and considers features to be of interest if they persist through a wide range of the distance parameter.

This article is at the intersection of computational geometry, geometric design, topology and algebraic geometry, linking all of these topics together. In what follows, we study the *true persistent homology* of the underlying variety, which we define to be the homology of its offsets. We show that the true persistent homology indicators (the barcodes) of a variety defined over the rational numbers are algebraic and thus can be computed exactly (Theorem **??**). Moreover, we connect persistent homology and algebraic optimization (Euclidean Distance Degree [**?**]) in a way that brings insights from each field to the other. Namely, we express the degree corresponding to the distance variable of the offset hypersurface in terms of the Euclidean Distance degree of the starting variety (Theorem **??**), obtaining a new way to compute these degrees. A consequence of this result is a bound on the degree of the *ED discriminant* (Corollary **??**). We describe the non-properness locus of the offset construction (Subsection 2.1) and use this to describe the set of points (Theorem **??**) in the ambient space that are topologically interesting (the medial axis and center points of the bounded components of the complement of the variety) and relevant to the computation of persistent homology. Lastly, we show that the reach of a manifold, the quantity used to ensure the correctness of persistent homology computations, is algebraic (Proposition **??**).

The article is structured as follows. Section **??** discusses offset hypersurfaces. We analyze the construction, dimension and degree of offsets and define the offset discriminant. Section **??** is about persistent homology. We review background material on persistent homology, define the true persistent homology of an algebraic variety and prove its algebraicity, connect the offset discriminant to topologically interesting points in the complement of the variety, and discuss the algebraicity of the reach.

2. Offset hypersurfaces of algebraic varieties

We devote this section to the algebraic study of offset hypersurfaces. Driven by real world applications, our starting variety $X_{\mathbb{R}} \subseteq \mathbb{R}^n$ is a real irreducible variety and we construct its ϵ-offset hypersurface, for any generic real positive ϵ. In order to be able to use techniques from algebraic geometry, we consider the variety $X \subseteq \mathbb{C}^n$ that is the complexification of $X_{\mathbb{R}}$ and let ϵ be any complex number. In what follows by the squared distance of two points $x, y \in \mathbb{C}^n$ we will mean the complex value of the function $d(x,y) = \sum_{i=1}^n (x_i - y_i)^2$. This is not the usual Hermitian distance function on $\mathbb{C}^n$, but rather the complexification of the real Eucliean distance function. It is not a metric on $\mathbb{C}^n$, but it is a metric when restricted to $\mathbb{R}^n$.

2.1. Offset construction

Let $X \subseteq \mathbb{C}^n$ be an irreducible variety of codimension c and let ϵ be a fixed (generic) complex number. By an ϵ-hyperball centered at a point $y \in \mathbb{C}^n$ we mean the variety $V(d(x,y) - \epsilon^2)$.

Definition 2.1. The **ϵ-offset hypersurface** is defined to be the union of the centers of ϵ-hyperballs that intersect the variety X non-transversally at some point $x \in X$. Equivalently the ϵ-offset hypersurface is the envelope of a family of ϵ-hyperballs centered on the variety. For a fixed ϵ we denote the ϵ-offset hypersurface by $\mathcal{O}_\epsilon(X)$.

Let $y \in \mathcal{O}_\epsilon(X)$, the above-defined ϵ-offset hypersurface. Then there exists an $x \in X_{reg}$, that is a regular point of the variety (non-singular point), such that the squared distance $d(x,y)$ is exactly ϵ^2 and by the non-transversality $T_xX \subseteq T_xV(d(x,y) - \epsilon^2)$. Hence

$$x - y \perp T_xX,$$

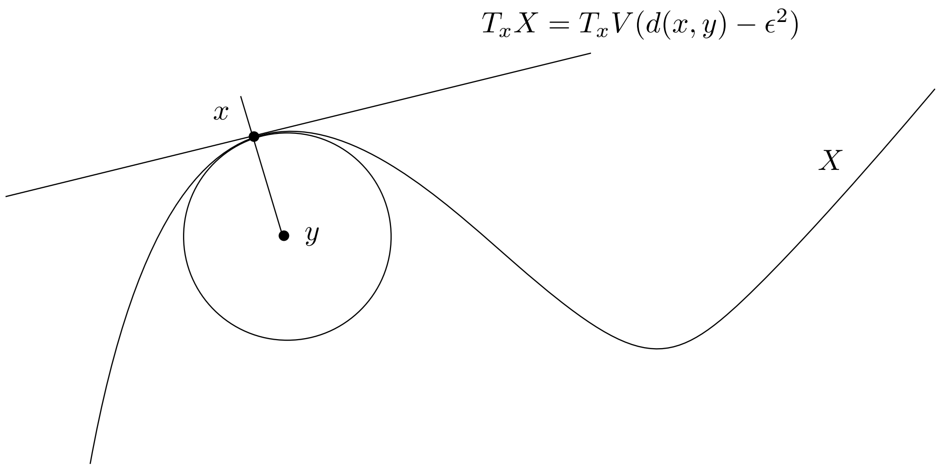

Figure 1: Non-transversal intersection of the variety with the ϵ-hyperball.

where T_xX is the tangent space at x to X and $T_xV(d(x,y)-\epsilon^2)$ is the tangent space at x to $V(d(x,y)-\epsilon^2)$, the variety defined by the vanishing of the polynomial $d(x,y)-\epsilon^2$, which is the ϵ-hyperball centered at y.

The latter condition can be described by polynomial equations as follows. The condition $x - y \perp T_xX$ is satisfied if and only if the rank of

$$\begin{pmatrix} x-y \\ \mathrm{Jac}_x(I) \end{pmatrix}$$

is less then $c+1$, where $\mathrm{Jac}_x(X)$ is the Jacobian of the defining radical ideal of the variety X, at the point x (the matrix of all the partial derivatives of all the minimally defining polynomials of X). Namely $x - y \perp T_xX$ if and only if all the $(c+1)\times(c+1)$ minors of the matrix above vanish.

To capture the entire geometry behind the construction of the offset hypersurface we consider the closure of the set of all pairs $(x,y) \in \mathbb{C}^n \times \mathbb{C}^n$ such that $x \in X_{reg}$ and y satisfies the conditions above. We name this variety the **offset correspondence** of X and denote it by $\mathcal{OC}_\epsilon(X)$. This correspondence is a variety in $\mathbb{C}^n_x \times \mathbb{C}^n_y$ and is equal to the closure of the intersection

$$(X_{reg} \times \mathbb{C}^n) \cap V\left((c+1)\times(c+1) \text{ minors of } \begin{pmatrix} x-y \\ \mathrm{Jac}_x(I) \end{pmatrix}\right) \cap V(d(x,y)-\epsilon^2).$$

Observe that the intersection of the first two varieties is equal to the Euclidean Distance Degree correspondence, $\mathcal{E}(X)$, that is the closure of the pair of points (x,y) in $\mathbb{C}^n_x \times \mathbb{C}^n_y$, such that $x \in X_{reg}$ and $x - y \perp T_xX$. This correspondence contains pairs of "data points" $y \in \mathbb{C}^n_y$ and corresponding points on the variety $x \in X_{reg}$, such that x is a constrained critical point of the Euclidean distance function $d_y(x) = d(x,y)$ with respect to the constraint that $x \in X_{reg}$. For more details on this problem we direct the reader to [**?**, Section 2]. Using the terminology of the Euclidean distance degree problem, we have

$$\mathcal{OC}_\epsilon(X) = \mathcal{E}(X) \cap V(d(x,y) - \epsilon^2).$$

From the offset correspondence, we have the natural projections $\mathrm{pr}_x : \mathcal{OC}_\epsilon(X) \to \mathbb{C}^n_x$ and $\mathrm{pr}_y : \mathcal{OC}_\epsilon(X) \to \mathbb{C}^n_y$. The closure of the first projection is the variety X and the closure of the second projection is the offset hypersurface $\mathcal{O}_\epsilon(X)$.

It follows that the **offset hypersurface** is

$$\mathcal{O}_\epsilon(X) = \overline{\mathrm{pr}_y(\mathcal{OC}_\epsilon(X))} \subseteq \mathbb{C}^n_y.$$

Remark 2.2. When X is a real variety, note that by the Tarski-Seidenberg theorem ([**?**, Theorem 1.4.2]) the offset hypersurface is defined over the same closed real (sub)field as X and ϵ are defined.

In the following example, we illustrate an algorithm to compute the defining polynomial of the offset hypersurface of an ellipse using `Macaulay2` [**?**].

Example 2.3 (**Computing the offset hypersurface of the ellipse**). Consider the ellipse $X \subseteq \mathbb{C}^2$ defined by the vanishing of the polynomial $f = x_1^2 + 4x_2^2 - 4$. The code below will output the defining ideal of the offset hypersurface, with ϵ a parameter.

```
n=2;
kk=QQ[x_1..x_n,y_1..y_n,e];
f=x_1^2+4*x_2^2-4;
I=ideal(f);
c=codim I;
Y=matrix{{x_1..x_n}}-matrix{{y_1..y_n}};
Jac= jacobian gens I;
S=submatrix(Jac,{0..n-1},{0..numgens(I)-1});
Jbar=S|transpose(Y);
EX = I + minors(c+1,Jbar);
SingX=I+minors(c,Jac);
EXreg=saturate(EX,SingX);
distance=Y*transpose(Y)-e^2;
Offset_Correspondence=EXreg+ideal(distance);
Off_hypersurface=eliminate(Offset_Correspondence,toList(x_1..x_n))
```

The result is that $\mathcal{O}_\epsilon(X)$ is the zero locus of the polynomial

$$y_1^8 + 10y_1^6y_2^2 + 33y_1^4y_2^4 + 40y_1^2y_2^6 + 16y_2^8 + 4y_1^6\epsilon^2 - 30y_1^4y_2^2\epsilon^2 - 90y_1^2y_2^4\epsilon^2$$
$$-56y_2^6\epsilon^2 - 2y_1^4\epsilon^4 + 62y_1^2y_2^2\epsilon^4 + 73y_2^4\epsilon^4 - 12y_1^2\epsilon^6 - 42y_2^2\epsilon^6 + 9\epsilon^8 - 14y_1^6$$
$$-90y_1^4y_2^2 - 120y_1^2y_2^4 + 64y_2^6 - 62y_1^4\epsilon^2 + 140y_1^2y_2^2\epsilon^2 - 248y_2^4\epsilon^2 - 90y_1^2\epsilon^4$$
$$+270y_2^2\epsilon^4 - 90\epsilon^6 + 73y_1^4 + 248y_1^2y_2^2 - 32y_2^4 + 270y_1^2\epsilon^2 - 360y_2^2\epsilon^2$$
$$+297\epsilon^4 - 168y_1^2 - 192y_2^2 - 360\epsilon^2 + 144.$$

The code above is designed to work in arbitrary dimensions and for any variety. For this reason, we saturate by the singular locus of the variety, even though this step is unnecessary in this example as the ellipse is smooth.

Example 2.4 (Offset hypersurface of a space curve). Let the variety X be the Viviani curve in $\mathbb{C}^3$, defined by the intersection of a sphere with a cylinder tangent to the sphere and passing through the center of the sphere. So X is defined by the vanishing of $f_1 = x_1^2 + x_2^2 + x_3^2 - 4$ and $f_2 = (x_1 - 1)^2 + x_2^2 - 1$. In Figure **??** the reader can see (on the left) the real part of the Viviani curve and (on the right) the $\epsilon = 1$ offset surface of the curve. This surface is defined by a degree 10 irreducible polynomial consisting of 175 monomials.

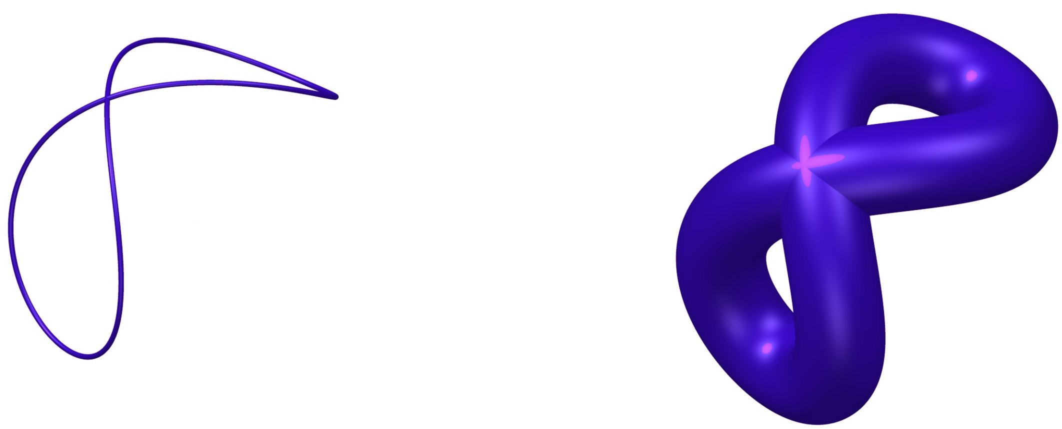

Figure 2: The Viviani curve (left) and its offset surface (right).

One could consider the family of all ϵ- offset hypersurfaces $\mathcal{O}_\epsilon(X) \subseteq \mathbb{C}^n$ as ϵ varies over $\mathbb{C}$. This family is again a hypersurface in $\mathbb{C}^n_y \times \mathbb{C}^1_\epsilon$ defined by the same ideal as is $\mathcal{O}_\epsilon(X)$, but now ϵ is a variable.

Define the **offset family**, $\mathcal{O}(X)$, to be the closure of all offset hypersurfaces of X in the $n+1$ dimensional space $\mathbb{C}^n_y \times \mathbb{C}^1_\epsilon$ defined by the same ideal as $\mathcal{O}_\epsilon(X)$. More precisely, let

$$\mathcal{O}(X) = \overline{\{(y, \epsilon),\ y \in \mathcal{O}_\epsilon(X)\}} \subseteq \mathbb{C}^n_y \times \mathbb{C}^1_\epsilon.$$

Example 2.5 (The offset family of an ellipse). A picture of the real part of $\mathcal{O}(X)$, where X is the ellipse defined by the vanishing of $x_1^2 + 4x_2^2 - 4 = 0$, can be seen below in Figure **??**. It is the set of all points $(y_1, y_2, \epsilon) \in \mathbb{C}^3$ that are zeros of the polynomial in **??**.

A horizontal cut (by a plane $\epsilon = \epsilon_0$) of the surface above is the ϵ_0-offset curve of the ellipse.

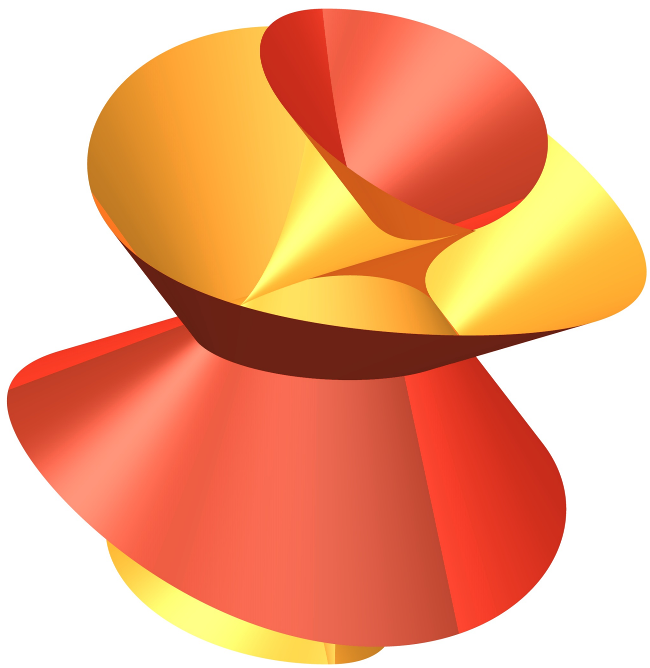

Figure 3: Offset family of an ellipse.

2.2. Offset dimension and degree

The degree and dimension are important invariants of an algebraic variety. These invariants of the offset hypersurface have been studied by many authors (for example by San Segundo and Sendra in [**?**, **?**, **?**]) in both the implicit and the parametric situations. To supplement the existing literature, in this subsection we relate the ϵ-degree of generic offset hypersurfaces to the ED degree of the starting variety, for any irreducible variety X. In this way, we achieve a new method for computing both ED degrees of varieties and degrees of offsets.

We remind the reader that (see [**?**, Theorem 4.1]) the Euclidean Distance correspondence $\mathcal{E}(X)$ is an irreducible variety of dimension n inside $\mathbb{C}^n_x \times \mathbb{C}^n_y$. The first projection $\mathrm{pr}_x : \mathcal{E}(X) \to X \subseteq \mathbb{C}^n_x$ is an affine vector bundle of rank c over X_{reg}. Over generic $y_0 \in \mathbb{C}^n_y$, the second projection $\mathrm{pr}_y : \mathcal{E}(X) \to \mathbb{C}^n_y$ has finite fibers $\mathrm{pr}_y^{-1}(y_0)$ of cardinality equal to the *Euclidean Distance Degree* (ED degree) of X.

$$\begin{array}{ccc} & \mathcal{E}(X) & \\ {}^{\mathrm{pr}_x}\swarrow & & \searrow^{\mathrm{pr}_y} \\ \mathbb{C}^n_x & & \mathbb{C}^n_y \end{array}$$

The second projection, pr_y, has a branch locus, which will play an important role in our future investigation. This ramification locus is generically a hypersurface in $\mathbb{C}^n_y$, by the Nagata-Zariski Purity Theorem [**?**],[**?**]. The *Euclidean Distance discriminant (ED discriminant)* is the closure of the image of the ramification locus of pr_y, (i.e. the points where the derivative of pr_y is not of full rank, under the projection pr_y). As in [?, Section 7], we denote the ED discriminant of the variety X by $\Sigma(X)$.

The offset correspondence is the intersection of the Euclidean Distance correspondence with the hypersurface $V(d(x,y)-\epsilon^2)$ in $\mathbb{C}^n_x \times \mathbb{C}^n_y$. This intersection is $n-1$ dimensional because $\mathcal{E}(X)$ is not a subvariety of $V(d(x,y)-\epsilon^2)$ (because not all pairs $(x,y) \in \mathcal{E}(X)$ are at ϵ^2 squared distance from each other). So the offset correspondence, $\mathcal{OC}_\epsilon(X)$, is an $n-1$ dimensional variety in $\mathbb{C}^n_x \times \mathbb{C}^n_y$. But over generic $y_0 \in \mathbb{C}^n_y$, the projection pr_y has finite fibers, so the $\mathcal{O}_\epsilon(X)$ is $n-1$ dimensional as well, hence the name offset *hypersurface*. For a more detailed analysis of the dimension degeneration of components of the offset hypersurface see [?].

Remark 2.6. Observe that a fixed generic y_0 (outside a hypersurface, see Proposition **??**) is an element of the offset hypersurface $\mathcal{O}_\epsilon(X)$ for precisely two times ED degree many distinct ϵ. This is because y_0 has ED degree many critical points to X, say $\{x_1, \ldots, x_{EDdegree(X)}\}$ and then the corresponding offset hypersurfaces that include y_0, are the ones where ϵ is in

$$\left\{\pm\sqrt{d(x_1,y_0)}, \ldots, \pm\sqrt{d(x_{EDdegree(X)},y_0)}\right\}.$$

Theorem 2.7. *The ϵ-degree of the defining polynomial of $\mathcal{O}(X)$ is equal to two times the Euclidean Distance degree of the variety X.*

Proof. Suppose that $\mathcal{O}(X)$ is defined by $f(y,\epsilon)$. By Remark **??**, a generic y_0 is an element of $\mathcal{O}_\epsilon(X)$ for precisely two times *ED* degree many ϵ. This is equivalent to $f(y_0,\epsilon)$ having exactly two times ED degree many roots. And these roots are

$$\left\{\pm\sqrt{d(x_1,y_0)}, \ldots, \pm\sqrt{d(x_{EDdegree(X)},y_0)}\right\},$$

where x_i are critical points of the distance from y_0 to the variety. □

We note that San Segundo and Sendra [?] derived the ϵ-degree of plane offset curves in terms of resultants. In the light of Theorem **??** their result says the following.

Proposition 2.8 (Theorem 35 from [?]). *Let X be a plane curve defined by the polynomial $f(x_1,x_2)$ of degree d. The ED degree of X equals*

$$\deg_{x_1,x_2}\left(PP_{y_1,y_2}(\mathrm{Res}_{x_3}(F(x_H), N(x_H,y)))\right),$$

where $x_H = (x_1,x_2,x_3)$, $y=(y_1,y_2)$, $F(x_H)$ is the homogenization of f with respect to a new variable x_3, $N(x_H,y) = -F_2(x_H)(y_1x_3 - x_1) + F_1(x_H)(y_2x_3 - y_1)$, where F_1 and F_2 are the homogenized partial derivatives of f and PP_{y_1,y_2} denotes the primitive part of the given polynomial with respect to $\{y_1,y_2\}$.

Example 2.9 (**Determinantal varieties**). Let $M_{n,m}^{\leq r}$ be the variety of $n \times m$ matrices (suppose $n \leq m$) over $\mathbb{C}$ of rank at most r. This variety is defined by the vanishing of all $(r+1) \times (r+1)$ minors of the matrix. For a fixed ϵ the construction of the offset hypersurface reduces to determining the set of matrices that have at least one critical rank r approximation at squared distance ϵ^2. By [**?**, Example 2.3] all the critical rank r approximations to a matrix U look like

$$T_1 \cdot \mathrm{Diag}(0, 0, ..., \sigma_{i_1}, 0, ..., 0, \sigma_{i_r}, 0, ..., 0) \cdot T_2,$$

where the singular value decomposition of U is equal to $U = T_1 \cdot \mathrm{Diag}(\sigma_1, ..., \sigma_n) \cdot T_2$, with $\sigma_1 > ... > \sigma_n$ singular values and T_1, T_2 orthogonal matrices of size $n \times n$ and $m \times m$. Now by [**?**, Corollary 2.3] the squared distance of such a critical approximation from U is exactly

$$\sigma_{i_1}^2 + \ldots + \sigma_{i_r}^2.$$

Recall that σ_i^2 are the eigenvalues of $U \cdot U^T$, so what we seek is that the sum of an r-tuple of the eigenvalues of $U \cdot U^T$ equals ϵ^2. Let us denote by $\bigwedge^{(r)}(U \cdot U^T)$ the r-th **additive compound matrix** of $U \cdot U^T$. For the construction of this object we refer to [**?**, P14]. The additive compound matrix is an $\binom{n}{r} \times \binom{n}{r}$ matrix with the property that its eigenvalues are the sums of r-tuples of eigenvalues of the original matrix [**?**, Theorem 2.1]. So the eigenvalues of $\bigwedge^{(r)}(U \cdot U^T)$ are exactly $\sigma_{i_1}^2 + \ldots + \sigma_{i_r}^2$. Putting this together we get that the offset hypersurface of $M_{n \times m}^{\leq r}$ is defined by the vanishing of

$$\det\left(\bigwedge^{(r)}(U \cdot U^T) - \epsilon^2 \cdot I_{\binom{n}{r}}\right).$$

Observe that the ϵ degree of this polynomial is $2 \cdot \binom{n}{r}$, which is indeed two times the ED degree of $M_{n \times m}^{\leq r}(see$[**?**, Example 2.3]).

2.3. Offset discriminant

We now consider the restriction $\mathrm{pr}_y|_{\mathcal{OC}_\epsilon(X)} : \mathcal{OC}_\epsilon(X) \rightarrow \mathcal{O}_\epsilon(X)$. We claim that, for generic ϵ, this restriction is one-to-one, outside its branch locus. Indeed if we pick a generic $y_0 \in \mathcal{O}_\epsilon(X)$, then the fiber above it equals

$$\mathrm{pr}_y^{-1}(y_0) = \left(V(d(x,y) - \epsilon^2) \cap (\mathbb{C} \times \{y_0\})\right) \cap (\mathcal{E}(X) \cap (\mathbb{C} \times \{y_0\})).$$

By the definition of ED degree we have that

$$\mathcal{E}(X) \cap (\mathbb{C} \times \{y_0\}) = \{(x_1, y_0), \ldots, (x_{EDdegree(X)}, y_0)\}.$$

Combining this information,

$$\mathrm{pr}_y^{-1}(y_0) = \{(x_1, y_0), \ldots, (x_{EDdegree(X)}, y_0)\} \cap \left(V(d(x,y) - \epsilon^2) \cap (\mathbb{C} \times \{y_0\})\right).$$

This means that the fiber consists of pairs (x, y_0), such that x is a critical point of the squared distance function from y_0 and is of squared distance ϵ^2 from y_0. For generic $y_0 \in \mathcal{O}_\epsilon(X)$ and generic ϵ, there is exactly one such critical point. Otherwise a y_0 with at least two elements in the fiber would be a doubly covered point of the offset hypersurface, hence part of its singular locus, which is of strictly lower dimension than the offset hypersurface itself. Indeed the branch locus of the restriction of pr_y is (generically) a hypersurface inside $\mathcal{O}_\epsilon(X)$ (by the Nagata-Zariski Purity theorem [**?**, **?**]), hence a codimension two variety in $\mathbb{C}^n_y$ and consists of points y for which there exist at least two $x_1, x_2 \in X_{reg}$, such that $(x_1, y), (x_2, y) \in \mathcal{OC}_\epsilon(X)$, or one $(x_1, y) \in \mathcal{OC}_\epsilon(X)$ with multiplicity greater than one. We denote the closure of the union of all branch loci, over ϵ in $\mathbb{C}$, by $B(X, X)$, which is the **bisector hypersurface of the variety** X (see for instance [**?**, **?**]). Note that the variety itself is a component of $B(X, X)$ because for $\epsilon = 0$ the variety is covered doubly under the projection pr_y. We call the set of doubly covered points such that $(x_1, y) \neq (x_2, y) \in \mathcal{OC}_\epsilon(x)$, the **proper bisector locus**, and we denote it by $B_0(X, X)$. In summary, we have the following result.

Proposition 2.10. *For a fixed generic ϵ the projection $\mathrm{pr}_y|_{\mathcal{OC}_\epsilon(X)} : \mathcal{OC}_\epsilon(X) \to \mathcal{O}_\epsilon(X)$ is one-to-one outside the bisector hypersurface $B(X, X)$.*

Let us see how this relates to (not the union but) the collection of offset hypersurfaces for all ϵ. This collection is the offset family, $\mathcal{O}(X)$, and it is a hypersurface in $\mathbb{C}^n_y \times \mathbb{C}^1_\epsilon$. Its defining polynomial is the same as of $\mathcal{O}_\epsilon(X)$. Let us denote this polynomial by $f(y, \epsilon)$. Now if we consider $f(y, \epsilon)$ to be a univariate polynomial in the variable ϵ, then we can compute its discriminant $\mathrm{Discr}_\epsilon(f)$, which is a polynomial in the variables y, with the property that $f(y_0, \epsilon)$ has a double root (in ϵ) if and only if y_0 is in the zero set of $\mathrm{Discr}_\epsilon(f)$.

Now $y_0 \in \mathrm{Discr}_\epsilon(f)$ if and only there are fewer than two times ED degree many distinct roots, not counting multiplicities, of $f(y_0, \epsilon)$. By Theorem **??** this means that either y_0 has a non-generic number of critical points, meaning that y_0 is an element of the ED discriminant, or there are two critical points $x_i \neq x_j$, such that

$$d(x_i, y_0) = d(x_j, y_0),$$

meaning that the projection $\mathrm{pr}_y : \mathcal{OC}_\epsilon(X) \to \mathcal{O}_\epsilon(X)$ is not one-to-one over y_0, so y_0 in an element of the branch locus. To summarize this we have the following proposition.

Proposition 2.11. *Suppose that $\mathcal{O}(X)$ is defined by the vanishing of $f(y, \epsilon)$. Then the zero locus of the ϵ-discriminant of f is the union of the ED discriminant of X and the bisector hypersurface of X. So we have that*

$$\mathrm{Discr}_\epsilon(f) = \Sigma(X) \cup B(X, X).$$

Throughout the rest of the article we call the union of the ED discriminant and the bisector hypersurface the **offset discriminant**, denoted $\Delta(X)$. And we recall that by the way it was constructed, it is *the envelope of all the offset hypersurfaces* to X.

Example 2.12 (**Offset discriminant of an ellipse).** Let X be the ellipse defined by $x_1^2+4x_2^2-4$. The offset family of the ellipse is defined by the vanishing of the polynomial from Example **??**. The ϵ-discriminant of this polynomial factors into five irreducible components. One of them is the defining polynomial of the sextic *Lamé curve*

$$64y_1^6 + 48y_1^4y_2^2 + 12y_1^2y_2^4 + y_2^6 - 432y_1^4 + 756y_1^2y_2^2 - 27y_2^4 + 972y_1^2 + 243y_2^2 - 729,$$

with zero locus $\Sigma(X)$, the ED discriminant (evolute) of X. The remaining four components comprise the bisector curve $B(X, X)$ of the ellipse. Two out of these four components of $B(X, X)$ are the x- and y- axes(the proper bisector locus $B_0(X, X)$), one of the components is the ellipse itself (because for $\epsilon = 0$ the variety is doubly covered under the projection pr_y) and the remaining component is fully imaginary. A cartoon of the real part of $\Delta(X)$ can be seen in Figure **??**. The ellipse is black, the proper bisector locus (the axis) is blue and the ED discriminant is red.

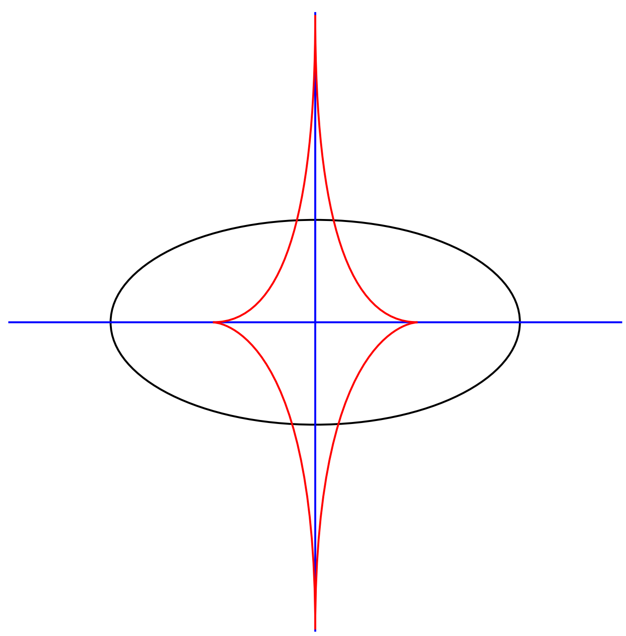

Figure 4: The ED discriminant and the bisector curve of the ellipse.

Corollary 2.13. *Let X be an irreducible variety in $\mathbb{C}^n$. The degree of its offset discriminant $\Delta(X)$ (hence also the degree of its ED discriminant $\Sigma(X)$ and the degree of the bisector hypersurface $B(X, X)$) is bounded from above by*

$$2 \cdot \deg_y(\mathcal{O}(X)) \cdot (4 \cdot \mathrm{EDdegree}(X) - 2)\,.$$

Proof. Suppose that the offset family $\mathcal{O}(X)$ is the zero set of the polynomial $f(y, \epsilon)$. The offset discriminant $\Delta(X)$ is the discriminant of the univariate polynomial

$$f(y, \epsilon) = a_0(y) + \ldots + a_{d-1}(y) \cdot \epsilon^{d-1} + a_d(y)\epsilon^d$$

in the variable ϵ of degree $d = \deg_\epsilon(f)$. So $\Delta(X)$ is a homogeneous polynomial in the coefficients $a_0(y), \ldots, a_d(y)$ of degree equal to $2 \cdot d - 2$. By Theorem **??** we have that $d = 2 \cdot \text{EDdegree}(X)$. Now because the discriminant is a homogeneous polynomial in the coefficients we get the desired degree bound. □

Example 2.14 (**Degree bounds of the offset discriminant**)**.** The following table contains degree bounds of the offset discriminant based on the formula above and the total degree formulae, $\deg_y(\mathcal{O}(X))$, by San Segundo and Sendra [**?**, Appendix. Table of offset degrees].

Name of X	Defining poly. of X	$\deg_y \mathcal{O}(X)$	$\deg_\epsilon \mathcal{O}(X)$	$\deg_y \Delta(X) \leq$
Circle	$x_1^2 + x_2^2 - 1$	4	4	24
Parabola	$x_2 - x_1^2$	6	6	60
Ellipse	$x_1^2 + 4x_2^2 - 4$	8	8	112
Cardioid	$(x_1^2 + x_2^2 + x_1)^2 - x_1^2 - x_2^2$	10	8	140
Rose(3 petals)	$(x_1^2 + x_2^2)^2 + x_1(3x_2^2 - x_1^2)$	14	12	308

3. Algebraicity of persistent homology

As an application of this knowledge of offset hypersurfaces, we study the true persistent homology of an algebraic variety. We define this to be the homology of the set of points within distance ϵ of the variety, which is bounded by the offset hypersurface. First, we review background material on persistent homology. Next, we define the true persistent homology in terms of the offset hypersurface. We then prove the algebraicity of true persistent homology. Finally, we discuss the relevance of the offset discriminant to persistent homology.

3.1. Background on Persistent Homology

The persistent homology of a point cloud at parameter ϵ is defined as the homology of a simplicial complex, called the Čech complex, associated to a covering of the point cloud by hyperballs of radius ϵ. By the nerve theorem, the Čech complex has the same homology as the covering.

Definition 3.1. Let (X, d) be a finite metric space and $\epsilon > 0$ a parameter. The **Čech complex** of X at radius ϵ is

$$C_X(\epsilon) = \left\{ \sigma \subset 2^X \text{ s.t. } \bigcap_{x \in \sigma} B_\epsilon(x) \neq 0 \right\},$$

a simplicial complex where the $n-$faces are the subsets of size n of X with nonempty n-wise intersection.

From these simplicial complexes, we obtain a filtration for which we can define persistent homology.

Following [**?**], consider a simplicial complex, K, and a function $f : K \to \mathbb{R}$. We require that f be *monotonic* by which we mean it is non-decreasing along chains of faces, that is, $f(\sigma) \leq f(\tau)$ whenever σ is a face of τ. Monotonicity implies that the sublevel set, $K(a) = f^{-1}(-\infty, a]$, is a subcomplex of K for every $a \in \mathbb{R}$. Letting m be the number of simplices in K, we get $n + 1 \leq m + 1$ different subcomplexes, which we arrange as an increasing sequence,

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K.$$

In other words, if $a_1 < a_2 < \cdots < a_n$ are the function values of the simplices in K and $a_0 = -\infty$ then $K_i = K(a_i)$ for each i. We call this sequence of complexes the *filtration* of f.

For every $i \leq j$ we have an inclusion map from the underlying space of K_i to that of K_j and therefore an induced homomorphism, $f_q^{i,j} : H_q(K_i) \to H_q(K_j)$, for each dimension q.

Definition 3.2. The **q-th persistent homology groups** are the images of the homomorphisms induced by inclusion, $H_q^{i,j} = \text{im } f_q^{i,j}$, for $0 \leq i \leq j \leq n$. The corresponding **q-th persistent Betti numbers** are the ranks of these groups, $\beta_q^{i,j} = \text{rank } H_q^{i,j}$.

As a consequence of the Structure Theorem for PIDs, the family of modules $H_q(K_i)$ and homomorphisms $f_q^{i,j} : H_q(K_i) \to H_q(K_j)$ over a field F yields a decomposition

$$H_q(K_i; F) \cong \bigoplus_i x^{t_i} \times F[x] \bigoplus \left(\bigoplus_j x^{r_j} \cdot (F[x]/(x^{s_j} \cdot F[x])) \right), \tag{1}$$

where t_i, r_j, and s_j are values of the persistence parameter ϵ [**?**].

The free portions of Equation **??** are in bijective correspondence with those homology generators which appear at parameter t_i and persist for all $\epsilon > t_i$, while the torsional elements correspond to those homology generators which appear at parameter r_j and disappear at parameter $r_j + s_j$.

To encode the information given by this decomposition, we create a graphical representation of the q-th persistent homology group called a **barcode** [**?**]. For each parameter interval $[r_j, r_j + s_j]$ corresponding to a homology generator, there is a horizontal line segment (bar), arbitrarily ordered along a vertical axis. The persistent Betti number $\beta_q^{i,j}$ equals the number of intervals in the barcode of $H_q(K_i; F)$ spanning the parameter interval $[i, j]$.

We include here an example of the real variety defined by the Trott curve and a barcode representing its persistent homology, computed by taking a sample of points on the variety.

Example 3.3 (The barcodes of the Trott curve). In dimension 1, the first four bars correspond to the cycles in each of the four components of the real variety. As epsilon increases, these cycles fill in, and then the components join together in one large circle. This demonstrates how persistent homology can detect the global arrangement of the components of a variety. The barcodes were computed using Ripser [**?**].

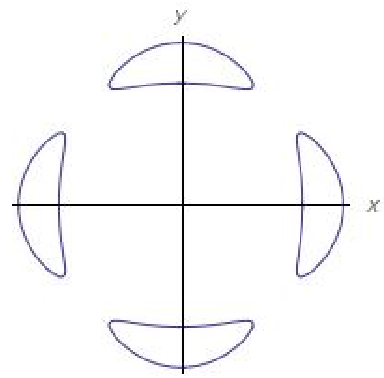

Figure 5: The Trott curve.

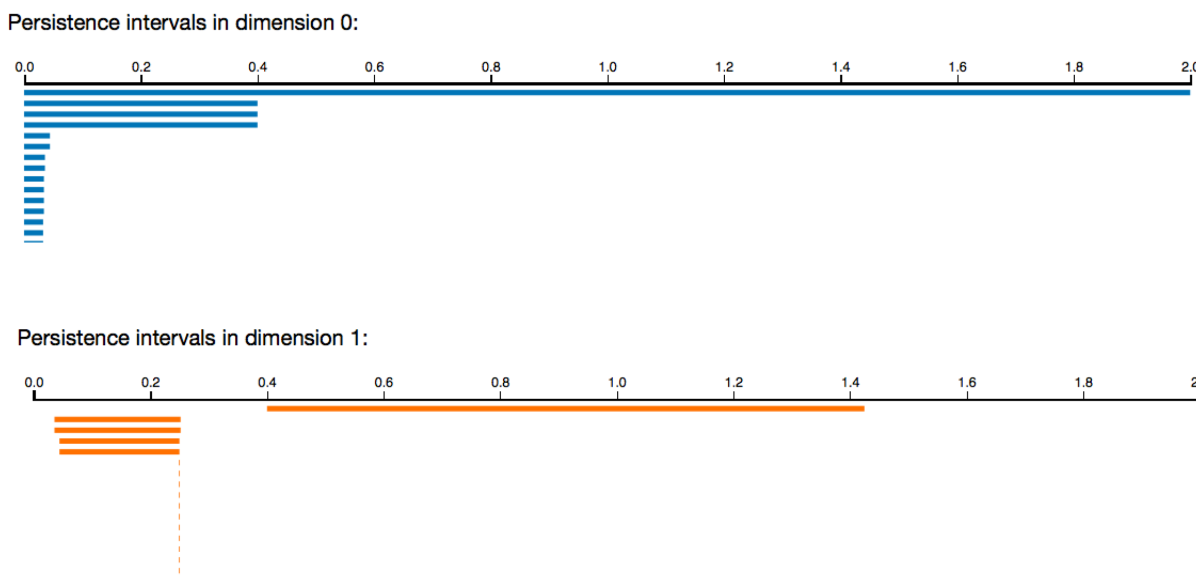

Figure 6: Barcodes for the Trott curve in homological dimensions 0 and 1.

3.2. True Persistent Homology of a Variety

Persistent homology is typically defined for a finite metric space. To compute the persistent homology of a variety X, one might sample a finite set of points from the variety and compute the Čech complex of those points. The equivalent of the Čech complex $C_X(\epsilon)$ obtained from sampling every point on the variety would be the set of all points within ϵ of the variety. For this reason, we define the **true persistent homology of a variety X at parameter ϵ** as the homology of the subset

$$X_\epsilon = \{x| \text{ there exists } y \in V \text{ with } ||x - y|| \leq \epsilon\} \subset \mathbb{R}^n$$

consisting of all points within ϵ of the variety.

Since the ϵ-offset hypersurface is the envelope of a family of ϵ-hyperballs centered on the variety, we can define the true persistent homology of a variety at parameter ϵ equivalently as the homology of the set bounded by $\mathcal{O}_\epsilon(X)$.

Let $S = \{(X_\epsilon, \epsilon)|\epsilon \in [0, \infty)\} \subset \mathbb{R}^{n+1}$ and let $\text{pr}_\epsilon : S \to \mathbb{R}$ be the projection to ϵ. By Hardt's theorem [**?**, Theorem 9.3.2], there is a partition of $\mathbb{R}$ into finitely many intervals $I_l = [\delta_l, \epsilon_l]$ for $l \in \{1, \ldots, j\}$ such that the fibers $\text{pr}_\epsilon^{-1}(\epsilon) = X_\epsilon$ for all $\epsilon \in [\delta_l, \epsilon_l]$ are homeomorphic. Thus we can create the **true barcode** of X.

We show that $\{\delta_l\} \cup \{\epsilon_l\}$ for $l \in \{1, \ldots, j\}$, the values of the persistence parameter ϵ at which a bar in the true barcode appears or disappears are alegbraic over the field of definition of a real affine variety X. As a consequence, the true persistent homology of X can be computed exactly.

Theorem 3.4. *(Algebraicity of persistent homology barcodes.) Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ with $X_\mathbb{R} = V_\mathbb{R}(f_1, \ldots, f_s)$. Then the values of the persistence parameter ϵ at which a bar in the true barcode appears or disappears are real numbers algebraic over $\mathbb{Q}$.*

Proof. Let $\mathbb{R}_{alg}$ denote the closed subfield of real algebraic numbers over $\mathbb{Q}$. Let $S_{alg} = \{(X_\epsilon \cap \mathbb{R}^n_{alg}, \epsilon)|\epsilon \in [0, \infty)\}$. Then S_{alg} is a semialgebraic subset of $\mathbb{R}^{n+1}_{alg}$ in the sense of [**?**, Definition 2.4.1] since S_{alg} is defined by polynomial equalities and inequalities with coefficients in $\mathbb{Q}$. Let $\text{pr}_\epsilon : S_{alg} \to \mathbb{R}_{alg}$ be the projection to ϵ.

Since $\mathbb{R}_{alg}$ is a closed subfield of $\mathbb{R}$, by Hardt's theorem [**?**, Theorem 9.3.2] there is a partition of $\mathbb{R}_{alg}$ into finitely many sets $I_l = [\delta_l, \epsilon_l] \cap \mathbb{R}_{alg}$ with $\delta_l, \epsilon_l \in \mathbb{R}_{alg}$ for $l \in \{1, \ldots, j\}$ such that $\text{pr}_\epsilon^{-1}(\epsilon) = X_\epsilon \cap \mathbb{R}^n_{alg}$ for all $\epsilon \in [\delta_l, \epsilon_l]$ are homeomorphic.

By Tarski-Seidenberg's theorem ([**?**, Theorem 1.4.2]) and [**?**],there is an isomorphism of homology groups

$$H_q(X_\epsilon \cap \mathbb{R}^n_{alg}) \cong H_q(X_\epsilon)$$

for each q and ϵ. So the partition by the sets $I_l = [a_l, b_l] \cap \mathbb{R}_{alg}$ given by Hardt's theorem corresponds to a partition of $\mathbb{R}$ by intervals $\tilde{I}_l = [\delta_l, \epsilon_l] \subset \mathbb{R}$ such that X_ϵ for all $\epsilon \in [\delta_l, \epsilon_l]$ with $\delta_l, \epsilon_l \in \mathbb{R}_{alg}$ are homeomorphic. Thus $\{\delta_l\}_{l \in \{1,\ldots,j\}} \cup \{\epsilon_l\}_{l \in \{1,\ldots,j\}} \subset \mathbb{R}_{alg}$. □

3.3. Using the offset discriminant to identify points of interest for persistent homology

We now discuss the bisector hypersurface (a component of the offset discriminant) in the context of true persistent homology. We first show how the bisector hypersurface can help identify points where homological events occur. Then we discuss the medial axis, a subset of the proper bisector locus of X, which gives information about the density of sampling required to compute the persistent homology accurately.

Consider a bar in the true barcode corresponding to the top dimension Betti number. To each such bar, there corresponds a $y \in \Delta(X)$. Informally, this y is the center of the n-dimensional hole corresponding to the bar. We illustrate with the example of the circle $x_1^2 + x_2^2 = r^2 \subset \mathbb{R}^2$ in Figure **??**. The persistent homology of the circle has $\beta_1 = 1$ for all $\epsilon < r$, and a real component of the offset hypersurface is a smaller circle inside the circle. When $\epsilon = r$, the offset hypersurface is simply the point at the center of the circle, and $\beta_1 = 0$.

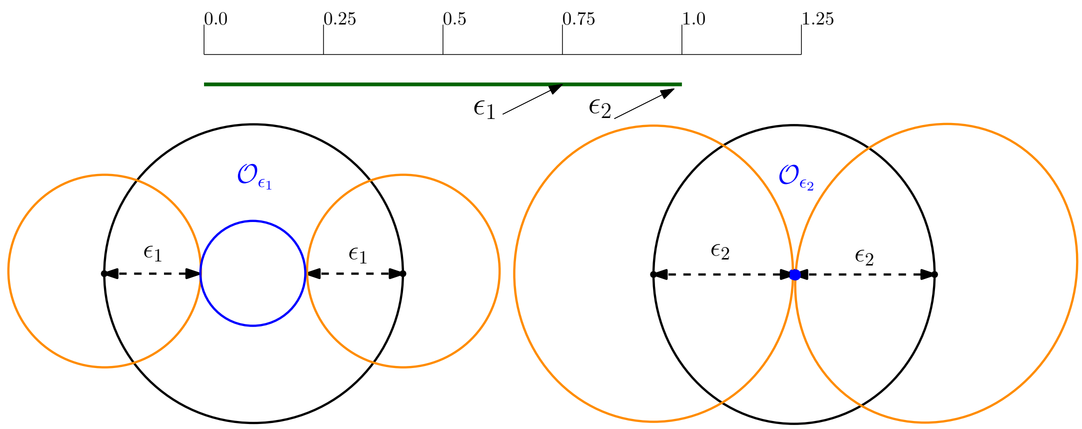

Figure 7: These pictures illustrate how the offset variety provides a geometric interpretation of the endpoints of a bar. The black circle is the variety X and the orange circles are ϵ-balls around X. When ϵ reaches the radius of the black circle, the blue offset hypersurface $\mathcal{O}_\epsilon$ has an isolated real point.

Theorem 3.5. *(Geometric interpretation of endpoints in barcode.) Let $X \subset \mathbb{R}^{n+1}$ be a hypersurface. Let $J = \{[\delta_l, \epsilon_l] | l \in \{1, \dots, m\}\}$ be the set of intervals in the barcode for the top dimensional Betti number β_n. Then each interval endpoint ϵ_l corresponds to a point $y_l \in \mathcal{O}_{\epsilon_l}(X)$ on the bisector hypersurface $B(X, X)$. Furthermore, y_l is the limit of a sequence of centers of hyperballs contained in the complement of $\mathcal{O}_\epsilon(X)$ as $\epsilon \to \epsilon_l$.*

We make the following observations. First, the correspondence does not assign each interval to a unique point on the offset discriminant. Consider the persistent homology of the Trott curve. In dimension 1, there is one interval corresponding to four cycles, and we do not specify to which cycle to assign the interval. Second, we note that the set of y_l corresponding to endpoint intervals may not be 0-dimensional. For example, let X be the torus. Then the set of y_l contains a circle.

We also comment on the topology of the real algebraic varieties involved. Suppose $X \subset \mathbb{R}^2$ is a curve. Then $\mathcal{O}_\epsilon$ for $\epsilon \in (\epsilon_l - \epsilon', \epsilon_l)$ will have an oval component not present in $\mathcal{O}_\epsilon$ for $\epsilon > \epsilon_l$, so y_l is an isolated real point of $\mathcal{O}_{\epsilon_l}$.

Proof. Fix $[\delta_l, \epsilon_l] \in J$. Then there exists $\epsilon' > 0$ such that $\beta_n(X_{\epsilon_l}) < \beta_n(X_\epsilon)$ for all $\epsilon \in (\epsilon_l - \epsilon', \epsilon_l)$.

Since $\beta_n(X_{\epsilon_l}) < \beta_n(X_\epsilon)$ for all $\epsilon \in (\epsilon_l - \epsilon', \epsilon_l)$, there is some $\epsilon_1 \in (\epsilon_l - \epsilon', \epsilon_l)$ such that there is a maximum $\delta_1 > 0$ such that there is a ball $B^{n+1}_{\delta_1}(z_{\epsilon_1})$ such that $B^{n+1}_{\delta_1}(z_{\epsilon_1}) \subset X_{\epsilon_l}$ but $B^{n+1}_{\delta}(z_\epsilon)$ is contained in a bounded connected component of $\mathbb{R}^{n+1} \setminus X_{\epsilon_1}$. Furthermore, since $X_{\epsilon_n} \supset X_{\epsilon_{n-1}}$ for $\epsilon_{n-1} < \epsilon_n$, there is a monotonically increasing sequence of $\{\epsilon_n\}_{n=1,2,\ldots}$ with $\epsilon_1 < \epsilon_{n-1} < \epsilon_n < \epsilon_l$ such that for each ϵ_n, there is a maximum δ_n such that there is a ball $B^{n+1}_{\delta_n}(z_{\epsilon_n}) \subset B^{n+1}_{\delta_{n-1}}(z_{\epsilon_{n-1}})$ with $B^{n+1}_{\delta_n}(z_{\epsilon_n})$ contained in a bounded connected component of $\mathbb{R}^{n+1} \setminus X_\epsilon$.

Let $\epsilon_n \to \epsilon_l$. The diameter of the bounded connected component of X_{ϵ_n} containing z_{ϵ_n} is less than that of $X_{\epsilon_{n-1}}$ containing $z_{\epsilon_{n-1}}$ for $\epsilon_{n-1} < \epsilon_n$ and $B^{n+1}_{\delta_1}(z_{\epsilon_1}) \subset X_{\epsilon_l}$, so $\delta_n \to 0$. So $\{z_{\epsilon_n}\}$ is a Cauchy sequence in $\mathbb{R}^{n+1}$, so it converges. Let $y = \lim_{\epsilon_n \to \epsilon_l} z_{\epsilon_n}$.

Since δ_n is the maximum radius of such a ball, $B^{n+1}_{\delta_n}(z_{\epsilon_n}) \cap \mathcal{O}_{\epsilon_n}$ contains at least two points $\{y_{1,\epsilon_n}, y_{2,\epsilon_n}\}$. Corresponding to these points in $\mathcal{O}_{\epsilon_n}$ are at least two points in the offset correspondence $\mathcal{OC}_{\epsilon_n}(X)$, say $\{(x_{1,\epsilon_n}, y_{1,\epsilon_n}), (x_{2,\epsilon_n}, y_{2,\epsilon_n})\}$.

As $\delta_n \to 0$, we have $||y_{1,\epsilon_n} - y_{2,\epsilon_n}|| \to 0$ since $||y_{1,\epsilon_n} - y_{2,\epsilon_n}|| \leq \delta_n$. Thus $y \in B(X, X)$. □

3.4. Algebraicity of the reach

The bisector hypersurface has further relevance to persistent homology because one of its components is the closure of the medial axis. The shortest distance from a manifold to its medial axis is called the reach. The reach of a manifold is a very important quantity in the computation of its persistent homology as it determines the density of sampling points required to obtain the correct homology. We now define the reach, describe its importance in the theory of persistent homology, and prove its algebraicity.

Definition 3.6. Let X be a real algebraic manifold in $\mathbb{R}^n$. The **medial axis** of X is the set M_X of all points $u \in \mathbb{R}^n$ such that the minimum Euclidean distance from X to u is attained by two distinct points in X. The **reach** $\tau(X)$ is the shortest distance between any point in the manifold X and any point in its medial axis M_X.

We now state the theorem showing that sampling density depends on the reach. In particular, the smaller the reach (and thus, the curvier the manifold), the higher the density of sampling points required to compute persistent homology accurately. We have adapted this from [**?**], where it is stated in terms of the reciprocal of a reach, a quantity which they call the condition number of the manifold.

Theorem 3.7. *(Theorem 3.1 from [**?**]) Let M be a compact submanifold of $\mathbb{R}^N$ of dimension k with reach τ. Let $\bar{x} = \{x_1, \ldots, x_n\}$ be a set of n points drawn in i.i.d fashion according to the uniform probability measure on M. Let $0 < \epsilon < \frac{1}{2\tau}$. Let $U = \bigcup_{x \in \bar{x}} B_\epsilon(x)$ be a corresponding random open subset of $\mathbb{R}^N$. Let $\beta_1 = \frac{vol(M)}{(\cos^k(\theta_1))vol(B^k_{\epsilon/4})}$ and $\beta_2 = \frac{vol(M)}{(\cos^k(\theta_2))vol(B^k_{\epsilon/8})}$ where $\theta_1 = \arcsin(\frac{\epsilon\tau}{8})$ and $\theta_2 = \arcsin(\frac{\epsilon\tau}{16})$. Then for all*

$$n > \beta_1 \left(\log(\beta_2) + \log\left(\frac{1}{\delta}\right)\right)$$

the homology of U equals the homology of M with high confidence (probability $> 1 - \delta$).

Studying a formulation for reach given by Federer with the tools of real algebraic geometry, we show the algebraicity of the reach.

Proposition 3.8. *(Algebraicity of reach). Let X be a real algebraic manifold in $\mathbb{R}^n$. Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ with $X_\mathbb{R} = V_\mathbb{R}(f_1, \ldots, f_s)$.*

Proof. Federer [**?**, Theorem 4.18] gives a formula for the reach τ of a manifold X in terms of points and their tangent spaces:

$$\tau(X) \; = \inf_{v \neq u \in X} \frac{||u-v||^2}{2\delta}, \qquad \text{where } \; \delta \; = \min_{x \in \mathrm{T}_v X} \|(u-v) - x\|. \tag{2}$$

Equation **??** gives the following system of polynomial equalities with rational coefficients

$$\begin{cases} x \in \mathrm{T}_v X \\ \delta^2 = \|(u-v) - x\|^2 \\ \delta > 0 \\ 2\delta\tau = ||u-v||^2 \end{cases}$$

in unknowns $\{x, \delta, \tau, u, v\}$. This defines a semialgebraic set in $\mathbb{R}^{3n+2}$ in the sense of [**?**, Definition 2.4.1]. Consider the projection on to $\mathbb{R}^2$ with coordinates (δ, τ). By Tarski-Seidenberg's theorem ([**?**, Theorem 1.4.2]), the image is a semialgebraic set S.

Project S onto $\mathbb{R}$ with coordinate δ. The minimum δ_0 is attained in the closure of the image. Then $\overline{S} \cap \{\delta = \delta_0\} \subset \mathbb{R}$ is semialgebraic over $\mathbb{Q}$. It is bounded below by 0. The reach is the infimum of this set, and thus is an algebraic number over $\mathbb{Q}$. □

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