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Stanley-Reisner rings for symmetric simplicial complexes, G-semimatroids and Abelian arrangements

by

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STANLEY-REISNER RINGS FOR SYMMETRIC SIMPLICIAL COMPLEXES, G-SEMIMATROIDS AND ABELIAN ARRANGEMENTS

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Abstract. We extend the notion of face rings of simplicial complexes and simplicial posets to the case of finite-length simplicial posets with a group action. The action on the complex induces an action on the face ring, and we prove that the ring of invariants is isomorphic to the face ring of the quotient simplicial poset when the group action is translative in the sense of [28]. When the acted-upon poset is the independence complex of a semimatroid, the h-polynomial of the ring of invariants can be read off the Tutte polynomial of the associated G-semimatroid. We thus recover the classical theory in the case of trivial group actions on finite simplicial posets and, in the special case of central toric arrangements, our rings are isomorphic to those defined by Martino [40] and Lenz [37].

We also describe a further condition on the group action ensuring that the topological Cohen-Macaulay property is preserved under quotients. In particular, we prove that the independence complex and the Stanley-Reisner ring of any Abelian arrangement are Cohen-Macaulay over every field.

As a byproduct, we prove that posets of connected components (also known as posets of *layers*) of Abelian arrangements are (homotopically) Cohen-Macaulay.

1. Introduction

1.1. **Background.** A classical construction associates a commutative ring, called *Stanley-Reisner ring*, to every finite simplicial complex. In the wake of pioneering work in the 1970s by R. Stanley, M. Hochster and G. Reisner, a rich research activity has blossomed around this bridge between combinatorics and topology on the one side and commutative algebra on the other, leading to major advances such as Stanley's proof of the Upper Bound Conjecture [52].

A recurring theme in this research area is to investigate properties of the class of Stanley-Reisner rings associated to particular (combinatorially defined) families of simplicial complexes. A good example is given by simplicial complexes that arise as the complex of independent sets of a matroid ("matroid complexes" [12]). Such complexes are defined by abstract properties modeled on the family of independent subsets of a given set of vectors in a vectorspace, and the associated Stanley-Reisner rings have attracted a large body of work. A topic of particular interest is a sequence of integers related to the Hilbert series of the defining ideal, namely the coefficients of the ring's h-polynomial, which is strongly related to classical polynomial invariants of matroids. For instance, Stanley-Reisner rings of matroids are *Cohen-Macaulay* [12], which implies positivity of said coefficients. A recent celebrated result in this context is the proof of a longstanding conjecture [2] implying log-concavity of this integer sequence [33]. Many further questions and conjectures remain open to date.

Simplicial posets are a generalization of posets of faces of simplicial complexes. (see Definition 2.15). Stanley [56] defined a "face ring" associated to any finite simplicial poset which, in the special case of posets of faces of simplicial complexes, is isomorphic to the classical Stanley-Reisner ring.

1.2. **Motivation.** The study of symmetries in the form of group actions on simplicial complexes has classical roots [16] and has come into the focus of growing interest over the last years. Significant results have been obtained in the combinatorial study of algebraically defined objects [30, 50] as well as in using symmetries in order to advance in combinatorial problems [39, 42, 46], a special mention being deserved by the strong impact of the study of group actions on topological combinatorics [1, 41].

Moreover, as we will discuss below in more detail, recent developments in the theory of arrangements lead to the study of structural aspects of group actions on matroids and posets. A peculiarity of the latter setup is that it does not meet the standard finiteness (or compactness) assumptions on which relies most of the extant literature (here, to the above-mentioned references we add some specific literature on group actions on posets, e.g., [54, 57]).

It is then natural to wonder about the algebraic implications of group actions on complexes or posets in terms of the associated Stanley-Reisner rings. In fact, this line of research has been pursued in the literature [32, 53] but, again, always under finiteness conditions.

1.3. **Aim and results.** We propose an enrichment of the Stanley-Reisner theory by considering group actions on finite-dimensional (but possibly infinite) simplicial complexes. In fact, in this context we find that the natural framework is that of finite-length simplicial posets. We associate a face ring $\mathcal{R}(P)$ to each such simplicial poset P (Definition 3.1) and, given an action of a group G on the poset, we study the ring $\mathcal{R}(P)^G$ of invariants of the induced action on the ring. We characterize precisely the group actions for which the quotient poset P/G is again simplicial: these turn out to be the type of actions called *translative* in [28] (Lemma 2.17). We prove that, given a translative action of a group G on a simplicial poset P, the ring of invariants $\mathcal{R}(P)^G$ is isomorphic to the ring $\mathcal{R}(P/G)$ associated to the quotient poset (Theorem 2).

We then consider *refined* actions (a condition strictly stronger than translativity, see Definition 2.3) and prove a general theorem stating that quotients of posets of cells of finite-dimensional homotopy Cohen-Macaulay simplicial complexes are again homotopy Cohen-Macaulay (Theorem 3).

Then we turn to the matroidal case, generalizing some of the properties of Stanley-Reisner rings of matroids to the case of semimatroids with group actions. We immediately obtain that, if P is the poset of independent sets of a semimatroid and the group action is refined, then the (finite) poset P/G is homotopy Cohen-Macaulay and, thus, so is the associated ring over every field. Moreover, the characteristic polynomial of P/G, and hence (Remark 2.19) the h-polynomial of the ring $\mathcal{R}(P/G)$, is an evaluation of the Tutte polynomial associated to any translative action on a semimatroid [28, §3.4].

As a byproduct, we prove that the quotient of any rank-finite geometric semilattice with respect to a translative and refined action is a (homotopically) Cohen-Macaulay poset (Theorem 4).

1.4. **Application: Abelian arrangements.** Many aspects of the classical theory of arrangements of hyperplanes are currently being extended to encompass *toric arrangements* and *elliptic arrangements*. The aim is a general topological and combinatorial theory of *Abelian arrangements*. In the following we give a quick primer in this subject and refer to, e.g., [27] for more.

An Abelian arrangement is a finite set $\mathscr{A}$ of level sets of group homomorphisms $\mathbb{G}^d \to \mathbb{G}$, where $\mathbb{G}$ is a complex algebraic group of rank one.

In this context, a main combinatorial invariant is the *poset of layers*, i.e., the set

$$\mathcal{C}(\mathscr{A}) := \{\text{conn. comp. of } \cap \mathscr{X} \mid \mathscr{X} \subseteq \mathscr{A}\} \tag{1}$$

of connected components of intersections of subsets of $\mathcal{A}$, partially ordered by reverse inclusion [60, 25]. There is as of yet little understanding of the structure of such posets beyond linear arrangements, except from the case of Weyl-type arrangements where the posets are known to be shellable [26] based on the explicit description given by Bibby [7].

In the case of hyperplanes ($\mathbb{G} = \mathbb{C}$) this poset has the structure of a geometric lattice, and is equivalent to the arrangement's *matroid* data. The case of toric arrangements ($\mathbb{G} = \mathbb{C}^*$) has recently been in the focus of a considerable amount of research that was at first motivated by applications to commutative algebra [6] and partition functions [25], but recently gained momentum as an independent topic. Research on topological [24, 22, 47] and combinatorial [29, 36, 44] aspects of toric (and elliptic, $\mathbb{G} = \mathbb{E}$, e.g, [8]) arrangements reaffirmed the importance of the poset $\mathcal{C}(\mathscr{A})$.

In particular, the theory of *arithmetic matroids* [15, 23] was developed as a combinatorial framework for toric arrangements, but the poset structure is not described by the arithmetic matroid (or by an even more refined invariant, the matroid over $\mathbb{Z}$ [31]): Pagaria [48] constructed two central toric arrangements with non-isomorphic posets of layers but isomorphic arithmetic matroid (resp. matroid over $\mathbb{Z}$).

An attempt at a structural characterization of posets of layers of Abelian arrangements (that distinguishes the examples of [48]) has been carried out in [28] along the following lines (see Section 7 for a more precise treatment).

The universal covering space of $\mathbb{G}^d$ is $\mathbb{C}^d$, and under the universal covering morphism the arrangement $\mathscr{A}$ lifts to a periodic arrangement $\mathscr{A}^\upharpoonright$ of affine hyperplanes. An affine arrangement such as $\mathscr{A}^\upharpoonright$ is customarily described by the associated *semimatroid* [4, 28, 34] or, equivalently, by the poset $\mathcal{C}(\mathscr{A}^\upharpoonright)$ which in this case is naturally a *geometric semilattice*. The periodicity group acts naturally on this poset, and the quotient poset is isomorphic to $\mathcal{C}(\mathscr{A})$ [28, Remark 2.3]. The approach of [28], then, is to study group actions on semimatroids (or, equivalently, on geometric semilattices) and to view the quotients of such actions as the natural framework for an abstract combinatorial theory of posets of layers of Abelian arrangements. In this context, our results imply the following.

- To every Abelian arrangement is naturally associated a Stanley-Reisner ring via the associated periodic semimatroid. This ring is Cohen-Macaulay and its h-polynomial is an evaluation of the action's Tutte polynomial.
- The poset of layers of every Abelian arrangement is topologically Cohen-Macaulay, and its homotopy type is determined by the Tutte polynomial of the associated action. In the special case of hyperplane arrangements, this recovers the classical theory. In the case of (central) toric arrangements our rings are isomorphic to those studied by Martino [40] and Lenz [37], and the action's Tutte polynomial is Moci's arithmetic Tutte polynomial [44].

1.5. **Structure of the paper.** Section 2 reviews some background material and states preliminary lemmas on posets, simplicial complexes and group actions thereon. In Section 3 we define Stanley-Reisner rings for general finite-length simplicial posets and prove that we recover the classical theory in the case of trivial actions on finite posets. The naturality of translative actions with respect to taking invariant rings, resp. poset quotients is discussed in Section 4. Our structural result about preservation of Cohen-Macaulayness under refined actions is the focus of Section 5.

Then we turn to the matroidal case in Section 6, where we study Stanley-Reisner rings of group actions on (semi)matroids as well as the associated quotients of posets of flats. The application to the case of Abelian arrangements is discussed in Section 7, after a quick review of the context.

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2. Group actions on posets

This section is devoted to recalling some basics and laying some groundwork for the rest of the article. To the best of our knowledge there is no textbook to which we could refer throughout due to our need to encompass the case of infinite structures. Nevertheless, we point out the book [3] for a treatment of finite-length posets, as well as the standard reference (for the finite case) by Stanley [55]. For an introduction to simplicial and Cohen-Macaulay complexes and posets we refer to Björner's survey [10] and Stanley's original paper [56].

2.1. **Generalities on posets.** A partially ordered set, for short *poset*, is a set P with a partial order relation $\leqslant$ (i.e., a reflexive, antisymmetric and transitive binary relation). As usual, $x < y$ means $x \leqslant y, x \neq y$. We write $x \lessdot y$ if $x < y$ and $x \leqslant z < y$ implies $z = x$ (in this case we say that "y *covers* x"). We often only mention P when the order relation is understood. A morphism of posets is an order-preserving function; it is an isomorphism if it has an order-preserving inverse.

Example 2.1. Let $n \in \mathbb{N}$ and let B_n denote the poset of all subsets of $\{1, 2, \dots, n\}$ partially ordered by inclusion. A *Boolean algebra on* n *elements* is any poset isomorphic to B_n.

Let Aut(P) denote the set of *automorphisms* of P, i.e., of all isomorphisms from P to itself; Aut(P) is a group with respect to composition of functions.

Definition 2.2. An *action* $G \circlearrowright P$ of a group G on a poset P is a group homomorphism $G \to \text{Aut}(P)$. As is customary, we write gp for the image of any $p \in P$ under the automorphism associated to $g \in G$.

We define the *quotient* P/G to be the set of all orbits of elements of P with a binary relation $Gp \leqslant Gq$ if $gp \leqslant q$ for some $g \in G$.

Remark 2.3.

(i) The binary relation defined on P/G is always reflexive and transitive, but in general it might fail to be antisymmetric.

(ii) If P/G is a poset then the canonical "quotient" map $P \to P/G$, $p \mapsto Gp$ is a well-defined order-preserving map.

A *chain* in a poset P is any totally ordered subset $X \subseteq P$. The length of a chain X is the cardinal number $|X| - 1$. The *length* of the poset P is the maximum length of a chain in P. The length of P is denoted $\ell(P)$ and in general is allowed to be infinite. The poset is called *of finite length* if $\ell(P) < \infty$.

Lemma 2.4. *Let* P *be a finite-length poset. Then, for every action* $G \circlearrowright P$ *the set* P/G *with the binary relation of Definition 2.2 is a partially ordered set.*

Proof. By contraposition: as noted in Remark 2.3.(i), the only way in which P/G can fail to be a poset is that there are $p, q \in P$ an $g, h \in G$ such that $gp \leqslant q$ and $hq \leqslant p$ but $Gp \neq Gq$. If $gp = q$ or $hq = p$ then $Gp = Gq$, thus it must be $gp < q$ and $hq < p$. But then, $...g^{-1}q > p > hq > hgp > hghq...$ is an infinite chain in P. $\square$

Given a poset P and an element $x \in P$ let $P_{\leqslant x} := \{p \in P \mid p \leqslant x\}$ and consider it as a poset with the partial order induced from P. Given $A \subseteq P$ let $P_{\leqslant A} := \bigcap_{a \in A} P_{\leqslant a}$ be the set of *lower bounds* of A. We define $P_{\geqslant x}$ and $P_{\geqslant A}$, the set of upper bounds, analogously. A *(lower) order ideal* (or *down-set*) of a poset P is a subset $\mathfrak{a} \subseteq P$ such that $x \in \mathfrak{a}$ and $y \leqslant x$ implies $y \in \mathfrak{a}$. Examples of lower order ideals include subsets of the type $P_{\leqslant x}$, which we call *principal* lower order ideals (generated by x). *Upper order ideals*, resp. up-sets, are defined accordingly.

The *(closed) interval* between two elements $x, y \in P$ is the set $[x, y] := P_{\geqslant x} \cap P_{\leqslant y}$ with the induced partial order. The corresponding *open interval* is $(x, y) := [x, y] \setminus \{x, y\}$.

If a poset P has a unique minimal element, this element is commonly denoted by $\hat{0}$. Then $\{\hat{0}\} = P_{\leqslant P}$ and we say that P is *bounded below*. Analogously, P is bounded above if it has a unique maximal element, usually denoted by $\hat{1}$. We will often have to modify a poset by adding or removing extremal elements, and thus we introduce the following notation. Given a bounded-below poset P,

$\overline{P} := P \setminus \{\hat{0}\}$ denotes the poset obtained by removing the minimal element;

$\overline{\overline{P}}$ denotes the poset obtained from by removing both $\hat{0}$ and $\hat{1}$ (if the latter exists);
$\widehat{P}$ denotes the poset obtained from P by adding a maximal element $\hat{1}$.

Moreover, $A(P)$ denotes the set of *atoms* of P, i.e., all $p \in P$ with $\hat{0} \lessdot p$.

Definition 2.5. We call a bounded-below poset *graded* if it possesses a *rank function*, i.e., a function rk: $P \to \mathbb{N}$ such that $\mathrm{rk}(\hat{0}) = 0$ and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ whenever $x \lessdot y$. If such a rank function exists, then it is uniquely determined by $\mathrm{rk}(x) = \ell(P_{\leqslant x})$.

To every graded, bounded-below poset P of finite length d is associated a *characteristic polynomial*

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{d - \mathrm{rk}(x)},$$

where μ_P denotes the Möbius function of P, see [55, §3.7].

Lemma 2.6. *Let P be a graded poset. Then, for every action $G \circlearrowright P$ the set P/G with the binary relation of Definition 2.2 is a partially ordered set.*

Proof. Automorphisms of graded posets preserve the rank of elements. If P/G is not a poset, then, as in the proof of Lemma 2.3, we would find elements $p, q \in P$ and $g, h \in G$ with $gp < q < h^{-1}p$, in particular $hgp < p$, implying that the rank of hgp is strictly smaller than that of p – a contradiction. □

2.2. **Translative actions.** We introduce a class of actions on posets that has been studied in [28] as a natural abstraction of the action induced by linear translations on the poset of intersections of a periodic hyperplane arrangement, whence the name. See Section 7 for a more precise discussion of this context.

Definition 2.7. An action $G \circlearrowright P$ is called *translative* if, for every $p \in P$ and $g \in G$, whenever the set $\{p, gp\}$ has an upper bound (i.e., if $P_{\geqslant \{p, gp\}} \neq \emptyset$) then $p = gp$.

Remark 2.8. If $G \circlearrowright P$ is a translative action, then

(i) the intersection of any G-orbit $X \in P/G$ with any lower interval $P_{\leqslant p}$ consists of at most one element;
(ii) if $y \geqslant x$, then $\mathrm{stab}(y) \subseteq \mathrm{stab}(x)$ (in fact, $g \in \mathrm{stab}(y)$ implies $x, gx \leqslant y$).

Remark 2.9. In particular, translative actions are related to actions on *scwols* in the sense of [17, Chapter III.C, Definition 1.11] as follows. A translative action on a finite length poset P induces an action on the *scwol* defined on the set P by putting an arrow $x \to y$ whenever $x \leqslant y$.

Lemma 2.10. *Let G be a group acting translatively on a poset P and suppose that P/G is a poset. Then, for every $p \in P$ the restriction*

$$\varphi_p : P_{\leqslant p} \to (P/G)_{\leqslant Gp}$$

of the quotient map is an isomorphism of posets.

Proof. By Remark 2.3 the function φ_p is well-defined and order-preserving. The definition of the ordering among orbits implies that every $X \leqslant Gp$ contains a representative $x \in X$, $x \leqslant p$. Then, Remark 2.8 shows that the function

$$\psi : (P/G)_{\leqslant Gp} \to P_{\leqslant p}, \quad X \mapsto X \cap P_{\leqslant p}$$

is well-defined. To see that it is order-preserving consider $X \leqslant Y \leqslant Gp$ in P/G and notice that $X = G\psi(X)$ and $Y = G\psi(Y)$. Then, $X \leqslant Y$ implies that there is $g \in G$ s.t. $g\psi(X) \leqslant \psi(Y)$. In particular $g\psi(X) \leqslant p$, and thus with Remark 2.8 we have $g\psi(X) = \psi(X)$: we conclude $\psi(X) \leqslant \psi(Y)$ as required.

We are left with proving that ψ and φ_p are inverses. For every $X \in (P/G)_{\leqslant Gp}$ we have $\varphi_p \circ \psi(X) = G\psi(X) = X$, thus $\varphi_p \circ \psi = \mathrm{id}_{(P/G)_{\leqslant Gp}}$. Moreover, for every $q \leqslant p$ we have $Gq \cap P_{\leqslant p} \supseteq \{q\}$ and by Remark 2.8 this inclusion is an equality. Hence we compute $\psi \circ \varphi_p(q) = \psi(Gq) = q$. Thus, $\psi \circ \varphi_p = \mathrm{id}_{P_{\leqslant p}}$ as required. □

Lemma 2.11. *Let P be a poset, consider a translative action $G \circlearrowright P$ such that P/G is a poset. Let $f\colon P \to P/G$ denote the quotient map as above. Then for every $X \in P/G$*

(i) $f^{-1}((P/G)_{\geqslant X}) = \coprod_{x \in f^{-1}(X)} P_{\geqslant x}$.

Moreover, for every $x \in P$ the following hold.

(ii) There is an isomorphism of posets

$$(P/G)_{\geqslant Gx} \simeq P_{\geqslant x}/\operatorname{stab}(x)$$

and the action $\operatorname{stab}(x) \circlearrowright P_{\geqslant x}$ is translative.

(iii) If $\operatorname{stab}(x)$ is normal in G we can consider the group $H := G/\operatorname{stab}(x)$. Then, H acts translatively on $P_{\leqslant Gx} = f^{-1}((P/G)_{\leqslant Gx})$, and

$$(P/G)_{\leqslant Gx} \simeq P_{\leqslant Gx}/H.$$

Proof. Part (i) follows immediately by translativity (see Remark 2.8). For part (ii) write $X = Gx$ and compare the definitions:

$$P_{\geqslant x}/\operatorname{stab}(x) = \{\operatorname{stab}(x)y \mid y \geqslant_P x\}, \quad (P/G)_{\geqslant X} = \{Gy \mid y \geqslant_P gx \text{ for some } g\}$$

Now consider the map

$$P_{\geqslant x}/\operatorname{stab}(x) \to (P/G)_{\geqslant X}, \qquad \operatorname{stab}(x)y \mapsto Gy.$$

It is clearly well-defined and order-preserving. We will provide an order-preserving inverse. Consider $Y \in (P/G)_{\geqslant X}$. By definition, there is $y \in Y$ such that $y \geqslant_P x$. By part (i) this y is unique up to the action of $\operatorname{stab}(x)$. Thus the function

$$(P/G)_{\geqslant X} \to P_{\geqslant x}/\operatorname{stab}(x), \qquad Y \mapsto \operatorname{stab}(x)y$$

is well-defined. A straightforward check proves that this function is also order-preserving and it is indeed inverse to the previous. Translativity of $\operatorname{stab}(x) \circlearrowright P_{\geqslant x}$ is also easily verified.

Let us now consider part (iii). By definition of the quotient poset, every $Y \in (P/G)_{\leqslant Gx}$ is of the form $Y = Gy_Y$ for some $y_Y \leqslant x$. Translativity of $G \circlearrowright P$ implies uniqueness of such an y_Y, thus we have defined an order-preserving map $\varphi : (P/G)_{\leqslant Gx} \to P_{\leqslant Gx}/H$, $Y \mapsto Hy_Y$. A straightforward check shows that the obvious order-preserving map $\psi : P_{\leqslant Gx}/H \to (P/G)_{\leqslant Gx}$, $Hy \mapsto Gy$ is inverse to φ. An easy check of the definition verifies the translativity claim and concludes the proof. □

2.3. **Refined actions.** Let P be a graded poset of (finite) length d, and let G be a free Abelian group. Suppose that G acts on P so that there is some $k \in \mathbb{N}$ satisfying

($\star$) for all $x \in P$, $\mathrm{stab}(x)$ is a direct summand of G of rank $k(d - \mathrm{rk}(x))$,

where rk is the poset's rank function (see Definition 2.5).

Remark 2.12.

(i) Since G is a finitely generated free Abelian group, the condition for a subgroup H of G to be a direct summand of G is equivalent to H being a *pure* subgroup, meaning that G/H has no torsion elements (equivalently, $nh \in H$ implies $h \in H$ for every $h \in G$ and every $n > 0$). See [21, §16A].

(ii) For every $x \in P$ of maximal rank, ($\star$) implies $\mathrm{stab}(x) = \{0\}$. Moreover, ($\star$) implies also that $G \simeq \mathbb{Z}^{kd}$.

Definition 2.13 (Refined actions)**.** We call a group action on a graded poset P *refined* if it is translative and satisfies ($\star$) for some $k \in \mathbb{N}$. If we wish to specify the number k, we will call the action k-*refined*.

Lemma 2.14. *Suppose that the action of G on P is k-refined for some $k \in \mathbb{N}$.*

(i) For every $x \in P$ the action of the group $\mathrm{stab}(x)$ on $P_{\geqslant x}$ is k-refined.

(ii) For every $x \in P$ the action of the group $G/\mathrm{stab}(x)$ on $P_{\leqslant Gx}$ is k-refined.

Proof. By Lemma 2.11 we immediately know that both actions $\mathrm{stab}(x) \circlearrowright P_{\geqslant x}$ and $G/\mathrm{stab}(x) \circlearrowright P_{\leqslant Gx}$ are translative. Thus we only have to check condition ($\star$).

We start with (i). First, notice that $P_{\geqslant x}$ is ranked of rank $d' := d - \mathrm{rk}(x)$. Call $\mathrm{rk}_{\geqslant x}$ the rank function of $P_{\geqslant x}$. Call $G' := \mathrm{stab}(x)$ and consider $y \in P_{\geqslant x}$. By Remark 2.8.(ii), translativity of the action implies that $\mathrm{stab}_G(y) \subseteq G'$. Hence $\mathrm{stab}_{G'}(y) = \mathrm{stab}_G(y)$ and, by assumption, this group has rank $k(d - \mathrm{rk}(y)) = k(d - \mathrm{rk}_{\geqslant x}(y) - \mathrm{rk}(x)) = k(d' - \mathrm{rk}_{\geqslant x}(y))$ as required. Moreover, recall that G' is a direct summand of G, say $G = G' \oplus H$. In particular, G' is free Abelian. Finally, every torsion element in $G'/\mathrm{stab}_{G'}(y) = \mathrm{stab}_G(x)/\mathrm{stab}_G(y)$ is a torsion element in $G/\mathrm{stab}_G(y) = (G' \oplus H)/\mathrm{stab}_G(y) = G'/\mathrm{stab}_{G'}(y) \oplus H$. Since by assumption $\mathrm{stab}_G(y)$ is pure in G, we conclude that $\mathrm{stab}_{G'}(y)$ is pure in G'.

Now let us turn to (ii). Notice that $P_{\leqslant Gx}$ is ranked of length $d'' := \mathrm{rk}(x)$, and the rank function $\mathrm{rk}_{\leqslant Gx}$ is the restriction of the rank function rk of P. Since the original action satisfies ($\star$) we can write $G = H \oplus \mathrm{stab}(x)$ for some subgroup H, so $G/\mathrm{stab}(x) \simeq H$ is free Abelian. Now fix $y \in P_{\leqslant Gx}$ and consider $\mathrm{stab}_H(y)$. By definition there is $g \in G$ with $gx \geqslant y$. Hence, by commutativity of G and with Remark 2.8.(ii), $\mathrm{stab}_G(x) = \mathrm{stab}_G(gx) \subseteq \mathrm{stab}_G(y)$. Thus

$$\mathrm{stab}_H(y) \simeq \mathrm{stab}_G(y)/\mathrm{stab}_G(x). \tag{\dagger}$$

From this we can prove purity of $\mathrm{stab}_H(y)$ as a subgroup of H by writing

$$H/\mathrm{stab}_H(y) \simeq (G/\mathrm{stab}_G(x))/(\mathrm{stab}_G(y)/\mathrm{stab}_G(x)) \simeq G/\mathrm{stab}_G(y)$$

and noticing the latter group is torsion-free by assumption. Moreover, using Equation (†) and property ($\star$) for $G \circlearrowright P$ we can compute the rank of $\mathrm{stab}_H(y)$ to be $(d - \mathrm{rk}(y)) - (d - \mathrm{rk}(x)) = \mathrm{rk}(x) - \mathrm{rk}(y) = d'' - \mathrm{rk}_{\leqslant Gx}(y)$, as required. □

2.4. **Simplicial posets.**

Definition 2.15 (Compare [55])**.** A finite-length, countable poset P is called *simplicial* if it has a unique minimal element, and for all $p \in P$ the lower interval $P_{\leqslant p}$ is a Boolean algebra.

Remark 2.16. Every simplicial poset is graded in the sense of Definition 2.5. In particular, if P is a simplicial poset and $p \in P$, then $\mathrm{rk}(p)$ equals the number of elements of the Boolean algebra $P_{\leqslant p}$, i.e., $P_{\leqslant p}$ is isomorphic to $B_{\mathrm{rk}(p)}$ (cf. Example 2.1).

Lemma 2.17. *Let G be a group acting on a simplicial poset P. Then P/G is a simplicial poset if and only if the action is translative.*

Proof. That quotients of simplicial posets by translative actions are simplicial is an immediate consequence of Lemma 2.6 and Lemma 2.10.

For the reverse implication, consider a group G acting by automorphisms on a poset P and suppose that P/G is a simplicial poset. Given $y \in P$, since automorphisms preserve poset rank, we know that the rank of y in P equals the rank of Gy in P/G. Simpliciality of P and P/G then implies that $|P_{\leqslant y}| = |(P/G)_{\leqslant Gy}|$.

By way of contradiction suppose now that the action is not translative. This means that we can choose y so that there are $x \in P$, $g \in G$ with $x < y$, $gx < y$ and $gx \neq x$. In particular, the quotient map $P_{\leqslant y} \to P/G_{\leqslant Gy}$ is not injective. Since this map is surjective by definition, we conclude $|P_{\leqslant y}| > |(P/G)_{\leqslant Gy}|$ – a contradiction. □

Definition 2.18. We also recall from [56] the definition of the f-vector of a finite simplicial poset P of length d,

$$f(P) := (f_{-1}(P), \dots, f_{d-1}(P)), \quad \text{where } f_i(P) := |\{x \in P \mid \mathrm{rk}(x) = i+1\}|,$$

and of the associated h-polynomial

$$h_P(t) := t^d \sum_{i=0}^{d} f_{i-1}(P) \left(\frac{1-t}{t}\right)^{d-i},$$

where it is customary to set $f_{-1}(P) = 1$ for every P.

Remark 2.19. The h- and the characteristic polynomial of a simplicial poset P are related as follows:

$$\chi_P(t) = \sum_{i=0}^{d} f_{i-1}(P)(-1)^i t^{d-i} = (-1)^d h_P\left(\frac{1}{1-t}\right).$$

2.5. **Simplicial complexes.** Let V be a set. An abstract simplicial complex on the vertex set V is a family Σ of finite subsets of V that is closed under taking subsets (i.e., $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies $\tau \in \Sigma$). We will assume that every one-element subset of V is in Σ.

Elements of Σ are called *faces* (or *simplices*), and every face $\sigma \in \Sigma$ has a dimension $\dim \sigma := |\sigma| - 1$. The dimension of Σ is then the maximum of the dimensions of its faces: this can be an infinite cardinal, and we call Σ finite-dimensional if its dimension is finite. It is customary to call Σ *pure* (or pure-dimensional) if all maximal faces of Σ have the same dimension.

The set Σ partially ordered by inclusion is a simplicial poset P_Σ, ranked by cardinality of its elements. The atoms of P_Σ correspond to the single-element subsets of V. Every action $G \circlearrowright P_\Sigma$ induces an action on V and, vice-versa, the action on the whole poset is determined by the action on the vertices.

Remark 2.20. Given a translative action $G \circlearrowright P_\Sigma$, for every $x \in P_\Sigma$ we have $\mathrm{stab}(x) = \bigcap_{v \in x} \mathrm{stab}(v)$. (In fact, translativity implies $\mathrm{stab}(x) \subseteq \mathrm{stab}(v)$ for every $v \in x$ (Remark 2.8.(ii)). Since Σ is a simplicial complex, every $g \in \bigcap_{v \in x} \mathrm{stab}(v)$ fixes x because it fixes all its vertices.)

In particular, for every translative action $G \circlearrowright P_\Sigma$ the associated action on Σ satisfies Bredon's condition (A), see [16, §III.1].

Remark 2.21. Given $\sigma \in P_\Sigma$, then the poset $P_{\geqslant \sigma}$ is again the poset of faces of a simplicial complex. More precisely, consider the set $V' := \{\sigma' \in P_\Sigma \mid \sigma \lessdot \sigma'\}$. Then, $P_{\geqslant x} \simeq P_{\Sigma'}$ where $\Sigma' = \{X \subseteq V' \mid \cup X \in \Sigma\}$, which is isomorphic to the *link* of x in Σ. (Recall [10, §9.9] that the *link* of a face σ in a simplicial complex Σ is the simplicial complex $\mathrm{lk}(\sigma) := \{\tau \in \Sigma \mid \tau \cap \sigma = \varnothing \text{ and } \tau \cup \sigma \in \Sigma\}$.)

2.6. **Topology.** Every abstract simplicial complex as defined in the previous section has a geometric realization [45, Chapter 1, §2] which is unique up to homeomorphism. Hence, every abstract simplicial complex has a well-defined homotopy type.

Moreover, to every partially ordered set P we can associate the abstract simplicial complex of all finite chains in P. This is called the *order complex* of P (notice that its dimension equals the length of P). Thus a well-defined homotopy type can be associated to every partially ordered set. Order-preserving maps induce simplicial maps of order complexes and, thus, continuous maps between geometric realizations. When we will discuss topological attributes of a poset we will always think of them as referred to the order complex. For instance, with $H_i(P)$, $\pi_i(P)$ etc. we will mean the homology or homotopy groups of the order complex. For a more careful introduction and a broader account of the scope of combinatorial algebraic topology see, e.g., [35].

2.6.1. *Connectivity.* Given an integer $t \in \mathbb{N}$ we call a poset P t-*connected* if it is nonempty, connected and the homotopy groups $\pi_i(P)$ are trivial for all $i = 1, \dots, t$. Analogously we call P t-*acyclic* if the reduced homology $\widetilde{H}_i(P, \mathbb{Z})$ is trivial for $i = 0, \dots, t$. We extend these definition by saying any nonempty P to be "(-1)-acyclic" and "(-1)-connected".

Remark 2.22 (On shellability)**.** A simplicial complex Σ is called *shellable* if there exists a well-ordering $\prec$ on its set $\mathcal{M}$ of maximal simplices so that for all $\sigma \in \mathcal{M} \setminus \min_\prec \mathcal{M}$, the intersection of σ with the subcomplex $\Sigma_{\prec\sigma}$ induced by the simplices in $\mathcal{M}_{\prec\sigma}$ is a pure simplicial complex of dimension $\dim \sigma - 1$. If a pure, d-dimensional simplicial complex Σ is shellable, then it is $d-1$-connected, see [11, Remark 4.21].

We state for later reference the following lemma proved by Mirzaii and van der Kallen (independently published by Björner, Wachs and Welker for finite posets, although their proof can be easily adapted to the finite-length case).

Lemma 2.23 (cf. Theorem 3.8 of [43] and Corollary 3.2 of [14])**.** *Let* $f : P \to Q$ *be a poset map. Fix* $t \geqslant 0$ *and suppose that for all* $q \in Q$

(1) $Q_{>q}$ *is* $(t - \ell(Q_{<q}) - 2)$*-connected, and*
(2) the fiber $f^{-1}(Q_{\leqslant q})$ *is* $\ell(Q_{<q})$*-connected.*

Then the homotopy groups of P *and* Q *agree up to (and including) degree* t*. In particular,* P *is* t*-connected if and only if* Q *is* t*-connected.*

2.6.2. *Cohen-Macaulay complexes and posets.* We will be concerned with a well-known property of simplicial complexes with strong commutative-algebraic implications.

Definition 2.24. We will call a simplicial complex Σ of dimension d *Cohen-Macaulay* if for every face $\sigma \in \Sigma$ (including the case $\sigma = \emptyset$) the link $\mathrm{lk}(\sigma)$ of σ in Σ is $(\dim(\mathrm{lk}(\sigma)) - 1)$-connected. Accordingly, we call a poset P Cohen-Macaulay if the order complex of P is a Cohen-Macaulay simplicial complex.

Remark 2.25. The property defined above is usually referred to as *homotopy Cohen-Macaulay* property if one wishes to differentiate it from weaker (i.e., homological) versions. Since we will not use any of the other variations we call it simply "Cohen-Macaulay" in order to streamline the terminology.

2.6.3. *Euler characteristic.* As a last piece of preparation let us consider Euler characteristics of posets. We let $\epsilon(P)$ denote the *reduced* Euler characteristic of the order complex of P (from this the standard Euler characteristic can be recovered by adding 1, see [55]). We give for completeness a proof of the following elementary lemma.

Lemma 2.26. *Let* P *be a bounded-below poset. Then*

$$\epsilon(\overline{P}) = -\chi_P(1).$$

If P *is also bounded above, then*

$$\epsilon(\overline{\overline{P}}) = \chi_P(0).$$

Proof. Key is the following interpretation of the Möbius function of a bounded poset P known as "Hall's theorem" [55, Proposition 3.8.5]:

$$\mu_P(\hat{0}, \hat{1}) = \epsilon(\overline{\overline{P}}).$$

Recall that, since P is bounded below, $\widehat{P}$ denotes the poset P with a unique maximal element $\hat{1}$ adjoined. Then $\epsilon(\overline{P}) = \epsilon\left(\overline{\overline{\widehat{P}}}\right) = \mu_{\widehat{P}}(\hat{0}, \hat{1})$.

On the other hand, by definition of the Möbius function [55, Chapter 3, §7]

$$\mu_{\widehat{P}}(\hat{0}, \hat{1}) = -\sum_{\hat{0} \leqslant x < \hat{1}} \mu_{\widehat{P}}(\hat{0}, x) = -\sum_{x \in P} \mu_P(\hat{0}, x) = -\chi_P(1).$$

We conclude that $\epsilon(\overline{P}) = -\chi_P(1)$.

If P is also bounded above, then immediately $\chi_P(0) = \mu_P(\hat{0}, \hat{1}) = \epsilon(\overline{\overline{P}})$. ☐

2.7. **Algebra.** Given a finite simplicial complex Σ and a field $\Bbbk$, consider the polynomial ring $\Bbbk[x_v \mid v \in V]$ whose variables are indexed by vertices of Σ. Therein define the ideal

$$\mathcal{I}_\Sigma := (\prod_{v \in \sigma} x_v \mid \sigma \notin \Sigma)$$

generated by all monomials corresponding to non-faces of Σ.

Definition 2.27. The *Stanley-Reisner ring* of a finite simplicial complex Σ is the quotient

$$\mathcal{R}(\Sigma) := \Bbbk[x_v \mid v \in V]/\mathcal{I}_\Sigma$$

Remark 2.28. One of the basic facts about Stanley-Reisner rings is that the Cohen-Macaulay property for Σ (see Definition 2.24) implies the (algebraic) Cohen-Macaulay property for the ring $\mathcal{R}(\Sigma)$ over every field (the latter algebraic property is in fact equivalent to a homological version of the Cohen-Macaulay property for Σ, obtained by replacing "connected" by "acyclic" in Definition 2.24, see [49]).

Remark 2.29. Stanley defined an analogous ring associated to every finite simplicial poset. We will review this definition in Section 3.2.

3. Stanley-Reisner rings of finite-length simplicial posets

Throughout the section, P will be a simplicial poset with atoms $A(P)$ and G will be a group acting on P by automorphisms. Recall that, under these hypotheses, the quotient P/G is again a simplicial poset. Let us denote by $f\colon P \to P/G$ the standard projection. Given $p \in P$, as usual we use the notation Gp to denote $f(p) \in P/G$.

3.1. **The definition.** Let $\max(P)$ denote the set of maximal elements of P. Given a collection $\tau \subseteq \max(P)$, we denote by $\lceil\tau\rceil$ the order ideal of P given by $\bigcap_{p\in\tau} P_{\leqslant p}$. Note that $\lceil\tau\rceil$ is the face poset of a simplicial complex. The associated Stanley-Reisner ring will be denoted by $R_{\lceil\tau\rceil}$. When $\lceil\tau\rceil = \{\hat{0}\}$, one has that $R_{\lceil\tau\rceil} = \Bbbk$. For any set bounded above inside a given $\lceil\tau\rceil$, the join in $\lceil\tau\rceil$ is well-defined.

Following Yuzvinsky [59], we call $X(P)$ the set

$$\{\lceil\tau\rceil \mid \varnothing \neq \tau \subseteq \max(P)\}$$

of all (lower) order ideals $\lceil\tau\rceil \subseteq P$ coming from nonempty collections $\tau \subseteq \max(P)$, with the partial order given by

$$\lceil\tau\rceil \leqslant_{X(P)} \lceil\tau'\rceil \text{ if and only if } \lceil\tau\rceil \supseteq \lceil\tau'\rceil.$$

Any poset can be made into a topological space by considering the Alexandrov topology, where the open sets are the upper sets of the poset. Any sheaf (say, of commutative rings) on a poset is then completely determined by the assignment of a covariant functor from the poset (seen as a category as in Remark 2.9) to the category of commutative rings (see, e.g., [5]).

With this in mind, again following [59], we define the sheaf of commutative rings $Y(P)$ on the poset $X(P)$ by the assignments

$$X(P) \ni \lceil\tau\rceil \mapsto R_{\lceil\tau\rceil}$$
$$(\lceil\tau\rceil \leqslant_{X(P)} \lceil\tau'\rceil) \mapsto \pi^{\lceil\tau\rceil}_{\lceil\tau'\rceil}\colon R_{\lceil\tau\rceil} \twoheadrightarrow R_{\lceil\tau'\rceil},$$

where $\pi^{\lceil\tau\rceil}_{\lceil\tau'\rceil}\colon R_{\lceil\tau\rceil} \twoheadrightarrow R_{\lceil\tau'\rceil}$ is the natural projection.

Definition 3.1. The *Stanley-Reisner ring* of P is then the ring of (global) sections

$$\mathcal{R}(P) := \Gamma(Y(P)).$$

We view any $q \in \mathcal{R}(P)$ as an $X(P)$-tuple of polynomials, and for every $\lceil\tau\rceil \in X(P)$ we denote by $q_{\lceil\tau\rceil} \in R_{\lceil\tau\rceil}$ the component associated to $\lceil\tau\rceil$.

With a slight abuse of notation, we will reserve the letter σ to denote both a maximal element of P and a minimal element of $X(P)$, since every minimal element $\lceil\{\sigma\}\rceil$ of $X(P)$ is uniquely determined by the choice of a maximal element σ of P. In particular, we will write q_σ for $q_{\lceil\{\sigma\}\rceil}$.

Let us record here a simple observation that will come in handy in what follows.

Definition 3.2. Let $p \in P$. We say that a monomial $m = \prod_{\nu\in A(P)} x_\nu^{\alpha_\nu}$ is *supported on* p if $\alpha_\nu = 0$ for all $\nu \in A(P)\setminus P_{\leqslant p}$.

Lemma 3.3. *Let* $q \in \Gamma(Y(P))$, $p \in P$, $\sigma, \sigma' \in \max(P)$ *such that* $p \leqslant \sigma, \sigma'$. *Then the monomials supported on* p *appear with the same coefficients inside* q_σ *and* $q_{\sigma'}$.

Proof. Since $p \in P_{\leqslant\sigma} \cap P_{\leqslant\sigma'}$, any monomial supported on p appears inside q_σ (respectively $q_{\sigma'}$) with the same coefficient it has inside $q_{\lceil\tau\rceil} \in R_{\lceil\tau\rceil}$, where $\tau = \{\sigma, \sigma'\}$. □

3.2. **The finite case.** In this section we show that, in the finite case, we recover the classical constructions.

Fix a field $\Bbbk$. Given a finite simplicial poset P, we let $\widetilde{S}$ be the polynomial ring $\Bbbk[x_p \mid p \in P]$ and $S := \widetilde{S}/(x_{\hat{0}} - 1)$ be its dehomogenization with respect to $x_{\hat{0}}$. Given a order ideal $\mathfrak{a}$ inside P, we define $\widetilde{I^P_{\mathfrak{a}}}$ to be the following ideal of S:

$$\widetilde{I^P_{\mathfrak{a}}} := (x_p \mid p \notin \mathfrak{a}) + (x_p x_q - x_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z\in\mathfrak{a}}} x_z \mid p, q \in \mathfrak{a},\ p, q \text{ incomparable}).$$

Let then $I^P_{\mathfrak{a}}$ be the dehomogenization of $\widetilde{I^P_{\mathfrak{a}}}$ with respect to $x_{\hat{0}}$, i.e. the ideal of S obtained from $\widetilde{I^P_{\mathfrak{a}}}$ by setting to 1 all occurrences of the variable $x_{\hat{0}}$.

Definition 3.4 ([56, Section 3])**.** To every finite simplicial poset P there are two associated rings,

$$\widetilde{A_P} := \widetilde{S}/\widetilde{I^P_P}, \qquad A_P := S/I^P_P.$$

Proposition 3.5 ([56, Section 3])**.** *When* P *is the poset of faces of a finite simplicial complex* Σ, *then* A_P *coincides with* $\mathcal{R}(\Sigma)$. *In general, the* h*-polynomial of* A_P *coincides with* $h_P(t)$, *the* h*-polynomial of the simplicial poset* P *(see Definition 2.18).*

If the poset P *is Cohen-Macaulay, then so are* A_P *and* $\widetilde{A_P}$.

Let us record here a technical observation that will come in handy while proving Proposition 3.7 below.

Lemma 3.6. *Let* P *be a finite simplicial poset and* $\mathfrak{a}$ *a lower order ideal of* P. *Then* $I^P_{\mathfrak{a}}$ *is a radical ideal.*

Proof. By [20, Proposition 16.23] it is enough to check that $\widetilde{I^P_\mathfrak{a}}$ is radical, i.e. $\widetilde{S}/\widetilde{I^P_\mathfrak{a}}$ is reduced. Now note that $\widetilde{S}/\widetilde{I^P_\mathfrak{a}} \cong \widetilde{T}/\widetilde{I^\mathfrak{a}_\mathfrak{a}}$, where $\widetilde{T} = \Bbbk[x_p \mid p \in \mathfrak{a}]$. By [56, Lemma 3.4], $\widetilde{T}/\widetilde{I^\mathfrak{a}_\mathfrak{a}}$ is an algebra with straightening law on $\mathfrak{a}$ (seen as a poset on its own) and hence is reduced, as desired. □

Proposition 3.7. *Let* P *be a finite simplicial poset and let* $\mathfrak{a}$, $\mathfrak{b}$ *be order ideals of* P. *Then:*

(1) $I^P_\mathfrak{a} + I^P_\mathfrak{b} = I^P_{\mathfrak{a}\cap\mathfrak{b}}$ *(and hence the correspondence* $\mathfrak{a} \mapsto I^P_\mathfrak{a}$ *reverses inclusions);*
(2) $I^P_\mathfrak{a} \cap I^P_\mathfrak{b} = I^P_{\mathfrak{a}\cup\mathfrak{b}}$.

Proof. For brevity's sake, given elements p and q in P, we will use the symbol $y_{p\wedge q}$ defined in the following way:

$$y_{p\wedge q} := \begin{cases} 1 & \text{if } p \wedge q = \hat{0} \\ x_{p\wedge q} & \text{otherwise.} \end{cases}$$

We now proceed to the proof.

(1) By definition,

$$\begin{aligned} I^P_\mathfrak{a} + I^P_\mathfrak{b} &= (x_p \mid p \notin \mathfrak{a}) + (x_p x_q - y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{a}}} x_z \mid p, q \in \mathfrak{a},\ p, q \text{ incomparable}) \\ &\quad + (x_p \mid p \notin \mathfrak{b}) + (x_p x_q - y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{b}}} x_z \mid p, q \in \mathfrak{b},\ p, q \text{ incomparable}) \\ &= (x_p \mid p \notin \mathfrak{a}\cap\mathfrak{b}) + (x_p x_q - y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{a}}} x_z \mid p, q \in \mathfrak{a},\ p, q \text{ incomparable}) \\ &\quad + (x_p x_q - y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{b}}} x_z \mid p, q \in \mathfrak{b},\ p, q \text{ incomparable}). \end{aligned}$$

Now pick two incomparable elements p, q in $\mathfrak{a}$ and consider the generator $x_p x_q - y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z\in\mathfrak{a}}} x_z$.

- If at least one of p and q does not lie in $\mathfrak{b}$, then the whole generator is superfluous.
- If both p and q lie in $\mathfrak{b}$, one rewrites the generator as

$$x_p x_q - \left(y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{a}\cap\mathfrak{b}}} x_z \right) - \left(y_{p\wedge q} \sum_{\substack{z \geqslant p,q \\ z \in \mathfrak{a}\setminus\mathfrak{b}}} x_z \right),$$

where the last summand is now superfluous.

The claim now follows.

(2) It is enough to prove that $I^P_\mathfrak{a} = \bigcap_{q\in\mathfrak{a}} I^P_{(q)}$, where (q) denotes the principal order ideal generated by the element q. Note that, by part (1), $I^P_{(q)} \supseteq I^P_\mathfrak{a}$ for any $q \in \mathfrak{a}$. Moreover, for $\mathfrak{a} = \varnothing$ the claim holds trivially (taking the empty intersection of ideals to be the ring S). Let then $\mathfrak{a}$ be nonempty.

Since by Lemma 3.6 $I^P_\mathfrak{a}$ is radical, it is enough to prove that, whenever a prime ideal $\wp$ in S contains $I^P_\mathfrak{a}$, then it also contains the prime ideal $I^P_{(q)}$ for some $q \in \mathfrak{a}$.

To prove this, consider the (nonempty) set of maximal elements in $\mathfrak{a}$. There are two cases:

(i) There is exactly one maximal element M in $\mathfrak{a}$. In this case, $\wp \supseteq I^P_{\mathfrak{a}} = I^P_{(M)}$ and we are done.

(ii) There are at least two maximal elements M, M' in $\mathfrak{a}$. In this case the monomial $x_M x_{M'}$ must belong to $I^P_{\mathfrak{a}}$ and hence the prime ideal $\wp$ is forced to contain at least one of x_M and $x_{M'}$. Without loss of generality, say $x_M \in \wp$. Then $\wp \supseteq I^P_{\mathfrak{a}} + (x_M) = I^P_{\mathfrak{a}\setminus\{M\}}$.

Since P is finite and case (ii) gives us a reduction from $\mathfrak{a}$ to a strictly smaller order ideal $\mathfrak{a}\setminus\{M\}$, we are bound to meet case (i) at some point. Notice that the element we eventually find in case (i) will not in general be maximal in the original order ideal $\mathfrak{a}$.

□

Corollary 3.8. The ideals $I^P_{\mathfrak{a}}$ in S, ordered by inclusion, form a distributive lattice with respect to sum and intersection.

Proof. This is a direct consequence of Proposition 3.7, since order ideals of P form a distributive lattice with respect to intersection and union [55, §3.4]. □

Theorem 1. *For every finite simplicial poset* P*,*

$$\mathcal{R}(P) = A_P,$$

i.e., we recover Stanley's ring associated to P*.*

Proof. Let $\{p_1, \ldots, p_k\}$ denote the maximal elements of P. The sheaf $Y(P)$ on $X(P)$ satisfies the hypotheses of [18, Example 3.3] and hence we get a full description for the global sections of $Y(P)$:

$$\Gamma(Y(P)) = S/\bigcap_{i=1}^{k} I^P_{(p_i)} = S/I^P_P,$$

where the last equality comes from Proposition 3.7. □

Remark 3.9. After completing our work, we became aware of work of Lü and Panov [38] from which our Theorem 1 follows. Moreover, notice that Brun and Römer [19] prove the analogous statement for $\widetilde{A_P}$.

Notice that our proof also produces a minimal prime ideal decomposition of I^P_P into the $I^P_{(p_i)}$.

4. Induced actions and invariant rings

We now want to bring group actions into the picture, proving that every group action on a simplicial poset induces an action on the associated ring. Moreover, if the action is translative we will prove that the ring of invariants is isomorphic to the ring associated to the quotient poset.

4.1. **The induced action on $\mathcal{R}(P)$.** Consider an action $G \circlearrowright P$ of a group G by automorphisms of P. Given $g \in G$, let us define ω^g as the automorphism of $\Bbbk[x_v \mid v \in A(P)]$ obtained by sending x_v into x_{gv}. Given a collection $\tau \subseteq \max(P)$, one has that the assignment $p \mapsto gp$ induces an isomorphism of simplicial complexes $\lceil\tau\rceil \xrightarrow{\cong} \lceil g\tau\rceil$. Hence, ω^g induces a ring isomorphism $\omega^g \colon R_{\lceil\tau\rceil} \xrightarrow{\cong} R_{\lceil g\tau\rceil}$ between the corresponding Stanley-Reisner rings. Moreover, if $\tau \leqslant_{X(P)} \tau'$, then the following diagram

$$\begin{array}{ccc} R_{\lceil\tau\rceil} & \xrightarrow{\omega^g} & R_{\lceil g\tau\rceil} \\ {\scriptstyle \pi^{\lceil\tau\rceil}_{\lceil\tau'\rceil}}\downarrow & & \downarrow{\scriptstyle \pi^{\lceil g\tau\rceil}_{\lceil g\tau'\rceil}} \\ R_{\lceil\tau'\rceil} & \xrightarrow[\omega^g]{} & R_{\lceil g\tau'\rceil} \end{array} \tag{2}$$

commutes.

Definition 4.1. Consider a simplicial poset P with an action of a group G. Given any element $q = (q_{\lceil\tau\rceil})_{\lceil\tau\rceil \in X(P)}$ of $\mathcal{R}(P)$ and any $g \in G$ define the $X(P)$-tuple gq by

$$gq_{\lceil\tau\rceil} := \omega^g(q_{\lceil g^{-1}\tau\rceil}).$$

Lemma 4.2. *For any given action of a group G on a simplicial poset P by automorphisms, the assignment*

$$G \times \mathcal{R}(P) \to \mathcal{R}(P), \qquad (g, q) \mapsto gq$$

defines an action of G by (ring) automorphisms on $\mathcal{R}(P)$.

Proof. Let us first check that gq is indeed a global section of $Y(P)$: given $\tau \leqslant_{X(P)} \tau'$, one has that

$$\begin{aligned} \pi^{\lceil\tau\rceil}_{\lceil\tau'\rceil}(gq_{\lceil\tau\rceil}) &= \pi^{\lceil\tau\rceil}_{\lceil\tau'\rceil} \circ \omega^g(q_{\lceil g^{-1}\tau\rceil}) \\ &= \omega^g \circ \pi^{\lceil g^{-1}\tau\rceil}_{\lceil g^{-1}\tau'\rceil}(q_{\lceil g^{-1}\tau\rceil}) && \text{by the commutativity of (2)} \\ &= \omega^g(q_{\lceil g^{-1}\tau'\rceil}) && \text{since } q \text{ is a section} \\ &= gq_{\lceil\tau'\rceil}. \end{aligned}$$

Hence for every g we have a well-defined map $\varphi_g : \mathcal{R}(P) \to \mathcal{R}(P)$, $q \mapsto gq$. That φ_g is a ring homomorphism follows from the fact that ω^g is a ring homomorphism and that elements ($X(P)$-tuples) of $\mathcal{R}(P)$ are added and multiplied componentwise.

□

4.2. **Invariant rings for translative actions.** From now on we will require that the action of G be translative. Consider $\sigma \in \max(P)$, $\Sigma \in \max(P/G)$ such that $G\sigma = \Sigma$. By Lemma 2.10, translativity allows us to define the following ring isomorphism:

$$\Bbbk[x_v \mid v \in A(P) \cap P_{\leqslant\sigma}] \ \underset{\zeta^{\Sigma}_{\sigma}}{\overset{\zeta^{\sigma}_{\Sigma}}{\rightleftarrows}}\ \Bbbk[x_{\mathcal{V}} \mid \mathcal{V} \in A(P/G) \cap (P/G)_{\leqslant\Sigma}] \qquad (\cong)$$

For each $g \in G$, considering $\omega^g \colon \Bbbk[x_\nu \mid \nu \in A(P) \cap P_{\leqslant \sigma}] \xrightarrow{\cong} \Bbbk[x_\nu \mid \nu \in A(P) \cap P_{\leqslant g\sigma}]$ yields

$$\zeta_\Sigma^{g\sigma} \circ \omega^g = \zeta_\Sigma^\sigma \quad \text{and} \quad \omega^g \circ \zeta_\sigma^\Sigma = \zeta_{g\sigma}^\Sigma. \tag{3}$$

We now have all the ingredients for the main result of the section.

Theorem 2. *Let* G *be a group acting translatively on the simplicial poset* P*. Then there is a ring isomorphism* $\mathcal{R}(P)^G \cong \mathcal{R}(P/G)$.

Proof. We will define two mutually inverse ring homomorphisms φ and ψ between $\mathcal{R}(P)^G = \Gamma(Y(P))^G$ and $\mathcal{R}(P/G) = \Gamma(Y(P/G))$.

1. Definition of φ. Let $\varphi \colon \Gamma(Y(P))^G \to \Gamma(Y(P/G))$ be the map of rings defined for each $q \in \Gamma(Y(P))^G$ and $\lceil T \rceil \in X(P/G)$ by

$$(\varphi(q))_{\lceil T \rceil} := \pi_{\lceil T \rceil}^\Sigma \circ \zeta_\Sigma^\sigma(q_\sigma),$$

where $\Sigma \in \max(P/G)$ is such that $\lceil T \rceil \subseteq (P/G)_{\leqslant \Sigma}$ (in other words, Σ is any minimal element of $X(P/G)$ lying below $\lceil T \rceil$) and $\sigma \in f^{-1}(\Sigma)$.

2. The map φ *is well-defined.* First of all, once Σ is fixed, due to the G-invariance of q it makes no difference which representative σ we pick inside $f^{-1}(\Sigma)$. More precisely, one has that

$$\begin{aligned} \zeta_\Sigma^{g\sigma}(q_{g\sigma}) &= \zeta_\Sigma^{g\sigma} \circ \omega^g(q_\sigma) && \text{since } q \text{ is } G\text{-invariant} \\ &= \zeta_\Sigma^\sigma(q_\sigma) && \text{due to (3).} \end{aligned}$$

Let us now check that the definition of φ is independent on the choice of Σ. Let us pick $(\Sigma, \sigma), (\Sigma', \sigma')$ as in the definition above and let us show that

$$\pi_{\lceil T \rceil}^\Sigma \circ \zeta_\Sigma^\sigma(q_\sigma) = \pi_{\lceil T \rceil}^{\Sigma'} \circ \zeta_{\Sigma'}^{\sigma'}(q_{\sigma'}). \tag{4}$$

To do this, it is enough to check that any nonzero monomial in $R_{\lceil T \rceil}$ appears with the same coefficient on both sides of (4). We will use the notation $\langle m, f \rangle$ to denote the coefficient of the monomial m in the polynomial f. Let us fix a nonzero monomial M in $R_{\lceil T \rceil}$. By definition, M must be supported on an element $\Upsilon \in \lceil T \rceil$. By construction, one has that $\Upsilon \leqslant_{P/G} \Sigma$ and $\Upsilon \leqslant_{P/G} \Sigma'$. One can now choose $\upsilon \in f^{-1}(\Upsilon)$ such that $\upsilon \leqslant_P \sigma$ and $\upsilon \leqslant_P g\sigma'$ for some $g \in G$. Note that, by construction, $\zeta_\Sigma^\sigma M$ and $\zeta_{\Sigma'}^{g\sigma'} M$ represent the same monomial (supported on υ), that we will denote by m. By Lemma 3.3 we then have that $\langle m, q_\sigma \rangle = \langle m, q_{g\sigma'} \rangle$. We now get the desired result, since

$$\begin{aligned} \langle M, \pi_{\lceil T \rceil}^\Sigma \circ \zeta_\Sigma^\sigma(q_\sigma) \rangle &= \langle M, \zeta_\Sigma^\sigma(q_\sigma) \rangle = \langle m, q_\sigma \rangle = \langle m, q_{g\sigma'} \rangle \\ &= \langle M, \zeta_{\Sigma'}^{g\sigma'}(q_{g\sigma'}) \rangle = \langle M, \zeta_{\Sigma'}^{\sigma'}(q_{\sigma'}) \rangle = \langle M, \pi_{\lceil T \rceil}^{\Sigma'} \circ \zeta_{\Sigma'}^{\sigma'}(q_{\sigma'}) \rangle. \end{aligned}$$

Since the restriction maps behave well, one has that $\varphi(q)$ is indeed a global section of $Y(P/G)$ and thus φ is well-defined. △

3. Definition of ψ. Let

$$\psi\colon \Gamma(Y(P/G)) \to \Gamma(Y(P))^G$$

be the map of rings defined for each $Q \in \Gamma(Y(P/G))$ and $\lceil\tau\rceil \in X(P)$ by

$$(\psi(Q))_{\lceil\tau\rceil} := \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma}(Q_{G\sigma}),$$

where $\sigma \in \max(P)$ is such that $\lceil\tau\rceil \subseteq P_{\leqslant\sigma}$.

4. The map ψ *is well-defined.* We need to check that ψ is independent on the choice of σ, i.e. that, given σ and σ' as above,

$$\pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma}(Q_{G\sigma}) = \pi^{\sigma'}_{\lceil\tau\rceil} \circ \zeta^{G\sigma'}_{\sigma'}(Q_{G\sigma'}). \tag{5}$$

Let us consider a nonzero monomial m in $R_{\lceil\tau\rceil}$ supported on $p \in P$. Note that $\zeta^{\sigma}_{G\sigma}m$ and $\zeta^{\sigma'}_{G\sigma'}m$ represent the same monomial (supported on Gp), which we will denote by M. Since $Gp \leqslant_{P/G} G\sigma, G\sigma'$, by Lemma 3.3 one has that $\langle M, Q_{G\sigma}\rangle = \langle M, Q_{G\sigma'}\rangle$. This leads us to the desired result, since

$$\begin{aligned}\langle m, \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma}(Q_{G\sigma})\rangle &= \langle m, \zeta^{G\sigma}_{\sigma}(Q_{G\sigma})\rangle = \langle M, Q_{G\sigma}\rangle = \langle M, Q_{G\sigma'}\rangle \\ &= \langle m, \zeta^{G\sigma'}_{\sigma'}(Q_{G\sigma'})\rangle = \langle m, \pi^{\sigma'}_{\lceil\tau\rceil} \circ \zeta^{G\sigma'}_{\sigma'}(Q_{G\sigma'})\rangle.\end{aligned}$$

Again, $\psi(Q)$ is a global section of $Y(P)$ since restriction maps behave well. We still need to check that $\psi(Q)$ is G-invariant. This is indeed the case, since for each $Q \in \Gamma(Y(P/G))$, $\lceil\tau\rceil \in X(P)$, and $g \in G$ one has that

$$\begin{aligned} g\psi(Q)_{\lceil g\tau\rceil} &= \omega^g \circ \psi(Q)_{\lceil\tau\rceil} \\ &= \omega^g \circ \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma}(Q_{G\sigma}) \\ &= \pi^{g\sigma}_{\lceil g\tau\rceil} \circ \omega^g \circ \zeta^{G\sigma}_{\sigma}(Q_{G\sigma}) && \text{by the commutativity of (2)} \\ &= \pi^{g\sigma}_{\lceil g\tau\rceil} \circ \zeta^{G\sigma}_{g\sigma}(Q_{G\sigma}) && \text{due to (3)} \\ &= \psi(Q)_{\lceil g\tau\rceil}. \end{aligned}$$

It follows that ψ is well-defined. △

5. φ *and* ψ *are inverses.* Finally, it is easy to see that φ and ψ are inverse to each other. Given $q \in \Gamma(Y(P))^G$, for every $\lceil\tau\rceil \in X(P)$ one has that

$$\begin{aligned} \psi(\varphi(q))_{\lceil\tau\rceil} &= \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma}(\varphi(q)_{G\sigma}) \\ &= \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma} \circ \pi^{G\sigma}_{G\sigma} \circ \zeta^{\sigma}_{G\sigma}(q_{\sigma}) && \text{choosing } \sigma \text{ inside } f^{-1}(G\sigma) \\ &= \pi^{\sigma}_{\lceil\tau\rceil} \circ \zeta^{G\sigma}_{\sigma} \circ \zeta^{\sigma}_{G\sigma}(q_{\sigma}) \\ &= \pi^{\sigma}_{\lceil\tau\rceil}(q_{\sigma}) \\ &= q_{\lceil\tau\rceil}. \end{aligned}$$

Analogously, given $Q \in \Gamma(Y(P/G))$ one has that, for every $\lceil T\rceil \in X(P/G)$,

$$\begin{aligned}\varphi(\psi(Q))_{\lceil T \rceil} = \pi^{\Sigma}_{\lceil T \rceil} \circ \zeta^{\sigma}_{\Sigma}(\psi(Q)_{\sigma}) &= \pi^{\Sigma}_{\lceil T \rceil} \circ \zeta^{\sigma}_{\Sigma} \circ \pi^{\sigma}_{\sigma} \circ \zeta^{\Sigma}_{\sigma}(Q_{\Sigma}) \\ &= \pi^{\Sigma}_{\lceil T \rceil} \circ \zeta^{\sigma}_{\Sigma} \circ \zeta^{\Sigma}_{\sigma}(Q_{\Sigma}) = \pi^{\Sigma}_{\lceil T \rceil}(Q_{\Sigma}) = Q_{\lceil T \rceil}.\end{aligned}$$

□

5. Refined actions and the Cohen-Macaulay property

5.1. **Actions on simplicial complexes.** In this section let P denote the poset of cells of a finite-dimensional simplicial complex Σ on the set of vertices V. Call d the dimension of Σ, i.e., the length of P.

Our main result here will be Theorem 3, proving that refined quotients of posets of faces of (finite-dimensional) Cohen-Macaulay simplicial complexes are Cohen-Macaulay.

Lemma 5.1. *Let* $G \circlearrowright P$ *be a* k*-refined action, let* σ *be an element of* P *with* $\dim \sigma = d$ *and let* $v \in \sigma$*. Then,*

$$G = \mathrm{stab}(\sigma \backslash \{v\}) \oplus \mathrm{stab}(v).$$

Proof. First, notice that property $(\star)$ implies $\mathrm{stab}(v) \cap \mathrm{stab}(\sigma \backslash v) \subseteq \mathrm{stab}(\sigma) = \{0\}$. Hence, the sum $\mathrm{stab}(\sigma\backslash\{v\}) + \mathrm{stab}(v)$ is direct and has maximal rank in G.

Moreover, since $\mathrm{stab}(v)$ is pure, there is a rank k free subgroup $Z < G$ with

$$G = Z \oplus \mathrm{stab}(v).$$

Choose bases $\beta_{k+1}, \ldots, \beta_{kd}$ of $\mathrm{stab}(v)$, $\zeta_1, \ldots, \zeta_k$ of Z and $\beta_1, \ldots, \beta_k$ of $\mathrm{stab}(\sigma\backslash\{v\})$. Then, $\{\zeta_1, \ldots, \zeta_k, \beta_{k+1}, \ldots, \beta_{kd}\}$ is a basis of G and $\{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_{d-1}\}$ is a basis of $\mathrm{stab}(\sigma\backslash\{v\}) \oplus \mathrm{stab}(v)$. The matrix of the inclusion $\mathrm{stab}(\sigma\backslash\{v\}) \oplus \mathrm{stab}(v) \hookrightarrow G$ must be full-rank and is in fact the identity except for the first k columns. We can perform unimodular column operations in order to make this a block-diagonal $(k + (d-1)k) \times (k + (d-1)k)$-matrix whose bottom-right block is the identity and the upper-left $(k \times k)$ block is in Hermite normal form [51, §4.1], i.e., as in the left-hand side of Figure 1.

$$\begin{bmatrix} >1 & 0 & & \cdots & & 0 \\ * & \ddots & \ddots & & & \\ * & * & >1 & 0 & & \vdots \\ 1 & \cdots & \cdots & 1 & \ddots & \\ \vdots & & & & \ddots & 0 \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \qquad \begin{bmatrix} >1 & 0 & & \cdots & & 0 \\ * & \ddots & 0 & & & \\ * & * & >1 & 0 & & \vdots \\ 0 & \cdots & 0 & 1 & \ddots & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Figure 1

If the claimed equality does not hold, then, since both groups have the same rank, this matrix must have determinant different from 1: therefore, the upper-left block will have the form on the left-hand side of Figure 1 with at least one diagonal entry bigger than 1. Further unimodular column operations will then lead to a form such as that on the right-hand side of Figure 1, say with $t > 0$ columns having a diagonal entry different than 1.

This means, however, that there is an element $\gamma \in G$ (i.e., the t-th element of the basis obtained from $\{\beta_1, \dots, \beta_k, \beta_{k+1}, \dots, \beta_{kd}\}$ after the diagonalization) for which $\gamma \neq \mathrm{stab}(\sigma \backslash \{v\})$ but $m\gamma = \zeta_t \in \mathrm{stab}(\sigma \backslash \{v\})$ for some $m > 1$, contradicting purity of $\mathrm{stab}(\sigma \backslash \{v\})$. □

Lemma 5.2. *Let P be the poset of faces of a finite-dimensional simplicial complex Σ and suppose that P carries a refined action $G \circlearrowright P$ such that the induced action on $\max P$ has only one orbit. Then, $\hat{P}$ is chain-lexicographically shellable (in the sense of, e.g.,* [13]*). In particular, $\overline{P}$ is $(\ell(P) - 2)$-connected.*

Proof. We will show that $\widehat{P}$ admits a recursive coatom (well-)ordering (cf. [13, Theorem 5.11]). Choose $x = \{s_1, \dots, s_d\} \in \max P$ and for brevity write

$$M := \max P = Gx.$$

For every $s \in x$ we set $\Gamma(s) := G/\mathrm{stab}(s)$ and $\Gamma^x := \prod_{s \in x} \Gamma(s)$. We will use additive notation for Γ^x which is, by ($\star$), a free Abelian group of rank kd. The natural quotient maps $\varphi_s : G \to \Gamma(s)$ define a group homomorphism

$$\varphi : G \to \Gamma^x, g \mapsto (\varphi_s(g))_s.$$

Moreover, for every $s \in x$ the orbit Gs is in bijection with $\Gamma(s)$ via $gs \mapsto \varphi_s(g)$. In particular, for any fixed basis $\gamma_s^1, \dots, \gamma_s^k$ of $\mathrm{stab}(x \backslash \{s\})$, the images $\{\varphi_s(\gamma_s^i)\}_i$ are a basis of $\Gamma(s)$.

Hence we have an injective map

$$\lambda : M \to \Gamma^x, gx \mapsto \varphi(g)$$

with, for every $m, m' \in M$,

$$m \cap m' = \{s \in m \mid \lambda(m)_s = \lambda(m')_s\}. \tag{†}$$

It is easy to see that

$$\lambda(gm) = \varphi(g) + \lambda(m) \text{ for all } g \in G \text{ and all } m \in M. \tag{‡}$$

If we choose a basis $\gamma_s^1, \dots, \gamma_s^k$ of the group $\mathrm{stab}(x \backslash \{s\})$, by Lemma 5.1 we know that, for all $i = 1, \dots, k$,

$$\varphi_{s'}(\gamma_s^i) = \begin{cases} 0 & s \neq s' \\ \varphi_s(\gamma_s^i) & s = s' \end{cases}$$

For every $s \in x$ choose a well-ordering $<_s$ of the groups $\Gamma(s)$ that begins with the identity element 0_s. Let then $\prec$ be the total order on M induced via λ from the lexicographic ordering of Γ^x (i.e., we choose a total order of x, we write the elements of Γ^x as ordered x-tuples accordingly, and consider the lexicographic ordering where the s-th component is ordered according to $<_s$). This is a well-ordering and a linear extension of the partial order on Γ^x given by the product of the $<_s$. We will use the following property of $\prec$: if $\lambda(m)_{s_0} = 0 \neq \lambda(m')_{s_0}$ and $\lambda(m)_s = \lambda(m')_s$ for all $s \neq s_0$, then $m \prec m'$. The ordering $\prec$ is a well-ordering.

We want to show that $\prec$ is a recursive coatom well-ordering of $\widehat{P}$. Since lower intervals in P are Boolean (hence semimodular), by [13, Lemma 7.1] the following will be enough.

Consider $m_1, m_2 \in M$ with $m_1 \prec m_2$ and a $p \in P$ with $p < m_1, p < m_2$. We have to show that there is $m_3 \in M$ with $m_3 \prec m_2$ and a $p' \in P$ such that $\mathrm{rk}(p') = d-1$, $p' < m_2$, $p' < m_3$, $p' \geqslant p$. To this end, first notice that $\lambda(m_1)$ and $\lambda(m_2)$ must differ in at least one component, say the s_0-th, so that $\lambda(m_2)_{s_0} \neq 0_{s_0}$ and, more precisely, $\lambda(m_2)_{s_0} = \sum r_i \varphi_{s_0}(\gamma^i_{s_0})$ for some integers r_i that are not all zero. (Clearly, e.g., by Equation (†), $s_0 \notin p$).

Now, take $m_3 := (-\sum r_i \gamma^i_{s_0}) m_2$. Then, using (‡),

$$\lambda(m_3)_s = \begin{cases} \lambda(m_2)_s & s \neq s_0 \\ 0_{s_0} & s = s_0 \end{cases}$$

and so we conclude:

- $m_3 \prec m_2$
- $m_3 \cap m_2 = m_2 \backslash \{s_0\}$ and thus for $p' := m_2 \cap m_3$: $\mathrm{rk}(p') = d-1$, $p' < m_2$, $p' < m_3$, $p' \geqslant p$.

This shows that $\prec$ induces a recursive coatom ordering on $\widehat{P}$, hence the order complex of $\overline{P}$ is shellable and in particular $\ell(\overline{P}) - 1 = (\ell(P) - 2)$-connected as claimed. □

Proposition 5.3. *Let* $G \circlearrowright P$ *be a refined action on the simplicial poset* P *of faces of a simplicial complex. Then, for all* $x \in P$ *the poset* $\overline{P}_{\leqslant Gx}$ *is* $(\mathrm{rk}(x) - 2)$*-connected.*

Proof. Fix $x \in P$. By Lemma 2.14.(ii), the poset $P_{\leqslant Gx}$ carries a refined action of the group $G/\operatorname{stab}(x)$, and by definition the set of maximal elements has exactly one orbit. The claim then follows from Lemma 5.2. □

Lemma 5.4. *Let* $G \circlearrowright P$ *be a refined action on the poset* P *of faces of a finite-dimensional Cohen-Macaulay simplicial complex. Then,* $\overline{P}/G$ *is* $(\ell(P) - 2)$*-connected.*

Proof. We argue by induction, the case $\ell(P) = 1$ being trivial. Let then $\ell(P) = d > 1$, recall the quotient map

$$f : \overline{P} \to \overline{P}/G$$

and consider any $Gx \in \overline{P}/G$. On the one hand, by Proposition 5.3 we have that $f^{-1}((\overline{P}/G)_{\leqslant Gx}) = \overline{P}_{\leqslant Gx}$ is $(\mathrm{rk}(x) - 2)$-connected. On the other hand, $(\overline{P}/G)_{>Gx}$ is, by Lemma 2.11.(ii), Lemma 2.14.(ii), Remark 2.21 and induction assumption, $d - \mathrm{rk}(x) - 2$-connected.

This allows us to apply Lemma 2.23, which tells us that $\overline{P}/G$ is $(d-2)$ - connected if and only if $\overline{P}$ is. Since P was assumed to be Cohen-Macaulay, the claim follows. □

Theorem 3. *Let* P *be the poset of faces of a simplicial complex and let* $G \circlearrowright P$ *be a refined action. If* $\overline{P}$ *is Cohen-Macaulay, then* $\overline{P}/G$ *also is.*

Proof. By [10, §11.9], we have to prove that every open interval $(p, q) \subseteq \widehat{P}/G$ is $(\ell(p, q) - 1)$ - connected. If $q \neq \hat{1}$ this is true because Lemma 2.10 implies that bounded intervals in P/G are isomorphic to bounded intervals in P. But bounded intervals in P are Boolean since P is a simplicial poset. We are left with proving that, for every $p = Gx \in P/G$, the poset $(p, \hat{1}) = (P/G)_{>Gx}$ is $(\ell((P/G)_{>Gx}) - 1) = (\ell((P/G)_{\geqslant Gx}) - 2)$-connected.

But Lemma 2.11.(ii) and Lemma 2.14.(i) allow us to interpret $(P/G)_{\geqslant Gx}$ as the quotient of the simplicial poset $P_{\geqslant x}$ with respect to the refined action of $\mathrm{stab}(x)$. The poset $P_{\geqslant x}$ is again the poset of faces of a simplicial complex (Remark 2.21). Moreover, since P is Cohen-Macaulay, then so is $P_{\geqslant x}$ (which is isomorphic to the face poset of the link of x). The claim now follows from Lemma 5.4.

□

6. Semimatroids and geometric semilattices

In our context, a natural analogue to matroids in classical Stanley-Reisner theory seem to be (group actions on) semimatroids and geometric semilattices.

6.1. Semimatroids and geometric semilattices.

Semimatroids are abstract structures, introduced independently by Ardila and Kawahara [4, 34], that are intuitively best described as abstractions of the intersection pattern of a locally finite set $\mathscr{A}$ of affine hyperplanes (although the abstract theory is much more general [28, §4]). Given such a set, one can single out the family $\mathcal{K}$ of all subsets with nonempty intersection. The local finiteness assumption implies that $\mathcal{K}$ is an abstract simplicial complex on the vertex set $\mathscr{A}$. Moreover, every nonempty intersection of hyperplanes is an affine subspace with a well-defined codimension: this allows us to define a function $\rho : \mathcal{K} \to \mathbb{N}$ that associates to every element of $\mathcal{K}$ the codimension of the corresponding intersection.

Formally, a semimatroid is a triple $\mathscr{S} := (S, \mathcal{K}, \rho)$ consisting of a set S, a finite-dimensional simplicial complex $\mathcal{K}$ on the vertex set S and a function $\rho : \mathcal{K} \to \mathbb{N}$ satisfying a list of axioms that we will not need to specify (see [4, 34] for the original definition and [28] for the infinite case). The *rank* of the semimatroid $\mathscr{S}$ is the maximum value of ρ, which we denote by $\rho(\mathscr{S})$.

Associated to every semimatroid $\mathscr{S}$ are two posets.

- The poset of *independent sets* is the set
$$\mathcal{I}(\mathscr{S}) := \{I \in \mathcal{K} \mid \rho(I) = |I|\}$$
partially ordered by inclusion. This is the poset of faces of an abstract simplicial complex on the vertex set S.
- The poset of *closed sets* (or *flats*) is the set
$$\mathcal{L}(\mathscr{S}) := \{F \in \mathcal{K} \mid \rho(F') > \rho(F) \text{ for all } F' \supsetneq F\}$$
partially ordered by inclusion.

Both posets are *geometric semilattices* in the sense of Wachs and Walker [58]. Moreover, every geometric semilattice is the poset of flats of a semimatroid [28, Theorem E]. When the semimatroid is associated to an affine arrangement of hyperplanes, the poset of flats is isomorphic to the poset of all intersections ordered by reverse inclusion.

Definition 6.1 (Compare [28, §3])**.** Let G be a group. A G-*semimatroid* $\mathfrak{S} : G \circlearrowright \mathscr{S}$ is an action of G on a semimatroid $\mathscr{S} = (S, \mathcal{K}, \rho)$, i.e., an action of G by permutations of S that preserves $\mathcal{K}$ and ρ. We require that there is a finite number of orbits of elements of $\mathcal{K}$.

We can thus define a function

$$\begin{array}{rccl} m_{\mathfrak{S}}: & 2^{S/G} & \to & \mathbb{N} \\ & A & \mapsto & |\{\{s_1,\dots,s_k\} \in \mathcal{K} \mid \{Gs_1,\dots,Gs_k\} = A\}/G| \end{array}$$

counting the number of orbits on $\mathcal{K}$ whose representatives have a prescribed set of vertices.

Such a G-semimatroid is called *translative* if, for every $s \in S$, $\{g(s), s\} \in \mathcal{K}$ implies $g(s) = s$. The G-semimatroid is called *(k-)refined* if, in addition to being translative, G is a free Abelian group and there is $k \in \mathbb{N}$ such that, for every $x \in \mathcal{K}$, $\mathrm{stab}(x)$ is a direct summand of rank $k(\rho(\mathscr{S}) - \rho(x))$.

Every G-action on a semimatroid $\mathscr{S}$ induces an action of G by automorphisms on the posets $\mathcal{I}(\mathscr{S})$ and $\mathcal{L}(\mathscr{S})$. If the G-semimatroid is translative, resp. refined, then so are the induced actions on both posets. Conversely, every action on $\mathcal{I}(\mathscr{S})$ induces an action on $\mathscr{S}$ and every action on a geometric semilattice induces an action on the associated (simple) semimatroid.

Definition 6.2. Given a G-semimatroid $\mathfrak{S} : G \circlearrowright \mathscr{S}$ define the quotient posets

$$\mathcal{I}_{\mathfrak{S}} := \mathcal{I}(\mathscr{S})/G \qquad \mathcal{P}_{\mathfrak{S}} := \mathcal{L}(\mathscr{S})/G.$$

Remark 6.3.

(i) By the finiteness requirement in Definition 6.1, both quotient posets are finite.

(ii) Both $\mathcal{I}_{\mathfrak{S}}$ and $\mathcal{P}_{\mathfrak{S}}$ are bounded-below and ranked of length $\rho(\mathscr{S})$. The poset rank of an element X of either poset equals $\rho(X)$.

(iii) If $\mathfrak{S}$ is a translative G-semimatroid, then $\mathcal{I}_{\mathfrak{S}}$ is a simplicial poset.

The above definitions and terminology were motivated in [28] by the case of periodic affine hyperplane arrangements related to Abelian arrangements, as we will discuss later. However, these definitions are strictly more general, see [28].

6.2. **Tutte polynomials and h-polynomials.**

Definition 6.4. Let $\mathfrak{S} : G \circlearrowright \mathscr{S}$ denote a translative action of G on a semimatroid $\mathscr{S}$. The Tutte polynomial of $\mathfrak{S}$ is

$$T_{\mathfrak{S}}(x,y) := \sum_{A \subseteq S/G} m_{\mathfrak{S}}(A)(x-1)^{\rho(\mathscr{S})-\rho(A)}(y-1)^{|A|-\rho(A)}.$$

Lemma 6.5. *Let $\mathfrak{S}$ be a translative G-semimatroid of rank d. Then*

(i) The h-polynomial of the simplicial poset $\mathcal{I}_{\mathfrak{S}}$ is

$$h_{\mathcal{I}_{\mathfrak{S}}}(t) = t^d T_{\mathfrak{S}}(1/t, 1),$$

and the characteristic polynomial of $\mathcal{I}_{\mathfrak{S}}$ is $\chi_{\mathcal{I}_{\mathfrak{S}}}(t) = (-t)^d T_{\mathfrak{S}}(\frac{1-t}{t}, 1)$.

(ii) The characteristic polynomial of the poset $\mathcal{P}_{\mathfrak{S}}$ is

$$\chi_{\mathcal{P}_{\mathfrak{S}}}(t) = (-1)^d T_{\mathfrak{S}}(1-t, 0).$$

Proof. Item (ii) is [28, Theorem F]. In order to prove item (i), start by noticing that the number of orbits of independent sets of rank i in $\mathcal{I}_{\mathfrak{S}}$ is

$$f_{i-1}(\mathcal{I}_{\mathfrak{S}}) = \sum_{\substack{A\subseteq S/G \\ \rho(A)=|A|=i}} m_{\mathfrak{S}}(A).$$

Therefore, with Remark 6.3.(ii) we can write

$$h_{\mathcal{I}_{\mathfrak{S}}}(t) \overset{\text{df.}}{=} t^d \sum_{i=1}^{d} f_{i-1}(P)\left(\frac{1-t}{t}\right)^{d-i} = t^d \sum_{\substack{A\subseteq S/G \\ \text{rk}(A)=|A|}} m_{\mathfrak{S}}(A)\left(\frac{1-t}{t}\right)^{\rho(\mathscr{S})-\rho(A)}$$

$$= t^d \sum_{A\subseteq S/G} m_{\mathfrak{S}}(A)\left(\frac{1}{t}-1\right)^{\rho(\mathscr{S})-\rho(A)} 0^{|A|-\rho(A)} \overset{\text{df.}}{=} t^d T_{\mathfrak{S}}\left(\frac{1}{t},1\right).$$

The formula for the characteristic polynomial of $\mathcal{I}_{\mathfrak{S}}$ follows with Remark 2.19. □

6.3. **On Stanley-Reisner rings of G-semimatroids.** It is now natural to state the following definition.

Definition 6.6. Given a G-semimatroid $\mathfrak{S}$ let

$$\mathcal{R}_{\mathfrak{S}} := \mathcal{R}(\mathcal{I}_{\mathfrak{S}})$$

be the Stanley-Reisner ring of $\mathfrak{S}$.

From our results the following facts follow immediately.

Remark 6.7. Let $\mathfrak{S}$ be a G-semimatroid of rank d.

- If $\mathfrak{S}$ is translative, $\mathcal{R}_{\mathfrak{S}}$ is isomorphic to the Stanley ring associated to the (finite) simplicial poset $\mathcal{I}_{\mathfrak{S}}$.
- If G is the trivial group, $R_{\mathfrak{S}}$ is isomorphic to the classical Stanley-Reisner ring of the underlying (semi)matroid.
- If $\mathfrak{S}$ is refined, then the poset $\mathcal{I}_{\mathfrak{S}}$ is $(d-2)$-connected, and
$$\widetilde{H}_{d-1}(\mathcal{I}_{\mathfrak{S}}) \simeq \mathbb{Z}^{-T_{\mathfrak{S}}(0,1)}.$$
- If $\mathfrak{S}$ is refined, then the associated Stanley-Reisner ring is Cohen-Macaulay, with h-polynomial $h_{\mathcal{I}_{\mathfrak{S}}}(t) = t^d T_{\mathfrak{S}}(1/t, 1)$.

6.4. **On refined quotients of geometric semilattices.** As a byproduct of our previous considerations we can prove the following result on the topology of quotients of geometric semilattices.

Theorem 4. *If $\mathfrak{S}$ is a refined G-semimatroid, the poset $\overline{\overline{\mathcal{P}_{\mathfrak{S}}}}$ is Cohen-Macaulay.*

Remark 6.8. In particular, quotients of geometric semilattices by refined actions are Cohen-Macaulay.

Corollary 6.9. More precisely, if $\mathfrak{S} : G \circlearrowright \mathscr{S}$ is a refined action on a semimatroid of rank d,

$$\widetilde{H}_i(\overline{\mathcal{P}_{\mathfrak{S}}}, \mathbb{Z}) = \begin{cases} \{0\} & \text{if } i \leqslant d-2 \\ \mathbb{Z}^{T_{\mathfrak{S}}(0,0)} & \text{if } i = d-1 \end{cases}$$

If $\mathcal{P}/G$ is bounded above, then clearly $\mathcal{P}/G\setminus\{\hat{0}\}$ is contractible. In this case,

$$\widetilde{H}_i(\overline{\overline{\mathcal{P}_{\mathfrak{G}}}}, \mathbb{Z}) = \begin{cases} \{0\} & \text{if } i \leqslant d-2 \\ \mathbb{Z}^{-T_{\mathfrak{G}}(1,0)} & \text{if } i = d-1 \end{cases}$$

Proof. The Corollary's claim for $i \leqslant d-2$ is a reformulation of the connectivity claim in the Theorem. The claims for $i = d-1$ follow using Lemma 2.26 and Lemma 6.5. □

Proposition 6.10. *Let $\mathfrak{G}$ be a refined action on a semimatroid of rank* d*. Then $\overline{\mathcal{P}_{\mathfrak{G}}}$ is* $(d-2)$*-connected.*

Proof. The proof is by induction on d, the claim being trivial if $d = 0$ or $d = 1$. Then let $d > 1$ and suppose that the claim holds for all semimatroids of rank smaller than d.

The posets $\overline{\mathcal{I}_{\mathfrak{G}}}$ and $\overline{\mathcal{P}_{\mathfrak{G}}}$ are both ranked of the same length $(d-1)$. The equivariant and rank-preserving poset map $\mathrm{cl} : \mathcal{I}(\mathscr{S}) \to \mathcal{L}(\mathscr{S})$ given by semimatroid closure (see [28, Definition 3.27]) induces a rank preserving poset map

$$f : \overline{\mathcal{I}_{\mathfrak{G}}} \to \overline{\mathcal{P}_{\mathfrak{G}}}, \qquad GI \mapsto G\,\mathrm{cl}(I)$$

Since Lemma 5.4 ensures that $\overline{\mathcal{I}_{\mathfrak{G}}}$ is $(d-2)$-connected, the claim follows by Lemma 2.23 applied to f with $t = (d-2)$. Thus we only have to check that Lemma's assumptions. Let henceforth rk denote the rank function of the poset $\mathcal{P}_{\mathfrak{G}}$ and fix $p \in \overline{\mathcal{P}_{\mathfrak{G}}}$. Then, $\mathrm{rk}(p) > 0$ and $\ell((\overline{\mathcal{P}_{\mathfrak{G}}})_{<p}) = \mathrm{rk}(p) - 2$ (where we take the length of the empty poset to be -1).

(1) By Lemma 2.14.(i) and Lemma 2.11.(ii), the poset $(\overline{\mathcal{P}_{\mathfrak{G}}})_{>p}$ is the proper part of the quotient of the poset of flats of a semimatroid of rank $d - \mathrm{rk}(p)$ by a refined action, thus by induction hypothesis it is $(d - \mathrm{rk}(p) - 2) = (t - \ell((\overline{\mathcal{P}_{\mathfrak{G}}})_{<p}) - 2)$-connected.
(2) We are left with showing that $f^{-1}((\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p})$ is connected through codimension 1, i.e., that it is $(\mathrm{rk}(p) - 2)$-connected, and this will follow from the fact that it is isomorphic to the poset of (proper) faces of the independence complex of a rank $\mathrm{rk}(p)$ matroid, which is classically known to be $(\mathrm{rk}(p) - 2)$-connected [12]. This isomorphism is proved in the next claim which, then, concludes the proof of the theorem.

More precisely, we fix a representative $F := \{x_1, \dots, x_k\} \in p$ and consider the (rank $\mathrm{rk}(p)$) matroid $\mathscr{S}[F]$, the restriction of $\mathscr{S}$ to F (see [28, Definition 1.11]). We write $\mathcal{I}[F]$ for the poset of independent sets of this matroid and note that the poset of flats of $\mathscr{S}[F]$ is naturally isomorphic to $\mathcal{L}(\mathscr{S})_{\leqslant F}$.

Claim. We claim that the quotient map by the G-action induces a poset isomorphism

$$\gamma : \bar{\mathcal{I}}[F] \to f^{-1}((\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p}).$$

Proof of claim. Since both posets are finite, it will suffice to prove that γ is a bijective order-preserving map. Write $\mathcal{L}$ instead of $\mathcal{L}(\mathscr{S})$ for brevity, and consider the diagram

$$\begin{array}{ccc} f^{-1}((\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p}) & \xrightarrow{\quad f \quad} & (\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p} \\ \uparrow \gamma & & \alpha \uparrow \\ \bar{\mathcal{I}}[F] & \xrightarrow{\quad \mathrm{cl} \quad} & \overline{\mathcal{L}}_{\leqslant F} \end{array}$$

where cl is the closure map of the matroid $\mathscr{S}[F]$ and $\alpha : X \mapsto GX$ denotes the restriction of the quotient map of the action on $\mathcal{L}$. The maps f, cl and α are rank-preserving by definition, and α is a poset isomorphism because the group action is translative.
For every $I \in \bar{\mathcal{I}}[F]$, since $I \subseteq F$ and $\mathrm{cl}(F) = F$ we have $\mathrm{cl}(I) \in \overline{\mathcal{L}}_{\leqslant F}$. Unwrapping the definitions we see that $f\gamma(I) = f\,\mathrm{cl}(I) = \alpha\,\mathrm{cl}(I) \in (\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p}$, thus the map γ is well-defined and the diagram commutes. That γ is order-preserving follows because it is the restriction of the (order-preserving) quotient map on $\mathcal{I}(\mathscr{S})$.
Moreover, given any $q \in f^{-1}((\overline{\mathcal{P}_{\mathfrak{G}}})_{\leqslant p})$ consider the element $X := \alpha^{-1}(f(q))$ and let $I \subseteq X$ be such that $q = GI$. Then I is independent and $I \subseteq X \subseteq F$, hence $I \in \bar{\mathcal{I}}[F]$ and clearly $\gamma(I) = GI = q$, hence γ is surjective.
Finally, any $I' \in \bar{\mathcal{I}}[F]$ with $\gamma I' = q = GI$ satisfies $gI' = I$ for some $g \in I$ hence, by translativity of the action on $\mathcal{I}(\mathscr{S})$, we must have $I = I'$ and so γ is injective.
As a bijective order-reserving map between finite posets, γ is a poset-isomorphism as claimed.

□

Proof of Theorem 4. The proof is along the lines of the proof of Theorem 3. We explicit it for the reader's convenience.

By [10, §11.9], we have to prove that every open interval $(x, y) \subseteq \widehat{\mathcal{P}_{\mathfrak{G}}}$ is $(\ell(x, y) - 1)$ - connected. If $y \neq \hat{1}$ this is true because Lemma 2.10 implies that bounded intervals in $\mathcal{P}_{\mathfrak{G}}$ are isomorphic to bounded intervals in $\mathcal{L}(\mathscr{S})$. But bounded intervals in $\mathcal{L}(\mathscr{S})$ are geometric lattices [58, Theorem 2.1]. We are left with proving that, for every $Gx \in \mathcal{P}_{\mathfrak{G}}$, the poset $(\mathcal{P}_{\mathfrak{G}})_{>Gx}$ is $(\ell((\mathcal{P}_{\mathfrak{G}})_{>Gx}) - 1) = (\mathrm{rk}((\mathcal{P}_{\mathfrak{G}})_{\geqslant Gx}) - 2)$-connected.

Lemma 2.11 allows us to interpret $(\mathcal{P}_{\mathfrak{G}})_{\geqslant Gx}$ as the quotient of the geometric semilattice $\mathcal{L}(\mathscr{S})_{\geqslant x}$ with respect to the refined action of $\mathrm{stab}(x)$. The claim now follows from Proposition 6.10.

□

7. Stanley-Reisner rings and poset of layers of Abelian arrangements

As was briefly discussed in the introduction, one of our main motivations comes from the theory of arrangements, and in particular from the desire to uniformly describe *Abelian arrangements* in a way that generalizes the classical theory of hyperplane arrangements. To make the definition in Section 1.4 slightly more explicit, let $\mathbb{G}$ stand for one of $\mathbb{C}, \mathbb{C}^*$ or $\mathbb{E}$, an elliptic curve, seen as complex algebraic groups, and let Λ be the lattice of group homomorphisms $\mathbb{G}^d \to \mathbb{G}$.

Any choice of $a_1, \ldots, a_n \in \Lambda$ and $b_1, \ldots, b_n \in \mathbb{G}$ determines an arrangement

$$\mathscr{A} := \{H_i := a_i^{-1}(b_i) \mid i = 1, \ldots, n\}$$

of hypersurfaces in $\mathbb{G}^d$. It is called a *linear*, *toric*, *elliptic* arrangement if $\mathbb{G}$ is $\mathbb{C}$, respectively $\mathbb{C}^*$ or $\mathbb{E}$. The arrangement is called *essential* if the a_i's span a full-rank sublattice of Λ.

A *central* arrangement is one where $b_i = \mathrm{id}_G$ for all $i = 1, \ldots, n$. A deep enumerative-combinatorial study of central arrangements, with special attention to the linear and toric case, has led to the introduction of arithmetic Tutte polynomials [44] and arithmetic matroids [15, 23]. Questions about commutative-algebraic interpretations of some of the polynomials arising in this context led to attempts at modeling the poset $\mathcal{I}(\mathscr{A})$ of *"independent sets"* in the linear and (central) toric case, defined to be the set of pairs (X, c) where X is a ($\mathbb{Q}$-)linearly independent subset of $\{a_i\}_i$ and c is a connected component of the intersection of the corresponding hypersurfaces [37, 40].[1]

In the general (noncentral) case, one may still look at the poset of *layers* $\mathcal{C}(\mathscr{A})$ described in Section 1.4. The arithmetic matroid of the $\{a_i\}_i$ as well as – in the linear and toric case – the rational cohomology algebra of the arrangement's complement can be recovered from $\mathcal{C}(\mathscr{A})$ [22]. On the other hand, Pagaria [48] exhibited a pair of central toric arrangements with isomorphic arithmetic matroids (and matroid over $\mathbb{Z}$) but non-isomorphic posets of layers.

In order to model these posets we take the approach of [28], and consider the (topological) universal covering morphism

$$\mu : \mathbb{C}^d \to \mathbb{G}^d.$$

The lift of $\mathscr{A}$ through this universal covering is a set $\mathscr{A}^\upharpoonright$ of (affine) complex codimension 1 subspaces which is invariant under deck transformations. Now, the group of deck transformations acts by translations on $\mathbb{C}^d$ and is isomorphic to $\mathbb{Z}^{kd}$, with $k = 0, 1, 2$ according to whether we are in the linear, toric, respectively the elliptic case.

As is well-known [4, 28], every affine hyperplane arrangement such as $\mathcal{A}^\upharpoonright$ defines a semimatroid whose geometric semilattice is isomorphic to the arrangement's poset of intersections. In our case, associated to $\mathscr{A}^\upharpoonright$ we have a semimatroid $\mathscr{S} = (S, \mathcal{K}, \mathrm{rk})$ with $\mathcal{L}(\mathscr{S}) \simeq \mathcal{C}(\mathscr{A})$. On this semimatroid the group of deck transformations acts, defining a $\mathbb{Z}^{kd}$-semimatroid $\mathfrak{S}_{\mathscr{A}}$.

Lemma 7.1. *Let $\mathscr{A}$ be an Abelian arrangement. Then $\mathfrak{S}_{\mathscr{A}}$ is well-defined. Moreover,*

(i) $\mathcal{I}_{\mathfrak{S}_{\mathscr{A}}} \simeq \mathcal{I}(\mathscr{A})$; (ii) $\mathcal{P}_{\mathfrak{S}_{\mathscr{A}}} \simeq \mathcal{C}(\mathscr{A})$; (iii) $\mathfrak{S}_{\mathscr{A}}$ is refined if $\mathscr{A}$ is essential.

More precisely, $\mathfrak{S}_{\mathcal{A}}$ is $0, 1, 2$-refined according to whether $\mathscr{A}$ is a linear, toric or elliptic essential arrangement.

Remark 7.2. If $\mathscr{A}$ is central, then $T_{\mathfrak{S}}(x, y)$ corresponds to the arithmetic Tutte polynomial of the list of elements $a_1, \ldots, a_n$ of the Abelian group Λ, see [44].

---

[1]The definitions in [37, 40] are formally in terms of pairs (X, g) where g is a torsion element of the quotient group $\Lambda/\langle X \rangle_{\mathbb{Z}}$. Such torsion elements are however in (natural) bijection with the connected components of the intersection of the hypersurfaces determined by the elements of X (see, e.g., [44, 28]).

Proof of Lemma 7.1. We start with a general remark by recalling that $\mathcal{K}$ is given by all sets of hyperplanes with nonempty intersection. Orbits of $\mathcal{K}$ under the deck transformation group correspond bijectively to pairs (X, c) where $X \subseteq \mathscr{A}$ and c is a connected component of the intersection of the hypersurfaces in X.

Since $\mathscr{A}$ is finite and any intersection has only finitely many components, the finiteness-of-orbits condition in Definition 6.1 follows and so $\mathfrak{S}_{\mathscr{A}}$ is well-defined.

For (i) and (ii), notice that $\mathcal{I}(\mathscr{S})$ and $\mathcal{L}(\mathscr{S})$ are subsets of $\mathcal{K}$, and orbits of the induced action are, respectively,

- for $\mathcal{I}_{\mathfrak{S}_{\mathscr{A}}}$: pairs (X, c) where the characters defining the elements of X are linearly independent (over $\mathbb{Q}$);
- for $\mathcal{P}_{\mathfrak{S}_{\mathscr{A}}}$: pairs (X, c) where the characters defining the elements of X form a subset of $\{a_i\}_i$ that is closed under linear dependency (over $\mathbb{Q}$).

Comparing these descriptions with the definitions given above,claims (i) and (ii) follow.

For (iii) we separate the cases. In the linear case, the group is trivial, hence the action is clearly 0-refined. In the toric and elliptic case we can choose coordinates so that the action of the deck transformation group coincides with addition by elements of the sublattices $L_t := \mathbb{Z}^d \subseteq \mathbb{C}^d$, resp. $L_e := \mathbb{Z}^d + i\mathbb{Z}^d \subseteq \mathbb{C}^d$. The stabilizer subgroup of an affine subspace W equals the stabilizer subgroup of its translate at the origin, hence it is a direct summand of L_t, resp. L_e, of rank equal to the lattice rank of $W \cap L_t$, resp. $W \cap L_e$. This rank equals $\dim_{\mathbb{C}} W$ (resp. $2\dim_{\mathbb{C}} W$). Now, essentiality of $\mathscr{A}$ implies that the minimal intersections of $\mathscr{A}^{\upharpoonleft}$ have dimension 0, and so that the poset rank of W in $\mathcal{C}(\mathscr{A}^{\upharpoonleft})$ equals the codimension of W: $\rho(W) = d - \dim_{\mathbb{C}} W$. The stabilizer of W has then rank $d - \rho(W)$ (resp. $2(d - \rho(W))$). Via the isomorphism $\mathcal{C}(\mathscr{A}^{\upharpoonleft}) \simeq \mathcal{L}(\mathscr{S})$ we conclude that the stabilizer of every subset $X \in \mathcal{K}$, which coincides with the stabilizer of the intersection associated to X, has rank $d - \rho(X)$ (resp. $2(d - \rho(X))$). This proves that $\mathfrak{S}_{\mathscr{A}}$ is 1-refined, resp. 2-refined depending on whether we are in the toric or elliptic case. □

We are naturally led to the following definition.

Definition 7.3. Let $\mathscr{A}$ be an Abelian arrangement. The Stanley-Reisner ring of $\mathscr{A}$ is $\mathcal{R}(\mathscr{A}) := \mathcal{R}_{\mathfrak{S}_{\mathscr{A}}}$.

Our point of view allows us to also immediately deduce some properties of those rings for the general case of abelian arrangements, which we state in the following summary of our results in the general context of the theory of Abelian arrangements.

Theorem 5. *Let $\mathscr{A}$ be an Abelian arrangement.*

(i) The poset $\mathcal{C}(\mathscr{A})$ is Cohen-Macaulay. In particular, the homology of $\overline{\mathcal{C}(\mathscr{A})}$ is concentrated in top degree. Its (topological) Betti numbers are evaluations of the action's Tutte polynomial according to Corollary 6.9.

(ii) The simplicial poset $\mathcal{I}(\mathscr{A})$ is Cohen-Macaulay.

(iii) The arrangement's Stanley-Reisner ring $\mathcal{R}(\mathscr{A})$ is Cohen-Macaulay over $\mathbb{Z}$. This ring is isomorphic to the ring of invariants of the Stanley-Reisner ring associated to the periodic hyperplane arrangement $\mathscr{A}^{\upharpoonleft}$.

(iv) *The* h*-polynomial of* $\mathcal{R}(\mathscr{A})$ *is given by the action's Tutte polynomial as in Lemma 6.5.(i).*

Proof. Item (i) is Theorem 4, item (ii) follows from Theorem 3 and [9, 58], (iii) combines Theorem 2 and Theorem 1. Finally, (iv) folllows with Lemma 6.5.(i). □

Remark 7.4. Notice that when $\mathscr{A}$ is central and toric, via the case $k = 1$ of Lemma 7.1 we recover the ring of [37, 40], where item (iv) of Theorem 5 is proved in the corresponding situation.

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