Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Cores, shell indices and the degeneracy of a graph limit

by

Johannes Rauh

Preprint no.: 36 2018

CORES, SHELL INDICES AND THE DEGENERACY OF A GRAPH LIMIT

JOHANNES RAUH

ABSTRACT. The k-core of a graph is the maximal subgraph in which every node has degree at least k, the shell index of a node is the largest k such that the k-core contains the node, and the degeneracy of a graph is the largest shell index of any node. After a suitable normalization, these three concepts generalize to limits of dense graphs (also called graphons). In particular, the degeneracy is continuous with respect to the cut metric.

1. Introduction

The goal of this paper is to study the graph theoretic notions of cores, shell indices and degeneracy from the view point of graphons (i.e. limits of dense graphs). The first step is to normalize these quantities and generalize the definitions. The second step is to investigate measurability and continuity of the defined quantities.

Continuity (with respect to the cut metric) allows to transfer properties that have been proved for finite graphs to the domain of graphons. However, this goes against the idea behind graphons that some properties of graphs are easier to understand from an asymptotic viewpoint that is amenable to analytic approaches. Thus, the goal of this presentation is to provide direct proofs that do not rely on graph theoretic results. This works for all results, except for Lemma 11. Admittedly, the proofs for graphons often imitate the corresponding proofs for graphs.

A further result is the minimal and maximal edge density of a graphon for a given degeneracy. This generalizes results of Kim et al. [2016].

2. Preliminaries

2.1. **The case of finite graphs.** Let $G = (V, E)$ be a (simple, undirected) graph, and let $k \geq 0$ be an integer. The *k-core* of a graph is the maximal subgraph in which every node has degree at least k. It can be constructed algorithmically as follows:

$$K_k^0 := V,$$
$$K_k^{i+1} := \left\{x \in K_k^i : d_{G|_{K_k^i}}(x) \geq k\right\}.$$

Here, $d_{G|_{K_k^i}}(x)$ denotes the degree of x with the induced subgraph $G|_{K_k^i}$. Thus, K_k^{i+1} arises from K_k^i by dropping all nodes that are connected to less than k other nodes among K_k^i. The sequence $\left(K_k^i\right)_i$ stabilizes after finitely many steps, and the limit $K_k := \bigcap_i K_k^i$ equals the set of nodes of the k-core of G.

The *shell index* of a node x of G is the largest k such that $x \in K_k$. The *degeneracy* of a graph is the largest shell index of any node. Alternatively, the degeneracy $\delta(G)$ is the largest k such that K_k is non-empty.

As shown by Kim et al. [2016, Proposition 3.1], the degeneracy and the number of edges satisfy

$$\binom{\delta(G)+1}{2} \leq |E| \leq \binom{\delta(G)+1}{2} + (n-\delta(G)-1)\delta(G). \tag{1}$$

2.2. **Graphons.** For an introduction to graphons (also called graph limits), see Lovász and Szegedy [2006] and Diaconis and Janson [2008]. In this text, a *graphon* is a (Lebesgue)-measurable function $w : [0,1]^2 \to [0,1]$ from the unit square to the unit interval that is symmetric in the sense that $w(x,y) = w(y,x)$ for all $x, y \in [0,1]$.

A graphon is a limit object of dense graphs. In some sense it is analogous to an adjacency matrix. However, unlike an adjacency matrix, it can take values other than 0 or 1, and thus its entries do not mark edges, but edge densities. Similarly, the interval $[0,1]$ that constitutes the domain of each of the two arguments of a graphon w should not be interpreted as a set of nodes, but rather as a set of node classes.

To relate graphons with finite graphs, consider the following construction: for a finite graph G with N nodes, consider the random graph on $n \leq N$ nodes that arises by randomly sampling n nodes and looking at the induced subgraph. Similarly, a subgraph of arbitrary size can be sampled from a graphon w as follows: Take n i.i.d. samples $x_1, \dots, x_n$ from the uniform distribution on $[0,1]$. Then let $V_n = \{1, \dots, n\}$, and for each $1 \leq i < j \leq n$, add an edge $\{i,j\}$ independently with probability $w(x_i, x_j)$.

Different graphons are considered to be equivalent if the induced probabilities underlying the above sampling process are identical. One way to obtain equivalent graphons is by means of measure-preserving maps: A map $\sigma : [0,1] \to [0,1]$ is *measure-preserving* if it is measurable and satisfies $|A| = |\sigma^{-1}(A)|$ for any (Lebesgue)-measurable set $A \subseteq [0,1]$. Given a measure-preserving map σ, the transformed graphon w^σ is defined by

$$w^\sigma(x,y) = w(\sigma(x), \sigma(y)).$$

Measure preserving maps generalize permutations of the nodes of a finite graph. However, measure preserving maps need not be injective: For example, the map

$$\sigma_2 : x \mapsto \begin{cases} 2x, & \text{if } x \leq \frac{1}{2}, \\ 2x-1, & \text{if } x > \frac{1}{2}, \end{cases}$$

can be shown to be measure preserving.

Equivalence of graphons is related to the cut metric. For any pair of graphons w, w', define

$$d_\square(w,w') = \sup_{S,T\subseteq[0,1]} \left| \int_S \mathrm{d}x \int_T dy \big(w(x,y) - w'(x,y)\big) \right|.$$

The *cut metric* is then defined by

$$\delta_\square(w,w') = \inf_\sigma d_\square(w^\sigma, w'),$$

where the infimum is over all measure-preserving maps σ. The cut metric is a semi-metric on the set of all graphons, and it is a metric on the set of all equivalence classes of graphons.

3. The core and the degeneracy of a graphon

Let $w : [0,1]^2 \to [0,1]$ be a graphon. For any $x \in [0,1]$, let $d_w(x) = \int_0^1 w(x,y)\mathrm{d}y$ be the *degree* of x in w. For any measurable $K \subseteq [0,1]$, let $d_w^K(x) = \int_K w(x,y)\mathrm{d}y$ be the *degree* of x in w restricted to K. The following is a straight-forward generalization of the corresponding graph theoretic definitions:

Definition 1. The κ-*core* of w is the largest subset $K \subseteq [0,1]$ with $d_w^K(x) \geq \kappa$ for all $x \in K$. The *shell index* of $x \in [0,1]$ in w is

$$\delta_x(w) := \sup\big\{\kappa : x \in K_\kappa(w)\big\}.$$

The *degeneracy* is

$$\delta(w) := \sup\big\{\kappa : |K_\kappa(w)| > 0\big\},$$

where $|K_\kappa(w)|$ denotes the (Lebesgue)-volume of $K_\kappa(w)$.

In fact, as Lemma 6 below shows,

$$\delta(w) = \max\big\{\kappa : |K_\kappa(w)| \neq \emptyset\big\} = \max\big\{\delta_x(w) : x \in [0,1]\big\}.$$

The algorithmic definition of the k-core also generalizes to the graphon case: for $\kappa \in [0,1]$, let

$$K_\kappa^1(w) := \big\{x \in [0,1] : d_w(x) \geq \kappa\big\},$$
$$K_\kappa^{n+1}(w) := \Big\{x \in K_\kappa^n(w) : d_w^{K_\kappa^n(w)}(x) \geq \kappa\Big\}.$$

Then the κ-core is $K_\kappa(w) := \bigcap_{n=1}^\infty K_\kappa^n(w)$.

Example 2. Consider the graphon $w(x,y) = \min\{x,y\}$. Since $w(x,y)$ is a monotone function in x for any fixed y, it follows that $d_w(x)$ is monotone in x, and also $d_w^K(x)$ increases monotonically with x for any fixed K. Therefore, each set $K_\kappa^n(w)$ is a closed interval of the form $[k_\kappa^n(w), 1]$ (or empty). Thus, each κ-core is also of the form $K_\kappa(w) = [k_\kappa, 1]$ (or empty). The numbers k_κ^n satisfy the recursion

$$k_\kappa^{n+1} = 1 - \sqrt{1 - (k_\kappa^n)^2 - 2\kappa}, \qquad k_\kappa^0 = 0,$$

where the expression inside the squareroot is non-negative if and only if $K_\kappa^{n+1}(w)$ is non-empty. The sequence $(k_\kappa^n)_n$ increases (as long as it is defined). If the sequence does not abort, then it approaches a fixed point of the recursion. The recursion has a fixed point if and only $\kappa \leq \frac{1}{4}$, in which case the only stable fixed point lies at

$$k_\kappa = \lim_{n\to\infty} k_\kappa^n = \frac{1}{2}(1 - \sqrt{1-4\kappa}).$$

Thus, for $\kappa \leq \frac{1}{4}$, the κ-core is a non-empty interval $K_\kappa = [k_\kappa, 1]$. On the other hand, for $\kappa > \frac{1}{4}$, the sequence k_κ^n has no fixed point, and $K_\kappa = \emptyset$.

Thus, in this example, the cores are a family of intervals $[k_\kappa, 1]$ that are contained in each other and with boundaries that depend smoothly on κ for $\kappa \leq \frac{1}{4}$. The smallest non-empty core is the $\frac{1}{4}$-core $[\frac{1}{2}, 0]$. Thus, the degeneracy of w is $\delta(w) = \frac{1}{4}$.

The analysis can easily be generalized to other graphons $w(x,y)$ that are monotone in x for any fixed y. Such graphons arise, for example, in the study of large random graph models associated with a fixed degree sequences [Chatterjee et al., 2011]. In general, the sets $K_\kappa^n(w)$ and $K_\kappa(w)$ will always be intervals, but the boundaries need not depend continuously on κ. However, the next lemma shows that $\bigcap_{\kappa'<\kappa} K_{\kappa'}^n(w) = K_\kappa^n(w)$ and $\bigcap_{\kappa'<\kappa} K_{\kappa'}(w) = K_\kappa(w)$, which implies that the interval boundaries are upper semi-continuous in this case.

Lemma 3.

(1) *The sets* $K^n_\kappa(w)$ *and* $K_\kappa(w)$ *are measurable.*
(2) *If* $\kappa \geq \kappa'$, *then* $K^n_\kappa(w) \subseteq K^n_{\kappa'}(w)$ *and* $d_w^{K^n_\kappa}(x) \leq d_w^{K^n_{\kappa'}}(x)$ *for all* x.
(3) *Thus, if* $\kappa \geq \kappa'$, *then* $K_\kappa(w) \subseteq K_{\kappa'}(w)$.
(4) *For each* n *and* $\kappa > 0$, $K^n_\kappa(w) = \bigcap_{\kappa'<\kappa} K^n_{\kappa'}(w)$, *and thus* $K_\kappa(w) = \bigcap_{\kappa'<\kappa} K^n_{\kappa'}(w)$.
(5) *For each* n *and* $\kappa > 0$, $d_w^{K^n_\kappa(w)}(x) = \inf_{\kappa'<\kappa} d_w^{K^n_{\kappa'}(w)}(x)$ *for all* x

Proof. (1) follows from induction, since $d^K_w(x)$ is measurable if K is measurable. (2) and (3) follow by definition. The last two statements (4) and (5) can be proved simultaneously by induction: for each n, the statement about $d_w^{K^n_\kappa(w)}(x)$ follows from the statement about K^n_κ and the monotone convergence theorem for integrals.

In the case $n = 1$, $K^1_\kappa(w) = \bigcap_{\kappa'<\kappa} K^1_{\kappa'}(w)$ follows from the fact that a number d satisfies $d \geq \kappa$ if and only if $d \geq \kappa'$ for all $\kappa' < \kappa$. For $n > 1$, clearly,

$$\begin{aligned} K^{n+1}_\kappa(w) = \Big\{x \in K^n_\kappa(w) : d_w^{K^n_\kappa(w)}(x) \geq \kappa\Big\} \\ &= \bigcap_{\kappa'<\kappa} \Big\{x \in K^n_\kappa(w) : d_w^{K^n_\kappa(w)}(x) \geq \kappa'\Big\} \\ &\subseteq \bigcap_{\kappa'<\kappa} \Big\{x \in K^n_{\kappa'}(w) : d_w^{K^n_{\kappa'}(w)}(x) \geq \kappa'\Big\} = \bigcap_{\kappa'<\kappa} K^{n+1}_{\kappa'}(w). \end{aligned}$$

For the other containment, suppose that $x \notin K^{n+1}_\kappa(w)$. Then either $x \notin K^n_\kappa(w)$, or $d_w^{K^n_\kappa(w)}(x) < \kappa$. In the first case, by induction, there exists κ'_0 with $x \notin K^n_{\kappa'_0}(w)$, whence $x \notin \bigcap_{\kappa'<\kappa} K^{n+1}_{\kappa'}(w)$. In the second case, there exists κ'_0 with

$$\kappa > \kappa'_0 > d_w^{K^n_\kappa(w)}(x) = \inf_{\kappa'<\kappa} d_w^{K^n_{\kappa'}(w)}(x),$$

where the induction hypothesis was used. Thus, there exists κ' with $\kappa'_0 < \kappa' < \kappa$ with $d_w^{K^n_{\kappa'}(w)}(x) \leq \kappa'_0 < \kappa'$. Thus, in this case $x \notin K^{n+1}_{\kappa'}(w)$ either. This finishes the induction step. □

Lemma 4.

(1) $|K_\kappa(w)| = \lim_{n\to\infty} |K^n_\kappa(w)|$.
(2) *If* $|K_\kappa(w)| > 0$, *then* $|K_\kappa(w)| \geq \kappa$.
(3) *If* $|K_\kappa(w)| = 0$, *then* $K_\kappa(w) = \emptyset$. *In this case, the sequence* $K^n_\kappa(w)$ *stabilizes after a finite number of steps, i.e.* $K^n_\kappa(w) = \emptyset$ *for some finite* n.
(4) $\kappa \mapsto |K_\kappa(w)|$ *is upper semi-continuous:* $|K_\kappa(w)| = \lim_{\kappa'<\kappa} |K_{\kappa'}(w)| = \limsup_{\kappa'\to\kappa} |K_{\kappa'}(w)|$.

Proof. (1) follows from $K^{n+1}_\kappa(w) \subseteq K^n_\kappa(w)$ and the monotone convergence theorem. It follows that, if $|K_\kappa(w)| < \kappa$, then $|K^n_\kappa(w)| < \kappa$ for some n. Then, since $w(x, y) \leq 1$ for all $x, y \in [0, 1]$,

$$d_w^{K^n_\kappa(w)}(x) = \int_{K^n_\kappa(w)} w(x, y)\mathrm{d}y \leq |K^n_\kappa(w)| < \kappa$$

for all $x \in K^n_\kappa(w)$, whence $K^{n+1}_\kappa(w) = \emptyset$. From this, (2) and (3) follow.

The last statement follows from Lemma 3(4). □

As Example 2 shows, in general, the sequence $K^n_\kappa(w)$ for fixed κ does not stabilize after a finite number of steps.

Lemma 5.

(1) *The function* $x \mapsto \delta_x(w)$ *is measurable.*
(2) $\bigl|\{x \in [0,1] : d_w(x) \geq \delta(w)\}\bigr| \geq |K_{\delta(w)}| \geq \delta(w)$.
(3) $\inf_{x\in[0,1]} d_w(x) \leq \delta(w) \leq \sup_{x\in[0,1]} d_w(x)$.

Proof. For each $\kappa \in [0,1]$, $\delta_w^{-1}([\kappa,1]) = K_\kappa(w)$ is measurable. This implies (1).

The first inequality in (2) follows from $d_w(x) \geq \delta(w)$ for all $x \in K_{\delta(w)}$. The second inequality follows from Lemma 4.

For (3) observe that if $\kappa \leq \inf_{x\in[0,1]} d_w(x)$, then $[0,1] = K_\kappa(w)$. If $\kappa > \sup_{x\in[0,1]} d_w(x)$, then $K^1_\kappa(w) = \emptyset$. □

Lemma 6. *For any* $\kappa > 0$ *and* $x \in [0,1]$,

$$\delta_x(w) = \max\{\kappa : x \in K_\kappa(w)\},$$
$$\delta(w) = \max\{\kappa : K_\kappa(w) \neq \emptyset\} = \max\{\delta_x(w) : x \in [0,1]\}.$$

Proof. By Lemma 4, the conditions $|K_\kappa(w)| = 0$ and $K_\kappa(w) = \emptyset$ are equivalent. It remains to show that the suprema in the definitions of δ_x and δ are actually maxima. For δ_x, this follows from statement 4 in Lemma 3. For δ, suppose that $K_\kappa(w) = \emptyset$. By Lemma 4, $K^n_\kappa(w) = \emptyset$ for some n. By Lemma 3, there exists $0 < \kappa' < \kappa$ with $|K^n_{\kappa'}(w)| < \kappa'$, whence $K_{\kappa'}(w) = \emptyset$. □

Lemma 7. *Let* σ *be a Measure-preserving transformation, and let* w *be a graphon.*

(1) $K^n_\kappa(w^\sigma) = \sigma^{-1}(K^n_\kappa(w))$
(2) $K^n_\kappa(w) \subseteq \sigma(K^n_\kappa(w^\sigma))$, *with* $|\sigma(K^n_\kappa(w^\sigma)) \setminus K^n_\kappa(w)| = 0$.
(3) $\delta_x(w^\sigma) = \delta_{\sigma(x)}(w)$ *for all* $x \in [0,1]$.
(4) $\delta(w^\sigma) = \delta(w)$.

Proof. Statements (1), (3) and (4) follow by induction on n, using the following equivalence:

$$d^K_{w^\sigma}(x) \geq \kappa \iff \int_K w(\sigma(x),\sigma(y))\mathrm{d}y \geq \kappa$$
$$\iff \int_{\sigma^{-1}(K)} w(\sigma(x),y)\mathrm{d}y \geq \kappa \iff d_w^{\sigma^{-1}(K)}(\sigma(x)) \geq \kappa$$

that holds for all $\kappa \in [0,1]$ and $K \subseteq [0,1]$. The first part of (2) follows from (1), since $K^n_\kappa(w) \subseteq \sigma(\sigma^{-1}(K^n_\kappa(w))) = \sigma(K^n_\kappa(w^\sigma))$.

By (3), if $x \in \sigma(K^n_\kappa(w^\sigma)) \setminus K^n_\kappa(w)$, then $x \notin \sigma([0,1])$. Thus, $\sigma(K^n_\kappa(w^\sigma)) \setminus K^n_\kappa(w) \subseteq [0,1] \setminus \sigma([0,1])$, which is a set of measure zero. □

4. Continuity of the degeneracy

Lemma 8. $|\delta(w) - \delta(w')| \leq 2\sqrt{\delta_\square(w,w')}$.

Proof. Recall that $\delta_\square(w,w') = \inf_\sigma d_\square(w,\sigma w')$, where σ runs over all invertible measure-preserving transformations and where

$$d_\square(w,w') = \sup_{S,T\subseteq[0,1]} \Bigl|\int_S\int_T \bigl(w(x,y) - w'(x,y)\bigr)\mathrm{d}x\mathrm{y}\Bigr|.$$

Since measure-preserving transformations preserve the degeneracy δ, it suffices to show that

$$|\delta(w) - \delta(w')| \leq 2\sqrt{d_\Box(w,w')}.$$

We may assume that $\epsilon := \sqrt{d_\Box(w,w')} > 0$. By symmetry, it is enough to show that $\delta(w') \geq \delta(w) - 2\sqrt{d_\Box(w,w')}$. Since $\delta(w') \geq 0$, we may assume that $\epsilon = \sqrt{d_\Box(w,w')} < \frac{1}{2}\delta(w) < \delta(w)$.

Let K be the $\delta(w)$-core of w, and let

$$S_\epsilon := \Big\{x \in K : d^K_{w'}(x) < \delta(w) - \epsilon\Big\}.$$

Then

$$d_\Box(w,w') \geq \int_{S_\epsilon} \mathrm{d}x \int_K \mathrm{d}y \big(w(x,y) - w'(x,y)\big) > |S_\epsilon|\delta(w) - |S_\epsilon|(\delta(w) - \epsilon) = \epsilon|S_\epsilon|,$$

and so

$$|S_\epsilon| < \frac{d_\Box(w,w')}{\epsilon} = \sqrt{d_\Box(w,w')} < \delta(w) \leq |K|.$$

Let $K' := K \setminus S_\epsilon$. Then $|K'| > 0$. Moreover, for any $x \in K'$,

$$\begin{aligned}\int_{K'} w'(x,y)\mathrm{d}y &= \int_K w'(x,y)\mathrm{d}y - \int_{S_\epsilon} w'(x,y)\mathrm{d}y \\ &\geq d^K_{w'}(x) - |S_\epsilon| > (\delta(w) - \epsilon) - \frac{d_\Box(w,w')}{\epsilon} = \delta(w) - 2\sqrt{d_\Box(w,w')}.\end{aligned}$$

This implies that K' is part of the $(\delta(w) - 2\sqrt{d_\Box(w,w')})$-core. Thus,

$$\delta(w') \geq \delta(w) - 2\sqrt{d_\Box(w,w')}. \qquad \Box$$

Lemma 8 shows that the degeneracy is a continuous graph invariant. The square root suggests that the degeneracy is Hölder continuous, but not Lipschitz continuous. The following example illustrates this:

Example 9. Let $a, b, \alpha \in (0,1)$. Let $w(x,y) = a$ be the constant graphon, and let

$$w'(x,y) = \begin{cases} b, & \text{if } x \leq \alpha, y \leq \alpha, \\ a, & \text{otherwise.} \end{cases}$$

Since w is invariant under all measure-preserving transformations,

$$\delta_\Box(w,w') = d_\Box(w,w') = \alpha^2|a-b|.$$

It suffices to compute

$$\delta(w') = \begin{cases} (1-\alpha)a + \alpha b, & \text{if } b < a, \\ \max\{a, \alpha b\}, & \text{if } b \geq a. \end{cases} \tag{2}$$

This shows that $\delta(w') - \delta(w)$ varies linearly with α for $b < a$, while $\delta_\Box(w,w')$ is quadratic in α.

It remains to prove (2). Since

$$d_{w'}(x) = \begin{cases} (1-\alpha)a + \alpha b, & \text{if } x \leq \alpha, \\ a, & \text{if } x > \alpha, \end{cases}$$

it follows that $K_{\min\{(1-\alpha)a+\alpha b, a\}} = [0,1]$, whence $\delta(w') \geq \min\{(1-\alpha)a + \alpha b, a\}$. Assume that $b < a$, and let $\kappa > (1-\alpha)a + \alpha b$. Then $K^1_\kappa = [\alpha, 1]$ and $d^{K^1_\kappa(w')}(x) \leq$

$(1-\alpha)a < \kappa$ for all x. Hence, $K_\kappa^2(w') = \emptyset = K_\kappa(w')$, which proves that $\delta(w') = (1-\alpha)a + \alpha b$ in this case.

If $b \geq a$, then $\delta(w') \geq a$. Since $[0,\alpha] \subseteq K_{\alpha b}(w')$, it follows that $\delta(w') \geq \alpha b$. If $\kappa > a$, then $K_\kappa^1(w') \subseteq [0,\alpha]$, whence $d_{w'}^{K_\kappa^1(w')}(x) \leq \alpha b$. Thus, if $\kappa > \max\{\alpha b, a\}$, then $K_\kappa^2(w') = \emptyset$, whence $\delta(w') \leq \max\{a, \alpha b\}$.

5. Degeneracy and edge density

For any graphon w, let

$$e(w) = \iint_{[0,1]^2} \mathrm{d}x \, \mathrm{d}y \, w(x,y)$$

be the *edge density* of w.

Lemma 10. *$e(w) \geq \delta(w)^2$, with equality if and only if $\delta_\square(w, w') = 0$, where*

$$w'(x,y) = \begin{cases} 1, & \text{if } x, y \leq \delta(w), \\ 0, & \text{otherwise.} \end{cases}$$

This lemma follows from the corresponding result about graphs (see Kim et al. [2016, Proposition 3.1]). It is also insightfull to prove it directly, mimicking the proof of the result for graphs.

Proof. Let w be a graphon with $\delta(w)$-core K. The restricted graphon

$$w_K(x,y) := \begin{cases} w(x,y), & \text{if } x, y \in K, \\ 0, & \text{otherwise,} \end{cases}$$

has the same degeneracy as w, the same $\delta(w)$-core and satisfies $e(w_K) \leq e(w)$, with strict inequality if the set of pairs $x, y \in [0,1] \setminus K$ with $w(x,y) > 0$ has measure larger than zero. Hence, without loss of generality, we may assume that $w = w_K$; that is, $w(x,y) = 0$ if $x \notin K$ or $y \notin K$.

By Lemma 5,

$$e(w) = \int_K \mathrm{d}x \int_0^1 \mathrm{d}y \, w(x,y) = \int_K \mathrm{d}x \, d_w(x) \geq |K|\delta(w) \geq \delta(w)^2.$$

Equality holds if and only if (i) $|K| = \delta(w)$ and (ii) $d_w(x) = \delta(w)$ for almost all $x \in K$. Condition (i) implies $w(x,y) = 1$ for all $x, y \in K$. There is a measure-preserving transformation $\sigma : [0,1] \to [0,1]$ with $\sigma(K) = [0, \delta(w)]$ and $K = \sigma^{-1}([0, \delta(w)])$. Then $w = (w')^\sigma$, with w' as in the statement of the lemma. □

Lemma 11. *$e(w) \leq \delta(w)(2 - \delta(w))$. For any $\delta \in (0,1)$, equality holds for the graphon*

$$w_\delta(x,y) = \begin{cases} 1, & \text{if } \max\{x,y\} \geq 1 - \delta, \\ 0, & \text{otherwise.} \end{cases}$$

(with $\delta(w_\delta) = \delta$).

Lemma 11 follows from Proposition 3.1 by Kim et al. [2016]. Unlike for the other results, no alternative proof will be presented, as it seems to be difficult to directly prove this result in the graphon world. The proof of Kim et al. relies on Proposition 3.10 by Karwa et al. [2014], which starts by ordering the nodes of a graph according to their shell index. Among nodes with the same shell index k, nodes are ordered according to how quickly they disappear in the sequence $(K^i_{k+1})_i$. Generalizing this approach to graphons poses two problems. First, in the continuous case, a shell index $\kappa \in [0,1]$ does not have a "successor" $\kappa + \epsilon$, as the set of shell indices may be infinite. The appendix contains an example that illustrates the difficulty of ordering the nodes. Second, instead of constructing a permutation that orders the nodes one needs to construct an invertible measure preserving map.

Acknowledgments. Thanks goes to Dane Wilburne, whose questions motivated the results of this paper.

References

Sourav Chatterjee, Persi Diaconis, and Allan Sly. Random graphs with a given degree sequence. *Annals of Applied Probability*, 21(4):1400–1435, 2011.

Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica*, 28:33–61, 2008.

Vishesh Karwa, Michael Pelsmajer, Sonja Petrović, Despina Stasi, and Dane Wilburne. Statistical models for cores decomposition of an undirected random graph. *arXiv:1410.7357*, 2014.

Nicolas Kim, Dane Wilburne, Sonja Petrović, and Alessandro Rinaldo. On the geometry and extremal properties of the edge-degeneracy model. *arXiv:1602.00180*, 2016.

Lászlo Lovász and Balázs Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957, 2006.

Appendix A. Another example

Let $\epsilon = (\epsilon_i)_{i\in\mathbb{N}}, \epsilon' = (\epsilon'_i)_{i\in\mathbb{N}}$ be two monotonically decreasing sequences of positive real numbers with

$$\sum_{i\in\mathbb{N}} \epsilon_i = 1 = \sum_{i\in\mathbb{N}} \epsilon'_i.$$

Let $\alpha_i = \sum_{i'=1}^{i} \epsilon_{i'}$ and $\alpha'_i = \sum_{i'=1}^{i} \epsilon'_{i'}$. Define a graphon $w_{\epsilon,\epsilon'}(x,y)$ as follows: for all $x, y \in [0,1]$ with $x < y$, let

$$w_{\epsilon,\epsilon'}(x,y) = \begin{cases} 1, & \text{if } \frac{1}{5}(1-\alpha_{i+1}) \le x < \frac{1}{5}(1-\alpha_i) \\ & \quad \text{and } \frac{1}{5}(1-\alpha_i) \le y < \frac{1}{5}(1-\alpha_{i-1}), \\ 1-\epsilon_{i-1}, & \text{if } \frac{1}{5}(1-\alpha_i) \le x < \frac{1}{5}(1-\alpha_{i-1}) \\ & \quad \text{and } \frac{1}{5} \le y < \frac{2}{5}, \\ 1, & \text{if } \frac{1}{5} \le x < \frac{2}{5} \text{ and } \frac{2}{5} \le y < \frac{3}{5}, \\ 1, & \text{if } \frac{2}{5} \le x < \frac{3}{5} \text{ and } \frac{3}{5} \le y < \frac{4}{5}, \\ 1-\epsilon'_{i-1}, & \text{if } \frac{3}{5} \le x < \frac{4}{5} \\ & \quad \text{and } \frac{1}{5}(4+\alpha'_{i-1}) \le y < \frac{1}{5}(4+\alpha'_i), \\ 1, & \text{if } \frac{1}{5}(4+\alpha'_{i-1}) \le x < \frac{1}{5}(4+\alpha'_i) \\ & \quad \text{and } \frac{1}{5}(4+\alpha'_i) \le y < \frac{1}{5}(4+\alpha'_{i+1}), \\ 0, & \text{otherwise,} \end{cases}$$

where $\epsilon_0 = \epsilon'_0 = 0 = \alpha_0 = \alpha'_0$. In a picture:

$\ddots$ 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0	$\vdots$ $1-\epsilon_3$ $1-\epsilon_2$ $1-\epsilon_1$ 1	0	0	0
$1-\epsilon_{i-1}$	0	1	0	0
0	1	0	1	0
0	0	1	0	$1-\epsilon'_{i-1}$
0	0	0	1 $1-\epsilon'_1$ $1-\epsilon'_2$ $1-\epsilon'_3$ $\vdots$	0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0 $\ddots$

The calculation in the following will show that each $K^n_\kappa(w)$ is of the form $\big[l_n(\kappa); u_n(\kappa)\big)$ or empty, with strictly monotonically increasing sequences $l_n(\kappa)$, $u_n(\kappa)$. Claim: *It is possible to construct the sequences ϵ, ϵ' such that for any $\kappa_0 > \frac{1}{5}$ there are $\kappa, \kappa' > \frac{1}{5}$ with $\kappa, \kappa' < \kappa_0$ and*

$$l_n(\kappa) \le \frac{2}{5} < u_n(\kappa) < \frac{3}{5}, \quad \text{and} \quad \frac{2}{5} < l_{n'}(\kappa) \le \frac{3}{5} < u_{n'}(\kappa)$$

for some n, n'. This shows that it is difficult to generalize the idea of the proof Proposition 3.10 by Karwa et al. [2014] of ordering the nodes of a graph by how fast they disappear in the sequence K_κ^n.

One computes

$$d_w(x) = \begin{cases} \frac{1}{5}(1+\epsilon_{i+1}), & \text{if } \frac{1}{5}(1-\alpha_i) \le x < \frac{1}{5}(1-\alpha_{i-1}),\ i \ge 1, \\ \frac{1}{5}\big(1+\beta_\infty\big), & \text{if } \frac{1}{5} \le x < \frac{2}{5}, \\ \frac{2}{5}, & \text{if } \frac{2}{5} \le x < \frac{3}{5}, \\ \frac{1}{5}\big(1+\beta'_\infty\big), & \text{if } \frac{3}{5} \le x < \frac{4}{5}, \\ \frac{1}{5}(1+\epsilon'_{i+1}), & \text{if } \frac{1}{5}(4+\alpha'_{i-1}) \le x < \frac{1}{5}(4+\alpha'_i),\ i \ge 1, \end{cases}$$

where $\beta_j = \sum_{i=1}^j \epsilon_i(1-\epsilon_{i-1})$ and $\beta'_j = \sum_{i=1}^j \epsilon'_i(1-\epsilon'_{i-1})$. The assumptions on ϵ, ϵ' imply $0 \le \beta_j, \beta'_j \le 1$ and $\beta_j \le \beta_{j+1}$ and $\beta'_j \le \beta'_{j+1}$ for $j = 1, 2, \dots, \infty$. It follows that $K_\kappa(w) = [0,1]$ for $\kappa \le \frac{1}{5}$.

Let $\kappa > \frac{1}{5}$ be such that $\kappa \le \frac{1}{5}\big(1+\min\{\beta_1, \beta'_1\}\big)$. Denote by i_κ, i'_κ the smallest positive integers with $\frac{1}{5}(1+\epsilon_{i_\kappa+1}) < \kappa$ and $\frac{1}{5}(1+\epsilon'_{i'_\kappa+1}) < \kappa$. Then $K_\kappa^1(w) = \big[\frac{1}{5}(1-\alpha_{i_\kappa-1}), \frac{1}{5}(4+\alpha'_{i'_\kappa-1})\big)$, and

$$d_w^{K_\kappa^1(w)}(x) = \begin{cases} \frac{1}{5}, & \text{if } \frac{1}{5}(1-\alpha_{i_\kappa-1}) \le x < \frac{1}{5}(1-\alpha_{i_\kappa-2}), \\ \frac{1}{5}(1+\epsilon_{i+1}), & \text{if } \frac{1}{5}(1-\alpha_i) \le x < \frac{1}{5}(1-\alpha_{i-1}),\ 1 \le i < i_\kappa - 1 \\ \frac{1}{5}\big(1+\beta_{i_\kappa-1}\big), & \text{if } \frac{1}{5} \le x < \frac{2}{5}, \\ \frac{2}{5}, & \text{if } \frac{2}{5} \le x < \frac{3}{5}, \\ \frac{1}{5}\big(1+\beta'_{i'_\kappa-1}\big), & \text{if } \frac{3}{5} \le x < \frac{4}{5}, \\ \frac{1}{5}(1+\epsilon'_{i+1}), & \text{if } \frac{1}{5}(4+\alpha'_{i-1}) \le x < \frac{1}{5}(4+\alpha'_i),\ 1 \le i < i'_\kappa - 1, \\ \frac{1}{5}, & \text{if } \frac{1}{5}(4+\alpha_{i'_\kappa-2}) \le x < \frac{1}{5}(4+\alpha_{i'_\kappa-1}), \end{cases}$$

for $x \in K_\kappa^1(w)$.

Suppose that $i_\kappa > 1$. If $\frac{1}{5}(1-\alpha_{i_\kappa-1}) \le x < \frac{1}{5}(1-\alpha_{i_\kappa-2})$, then $x \notin K_\kappa^2(w)$. On the other hand, the interval $\big[\frac{1}{5}(1-\alpha_{i_\kappa-2}); \frac{1}{2}\big]$ belongs to $K_\kappa^2(w)$. Similarly, if $i'_\kappa > 1$, then $K_\kappa^2(w)$ contains $\big[\frac{1}{2}; \frac{1}{5}(4+\alpha'_{i'_\kappa-2})\big)$, but does not contain any x with $\frac{1}{5}(4+\alpha_{i_\kappa-1}) \le x < \frac{1}{5}(4+\alpha_{i_\kappa})$. Using induction, it follows that

$$K_\kappa^k(w) = \big[\frac{1}{5}(1-\alpha_{i_\kappa-k}), \frac{1}{5}(4+\alpha'_{i'_\kappa-k})\big]$$

for $k = 1, \dots, \min\{i_\kappa, i'_\kappa\}$.

Assume that $i_\kappa \le i'_\kappa$. Then $K_\kappa^{i_\kappa}(w) = \big[\frac{1}{5}, \frac{1}{5}(4+\alpha'_{i'_\kappa-i_\kappa})\big)$, and

$$d_w^{K_\kappa^{i_\kappa}}(x) = \begin{cases} \frac{1}{5}, & \text{if } \frac{1}{5} \le x < \frac{2}{5}, \\ \frac{2}{5}, & \text{if } \frac{2}{5} \le x < \frac{3}{5}, \\ \frac{1}{5}\big(1+\beta'_{i'_\kappa-i_\kappa}\big), & \text{if } \frac{3}{5} \le x < \frac{4}{5}, \\ \frac{1}{5}(1+\epsilon'_{i+1}), & \text{if } \frac{1}{5}(4+\alpha'_{i-1}) \le x < \frac{1}{5}(4+\alpha'_i),\ 1 \le i < i'_\kappa - i_\kappa, \\ \frac{1}{5}, & \text{if } \frac{1}{5}(4+\alpha_{i'_\kappa-i_\kappa-1}) \le x < \frac{1}{5}(4+\alpha_{i'_\kappa-i_\kappa}), \end{cases}$$

for $x \in K^{\kappa^{i_\kappa}}(w)$. If $i'_\kappa = i_\kappa$, then $K_\kappa^{i_\kappa+1}(w) = [\frac{2}{5}; \frac{3}{5})$. If $i'_\kappa = i_\kappa + 1$, then $K_\kappa^{i_\kappa+1}(w) = [\frac{2}{5}; \frac{4}{5})$. Thus, if $i'_\kappa \in \{i_\kappa, i_\kappa+1\}$, then $K_\kappa^{i_\kappa+2}(w) = \emptyset$.

If $i'_\kappa \geq i_\kappa + 2$, then $K_\kappa^{i_\kappa+1}(w) = \left[\frac{2}{5}; \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 1})\right)$, and thus

$$d_w^{K_\kappa^{i_\kappa+1}}(x) = \begin{cases} \frac{1}{5}, & \text{if } \frac{2}{5} \leq x < \frac{3}{5}, \\ \frac{1}{5}\left(1 + \beta'_{i'_\kappa - i_\kappa - 1}\right), & \text{if } \frac{3}{5} \leq x < \frac{4}{5}, \\ \frac{1}{5}(1 + \epsilon'_{i+1}), & \text{if } \frac{1}{5}(4 + \alpha'_{i-1}) \leq x < \frac{1}{5}(4 + \alpha'_i), \\ & \qquad 1 \leq i < i'_\kappa - i_\kappa - 1, \\ \frac{1}{5}, & \text{if } \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 2}) \leq x < \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 1}), \end{cases}$$

for $x \in K_\kappa^{i_\kappa+1}(w)$. Therefore, $K_\kappa^{i_\kappa+2} = \left[\frac{3}{5}; \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 2})\right)$. If $i'_\kappa = i_\kappa + 2$, then $K_\kappa^{i_\kappa+3} = \emptyset$.

Otherwise, if $i'_\kappa > i_\kappa + 2$, then

$$d_w^{K_\kappa^{i_\kappa+2}}(x) = \begin{cases} \frac{1}{5}\beta'_{i'_\kappa - i_\kappa - 2}, & \text{if } \frac{3}{5} \leq x < \frac{4}{5}, \\ \frac{1}{5}(1 + \epsilon'_{i+1}), & \text{if } \frac{1}{5}(4 + \alpha'_{i-1}) \leq x < \frac{1}{5}(4 + \alpha'_i), \\ & \qquad 1 \leq i < i'_\kappa - i_\kappa - 2, \\ \frac{1}{5}, & \text{if } \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 3}) \leq x < \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 2}), \end{cases}$$

whence $K_\kappa^{i_\kappa+3} = \left[\frac{4}{5}; \frac{1}{5}(4 + \alpha_{i'_\kappa - i_\kappa - 3})\right)$, and $K_\kappa^{i_\kappa+4} = \emptyset$.

In any case, if $i_\kappa < i'_\kappa$, then $[\frac{2}{5}; \frac{3}{5}) \subseteq K_\kappa^{i_\kappa+1} \setminus K_\kappa^{i_\kappa+2}$ and $[\frac{3}{5}; \frac{4}{5}) \subseteq K_\kappa^{i_\kappa+2}$ (that is, any $x \in [\frac{2}{5}; \frac{3}{5})$ disappears before any $y \in [\frac{3}{;}\frac{4}{5})$. Conversely, if $i_\kappa < i'_\kappa$, then $[\frac{3}{5}; \frac{4}{5}) \subseteq K_\kappa^{i'_\kappa} \setminus K_\kappa^{i'_\kappa+1}$ and $[\frac{2}{5}; \frac{3}{5}) \subseteq K_\kappa^{i'_\kappa+1}$.

To prove the claim, it suffices to construct the sequences ϵ, ϵ' such that for any $\kappa_0 > \frac{1}{5}$ there are $\kappa, \kappa' > \frac{1}{5}$ with $\kappa, \kappa' < \kappa_0$ and $i_\kappa < i'_\kappa$ and $i_{\kappa'} > i'_{\kappa'}$. Consider the function $f(n) = 1 - \frac{1}{n+1}$, and let

$$\alpha_n = \begin{cases} \frac{f(n-1)+f(n+1)}{2}, & \text{if } n \text{ is odd,} \\ f(n), & \text{if } n \text{ is even,} \end{cases} \qquad \alpha'_n = \begin{cases} f(n), & \text{if } n \text{ is odd,} \\ \frac{f(n-1)+f(n+1)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

As f is concave, the sequences $\epsilon_n = \alpha_n - \alpha_{n-1}$, $\epsilon'_n = \alpha'_n - \alpha'_{n-1}$ are decreasing. Moreover,

$$\epsilon'_1 > \epsilon_1 = \epsilon_2 > \epsilon'_2 = \epsilon'_3 > \epsilon_4 = \dots$$

Let $\kappa_i = \frac{1}{5}\left(+ \frac{\epsilon_i + \epsilon'_i}{2}\right)$. Then $\frac{1}{5}(1 + \epsilon'_i) > \kappa_i > \frac{1}{5}(1 + \epsilon_i)$ if i is odd, and $\frac{1}{5}(1 + \epsilon_i) > \kappa_i > \frac{1}{5}(1 + \epsilon'_i)$ if i is even. Thus,

$$i_{\kappa_i} = \begin{cases} i - 1, & \text{if } i \text{ is odd,} \\ i, & \text{if } i \text{ is even,} \end{cases} \quad \text{and} \quad i'_{\kappa_i} = \begin{cases} i, & \text{if } i \text{ is odd,} \\ i - 1, & \text{if } i \text{ is even.} \end{cases}$$

Thus, the sequences ϵ, ϵ' satisfy the claim.

MPI for Mathematics in the Sciences, Inselstrasse 23, 04103 Leipzig
E-mail address: jrauh@mis.mpg.de