Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Geometric analysis of a mixed elliptic-parabolic conformally invariant boundary value problem

(revised version: May 2019)

by

Jürgen Jost, Lei Liu, and Miaomiao Zhu

Preprint no.: 41 2018

GEOMETRIC ANALYSIS OF A MIXED ELLIPTIC-PARABOLIC CONFORMALLY INVARIANT BOUNDARY VALUE PROBLEM

JÜRGEN JOST, LEI LIU, AND MIAOMIAO ZHU

Abstract. In this paper, we show the existence of Dirac-harmonic maps from a compact spin Riemann surface with smooth boundary to a general compact Riemannian manifold via a heat flow method when a Dirichlet boundary condition is imposed on the map and a chiral boundary condition on the spinor. Technically, we solve a new elliptic-parabolic system arising in geometric analysis that is motivated by the nonlinear supersymmetric sigma model of quantum field theory. The corresponding action functional involves two fields, a map from a Riemann surface into a Riemannian manifold and a spinor coupled to the map. The first field has to satisfy a second order elliptic system, which we turn into a parabolic system so as to apply heat flow techniques. The spinor, however, satisfies a first order Dirac type equation. We carry that equation as a nonlinear constraint along the flow. In order to solve this system, we adapt the idea of Sacks-Uhlenbeck to raise the integrand of the harmonic map action to a power $\alpha > 1$; the solutions of the resulting Euler-Lagrange equations are called α-Dirac harmonic maps. Because of the (unchanged) spinor action, the analysis is more difficult than that of Sacks-Uhlenbeck. Nevertheless, we can carry out the limit $\alpha \searrow 1$ to solve our original problem.

Then we develop a general spectrum of methods (Pohozaev identity, three-circle method, blow-up analysis, energy identities, energy decay estimates etc.) for the compactness problem of the space of α-Dirac harmonic maps and for a further analysis of the limiting problem. We study the refined blow-up behaviour and asymptotic analysis for a sequence of α-Dirac harmonic maps from a compact Riemann surface with smooth boundary into a general compact Riemannian manifold with uniformly bounded energy. We prove generalized energy identities for both the map part and the spinor part. We also show that the map parts of the α-Dirac-harmonic necks converge to some geodesics on the target manifold. Moreover, we give a length formula for the limiting geodesic near a blow-up point. In particular, if the target manifold has a positive lower bound on the Ricci curvature or has a finite fundamental group and the sequence of α-Dirac harmonic maps has bounded Morse index, then the limit of the map part of the necks consists of geodesics of finite length which ensures the energy identities hold. In technical terms, these results are achieved by establishing a new decay estimate of the tangential energies of both the map part and the spinor part as well as a new decay estimate for the energy of the spinor as $\alpha \searrow 1$.

Contents

Date: May 12, 2019.

2010 *Mathematics Subject Classification.* 53C43 58E20.

Key words and phrases. Dirac-harmonic maps, α-Dirac-harmonic maps, α-Dirac-harmonic map flow, Dirichlet-chiral boundary, Existence, Blow-up, Energy identity, Neck analysis.

1. Introduction

Perhaps the most important core of geometric analysis consists of problems that are invariant under a non-compact group of local symmetries. Such problems, in particular, do not satisfy a Palais-Smale condition. Therefore, the standard methods of the calculus of variations and non-linear PDEs usually don't apply. The Palais-Smale condition for such problems, however, only barely fails, and more precisely, we are typically looking at limit cases of this condition. And this then requires a sophisticated asymptotic analysis that controls the formation of singularities and characterizes or even classifies the possible singularities. While the techniques employed need to carefully exploit the particular structure at hand, the techniques developed for one particular problem sometimes generalize to others, because the fundamental phenomena involving loss of compactness, blow-up of solutions, energy identities, are similar.

Such a precise asymptotic analysis is necessary to understand the space of solutions and to lay the ground for developing a Morse type theory for such variational problems, for instance. In this regard, after impressive advances over several decades in systematically investigating such problems that are invariant under a non-compact group of symmetries, programs of developing a Morse theory in such a setting are currently pursued for minimal hypersurfaces by Marques-Neves e.g. [39, 40, 41, 42, 43], with Sharp's recent analysis [57] being an important ingredient, and for minimal surfaces via a viscosity method by Rivière and his collaborators e.g. [51, 52, 53, 54]. For older results that developed a careful asymptotic analysis for a Morse type theory, see for instance [62, 15, 44, 47, 28].

Such advanced theories cannot succeed by analytical tools alone. They require a very precise and careful utilization of the geometric and algebraic structure of the particular setting. On the other hand, however, the approach should not be too specialized but rather also bring out some more general features, and in particular, develop methods of a more general scope. For this reason, we have systematically turned to another area, quantum field theory, where variational problems with noncompact symmetries naturally arise that have a rich and subtle geometric and algebraic structure and that have lead to spectacular mathematical insights. As an example of such a problem, motivated by the nonlinear supersymmetric sigma model [14], Dirac-harmonic maps are defined as solutions of a system consisting of a harmonic map type equation coupled with a Dirac type equation. They were introduced and first studied in [8, 9]. They combine and generalize harmonic maps and harmonic spinors, both of which have been widely and extensively studied, but only separately and not coupled as here.

The resulting variational problems therefore combines also essentially all the difficulties that one has to face in geometric analysis when dealing with problems with a noncompact local symmetry group. And since these difficulties are intertwined, they cannot be solved with the individual tools that have been developed for each of them in isolation, but we also need to extend the existing methods substantially. In some sense, this is the culmination of the work of ourselves and many colleagues over several years. Let us first informally describe the difficulties and point out our strategies to overcome them. We consider the functional

$$L(\phi,\psi) \;=\; \frac{1}{2}\int_M \Big(|d\phi|^2 + \langle\psi, \not{D}\psi\rangle_{\Sigma M\otimes\phi^\star TN}\Big)dM, \tag{1.1}$$

where ϕ is a map from a Riemann surface M (with or without boundary) into a Riemannian manifold N, and ψ is a spinor along that map. $\not{D}$ is a nonlinear Dirac operator that depends on ϕ, and through ϕ also on the geometry of N. Therefore, the two fields ϕ and ψ are coupled. While the precise notation will be explained shortly, hopefully this already suffices to understand the essential structure of the problem. So, here are the difficulties and how we address them:

(1) The functional is conformally invariant, and since the conformal group is noncompact, the Palais-Smale condition is violated. To handle this, there exist of course well established schemes. In particular, following Sacks-Uhlenbeck [55], we can modify (1.1) as

$$L_\alpha(\phi,\psi) = \frac{1}{2}\int_M \Big\{(1+|d\phi|^2)^\alpha + \langle\psi, \not{D}\psi\rangle\Big\}\,dM, \tag{1.2}$$

for $\alpha > 1$. This functional then satisfies Palais-Smale, and we can study the limit $\alpha \searrow 1$.

(2) However, neither (1.1) nor (1.2) is bounded from below, because of the spinor term which is strongly indefinite, as the spectrum of the Dirac operator is neither bounded from below nor from above. Therefore, classical variational methods (like direct minimization procedure or minimax scheme) don't apply to get the existence of critical points of (1.2), let alone (1.1). This is one crucial difference to the harmonic map problem studied by Sacks-Uhlenbeck [55] and many others, and therefore, we need to develop new tools.

(3) An alternative to variational methods are heat flow methods. But here the difficulty is that while the Euler-Lagrange equation for ϕ is second order and can therefore be easily parabolized, the equation for ψ is first order (it looks simple, $\not{D}\psi = 0$, but this is nonlinear, because the operator depends on ϕ, as already pointed out).

(4) Our solution consists in considering a compact domain surface with smooth boundary and studying an elliptic-parabolic system under appropriate boundary conditions that carries the nonlinear Dirac equation along as an elliptic side constraint for a harmonic type flow. To control that elliptic constraint, sharp estimates for solutions of Dirac equations under appropriate boundary conditions are needed. For the functional (1.1), the heat flow approach was carried out in [12, 26] and some partial existence results were obtained. In this paper, we solve the general existence problem. Our new scheme is to first investigate the heat flow for the functional (1.2) and get the existence of α-Dirac-harmonic maps for $\alpha > 1$ and close to 1, then we derive the existence of Dirac-harmonic maps by letting $\alpha \searrow 1$. Our scheme of converting the existence problem for a coupled system consisting of a second- and a first-order elliptic part into an elliptic-parabolic system should also be useful for other variational problems with a first-order side condition, like the conformality relations for minimal surfaces.

(5) In order to then control the limit $\alpha \searrow 1$, we need to carry out a blow-up analysis, because singularities may form and so-called bubbles may split off in the limit, see Section 5. While for the original Sacks-Uhlenbeck scheme for harmonic maps, this procedure is well

established, here we encounter new difficulties due to the limiting behavior of the spinor part.

(6) Also, the bubbles that split off need not be connected to the remaining solution. In order to show that in the limit, energy is only carried by that remaining solution and the bubbles, we need to control the necks that connect the forming bubbles with the rest for $\alpha > 1$, but disappear in the limit $\alpha \searrow 1$. In fact, the precise analysis reveals that they converge to geodesic lines and a length formula for these limit curves can be derived. When such a geodesic is of finite length, it appears as the image of a cylinder of infinite length and therefore carries no energy. To show this, we want to use techniques established for harmonic maps, more precisely the three-circle method [59]. In our situation, we first need to derive a new Pohozaev identity. Incidentally, the validity of a Pohozaev identity, rather than the stronger condition of conformal invariance, is what is equivalent to the removability of singularities in geometric variational problems [29], and therefore, our emphasis on such an identity fits well into the general scheme.

(7) Nevertheless, we cannot directly use a technique like the three-circle method as in [50] (or alternative methods developed for harmonic map type problems), because in that scheme, we would have to deal with two additional terms that are not uniformly bounded in L^2 (in fact they are not uniformly bounded in L^p for any $1 < p < 2$). In technical terms, to overcome this difficulty, we need to develop a new three-circle method for a class of integro-differential systems and control the decay of the tangential energies of both the map part and the spinor part on the neck domain as $\alpha \searrow 1$. This is perhaps the main technical achievement of the present paper. In fact, these decay estimates are new even for α-harmonic maps, that is, in the absence of the spinor part. In our case, however, we also need to obtain a new estimate for the energy decay of the spinor.

(8) Finally, we establish that under general and natural assumptions (lower Ricci bound on the target or finite fundamental group on the target and bounded variational Morse index of our subsequence of maps and spinors), the neck geodesics are indeed of finite length and hence do not carry energy in the limit.

Since we are using a heat flow approach to get the critical points of the functional L_α in (1.2), namely, α-Dirac-harmonic maps, in general, it is not clear whether the Morse index of the sequence $(\phi_\alpha, \psi_\alpha)$ is bounded or not. Therefore, to understand the compactness of the spaces of critical points as α approaches to 1, it is necessary for us to study the refined blow-up behaviour and asymptotic analysis for a general sequence of α-Dirac-harmonic maps as described in Theorems 2.6 and 2.8, which illustrate all possible blow-up phenomena that can occur.

On the other hand, we expect that a Floer type variational scheme should apply to derive the existence of critical points of the functional L_α in (1.2) and we could then also expect a sequence of α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha)$ with bounded Morse index, at which point our results in Theorem 2.12 can be applied. Remarkably, in the definitions of Morse index for Dirac-harmonic maps and α-Dirac-harmonic maps underlying Theorem 2.12, see Definition (2.10) and (2.11), it suffices to consider variations of the map part to get powerful convergence results. Assuming that this index is finite, as we do, is a very natural assumption. In contrast, if we define the Morse index by considering variations of both the map and the spinor, then it is likely that the Morse index is infinite, because the Dirac operator has infinitely many negative eigenvalues.

The rest of this paper is organized as follows. In Section 2, we introduce our functional in precise technical terms and state and explain our main results. In Section 3, we derive the Euler-Lagrange equations for α-Dirac-harmonic maps and prove the key estimate (2.6). In Section 4, we establish some properties of α-Dirac-harmonic map flow and obtain the global existence Theorem 2.1. In Section 5, we study the blow-up behaviour for a sequence of α-Dirac-harmonic maps. Theorem 2.2, Theorem 2.4 and Theorem 2.5 are proved in this section. In Section 6, we shall then prove some basic lemmas for α-Dirac-harmonic maps which will be used in this paper, such as the small energy regularity, the energy gap theorem and a new Pohozaev type identity. Then we establish the three-circle theorem for a system of integro-differential equations which can be applied to the α-Dirac-harmonic map system. In Section 7, we prove Theorem 2.6 about generalized energy identities. The limit behavior of α-Dirac-harmonic necks is studied in Section 8 and Theorem 2.8 is proved in this section. In Section 9, we derive the second variational formula for the functionals L_α and L and then prove Theorem 2.12.

2. Summary and main results

Let M be a compact Riemann surface with smooth boundary ∂M, equipped with a Riemannian metric g and with a fixed spin structure, ΣM be the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ be the natural Hermitian inner product on ΣM. Choosing a local orthonormal basis $e_\gamma, \gamma = 1,2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/_g := e_\gamma \cdot \nabla_{e_\gamma}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X \cdot \psi, \varphi\rangle_{\Sigma M} = -\langle \psi, X \cdot \varphi\rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\psi,\ \varphi \in \Gamma(\Sigma M)$.

Let ϕ be a smooth map from M to another compact Riemannian manifold (N, h) with dimension $n \geq 2$. Denote ϕ^*TN the pull-back bundle of TN by ϕ and then we get the twisted bundle $\Sigma M \otimes \phi^*TN$. There is a natural metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^*TN}$ on $\Sigma M \otimes \phi^*TN$ induced from the metrics on ΣM and ϕ^*TN. Likewise, there is a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^*TN$ induced from the connections on ΣM and ϕ^*TN. Let ψ be a section of the bundle $\Sigma M \otimes \phi^*TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a usual spinor on M and ∂_{y^i} is the nature local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k \otimes \partial_{y^i}(\phi), \tag{2.1}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by

$$D\!\!\!\!/\,\psi := e_\gamma \cdot \widetilde{\nabla}_{e_\gamma}\psi.$$

We consider the following functional

$$L(\phi,\psi) = \frac{1}{2}\int_M \Big(|d\phi|^2 + \langle\psi, D\!\!\!\!/\,\psi\rangle_{\Sigma M\otimes\phi^*TN}\Big)dM,$$

where $dM = dvol_g$.

The functional $L(\phi,\psi)$ is conformally invariant, see [9]. That is, for any conformal diffeomorphism $f: M\to M$, setting

$$\widetilde{\phi}=\phi\circ f \qquad and \qquad \widetilde{\psi}=\lambda^{-1/2}\psi\circ f,$$

where the positive function $\lambda>0$ is the conformal factor of the conformal map f, *i.e.* $f^*g=\lambda^2 g$, there holds $L(\widetilde{\phi},\widetilde{\psi})=L(\phi,\psi)$. Critical points (ϕ,ψ) are called Dirac-harmonic maps from M to N.

The Euler-Lagrange equations of the functional L are

$$\left(\Delta_g\phi^i+\Gamma^i_{jk}g^{\alpha\beta}\phi^j_\alpha\phi^k_\beta\right)\frac{\partial}{\partial y^i}(\phi(x)) = R(\phi,\psi), \tag{2.2}$$

$$\not{D}\psi = 0, \tag{2.3}$$

where $\Delta_g:=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}(\sqrt{g}g^{\beta\gamma}\frac{\partial}{\partial x^\gamma})$ is the Laplacian operator with respect to the Riemannian metric g, $R(\phi,\psi)$ is defined by

$$R(\phi,\psi)=\frac{1}{2}R^m_{lij}(\phi(x))\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle\frac{\partial}{\partial y^m}(\phi(x)).$$

Here R^m_{lij} is the Riemann curvature tensor of the target manifold (N,h).

By Nash's embedding theorem, we embed N isometrically into some $\mathbb{R}^K$. Then, critical points (ϕ,ψ) of the functional L satisfy the following extrinsic Euler-Lagrange equations

$$\Delta_g\phi = A(\phi)(d\phi,d\phi)+Re(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)), \tag{2.4}$$

$$\not{\partial}_g\psi = \mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi), \tag{2.5}$$

where $\not{\partial}_g$ is the usual Dirac operator on (M,g), $A(\cdot,\cdot)$ is the second fundamental form of N in $\mathbb{R}^K$, and

$$\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi):=(\nabla\phi^i\cdot\psi^j)\otimes A(\partial_{y^i},\partial_{y^j}),$$

$$Re(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)):=P(A(\partial_{y^l},\partial_{y^j});\partial_{y^i})Re(\langle\psi^i,d\phi^l\cdot\psi^j\rangle).$$

Here $P(\xi;\cdot)$ denotes the shape operator, defined by $\langle P(\xi;X),Y\rangle=\langle A(X,Y),\xi\rangle$ for $X,Y\in\Gamma(TN)$, and $Re(z)$ denotes the real part of $z\in\mathbb{C}$.

For $p>1$, we denote

$$W^{1,p}(M,N):=\left\{\phi\in W^{1,p}(M,\mathbb{R}^K)\mid\phi(x)\in N,\ a.e.\ x\in M\right\},$$

$$W^{1,p}(\Sigma M\otimes\phi^*TN):=\left\{\psi\in W^{1,p}(\Sigma M\otimes\phi^*\mathbb{R}^K)\mid\psi(x)\text{ is along the map }\phi,\ a.e.\ x\in M\right\}.$$

Here $\psi\in\Gamma(\Sigma M\otimes\phi^*TN)$ is along the map ϕ and should be understood as a K-tuple of spinors $(\psi^1,...,\psi^K)$ satisfying

$$\sum_{i=1}^K\nu_i\psi^i(x)=0$$

for any normal vector $\nu=(\nu_1,...,\nu_K)\in\mathbb{R}^K$ at $\phi(x)$. For more details and background on Dirac-harmonic maps, we refer to [8, 9, 66, 11, 58].

The blow-up theory for sequences of Dirac-harmonic maps, including the energy identity and the no neck property, i.e., bubble tree convergence, was explored in [8, 65, 38]. For the existence results of Dirac-harmonic maps, since the functional $L(\phi,\psi)$ does not have a lower bound due to the fact that the second term in the functional L does not have a fixed sign, classical variational methods developed for harmonic maps cannot be applied directly and hence the problem becomes very difficult. Up to now, there are only few results in this regard. See [10] for some attempt via the maximum principle, where some partial existence results were obtained. See [5] for a regularized heat flow approach for regularized Dirac-harmonic maps, which is different from ours to be introduced in a moment. See [1, 13] for some existence results of uncoupled Dirac-harmonic maps (here uncoupled means that the map part is harmonic) based on index theory and the Riemann-Roch theorem. For explicit constructions of solutions, see the recent [2].

In order to study the general existence problem, a heat flow approach for Dirac-harmonic maps from spin Riemannian manifolds with boundary was introduced in [12], and the short time existence of a solution was shown. (Recently, Wittmann [63] could show short time existence also in the case of a closed domain under certain conditions on the initial data.) Furthermore, the existence of a global weak solution to this flow in dimension two under some boundary-initial constraint was obtained in [26]. By studying the limit behaviour as time approaches infinity, they proved the existence results of Dirac-harmonic maps with Dirichlet-chiral boundary condition in a given homotopy class under the boundary-initial constraint. A technical difficulty stems from the fact that along the Dirac-harmonic map flow considered in [26], we only have that the energy of the map ϕ is uniformly bounded, i.e.,

$$E(\phi(\cdot,t)) = \int_M |\nabla\phi(\cdot,t)|^2 dM \le C < +\infty.$$

However, the Dirac type equation (2.5) for the spinor ψ does not control the energy of the spinor field

$$E(\psi(\cdot,t)) = \int_M |\psi(\cdot,t)|^4 dM,$$

as time approaches the first singular time $T_1 > 0$, even for the L^1-norm. This is the main difficulty and why we need to impose the additional boundary-initial constraint in [26] in order to obtain a global weak solution to the Dirac-harmonic map flow and show some existence results by letting time goes to infinity. The general question, however, is

Question I: Does there exist a Dirac-harmonic map from a compact Riemann surface with boundary to a compact Riemannian manifold with general Dirichlet-chiral boundary data?

In this paper, we give an affirmative answer to this question. To achieve this, we shall utilize a new parabolic-elliptic system.

In our new approach, one crucial observation is the following *key estimate* for the Dirac operator $\not{D}$ along a given map (see Lemma 3.4):

Key estimate: Let $\phi \in W^{1,q}(M,N)$ for some $q > 2$ and $\psi \in W^{1,p}(M, \Sigma M \otimes \phi^*TN)$ for some $1 < p < 2$, then there holds

$$\|\psi\|_{W^{1,p}(M)} \le C(p, M, N, \|\nabla\phi\|_{L^q(M)})(\|\not{D}\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}). \tag{2.6}$$

Here $\mathbf{B}$ is the chiral boundary operator for spinors along a map, see (2.13) for more details.

There are two key properties of the above estimate. The first one is that the positive constant $C = C(p, M, N, \|\nabla\phi\|_{L^q(M)}) > 0$ depends on the norm $\|\nabla\phi\|_{L^q(M)}$ with $q > 2$ of the map ϕ, which was observed in [12]. The second one is that the two numbers $q > 2$ and $1 < p < 2$ are independent of each other. This fact was not exploited in [12] while it will play an important role in this paper. In fact, such kind of estimate holds true for more general Dirac type systems, see Lemma 3.3.

Since the key estimate for the Dirac operator $\not{D}$ along a map in (2.6) requires that the map ϕ lies in $W^{1,q}(M,N)$ for some $q>2$, inspired by this fact, we introduce the following functional

$$L_\alpha(\phi,\psi)=\frac{1}{2}\int_M\Big\{(1+|d\phi|^2)^\alpha+\langle\psi,\not{D}\psi\rangle\Big\}\,dM, \tag{2.7}$$

where $\alpha>1$ is a constant. Critical points $(\phi_\alpha,\psi_\alpha)$ of L_α are called *α-Dirac-harmonic maps* from M to N. When the spinor field is vanishing, the above functional reduces to Sacks-Uhlenbeck's approximation for harmonic maps in [55].

By a direct computation, critical points $(\phi_\alpha,\psi_\alpha)$ of the functional L_α satisfy the following Euler-Lagrange equations (see Lemma 3.2)

$$\Delta_g\phi = -(\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2}+A(d\phi,d\phi)+\frac{Re\Big(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\Big)}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{2.8}$$

$$\not{\partial}_g\psi = \mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi). \tag{2.9}$$

One crucial step in our scheme is to get the existence result of Dirac-harmonic maps through studying the limit behaviour of a sequence of α-Dirac-harmonic maps as $\alpha\searrow 1$ [1]. If there exists a sequence of α-Dirac-harmonic maps $(\phi_\alpha,\psi_\alpha)$ with

$$E_\alpha(\phi_\alpha):=\int_M(1+|d\phi_\alpha|^2)^\alpha dM\le\Lambda<\infty,$$

then the key estimate (2.6) implies the following uniform control of the spinors:

$$\|\psi_\alpha\|_{W^{1,p}(M)} \text{ with } 1<p<2, \text{ is uniformly bounded as } \alpha\searrow 1.$$

Thus, we can do the blow-up analysis and we will show that the weak limit is just the desired Dirac-harmonic map. This is better than for the Dirac-harmonic map flow [12, 26], and so, here lies the advantage of considering α-Dirac-harmonic maps.

The remaining task is to show the existence of such an α-Dirac-harmonic map sequence. This is in fact one key step in our new scheme. Since the second term of the functional L_α is not bounded from below, the classical Ljusternik-Schnirelman theory may not be applied here to obtain critical points. Therefore, we need to develop a new method to proceed with our scheme.

In the present work, we shall consider the following new parabolic-elliptic system:

$$\partial_t\phi=\Delta_g\phi+(\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2}-A(d\phi,d\phi)-\frac{Re\Big(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\Big)}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{2.10}$$

$$\not{\partial}_g\psi=\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi), \tag{2.11}$$

[1] Here and in the sequel, for simplicity of notations, when talking about a sequence of $(\phi_\alpha,\psi_\alpha)$ for $\alpha\searrow 1$, we mean the sequence of $(\phi_{\alpha_k},\psi_{\alpha_k})$ for a given sequence of $\alpha_k\searrow 1$.

with the following boundary-initial data:

$$\begin{cases} \phi(x,t) = \varphi(x), & on \quad \partial M \times [0,T]; \\ \phi(x,0) = \phi_0(x), & in \quad M; \\ \mathbf{B}\psi(x,t) = \mathbf{B}\psi_0(x), & on \quad \partial M \times [0,T]; \\ \phi_0(x) = \varphi(x), & on \quad \partial M. \end{cases} \tag{2.12}$$

where $\mathbf{B} = \mathbf{B}^{\pm}$ is the chiral boundary operator defined as follows:

$$\mathbf{B}^{\pm} : L^2(\partial M, \Sigma M \otimes \phi^* TN|_{\partial M}) \to L^2(\partial M, \Sigma M \otimes \phi^* TN|_{\partial M})$$

$$\psi \mapsto \frac{1}{2}\left(Id \pm \overrightarrow{n} \cdot G\right) \cdot \psi, \tag{2.13}$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, $G = ie_1 \cdot e_2$ is the chiral operator defined using a local orthonormal frame $\{e_\gamma\}_{\gamma=1}^2$ on M and satisfying:

$$G^2 = Id, \quad G^* = G, \quad \nabla G = 0, \quad G \cdot X = -X \cdot G, \tag{2.14}$$

for any $X \in \Gamma(TM)$. Recall that the chiral boundary operator for usual spinors $\psi \in \Sigma M$ was first introduced by Gibbons-Hawking-Horowitz-Perry [19] to study positive mass theorems for black holes via Witten's approach through the spinor equation. Here, we consider its extension to spinors along a map, $\psi \in \Sigma M \otimes \phi^* TN$. In fact, one can also take $\mathbf{B}$ to be the MIT bag boundary operator $\mathbf{B}^{\pm}_{MIT}$ as considered in [12]. See e.g. [22, 4] for more detailed discussions on these boundary operators. For convenience, in the sequel, we shall only consider the case of chiral boundary conditions and omit the discussion of other case of boundary conditions, as the arguments for them are the same. We call (2.10)-(2.11) the *α-Dirac-harmonic map flow*.

Now, we state our first main result about the global existence of the α-Dirac-harmonic map flow with a Dirichlet-chiral boundary condition.

Theorem 2.1. *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N \subset \mathbb{R}^K$ be a compact Riemannian manifold. Suppose*

$$1 < \alpha < 1 + \min\{\epsilon_1, \epsilon_2\}$$

where ϵ_1 and ϵ_2 are the positive constants in Theorem 4.1 and Lemma 4.4 depending only on M, N. Then for any $\phi_0 \in C^{2+\lambda}(M,N)$, $\varphi \in C^{2+\lambda}(\partial M, N)$, $\psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^* TN)$ where $0 < \lambda < 1$ is a constant, there exists a unique global solution

$$\phi \in C^{2+\lambda, 1+\frac{\lambda}{2}}_{loc}(M \times [0,\infty), N)$$

and

$$\psi \in C^{\lambda, \frac{\lambda}{2}}_{loc}(M \times [0,\infty), \Sigma M \otimes \phi^* TN) \cap L^\infty([0,\infty), \|\psi(\cdot,t)\|_{C^{1+\lambda}(M)})$$

to the problem (2.10)-(2.11) *with boundary-initial data* (2.12)*, satisfying*

$$E_\alpha(\phi(t)) \leq E_\alpha(\phi_0) + 2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$$

and

$$\|\psi(\cdot,t)\|_{W^{1,p}(M)} \leq C(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}),$$

where $1 < p < 2$.

Moreover, there exist a time sequence $t_i \to \infty$ and an α-Dirac-harmonic map

$$(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi_\alpha^* TN)$$

with the boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that $(\phi(\cdot, t_i), \psi(\cdot, t_i))$ converges to $(\phi_\alpha, \psi_\alpha)$ in $C^2(M) \times C^1(M)$.

We remark that the harmonic map flow from a closed Riemann surface has been solved in [60], and from a compact Riemann surface with smooth boundary in [21, 6]. When the spinor field is vanishing and the domain is a closed surface, our flow reduces to the one in [23].

By Theorem 2.1, for any $\alpha > 1$ sufficiently close to 1, there exists an α-Dirac-harmonic map $(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi_\alpha^* TN)$ with the Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ and with the properties

$$E_\alpha(\phi_\alpha) \le E_\alpha(\phi_0) + 2\sqrt{2}\, \|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{2.15}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \le C(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\, \|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{2.16}$$

for any $1 < p < 2$. With this in hand, we can prove the existence of Dirac-harmonic maps by using the blow-up analysis.

Generally, we have the following existence and concentration compactness theorem of Dirac-harmonic maps corresponding to the previous **Question I**.

Theorem 2.2. *Let $(\phi_\alpha, \psi_\alpha) : M \to N$ be a sequence of α-Dirac-harmonic maps with Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ and with uniformly bounded energy*

$$E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(M)} \le \Lambda.$$

Denoting $E(\phi_\alpha; \Omega) := \int_\Omega |\nabla\phi_\alpha|^2 dvol_g$, $\Omega \subset M$ and the energy concentration set

$$\mathbf{S} := \left\{ x \in M \mid \liminf_{\alpha \to 1} E(\phi_\alpha; B_r^M(x)) \ge \frac{\epsilon_0}{2} \ for\ all\ r > 0 \right\},$$

where ϵ_0 is the positive constant in Lemma 5.1 and Lemma 5.2, $B_r^M(x)$ is the geodesic ball in M with center point x and radius r, then $\mathbf{S}$ is a finite set. Moreover, after selection of a subsequence of $(\phi_\alpha, \psi_\alpha)$ (without changing notation), there exists a Dirac-harmonic map

$$(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^* TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$, such that

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \text{ in } C^2_{loc}(M \setminus \mathbf{S}) \times C^1_{loc}(M \setminus \mathbf{S}).$$

Remark 2.3. Since we can impose nontrivial boundary conditions for both the map and the spinor, we shall obtain Dirac-harmonic maps with nontrivial map part and nontrivial spinor part.

Moreover, we show that at each singular point x_0, that is, when the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off. Here, however, we cannot employ the usual bubbling argument [8] for a blow-up sequence of Dirac-harmonic maps which are conformally invariant, since α-Dirac-harmonic maps are not conformally invariant. We need to develop a different type of rescaling argument by adding a new rescaling factor $\lambda_\alpha^{\alpha-1}$, with $\lambda_\alpha > 0$ being the blow-up radii, to the spinor part. Therefore, the blow-up analysis for α-Dirac-harmonic maps is more difficult and complicated than the case of Dirac-harmonic maps in [8, 65, 38]. To achieve this, we shall introduce the notation of *$\overline{\lambda}$-general α-Dirac-harmonic maps* and develop the appropriate analytical background, see Section 5.

Theorem 2.4. *Under the same assumption as in Theorem 2.2, suppose $x_0 \in \mathbf{S}$ is an energy concentration point, i.e.,*

$$\liminf_{\alpha\to 1} E(\phi_\alpha; B_r^M(x_0)) \geq \frac{\epsilon_0}{2} \; for \; all \; r > 0. \tag{2.17}$$

Then,

(1) *if $x_0 \in M \setminus \partial M$, there exist a subsequence of $(\phi_\alpha, \psi_\alpha)$ (still denoted by $(\phi_\alpha, \psi_\alpha)$) and sequences $x_\alpha \to x_0$, $\lambda_\alpha \to 0$ and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that as $\alpha \to 1$,*[2]

$$\left(\phi_\alpha(x_\alpha + \lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right) \to (\sigma(x), \xi(x)) \; in \; C^1_{loc}(\mathbb{R}^2) \times C^0_{loc}(\mathbb{R}^2).$$

(σ, ξ) has finite energy and conformally extends to a smooth Dirac-harmonic sphere [3].

(2) *if $x_0 \in \partial M$, then $\frac{dist(x_\alpha, \partial M)}{\lambda_\alpha} \to \infty$ and the same bubbling statement as in (1) holds.*

So far, we have answered the **Question I** about the existence of Dirac-harmonic maps with given Dirichlet-chiral boundary data. It is natural to ask whether the map component ϕ of the limit Dirac-harmonic map stays in the same homotopy class as ϕ_0.

Here we give a positive answer under some natural condition as in the harmonic map case. To see this, recall that we can actually choose a sequence of α-Dirac-harmonic maps satisfying the properties (2.15)-(2.16), for any fixed $1 < p < 2$. Therefore we are in a better situation than the case of $p = \frac{4}{3}$ considered in Theorem 2.2. In fact, we can take some p such that $\frac{4}{3} < p < 2$, then we can show that the bubbles in Theorem 2.4 are just nontrivial harmonic spheres, i.e., harmonic maps from S^2 to N. Thus, we have the following stronger version of the existence result

Theorem 2.5. *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N \subset \mathbb{R}^K$ be a compact Riemannian manifold. For any $\phi_0 \in C^{2+\lambda}(M, N)$, $\varphi \in C^{2+\lambda}(\partial M, N)$, $\psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^* TN)$ where $\phi_0|_{\partial M} = \varphi$ and $0 < \lambda < 1$ is a constant, if (N, h) dose not admit any nontrivial harmonic sphere, then there exists a Dirac-harmonic map*

$$(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^* TN)$$

[2]Compared to the usual rescaling, i.e. $\left(\phi_\alpha(x_\alpha + \lambda_\alpha x), \sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right)$, for a blow-up sequence of Dirac-harmonic maps given in [8], here the additional factor $\lambda_\alpha^{\alpha-1}$ comes from the fact that α-Dirac-harmonic maps are not conformally invariant, see Section 5.

[3]Here we have used the fact the unique spin structure on $\mathbb{S}^2 \setminus \{p\}$ extends to the unique spin structure on $\mathbb{S}^2$ and so does the associated spinor bundle.

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ *such that the map component* ϕ *is in the same homotopy class as* ϕ_0.

To understand the compactness of the spaces of α-Dirac-harmonic maps as $\alpha \searrow 1$, we shall carry out the most difficult step and study the finer blow-up behavior and asymptotic analysis for a general sequence of α-Dirac-harmonic maps near the interior blow-up points. More precisely, we shall investigate the following

Question II: Do some generalized energy identities hold for α-Dirac-harmonic maps? What are the limit curves of the α-Dirac-harmonic necks? How to compute the lengths of these limit neck curves?

In this paper, we will establish the generalized energy identities for α-Dirac-harmonic maps, including both the map part and the spinor part. We will show that the limit of the necks are geodesics on the target manifold and derive the length formula for these neck geodesics. Moreover, we find some natural geometric and topological conditions on the target manifold which ensures that energy identities hold true.

Since in general, multiple bubbles can split off at a blow-up point and the functional L_α is not conformally invariant, to better understand the multiple bubbling behavior for α-Dirac-harmonic maps, we shall consider the following more general α-energy functionals[4]

$$L_{\alpha,\sigma_\alpha}(\phi,\psi) = \frac{1}{2}\int_{D_1(0)} \left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2)^\alpha + \sigma_\alpha^{1-\alpha}\langle\psi, \not{D}\psi\rangle\right\} dvol_{g_\alpha},\ \alpha > 1,$$

where $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2 + (dx^2)^2\right)$, $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$, φ_α converges smoothly to $\varphi_0 \in C^\infty(D_1)$ and $\sigma_\alpha > 0$ is a constant.

Critical points of L_{α,σ_α} are called *general α-Dirac-harmonic maps*, and they satisfy the following Euler-Lagrange equations

$$\Delta_{g_\alpha}\phi = -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}\phi|^2\nabla_{g_\alpha}\phi}{\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2} + A(d\phi, d\phi) + \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2)^{\alpha-1}}, \tag{2.18}$$

$$\not{\partial}_{g_\alpha}\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi). \tag{2.19}$$

[4]One can check that a rescaled α-Dirac-harmonic map, e.g. $\left(\phi_\alpha(\lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(r_\alpha x)\right)$ is locally a critical point of this functional, we refer to Section 5 for details. One can also see the beginning of Section 2 in [35] for the analogous case of α-harmonic maps.

As (2.19) is conformally invariant, it is easy to see that the equations (2.18) and (2.19) are equivalent to [5]

$$\Delta\phi + (\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi|^2\nabla\phi}{\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2} - A(\phi)(d\phi, d\phi) - \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2)^{\alpha-1}} = 0, \tag{2.20}$$

$$\begin{aligned} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2)^{\alpha-1}\nabla\phi\right\} &- (\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2)^{\alpha-1}A(\phi)(d\phi, d\phi) \\ &- \frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right) = 0 \end{aligned} \tag{2.21}$$

and

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi), \tag{2.22}$$

where $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$, ∇ and $\partial\!\!\!/$ are operators corresponding to the standard Euclidean metric. $\{e_\gamma\}$ is a local orthonormal basis with respect to the standard Euclidean metric which is different from those in (2.18) and (2.19), for simplicity, we shall use the same notations. Although originally we consider a sequence of general α-Dirac-harmonic maps defined on a domain surface with a general Riemannian metric ((2.18) and (2.19)), however, as we will see later in this paper, the analysis involved in our problem can be localized and reduced in such a way that it is sufficient for us to work with solutions of the equivalent equations (2.20), (2.21) and (2.22) defined on a standard Euclidean domain, where g_α can be viewed as a smooth function on $(D_1(0), (dx^1)^2 + (dx^2)^2)$.

Before presenting our further results, we shall first give a general description of the blow-up procedure and the bubbling phenomena for general α-Dirac-harmonic maps. We follow the general scheme as in the case of α-harmonic maps [55, 35].

Denote

$$E_{\alpha,\sigma_\alpha}(\phi) = \int_M (\sigma_\alpha + |d\phi|^2)^\alpha dM, \quad E(\phi) = \int_M |d\phi|^2 dM,\ E(\psi) = \int_M |\psi|^4 dM,$$
$$E_\alpha(\phi) = \int_M (1 + |d\phi|^2)^\alpha dM, \quad E(\phi,\psi) = \int_M (|d\phi|^2 + |\psi|^4) dM.$$

Consider a sequence of general α-Dirac-harmonic maps $\{(\phi_\alpha, \psi_\alpha)\} : M \to N$ with Dirichlet-chiral boundary data $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_M = (\varphi, \mathbf{B}\psi_0)$, with $\sigma_\alpha > 0$ satisfying

$$0 < \beta_0 \le \liminf_{\alpha\searrow 1} \sigma_\alpha^{\alpha-1} \le 1$$

for some $\beta_0 > 0$ and with uniformly bounded energy

$$E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha) \le \Lambda.$$

From Theorem 2.2 and Theorem 2.4, we know that, by passing to a subsequence, $(\phi_\alpha, \psi_\alpha)$ converges strongly to some limit Dirac-harmonic map $(\phi, \psi) : M \to N$ with Dirichlet-chiral boundary

[5]More precisely, $(\phi\circ Id, e^{\frac{\varphi_\alpha}{2}}\psi\circ Id)$ satisfies (2.20)-(2.22), where $Id : (D_1(0), (dx^1)^2 + (dx^2)^2) \to (D_1(0), g_\alpha)$ is a conformal map defined by $Id(x) = x$. For simplicity of notation, we identify $(\phi\circ Id, e^{\frac{\varphi_\alpha}{2}}\psi\circ Id)$ with (ϕ,ψ) in the sequel. Although the energy $\int_M (1+|d\phi|^2)^\alpha dM$ is not conformally invariant for $\alpha > 1$, the limit $\lim_{\alpha\searrow 1}\int_M \left((1+|d\phi|^2)^\alpha - 1\right) dM$ is conformally invariant. Combining this with the fact that $\int_M |\psi|^4 dM$ is conformally invariant, this identification is indeed legitimate for the questions we are concerned with in this paper.

data $(\phi, \mathbf{B}\psi)|_M = (\varphi, \mathbf{B}\psi_0)$ away from at most finitely many blow-up points $\mathbf{S} = \{x_i\}_{i=1}^I$ as $\alpha \searrow 1$. Moreover, we show that at each blow-up point, that is, when the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off.

By the classical interior blow-up theory for α-harmonic maps [55, 35, 47], it is well known that at most finitely many bubbles can occur at a given interior blow-up point and the necks connecting the weak limit map and the bubbles or one bubble to the next all converge to geodesics, i.e., a bubble tree construction holds. Here, we shall show that this phenomenon also holds for a sequence of general α-Dirac-harmonic maps which blows up at an interior point. See Section 7 for the constructions of the first and the second bubble (if multiple bubbles occur at the blow-up point). The remaining bubbles (if any) can be constructed by a standard induction argument similar to the cases of harmonic map type problems considered in [35, 7]. For different schemes of constructing a bubble tree for harmonic maps and α-harmonic maps, we refer to [48, 47].

More precisely, for a fixed blow-up point x_i, $1 \leq i \leq I$, we may assume there are k_i bubbles occurring at this point, i.e. there are a sequence of points $\{x_\alpha^{ij}\}$, $j = 1, ..., k_i$, and a sequence of positive numbers $\{\lambda_\alpha^{ij}\}$ with $x_\alpha^{ij} \to x_i$, $\lambda_\alpha^{ij} \to 0$ as $\alpha \searrow 1$ and one of the following two alternatives holds true: if $1 \leq j_1, j_2 \leq k_i$ and $j_1 \neq j_2$,

(A1) for any fixed $R > 0$, $B^M_{R\lambda_\alpha^{ij_1}}(x_\alpha^{ij_1}) \cap B^M_{R\lambda_\alpha^{ij_2}}(x_\alpha^{ij_2}) = \emptyset$, whenever α is sufficiently close to 1.

(A2) $\frac{\lambda_\alpha^{ij_1}}{\lambda_\alpha^{ij_2}} + \frac{\lambda_\alpha^{ij_2}}{\lambda_\alpha^{ij_1}} = \infty$, as $\alpha \searrow 1$.

Moreover, the following two rescaled fields[6]

$$\sigma_\alpha^{ij} := \phi_\alpha(x_\alpha^{ij} + \lambda_\alpha^{ij} x), \quad \xi_\alpha^{ij} := (\lambda_\alpha^{ij})^{\alpha-1} \sqrt{\lambda_\alpha^{ij}}\, \psi_\alpha(x_\alpha^{ij} + \lambda_\alpha^{ij} x)$$

converge in $C^k_{loc}(\mathbb{R}^2 \backslash \{p_1^{ij}, ..., p_{s_j}^{ij}\})$ to a nontrivial Dirac-harmonic map (σ^{ij}, ξ^{ij}) defined on $\mathbb{R}^2$, which can be conformally extended to a nontrivial Dirac-harmonic map from $\mathbb{S}^2$. See the beginning of Section 7.

Now, we define two types of quantities:

$$\mu_{ij} = \liminf_{\alpha\searrow 1} (\lambda_\alpha^{ij})^{2-2\alpha}, \quad \nu_{ij} = \liminf_{\alpha\searrow 1} (\lambda_\alpha^{ij})^{-\sqrt{\alpha-1}}. \tag{2.23}$$

It is easy to see that $\nu_{ij} \in [1, \infty]$. Also, we can see that there exists a positive constant $\mu_{max} \geq 1$ such that $\mu_{ij} \in [1, \mu_{max}]$. In fact, for simplicity of notations, we may assume there is only one blow-up point denoted by $x \in M$, and there are k_1 bubbles occurring at this point, i.e., there are a sequence of points $\{x_\alpha^j\}$ and a sequence of positive numbers $\{\lambda_\alpha^j\}$, $1 \leq j \leq k_1$ satisfying (A1) or (A2). Without loss of generality, we assume λ_α^1 is the smallest one, i.e. $\frac{\lambda_\alpha^1}{\lambda_\alpha^j} \leq C < \infty$ for all $j = 2, ..., k_1$, as $\alpha \searrow 1$. We just need to show that

$$\mu_1 = \liminf_{\alpha\searrow 1} (\lambda_\alpha^1)^{2-2\alpha} \leq \mu_{max}.$$

[6]Let us explain the transformation of the spinor. It can be seen as a linear transformation (i.e. $\lambda_\alpha^{\alpha-1}\psi_\alpha$) composed with a conformal transformation (i.e. $\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)$). Since α-Dirac-harmonic maps are not conformally invariant, in order to get unified bubble equations, we need an additional factor $\lambda_\alpha^{\alpha-1}$ in the scale.

By applying the blow-up argument for general α-Dirac-harmonic maps (see Sec. 5 and Sec. 7), we have:

$$(\sigma^1_\alpha, \xi^1_\alpha) := \left(\phi_\alpha(x^1_\alpha + \lambda^1_\alpha x), (\lambda^1_\alpha)^{\alpha-1}\sqrt{\lambda^1_\alpha}\psi_\alpha(x^1_\alpha + \lambda^1_\alpha x)\right) \to (\sigma^1, \xi^1) \quad in\ C^k_{loc}(\mathbb{R}^2),$$

where (σ^1, ξ^1) can be conformally extended to a nontrivial Dirac-harmonic sphere. Therefore, we have

$$\begin{aligned}\Lambda \geq \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_{\lambda^1_\alpha R}(x^1_\alpha)} |\nabla_{g_\alpha}\phi_\alpha|^{2\alpha} dvol_{g_\alpha} &= \lim_{R\to\infty}\lim_{\alpha\searrow 1}(\lambda^1_\alpha)^{2-2\alpha}\int_{D_R(0)} |\nabla_{g_\alpha}\sigma^1_\alpha|^{2\alpha} dvol_{g_\alpha(x^1_\alpha+\lambda^1_\alpha x)} \\ &= \lim_{R\to\infty}\mu_1\int_{D_R(0)}|\nabla\sigma^1|^2 dx = \mu_1 E(\sigma^1).\end{aligned}$$

By the energy gap property for Dirac-harmonic spheres (see Lemma 6.2), we have

$$\mu_1 \leq \frac{\Lambda}{E(\sigma^1)} \leq \frac{\Lambda}{\epsilon_4}, \tag{2.24}$$

where $\epsilon_4 = \epsilon_4(N)$ is the positive constant in Lemma 6.2.

Now, we are able to state the generalized energy identities for a sequence of α-Dirac-harmonic maps that blows up at interior points.

Theorem 2.6. *Under the assumptions of Theorem 2.2, if we assume* $\mathbf{S}\cap\partial M = \emptyset$*, i.e. all the blow-up points are the interior points, then there are finitely many bubbles: a finite set of Dirac-harmonic spheres* $(\sigma^l_i, \xi^l_i) : S^2 \to N$*,* $l = 1, ...l_i$*, where* $l_i \geq 1$*,* $i = 1, ..., I$*, such that, the following generalized energy identities hold:*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i}\mu^2_{il}E(\sigma^l_i),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{l_i}\mu^2_{il}E(\xi^l_i),$$

where the quantities μ_{il} *are defined as in* (2.23)*.*

Remark 2.7. Here in Theorem 2.6, we only consider the interior blow-up for a sequence of α-Dirac-harmonic maps defined on a surface with boundary and it can be applied to a sequence of α-Dirac-harmonic maps defined on a closed surface (if there is such a sequence). The boundary blow-up case will be considered in a subsequent paper.

Furthermore, we shall show that the map parts of the α-Dirac-harmonic necks appearing during the interior blow-up process converge to geodesics in the target manifold N and then derive the length formula of these neck geodesics. More precisely, we have

Theorem 2.8. *Under the same assumptions as in Theorem 2.6, let* $x_1 \in \mathbf{S}$ *be an interior blow-up point. For simplicity, assume that there is only one bubble in* $B^M_r(x_1) \subset M$ *for some* $r > 0$*, for the sequence* $\{(\phi_{\alpha_k}, \psi_{\alpha_k})\}$*, denoted by* (σ^1, ξ^1)*, which is a Dirac-harmonic sphere. Let*

$$\nu^1 = \liminf_{\alpha\searrow 1}(\lambda^1_\alpha)^{-\sqrt{\alpha-1}}. \tag{2.25}$$

Then, by passing to subsequences, the map part of the Dirac-harmonic neck appearing during the blow-up process converges to a geodesic in the target manifold N. Moreover, we have the following alternatives:

(1) *when* $\nu^1 = 1$*, the set* $\phi(B_r^M(x_1)) \cup \sigma^1(S^2)$ *is a connected set in the target N;*
(2) *when* $\nu^1 \in (1,\infty)$*, then the set* $\phi(B_r^M(x_1))$ *and* $\sigma^1(S^2)$ *are connected by a geodesic of length*

$$L = \sqrt{\frac{E(\sigma^1)}{\pi}}\log\nu^1;$$

(3) *when* $\nu^1 = \infty$*, the map part of the Dirac-harmonic neck contains at least an infinite length curve which is a geodesic in N;*

Remark 2.9. Although Theorem 2.8 is stated and proved for the case that only one single bubble occurs at a given blow-up point, nevertheless, by following the arguments in Section 8, we can extend these results to the case of multiple bubbles at the blow-up point. In fact, the formulation for the multiple bubbles case is more complicated. For example, if we have three bubbles at a blow-up point: (σ^1,ξ^1), (σ^2,ξ^2), (σ^3,ξ^3) with blow-up positions and radii $(x_\alpha,\lambda_\alpha^i)$, $i=1,2,3$, i.e.

$$\left(\phi_\alpha(x_\alpha+\lambda_\alpha^i x), (\lambda_\alpha^i)^{\alpha-1}\sqrt{\lambda_\alpha^i}\psi_\alpha(x_\alpha+\lambda_\alpha^i x)\right) \rightharpoonup (\sigma^i,\xi^i)$$

weakly in $W^{1,2}_{loc}(\mathbb{R}^2)\times L^4_{loc}(\mathbb{R}^2)$, satisfying $\frac{\lambda_\alpha^1}{\lambda_\alpha^2}\to 0$, $\frac{\lambda_\alpha^2}{\lambda_\alpha^3}\to 0$ and $\nu^1,\nu^2,\nu^3 < \infty$, then the base map $\phi(B_\delta^M(x_\alpha))$ and the bubble $\sigma^3(S^2)$ are connected by a geodesic of length

$$L = \sqrt{\frac{E(\sigma^1)+E(\sigma^2)+E(\sigma^3)}{\pi}}\log\nu^3,$$

the two bubbles $\sigma^3(S^2)$ and $\sigma^2(S^2)$ are connected by a geodesic of length

$$L = \sqrt{\frac{E(\sigma^1)+E(\sigma^2)}{\pi}}\log\frac{\nu^2}{\nu^3},$$

the two bubbles $\sigma^2(S^2)$ and $\sigma^1(S^2)$ are connected by a geodesic of length

$$L = \sqrt{\frac{E(\sigma^1)}{\pi}}\log\frac{\nu^1}{\nu^2}.$$

A crucial step in proving Theorem 2.8 is to establish a key lemma about the decay of the tangential energies of both the map part and the spinor part on the neck domain as $\alpha \searrow 1$. In the absence of the spinor part, i.e., when ϕ_α are α-harmonic maps, this is achieved in [35] where the authors used some idea in [16] to derive a differential inequality on the neck. For a different proof of the length formula in the case that the neck is of finite length L, see [46, 47]. However, these techniques require the structure of harmonic map type equations and hence cannot be applied to our situation where there is a Dirac type equation coupled with an α-harmonic map type equation.

Recall that another powerful tool in deriving exponential decay on the long cylinders is the three-circle method. For maps $\{\phi_n\}$ with L^2-uniformly bounded tension fields $\tau(\phi_n)$, or equivalently, solutions to the harmonic map system up to some error terms $\tau(\phi_n)$ that are L^2-uniformly bounded, the corresponding three-circle theorem was derived in [50], which used a special case of the three-circle theorem due to [59] to show that the tangential energy of the sequence of solutions on the long cylinder decays exponentially. In fact, the condition that the error terms $\tau(\phi_n)$ are L^2-uniformly bounded ensures the following decay estimate

$$\| \, |x - x_n| \cdot \tau(\phi_n)\|_{L^2(D_{2t}(x_n)\setminus D_t(x_n))} \leq Ct,$$

which is crucial in [50] to get the exponential decay.

However, in view of the equation (2.18) for the map part ϕ_α in the general α-Dirac-harmonic map system, the error term contains the following two terms: the second derivative term

$$(\alpha - 1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2}, \tag{2.26}$$

and the curvature term

$$\frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma \cdot \psi_\alpha); \psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_g\phi_\alpha|^2)^{\alpha-1}}, \tag{2.27}$$

both of which are not uniformly bounded in L^2 anymore (in fact, they are not bounded in L^p for any $1 < p < 2$) and hence the results in [50] can not be applied.

Therefore, we need to develop new methods. Instead, in this paper, to overcome the new difficulties occurring, we shall first establish a type of three-circle theorem for the integro-differential equations corresponding to α-Dirac-harmonic map type systems, see Theorem 6.7. Then, we shall derive the decay of the tangential energies of both the map part and the spinor part on the neck domain, see Lemma 8.2. To achieve this, we shall handle the two error terms separately. The treatments are involved. For the second derivative term (2.26), although it is not uniformly bounded in L^p for any $p > 1$ and the exponential decay will not hold [7], however, our new observation is that one can still get some decay at the speed of $\alpha - 1$. For the curvature term (2.27), we need to decompose it into several forms which can be used in the three-circle theorem. We will see that the proof is still subtle. In particular, our result applies to the case of α-harmonic maps, i.e. when $\psi_\alpha \equiv 0$ and hence leads to decay estimates for α-harmonic map type systems, which is still new in the existing literature. Moreover, in the case of α-harmonic maps, by applying the results in Lemma 8.2, we can show that the tangential energies decay at a speed of $\alpha - 1$, which is a small improvement of the speed of $\sqrt{\alpha - 1}$ given by Proposition 4.2 in [35].

In the case of Dirac-harmonic maps [38, 27], the decay of the tangential energy of the spinor is sufficient for the neck analysis, whereas here in the case of α-Dirac-harmonic maps, in order to study the limit behaviour of the necks, we need to get the decay of some weighted energy of the spinor as $\alpha \searrow 1$. Moreover, in the case considered in this paper, we do not have the exponential decay of the energy of the spinor as in the case of Dirac-harmonic maps [38, 27], since the map part ϕ_α satisfies an α-harmonic map type system rather than the harmonic map type system.

[7] One can see that it is controlled by $(\alpha - 1)|\nabla^2\phi_\alpha|$ and by Lemma 5.1, we have

$$\||x - x_\alpha|^{2(1-\frac{1}{p})}|\nabla^2\phi_\alpha|\|_{L^p(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))} \leq C\|\nabla\phi_\alpha\|_{L^2(D_{4t}(x_\alpha)\setminus D_{\frac{t}{2}}(x_\alpha))}.$$

To overcome this new difficulty, we need some new observations. Instead, we shall prove that the energy $\|\psi_\alpha\|_{L^4}$ of the spinor decays at a speed of $(\alpha-1)^{\frac{1}{3}}$. More precisely, the following holds:

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\frac{4}{3}}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\psi_\alpha|^4dx=0. \tag{2.28}$$

See Lemma 8.4 and Lemma 8.5. Such a decay estimate plays a key role in the present paper, as it is not only important in the proof of Theorem 2.8 (see Proposition 8.9), but also crucial in the proof of Theorem 2.12 (see Lemma 9.5) which will be discussed later. In technical terms, the above decay estimate (2.28) is achieved by applying a Hardy-type inequality to derive a differential inequality on the neck to get the decay of some weighted energy of a spinor.

According to Theorem 2.6, it is easy to see that the energy identities hold if and only if the following analytical condition:

$$\mu_{il}=1,\quad i=1,...,I,\ l=1,...,l_i,$$

where the quantities μ_{il} are defined as in (2.23). Moreover, by definitions of the two types of quantities μ_{il} and ν_{il} in (2.23), it is easy to check that if all neck geodesics are of finite length, then the energy identities hold.

From the perspectives of differential geometry and topology, it is natural and interesting to find some geometric or topological condition on the target manifold to ensure that the energy identities hold. In particular, a natural question is whether one can exploit some geometric or topological condition to ensure that the limit of the map parts of the α-Dirac-harmonic necks consists of some geodesics of finite length so that the energy identities follow immediately. When the target is a sphere, the energy identity and no neck property for a sequence of α-harmonic maps was proved in [32].

In view of the research on minimal hypersurfaces (see [57] for the current state), it seems reasonable to impose the assumptions that the Ricci curvature of the target has a positive lower bound and that there is a suitable notion of a Morse index for α-Dirac-harmonic maps that is bounded on the sequence. See also [34] for a similar assumption. Alternatively, one can require that the target manifold has a finite fundamental group and the sequence has bounded Morse index, as in the case of α-harmonic maps considered in [47]. For this goal, we need to define the Morse index of α-Dirac-harmonic maps and Dirac-harmonic maps.

Let $(\phi,\psi): M\to N$ be an α-Dirac-harmonic map or a Dirac-harmonic map. Let $\phi^*(TN)$ be the pull-back bundle over M and $\Gamma(\phi^*TN)$ denote the linear space of the smooth sections of ϕ^*TN. $V\in\Gamma(\phi^*(TN))$ can be used to vary (ϕ,ψ) by

$$\phi_\tau(x)=exp_{\phi(x)}(\tau V),\ \psi_\tau(x)=\psi^i(x)\otimes\frac{\partial}{\partial y^i}(\phi_\tau(x)). \tag{2.29}$$

Definition 2.10. The Morse index of an α-Dirac-harmonic map (ϕ,ψ), denoted by Index $(\phi,\psi;L_\alpha)$, is defined as the maximal dimension of a linear subspace Ξ of $\Gamma(\phi^*TN)$ on which the second variation of L_α with respect to the variations (2.29) is negative, i.e., for any $V\in\Xi\subset\Gamma(\phi^*TN)$, there holds

$$\delta^2L_\alpha(\phi,\psi)(V,V)<0,$$

where

$$\begin{aligned}
\delta^2 L_\alpha(\phi,\psi)(V,V) &= \frac{d^2}{d\tau^2}\bigg|_{\tau=0} L_\alpha(\phi_\tau,\psi_\tau)\\
&= 2\alpha\int_M (1+|\nabla_g\phi|^2)^{\alpha-1}\Big(\langle\nabla_g V,\nabla_g V\rangle - R(V,\nabla_g\phi,\nabla_g\phi,V)\Big)dM\\
&+ 4\alpha(\alpha-1)\int_M (1+|\nabla_g\phi|^2)^{\alpha-2}\langle\nabla_g\phi,\nabla_g V\rangle^2 dM\\
&+ 2\int_M \left\langle \psi^j\otimes\nabla_V\frac{\partial}{\partial y^j}, e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_V\frac{\partial}{\partial y^i}\right) + e_\beta\cdot\psi^i\otimes R(V,e_\beta)\frac{\partial}{\partial y^i}\right\rangle dM\\
&+ \int_M \left\langle \psi, e_\beta\cdot\psi^i\otimes\left(R^j_{ikl;p}V^pV^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j} + R(V,e_\beta)\nabla_V\frac{\partial}{\partial y^i} + R(V,\nabla_{e_\beta}V)\frac{\partial}{\partial y^i}\right)\right\rangle dM\\
&+ 2\alpha\int_{\partial M}\left\langle (1+|\nabla_g\phi|^2)^{\alpha-1}\frac{\partial\phi}{\partial\overrightarrow{n}},\nabla_V V\right\rangle - \int_{\partial M}\left\langle \overrightarrow{n}\cdot\psi,\psi^i\otimes\nabla_V\nabla_V\frac{\partial}{\partial y^i}\right\rangle.
\end{aligned}$$

Definition 2.11. The Morse index of a Dirac-harmonic map (ϕ,ψ), denoted by Index $(\phi,\psi;L)$, is defined as the maximal dimension of a linear subspace Ξ of $\Gamma(\phi^*TN)$ on which the second variation of L with respect to the variations (2.29) is negative, i.e., for any $V\in\Xi\subset\Gamma(\phi^*TN)$, there holds

$$\delta^2 L(\phi,\psi)(V,V) < 0,$$

where

$$\begin{aligned}
\delta^2 L(\phi,\psi)(V,V) &= \frac{d^2}{d\tau^2}\bigg|_{\tau=0} L(\phi_\tau,\psi_\tau)\\
&= 2\int_M \Big(\langle\nabla_g V,\nabla_g V\rangle - R(V,\nabla_g\phi,\nabla_g\phi,V)\Big)dM\\
&+ 2\int_M \left\langle \psi^j\otimes\nabla_V\frac{\partial}{\partial y^j}, e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_V\frac{\partial}{\partial y^i}\right) + e_\beta\cdot\psi^i\otimes R(V,e_\beta)\frac{\partial}{\partial y^i}\right\rangle dM\\
&+ \int_M \left\langle \psi, e_\beta\cdot\psi^i\otimes\left(R^j_{ikl;p}V^pV^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j} + R(V,e_\beta)\nabla_V\frac{\partial}{\partial y^i} + R(V,\nabla_{e_\beta}V)\frac{\partial}{\partial y^i}\right)\right\rangle dM\\
&+ 2\int_{\partial M}\left\langle \frac{\partial\phi}{\partial\overrightarrow{n}},\nabla_V V\right\rangle - \int_{\partial M}\left\langle \overrightarrow{n}\cdot\psi,\psi^i\otimes\nabla_V\nabla_V\frac{\partial}{\partial y^i}\right\rangle.
\end{aligned}$$

For the second variation formulas of the functionals L_α and L, see Section 9. With the help of the notions of Morse index of α-Dirac-harmonic maps and Dirac-harmonic maps, by applying the limiting asymptotic behavior of the necks in Theorem 2.8 and the new decay estimate (2.28) for the spinor's energy (see Lemma 8.4 and Lemma 8.5), we now state our last theorem:

Theorem 2.12. *Under the assumption of Theorem 2.6, assume that the sequence $(\phi_\alpha,\psi_\alpha)$ has bounded Morse index*

$$\mathit{Index}\,(\phi_\alpha,\psi_\alpha;L_\alpha)\le\Lambda_{index},$$

where Λ_{index} is an integer, then

(1) *the weak limit* (ϕ, ψ) *is a Dirac-harmonic map with given Dirichlet-chiral boundary data* $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ *and with finite Morse index*

$$\textit{Index }(\phi, \psi; L) \leq \Lambda_{index}.$$

(2) *all the bubbles we get are of finite Morse index, i.e. the Dirac-harmonic spheres* $(\sigma_i^l, \xi_i^l) : S^2 \to N$, $l = 1, ..., l_i$, $i = 1, ..., I$, *satisfy*

$$\textit{Index }(\sigma_i^l, \xi_i^l; L) \leq \Lambda_{index}.$$

(3) *if the Ricci curvature of the target manifold* (N, g) *has a positive lower bound, i.e. there exists a positive constant* $\lambda_0 > 0$ *such that* $Ric_N \geq \lambda_0 > 0$*, then the limit of the necks consist of geodesics of finite length. Moreover, the energy identities hold, i.e.*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i} E(\sigma_i^l),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{l_i} E(\xi_i^l).$$

(4) *if the target manifold* N *has a finite fundamental group, then the same conclusions as in statement* (3) *hold.*

Notations: Δ_g, ∇_g and $\partial\!\!\!/_g$ are operators corresponding to the Riemannian metric g. Δ, ∇ and $\partial\!\!\!/$ are operators corresponding to the standard Euclidean metric. Throughout the paper, the uppercase letter C denotes a positive constant which may vary from line to line.

3. Euler-Lagrange equations and key estimate for Dirac type equations

In this section, we shall derive the Euler-Lagrange equations for α-Dirac-harmonic maps and then prove the key estimate (2.6) for Dirac type equations.

Lemma 3.1. *The Euler-Lagrange equations for* L_α *are*

$$\tau_\alpha = \frac{1}{\alpha}\mathrm{R}(\phi, \psi) \tag{3.1}$$

$$D\!\!\!\!/\,\psi = 0 \tag{3.2}$$

where $\tau_\alpha = (\tau_\alpha^1, ..., \tau_\alpha^n)$ *and* $\mathrm{R}(\phi, \psi)$ *are defined respectively by*

$$\tau_\alpha^i(\phi) := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left((1 + |d\phi|^2)^{\alpha-1}\sqrt{g}g^{\beta\gamma}\frac{\partial\phi^i}{\partial x^\gamma}\right) - (1 + |d\phi|^2)^{\alpha-1}g^{\beta\gamma}\Gamma^i_{jk}\frac{\partial\phi^j}{\partial x^\beta}\frac{\partial\phi^k}{\partial x^\gamma} \tag{3.3}$$

and

$$\mathrm{R}(\phi, \psi)(x) := \frac{1}{2}\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle R^m_{lij}(\phi(x))\frac{\partial}{\partial y^m}(\phi(x)). \tag{3.4}$$

Proof. Let ψ_t be a variation of ψ with $\frac{d\psi_t}{dt}\big|_{t=0} = \eta$ and fix ϕ. By Proposition 2.1 in [9], we know

$$\frac{dL_\alpha(\psi_t)}{dt}\bigg|_{t=0} = \int_M \langle\eta, D\!\!\!\!/\,\psi\rangle dvol_g.$$

Then (3.2) follows immediately.

For the equation of ϕ, let ϕ_t be a variation of ϕ such that $\frac{d\phi_t}{dt}\Big|_{t=0} = \xi$ and ψ^i $(i = 1, ..., n)$ in $\psi(x) = \psi^i(x) \otimes \frac{\partial}{\partial y^i}(\phi(x))$ are independent of t. Also, by Proposition 2.1 in [9], we get

$$\frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_M \langle\psi_t, \not{D}\psi_t\rangle dvol_g = \frac{1}{2}\int_M \langle\psi^i, \nabla\phi^l \cdot \psi^j R_{mlij}\xi^m\rangle dvol_g.$$

Finally, it is easy to check that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_M (1 + |d\phi_t|^2)^\alpha dvol_g \\
&= \alpha \int_M \left\{-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left((1 + |d\phi|^2)^{\alpha-1}\sqrt{g}g^{\beta\gamma}\frac{\partial\phi^i}{\partial x^\gamma}\right) + (1 + |d\phi|^2)^{\alpha-1}g^{\beta\gamma}\Gamma^i_{jk}\frac{\partial\phi^j}{\partial x^\beta}\frac{\partial\phi^k}{\partial x^\gamma}\right\} h_{im}\xi^m dvol_g \\
&:= \alpha \int_M -\tau^i_\alpha h_{im}\xi^m dvol_g.
\end{aligned}$$

Thus, we obtain

$$\frac{dL_\alpha(\phi_t)}{dt}\Big|_{t=0} = \int_M \left\{-\alpha\tau^i_\alpha h_{im} + \frac{1}{2}\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle R_{mlij}\right\}\xi^m dvol_g,$$

which implies the equation (3.1). □

By Nash's embedding theorem, we embed N isometrically into some $\mathbb{R}^K$, which is denoted by $f : N \to \mathbb{R}^K$. Set

$$\phi' = f \circ \phi \quad \textit{and} \quad \psi' = f_*\psi.$$

If we identify ϕ with ϕ' and ψ with ψ', we can get the following extrinsic form of the Euler-Lagrange equations:

Lemma 3.2. *Let $(\phi, \psi) : M \to N$ be an α-Dirac-harmonic map. Then, (ϕ, ψ) satisfies*

$$\Delta_g\phi = -(\alpha - 1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1 + |\nabla_g\phi|^2} + A(d\phi, d\phi) + \frac{Re\left(P(\mathcal{A}(d\phi(e_\beta), e_\beta \cdot \psi); \psi)\right)}{\alpha(1 + |\nabla_g\phi|^2)^{\alpha-1}}, \tag{3.5}$$

$$\not{\partial}_g\psi = \mathcal{A}(d\phi(e_\beta), e_\beta \cdot \psi). \tag{3.6}$$

Proof. Firstly, it is easy to see that $\tau'_\alpha(\phi')$ and $\tau_\alpha(\phi)$ satisfy

$$\tau'_\alpha(\phi') = (1 + |d\phi|^2)^{\alpha-1}A(d\phi(e_\beta), d\phi(e_\beta)) + df(\tau_\alpha(\phi)). \tag{3.7}$$

Secondly, by similar arguments as in [8, 9, 11], we know

$$\not{D}'\psi' = f_*(\not{D}\psi) + \mathcal{A}(d\phi(e_\beta), e_\beta \cdot \psi)$$

and

$$df(\tau_\alpha(\phi)) = \frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\beta), e_\beta \cdot \psi); \psi)\right).$$

Then the conclusion of the lemma follows from the fact that $\not{D}' = \not{\partial}_g$ (here, $\not{D}'$ is the Dirac operator along the map ϕ') and

$$\tau'_\alpha(\phi) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left((1 + |d\phi|^2)^{\alpha-1}\sqrt{g}g^{\beta\gamma}\frac{\partial\phi}{\partial x^\gamma}\right).$$

□

In the end of this section, we shall prove the key estimate (2.6). The idea is to use a contradiction argument, where a crucial ingredient is the uniqueness of the Dirac equation, i.e.

$$\begin{cases} \not{D}\psi &= 0, \ M, \\ \mathbf{B}\psi &= 0, \ \partial M \end{cases}$$

has only the trivial solution $\psi \equiv 0$ when $\phi \in W^{1,p}(M, N)$ for some $p > 2$, see Theorem 1.2 and Theorem 4.1 in [12].

Lemma 3.3. *Let M be a compact spin Riemann surface with smooth boundary ∂M and $\psi = (\psi^1, ..., \psi^K)$, $\psi^A \in \Sigma M$, $A = 1, ..., K$. Let $\Omega \in \Gamma(\Lambda^1 T^*M \otimes so(K))$, i.e. $\Omega^A_B = -\Omega^B_A$ and $\Omega \in L^{2p'}(M)$, $d\Omega \in L^{p'}(M)$ for some $p' > 1$. Suppose $\psi \in W^{1,p}(M, \mathbb{R}^K)$ and $\eta = (\eta^1, ..., \eta^K) \in L^p(M, \mathbb{R}^K)$, $1 < p < 2$ satisfy*

$$\not{\partial}_g \psi^A + \Omega^A_B \cdot \psi^B = \eta^A$$

then there exists a positive constant $C = C(p, M, K, \|\Omega\|_{L^{2p'}(M)} + \|d\Omega\|_{L^{p'}(M)}) > 0$ such that

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|\not{\partial}_g \psi + \Omega \cdot \psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}). \tag{3.8}$$

Our proof will follow the scheme of Remark 3.3, Theorem 3.11 and Remark 3.7 in [12]. The main difference is that, on a two dimensional domain considered in Lemma 3.3, the two real numbers $p' > 1$ and $1 < p < 2$ can be arbitrary and be independent of each other, while Theorem 3.11 in [12] requires that $1 < p < p'$, which is too strong and hence can not be applied to our blow-up analysis of a sequence of α-Dirac-harmonic map as $\alpha \searrow 1$. This is a new and crucial observation in the present paper.

Proof. First, by Theorem 3.3 in [12], we have

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|\not{\partial}_g \psi + \Omega \cdot \psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)} + \|\psi\|_{L^p(M)}), \tag{3.9}$$

where $C = C(p, M, K, \Omega)$ is a positive constant.

Next, we claim:

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|\not{\partial}_g \psi + \Omega \cdot \psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}), \tag{3.10}$$

where $C = C(p, M, K, \Omega)$ is a positive constant.

In fact, if (3.10) does not hold, then there exists $\psi_i \in W^{1,p}(M, \mathbb{R}^K)$, such that

$$\|\psi_i\|_{W^{1,p}(M)} \geq i\,(\|\not{\partial}_g \psi_i + \Omega \cdot \psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}). \tag{3.11}$$

Without loss of generality, we may assume $\|\psi_i\|_{L^p} = 1$. Then by (3.9) and (3.11), we have

$$\|\not{\partial}_g \psi_i + \Omega \cdot \psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)} \leq \frac{C}{i} \tag{3.12}$$

and

$$\|\psi_i\|_{W^{1,p}(M)} \leq C. \tag{3.13}$$

Then there exists a subsequence of $\{\psi_i\}$ (also denoted by $\{\psi_i\}$) with $\psi \in W^{1,p}(M, \mathbb{R}^K)$, such that,

$$\psi_i \rightharpoonup \psi \text{ weakly in } W^{1,p}(M) \text{ and } \psi_i \to \psi \text{ strongly in } L^p(M).$$

Moreover, it is easy to see that ψ is a weak solution of

$$\partial\!\!\!/_g\psi + \Omega \cdot \psi = 0$$

with boundary condition

$$\mathbf{B}\psi = 0.$$

Since $p' > 1$, by Theorem 4.1 in [12], there must hold $\psi \equiv 0$. However, the fact that $\|\psi_i\|_{L^p(M)} = 1$ tells us $\|\psi\|_{L^p(M)} = 1$. This is a contradiction and hence (3.10) holds.

For (3.8), we can also prove it by a contradiction argument. In fact, if it does not hold, then we can find a sequence $\Omega_i \in \Gamma(\Lambda^1 T^*M \otimes so(K))$ and $\psi_i \in W^{1,p}(M, \mathbb{R}^K)$, such that

$$1 = \|\psi_i\|_{W^{1,p}(M)} \geq i\,(\|\partial\!\!\!/_g\psi_i + \Omega_i \cdot \psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}), \tag{3.14}$$

and

$$\|\Omega_i\|_{L^{2p'}(M)} + \|d\Omega_i\|_{L^{p'}(M)} \leq C.$$

By the weak compactness and compact embedding, there exists a subsequence of (Ω_i, ψ_i) (without changing notation) and $\psi \in W^{1,p}(M, \mathbb{R}^K)$, $\Omega \in W^{1,p'}(M) \cap L^{2p'}(M)$, such that

$$\Omega_i \rightharpoonup \Omega \textit{ weakly in } L^{2p'}(M) \textit{ and } d\Omega_i \rightharpoonup d\Omega \textit{ weakly in } L^{p'}(M)$$

and

$$\psi_i \rightharpoonup \psi \textit{ weakly in } W^{1,p}(M) \textit{ and } \psi_i \to \psi \textit{ strongly in } L^{p^*}(M),$$

for any p^* satisfying $\frac{1}{p^*} > \frac{1}{p} - \frac{1}{2}$.

Then it is easy to see that ψ is a weak solution of $\partial\!\!\!/\psi + \Omega \cdot \psi = 0$ with boundary condition $\mathbf{B}\psi = 0$ which implies $\psi \equiv 0$ by Theorem 4.1 in [12], since $p' > 1$. Thus

$$\lim_{i\to\infty} \|\psi_i\|_{L^{p^*}(M)} = 0.$$

Therefore, we have

$$\begin{aligned}
\|\partial\!\!\!/_g\psi_i + \Omega \cdot \psi_i\|_{L^p(M)} &\leq \|\partial\!\!\!/_g\psi_i + \Omega_i \cdot \psi_i\|_{L^p(M)} + \|(\Omega - \Omega_i)\psi_i\|_{L^p(M)} \\
&\leq \|\partial\!\!\!/_g\psi_i + \Omega_i \cdot \psi_i\|_{L^p(M)} + \|\Omega - \Omega_i\|_{L^{2p'}(M)}\|\psi_i\|_{L^{p^*}(M)} \\
&\leq \frac{1}{i} + C(N)\||\Omega| + |\Omega_i|\|_{L^{2p'}(M)}\|\psi_i\|_{L^{p^*}(M)} \to 0
\end{aligned}$$

as $i \to \infty$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2p'} > \frac{1}{p} - \frac{1}{2}$.

But, (3.10) tells us

$$1 = \|\psi_i\|_{W^{1,p}(M)} \leq C(p, M, K, \Omega)(\|\partial\!\!\!/_g\psi_i + \Omega \cdot \psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}) \to 0, \tag{3.15}$$

as $i \to \infty$, which is a contradiction. We proved this lemma. □

As a direct application of Lemma 3.3, we have

Lemma 3.4. *Let M be a compact spin Riemann surface with boundary ∂M, N be a compact Riemannian manifold. Let $\phi \in W^{1,2\alpha}(M, N)$ for some $\alpha > 1$ and $\psi \in W^{1,p}(M, \Sigma M \otimes \phi^*TN)$, $1 < p < 2$, then there exists a positive constant $C = C(p, M, N, \|\nabla\phi\|_{L^{2\alpha}(M)})$, such that*

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|D\!\!\!/\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}). \tag{3.16}$$

Proof. Firstly, let us assume that there exists a global orthonormal basis of the normal bundle $T^\perp N$ denoted by $\{\nu^i\}_{i=n+1}^K$ where $\nu^i = ((\nu^i)^1, ..., (\nu^i)^K)$. By Remark 2.1 in [12], we can rewrite the term $\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi)$ as $\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) = -\Omega \cdot \psi$ where

$$\Omega = \sum_{i=n+1}^K [\nu^i(\phi), d\nu^i(\phi)] = \sum_{i=n+1}^K \left((\nu^i)^A (\nabla_{e_\gamma} \nu^i)^B e_\gamma - (\nu^i)^B (\nabla_{e_\gamma} \nu^i)^A e_\gamma\right),$$

thus

$$\not{D}\psi = \not{\partial}_g \psi - \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) = \not{\partial}_g \psi + \Omega(\phi) \cdot \psi. \tag{3.17}$$

The conclusion of the lemma follows immediately from Lemma 3.3 and the fact that $d\Omega = [d\nu(\phi), d\nu(\phi)]$.

In general, the global orthonormal basis of the normal bundle $T^\perp N$ may not exist. In this case, we can apply a method analogous to the case of harmonic map equation in [45]. Let

$$1 = \sum_{j=1}^J \chi_j$$

be a smooth partition of unity such that for every $j = 1, ..., J$, there exists a local orthonormal basis of the normal bundle $T^\perp N|_{supp(\chi_j)}$, denoted by $\left\{\widetilde{\nu}^i_j\right\}_{i=n+1}^K$. Setting $\nu^i_j = \widetilde{\nu}^i_j \chi_j$ and noting that $\langle \psi, \nu^i_j \rangle = 0$, one can easily find that (3.17) still hold for

$$\Omega = \sum_{j=1}^J \sum_{i=n+1}^K [\nu^i_j(\phi), d\widetilde{\nu}^i_j(\phi)] = \sum_{j=1}^J \sum_{i=n+1}^K \left((\nu^i_j)^A (\nabla_{e_\gamma} \widetilde{\nu}^i_j)^B e_\gamma - (\nu^i_j)^B (\nabla_{e_\gamma} \widetilde{\nu}^i_j)^A e_\gamma\right).$$

By Lemma 3.3 and the fact that $d\Omega = \sum_{j=1}^J \sum_{i=n+1}^K [d\nu^i_j(\phi), d\widetilde{\nu}^i_j(\phi)]$, we finished the proof of this lemma. □

4. Global existence of α-Dirac-harmonic map flow

In this section, we will prove the global existence result for the α-Dirac-harmonic map flow and show that the limit at infinity time is an α-Dirac-harmonic map.

The equations (2.10)-(2.11) have the following equivalent intrinsic form

$$\begin{aligned} &\partial_t \phi = \frac{1}{(1+|d\phi|^2)^{\alpha-1}} \left(\tau_\alpha(\phi) - \frac{1}{\alpha} R(\phi, \psi)\right), \\ &\not{D}\psi = 0, \end{aligned}$$

where we regard ϕ as a map into N and ψ as a section of $\Sigma M \otimes \phi^* TN$, $\tau_\alpha(\phi)$ is given in (3.1). This leads us to consider another isometrical embedding. In fact, as in [21] (Page 108), we can embed (N, h) isometrically into some $\mathbb{R}^L$ with some non-flat metric denoted by h_L, such that this isometric embedding is totally geodesic and there exist a tubular neighborhood $\mathcal{N}$ of N and an isometric involution $i : \mathcal{N} \to \mathcal{N}$ which making precisely N being its fixed point set. Since N is a totally geodesic submanifold, then $\tau_\alpha(\phi) = \tau_\alpha^{\mathbb{R}^L}$ and it suffices to study the following system [8]

$$\partial_t\phi = \Delta_g\phi + (\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} + \Gamma(\phi)\#\nabla\phi\#\nabla\phi + R(\phi)\#\nabla\phi\#\psi\#\psi, \tag{4.1}$$

$$\not{D}\psi = 0, \tag{4.2}$$

where Γ is the Levi-Civita connection of $(\mathbb{R}^L, h_L)$, R is the curvature of $(\mathbb{R}^L, h_L)$ and

$$\not{D}\psi = \not{\partial}_g\psi + \Gamma(\phi)\#\nabla\phi\#\psi.$$

Next, in order to emphasize the Dirac operator $\not{D}$ depends on the map ϕ, we sometimes use the notation

$$\not{D}_\phi := \not{D}.$$

Noting that

$$|\nabla\phi|^2 = (h_L)_{ij}(\phi)\nabla\phi^i\nabla\phi^j,$$

if we expand $\nabla|\nabla\phi|^2$, there is an additional term like $(h_L)_{ij,k}\nabla\phi^i\nabla\phi^j\nabla\phi^k$. This term and $\Gamma(\phi)\#\nabla\phi\#\nabla\phi$ will be put together into the term $\Gamma(\phi)\#\nabla\phi\#\nabla\phi\#\nabla\phi$. Therefore, the equations can be rewritten as

$$\begin{aligned}\partial_t\phi &= \Delta_g\phi + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi^i\nabla_\beta\phi^i\nabla_\gamma\phi}{1+|\nabla_g\phi|^2} + \Gamma(\phi)\#\nabla\phi\#\nabla\phi\#\nabla\phi \\ &\quad + R(\phi)\#\nabla\phi\#\psi\#\psi,\end{aligned} \tag{4.3}$$

$$\not{D}_\phi\psi = 0. \tag{4.4}$$

Firstly, we have the following short-time existence result for the α-Dirac-harmonic map flow with Dirichlet-chiral boundary condition. The argument here is not the same as in the case of Dirac-harmonic map flow in [12]. For the case of α-harmonic map flow from a closed Riemann surface, see the Appendix in [23].

Theorem 4.1. *Let (M, g) be a compact spin Riemann surface with a smooth boundary ∂M and (N, h) be another compact Riemannian manifold. Then there exists a positive constant ϵ_1 depending only on M, N, such that, for any $1 < \alpha < 1 + \epsilon_1$ and any*

$$\phi_0 \in C^{2+\lambda}(M, N),\ \varphi \in C^{2+\lambda}(\partial M, N),\ \psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^*TN),$$

where $0 < \lambda < 1$, the problem (2.10)-(2.11) and (2.12) admits a unique solution

$$\phi \in C^{2+\lambda,1+\lambda/2}(M \times [0, T], N),$$

and

$$\psi \in C^{\lambda,\lambda/2}(M \times [0, T], \Sigma M \otimes \phi^*TN),\ \psi \in L^\infty([0, T]; C^{1+\lambda}(M)),$$

for some time $T > 0$.

[8]Here and in the sequel, # denotes a multi-linear map with smooth coefficients.

Proof. We shall adapt the classical methods developed for harmonic map flows [18, 21, 33].

Step 1: Short-time existence of (4.3)-(4.4).

For every $T > 0$, we define

$$\mathcal{U} := \left\{u, du \in C^{\lambda,\lambda/2}(M \times [0,T]) \,\middle|\, \|u\|_{C^{\lambda,\lambda/2}(M\times[0,T])} + \|du\|_{C^{\lambda,\lambda/2}(M\times[0,T])} \le 1, u|_{M\times\{0\}\cup\partial M\times[0,T]} = 0\right\}.$$

Consider the following linear parabolic-elliptic system:

$$\begin{aligned} \partial_t\phi &= \Delta_g\phi + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi^i\nabla_\beta u^i\nabla_\gamma u}{1+|\nabla_g u|^2} + \Gamma(u)\#\nabla u\#\nabla u\#\nabla u \\ &\quad + R(u)\#\nabla u\#\psi\#\psi + \Delta_g\phi_0 + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi^i_0\nabla_\beta u^i\nabla_\gamma u}{1+|\nabla_g u|^2}, \end{aligned} \tag{4.5}$$

$$\not{D}_u\psi = 0. \tag{4.6}$$

Now, let us begin a routine iteration argument as in [33] to show the local existence. For every $u \in \mathcal{U}$, on one hand, by Theorem 4.6 in [12], there exists a unique solution $v_1 \in C^{1+\lambda}(M, \Sigma M\otimes u^*\mathbb{R}^L)$ to the problem (4.6) with boundary condition $\mathbf{B}\psi = \mathbf{B}\psi_0$, satisfying

$$\|v_1\|_{C^{1+\lambda}(M)} \le C(\lambda, M, N, \|u\|_{C^{1+\lambda}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}.$$

Moreover, for any $0 < t, s < T$, it is easy to see that $v_1(\cdot,t) - v_1(\cdot,s)$ satisfy the following equation

$$\begin{aligned} \not{\partial}_g(v_1(\cdot,t) - v_1(\cdot,s)) = &-\Gamma(u(t))\#\nabla u(t)\#(v_1(\cdot,t) - v_1(\cdot,s)) \\ &- \Gamma(u(t))\#\nabla(u(t) - u(s))\#v_1(\cdot,s) \\ &- (\Gamma(u(t)) - \Gamma(u(s)))\#\nabla u(s)\#v_1(\cdot,s) \text{ in } M, \end{aligned}$$

i.e.

$$\begin{aligned} \not{D}_{u(t)}(v_1(\cdot,t) - v_1(\cdot,s)) = &-\Gamma(u(t))\#\nabla(u(t) - u(s))\#v_1(\cdot,s) \\ &- (\Gamma(u(t)) - \Gamma(u(s)))\#\nabla u(s)\#v_1(\cdot,s) \text{ in } M, \end{aligned} \tag{4.7}$$

with boundary data

$$\mathbf{B}(v_1(\cdot,t) - v_1(\cdot,s)) = 0 \text{ on } \partial M.$$

By Theorem 1.2 in [12] and Sobolev embedding, for any $\delta \in (0,1)$, we have

$$\begin{aligned} &\|v_1(\cdot,t) - v_1(\cdot,s)\|_{C^\delta(M)} \\ &\le C(\delta, M, N, \|u\|_{C^1(M)})\|v_1\|_{L^\infty(M)}(\|u(\cdot,t) - u(\cdot,s)\|_{L^\infty(M)} + \|du(\cdot,t) - du(\cdot,s)\|_{L^\infty(M)}) \\ &\le C(\delta, M, N, \|u\|_{C^1(M)})\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}|t-s|^{\frac{\lambda}{2}}. \end{aligned}$$

Therefore,

$$\|v_1\|_{C^{\lambda,\frac{\lambda}{2}}(M\times[0,T])} \le C(\lambda, M, N, \|u\|_{C^{1+\lambda}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}.$$

On the other hand, when $\alpha - 1$ is sufficiently small, by the standard theory of linear parabolic systems, for above (u, v_1), there exists a unique solution $u_1 \in C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[0,T], \mathbb{R}^L)$ to the problem (4.5) with the initial-boundary data $\phi|_{M\times\{0\}\cup\partial M\times[0,T]} = 0$, such that

$$\|u_1\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[0,T])} \le C(\lambda, M, N)(\|u_1\|_{C^0(M\times[0,T])} + \|\phi_0\|_{C^{2+\lambda}(M)} + \|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)} + 1).$$

Noting that $u_1(\cdot,0)=0$, we have

$$\|u_1\|_{C^0(M\times[0,T])}\le C(\mu,M,N)T(\|u_1\|_{C^0(M\times[0,T])}+\|\phi_0\|_{C^{2+\lambda}(M)}+\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}+1).$$

Taking $T>0$ small enough, we obtain

$$\|u_1\|_{C^0(M\times[0,T])}\le CT(\|\phi_0\|_{C^{2+\lambda}(M)}+\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}+1).$$

By the interpolation inequality for Hölder spaces (see Proposition 4.2 in [36]), we have

$$\|u_1\|_{C^{\lambda,\frac{\lambda}{2}}(M\times[0,T])}+\|\nabla u_1\|_{C^{\lambda,\frac{\lambda}{2}}(M\times[0,T])}\le C\|u_1\|^{\frac{1-\lambda}{2}}_{C^0(M\times[0,T])}\|u_1\|^{\frac{1+\lambda}{2}}_{C^{2,1}(M\times[0,T])}.$$

Thus, if we choose $T>0$ sufficiently small, then $u_1\in\mathcal{U}$. This is the first step. Similarly, we can get (u_2,v_2) by using the above argument and substituting u with $u_1+\phi_0$. After a standard induction procedure, we will get a solution (u_{k+1},v_{k+1}) of (4.5) and (4.6) with $u=u_k+\phi_0$, satisfying

$$\|v_{k+1}\|_{C^{\lambda,\frac{\lambda}{2}}(M\times[0,T])}\le C(\lambda,M,N,\|\phi_0\|_{C^{1+\lambda}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}$$

and

$$\|u_{k+1}\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[0,T])}\le C(\lambda,M,N)(\|\phi_0\|_{C^{2+\lambda}(M)}+\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)}+1).$$

By passing to a subsequence, we know that u_k converges to some ϕ in $C^{2,1}(M\times[0,T])$ and v_k converges to some ψ in $C^0(M\times[0,T])$. Then is is easy to see that $(\phi+\phi_0,\psi)$ is a solution of the system (4.3)-(4.4) with boundary-initial data (2.12). Since $\phi\in C^{2,1}(M\times[0,T],\mathbb{R}^L)$ and $\psi\in C^0(M\times[0,T],\Sigma M\otimes(\phi+\phi_0)^*T\mathbb{R}^L)$, by standard theory of Dirac-harmonic maps (see Lemma 3.6 in [26] or Lemma 4.4 below), we conclude that $\phi\in C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[0,T],\mathbb{R}^L)$ and $\psi\in C^{\lambda,\frac{\lambda}{2}}(M\times[0,T],\Sigma M\otimes(\phi+\phi_0)^*T\mathbb{R}^L)\cap L^\infty([0,T];C^{1+\lambda}(M))$.

Step 2: Uniqueness.

If there are two solutions (u_1,v_1) and (u_2,v_2) to equation (4.3)-(4.4) with boundary-initial data (2.12), subtracting the equations of u_1 and u_2, then multiplying by u_1-u_2 and integrating over M, we have

$$\begin{aligned}&\int_M\partial_t(u_1-u_2)(u_1-u_2)\\&\le\int_M\Delta_g(u_1-u_2)(u_1-u_2)+2(\alpha-1)\int_M\frac{\nabla^2_{\beta\gamma}(u_1^i-u_2^i)\nabla_\beta u_1^i\nabla_\gamma u_1}{1+|\nabla_g u_1|^2}(u_1-u_2)\\&\quad+C\int_M|u_1-u_2|^2+C\int_M|\nabla u_1-\nabla u_2||u_1-u_2|+C\int_M|v_1-v_2||u_1-u_2|.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_M|u_1-u_2|^2\\&\le\int_M-|\nabla u_1-\nabla u_2|^2-2(\alpha-1)\int_M\frac{\nabla_\beta(u_1^i-u_2^i)\nabla_\beta u_1^i\nabla_\gamma u_1}{1+|\nabla_g u_1|^2}\nabla_\gamma(u_1-u_2)\\&\quad+C\int_M|u_1-u_2|^2+C\int_M|\nabla u_1-\nabla u_2||u_1-u_2|+C\int_M|v_1-v_2||u_1-u_2|.\end{aligned}$$

By Young's inequality and noting that the second term on the right hand side of the above inequality is nonpositive, we obtain

$$\begin{aligned}\frac{d}{dt}\int_M |u_1-u_2|^2 \le -\frac{1}{2}\int_M |\nabla u_1-\nabla u_2|^2 + C\int_M |u_1-u_2|^2 \\ + C\int_M |v_1-v_2||u_1-u_2|.\end{aligned} \tag{4.8}$$

Similarly to the derivation of (4.7), we know $v_1 - v_2$ satisfies the following equation

$$\begin{aligned}\not{D}_{u_1}(v_1-v_2) = -\Gamma(u_1)\#\nabla(u_1-u_2)\#v_2 \\ -(\Gamma(u_1)-\Gamma(u_2))\#\nabla u_2\#v_2 \ in\ M,\end{aligned} \tag{4.9}$$

with the boundary data

$$\mathbf{B}(v_1-v_2) = 0 \quad on\ \partial M.$$

By Theorem 1.2 in [12], we have

$$\|v_1-v_2\|_{W^{1,2}(M)} \le C(\|u_1-u_2\|_{L^2(M)} + \|\nabla u_1-\nabla u_2\|_{L^2(M)}). \tag{4.10}$$

Therefore, by (4.8) and Young's inequality, we have

$$\frac{d}{dt}\int_M |u_1-u_2|^2 \le C\int_M |u_1-u_2|^2, \tag{4.11}$$

which implies $u_1 \equiv u_2$ on $M\times[0,T]$, if $u_1 = u_2$ for $t = 0$. Then $v_1 \equiv v_2$ follows immediately from (4.10).

Step 3: $\phi(x,t) \in N$ for all $(x,t) \in M\times[0,T]$.

Since $i : \mathcal{N} \to \mathcal{N}$ is an isometric involution and $\phi_0 \in N$, $\varphi \in N$, then $(i\circ\phi, i_*\psi)$ is also a solution to (4.3)-(4.4) with the same boundary-initial data (2.12). By the uniqueness, $i\circ\phi = \phi$ which implies $\phi(x,t) \in N$. We finished the proof of this theorem. □

Next, we shall control the α-energy of the map part, i.e. $E_\alpha(\phi)$, along the α-Dirac-harmonic map flow. Precisely, we have

Lemma 4.2. *Suppose* (ϕ,ψ) *is a solution of* (2.10)-(2.11) *with the boundary-initial data* (2.12)*, then there holds*

$$E_\alpha(\phi(t)) + 2\alpha\int_{M^t}(1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt \le E_\alpha(\phi_0) + 2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.$$

Moreover, $E_\alpha(\phi(t)) + \int_{\partial M}\langle\vec{n}\cdot\mathbf{B}\psi_0, \psi(t)\rangle$ *is absolutely continuous on* $[0,T]$ *and non-increasing.*

Proof. Firstly, it is easy to see that the equation (2.10) can be written as follows:

$$\begin{aligned}(1+|\nabla_g\phi|^2)^{\alpha-1}\partial_t\phi = div\left((1+|\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right) - (1+|\nabla_g\phi|^2)^{\alpha-1}A(d\phi,d\phi) \\ -\frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right).\end{aligned} \tag{4.12}$$

Multiplying the above equation by $\alpha\partial_t\phi$ and using the Lemma 3.1 in [26] that,

$$\int_{M_s^t}\langle P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi), \frac{\partial\phi}{\partial t}\rangle dMdt = -\frac{1}{2}\int_s^t\frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle(t)dt, \tag{4.13}$$

we have

$$\begin{aligned}&\alpha\int_{M_s^t}(1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt - \alpha\int_{M_s^t} div\left((1+|\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right)\partial_t\phi dMdt\\ &= -\int_{M_s^t}\langle P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi), \partial_t\phi\rangle dMdt = \frac{1}{2}\int_s^t\frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle dt,\end{aligned}$$

for any $0\le s\le t\le T$. Integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\int_s^t\frac{d}{dt}\int_M(1+|\nabla_g\phi|^2)^{\alpha}dMdt + \alpha\int_{M_s^t}(1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt\\ &= \frac{1}{2}\int_s^t\frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\Psi\rangle dt.\end{aligned} \tag{4.14}$$

So, we have

$$\begin{aligned}&E_\alpha(\phi(t)) + 2\alpha\int_{M^t}(1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt\\ &\le E_\alpha(\phi_0) + \left|\int_{\{0\}\times\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle\right| + \left|\int_{\{t\}\times\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle\right|\\ &\le E_\alpha(\phi_0) + 2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)},\end{aligned}$$

where the last inequality follows from Proposition 2.5 in [26] that

$$\|\psi\|_{L^2(\partial M)} = \sqrt{2}\|\mathbf{B}\psi\|_{L^2(\partial M)} = \sqrt{2}\|\mathbf{B}\psi_0\|_{L^2(\partial M)},$$

since $\not{D}\psi\equiv 0$. Also, we have

$$\begin{aligned}&\int_s^t\frac{d}{dt}\left(\int_M(1+|\nabla_g\phi|^2)^{\alpha}dM + \int_{\partial M}\langle\overrightarrow{n}\cdot\mathbf{B}\psi_0, \Psi\rangle\right)dt\\ &= -2\alpha\int_{M_s^t}(1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt,\end{aligned} \tag{4.15}$$

and the claims follow. □

Consequently, using the key estimate for the Dirac operator along a map in Lemma 3.4, we are able to control the spinor part along the α-Dirac-harmonic map flow. For the Dirac-harmonic map flow studied in [12, 26], however, there is in general no such a nice property.

Lemma 4.3. *Suppose* (ϕ,ψ) *is a solution of* (2.10)-(2.11) *with the boundary-initial data* (2.12)*, then for any* $1<p<2$*, there holds*

$$\|\psi(\cdot,t)\|_{W^{1,p}(M)}\le C\|\mathbf{B}\psi_0\|_{C^1(\partial M)},\ \forall\ 0\le t\le T, \tag{4.16}$$

where C is a positive constant depending only on $p,\ M,\ N,\ E_\alpha(\phi_0)+2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$.

Proof. According to Lemma 4.2, for any $0 \le t \le T$, we get

$$E_\alpha(\phi(\cdot,t)) \le E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.$$

Then by Lemma 3.4, we have

$$\begin{aligned}\|\psi(\cdot,t)\|_{W^{1,p}(M)} &\le C(p,M,N,\|\nabla\phi\|_{L^{2\alpha}(M)})\|\mathbf{B}\psi_0\|_{C^1(\partial M)}\\ &\le C(p,M,N,E_\alpha(\phi_0)+2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)})\|\mathbf{B}\psi_0\|_{C^1(\partial M)},\ \forall\, 0\le t\le T.\end{aligned}$$

□

Next, we derive a small energy regularity theory for the α-Dirac-harmonic map flow.

Lemma 4.4. *Suppose that $\phi_0 \in C^{2+\lambda}(M,N)$, $\varphi \in C^{2+\lambda}(\partial M,N)$ and $\psi_0 \in C^{1+\lambda}(\partial M,\Sigma M\otimes\varphi^*TN)$, where $0<\lambda<1$ is a positive constant. Let (ϕ,ψ) be a solution of* (2.10)-(2.11) *in $M\times[0,T]$ with boundary-initial data* (2.12)*. Given $z_0=(x_0,t_0)\in M\times(0,T]$, denote*

$$P^M_R(z_0) := B^M_R(x_0)\times[t_0-R^2,t_0].$$

Then there exist three positive constants $\epsilon_2=\epsilon_2(M,N)>0$, $\epsilon_3=\epsilon_3(M,N,\phi_0,\varphi,\psi_0)>0$ and $C=C(\lambda,R,M,N,E_\alpha(\phi_0),\|\phi_0\|_{C^{2+\lambda}(M)},\|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)})>0$ such that if

$$1<\alpha<1+\epsilon_2 \ \ \text{and} \ \ \sup_{[t_0-4R^2,t_0]} E(\phi(t);B^M_{2R}(x_0)) \le \epsilon_3,$$

then

$$\sqrt{R}\|\psi\|_{L^\infty(P^M_R(z_0))} + R\|\nabla\phi\|_{L^\infty(P^M_R(z_0))} \le C \tag{4.17}$$

and for any $0<\beta<1$,

$$\sup_{t_0-\frac{R^2}{4}\le t\le t_0} \|\psi(t)\|_{C^{1+\lambda}(B^M_{R/2}(z_0))} + \|\nabla\phi\|_{C^{\beta,\frac{\beta}{2}}(P^M_{R/2}(z_0))} \le C(\beta), \tag{4.18}$$

Moreover, if

$$\sup_{x_0\in M}\sup_{[t_0-4R^2,t_0]} E(\phi(t);B^M_{2R}(x_0)) \le \epsilon_3,$$

then

$$\|\phi\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[t_0-\frac{R^2}{8},t_0])} + \|\psi\|_{C^{\lambda,\frac{\lambda}{2}}(M\times[t_0-\frac{R^2}{8},t_0])} + \sup_{t_0-\frac{R^2}{8}\le t\le t_0}\|\psi\|_{C^{1+\lambda}(M)} \le C. \tag{4.19}$$

Proof. For simplicity of notation and a better expression of the idea of proof, without loss of generality, we assume $M\subset\mathbb{R}^2$ is a bounded closed domain with the standard Euclidean metric.

Step 1: We derive (4.18) and (4.19) from (4.17).

Take a cut-off function $\eta\in C^\infty_0(P^M_R(z_0))$ such that $0\le\eta\le1$, $\eta|_{P^M_{3R/4}(z_0)}\equiv1$, $|\nabla^j\eta|\le\frac{C}{R^j}$, $j=1,2$ and $|\partial_t\eta|\le\frac{C}{R^2}$. Set $U=\eta\phi$, then

$$\begin{cases} U_t - a_{\beta\gamma}\frac{\partial^2 U}{\partial x^\beta\partial x^\gamma} = f, & in \quad P^M_R(z_0);\\ U(x,t)=0, & on \quad B^M_R(z_0)\times\{t=t_0-R^2\};\\ U(x,t)=\eta\varphi, & on \quad \partial M\times(t_0-R^2,t_0),\end{cases}$$

where $f = f(\nabla\phi, \phi, \psi, \partial_t\eta, \nabla^2\eta, \nabla\eta, \eta)$ and

$$a_{\beta\gamma} = \delta_{\beta\gamma} + 2(\alpha - 1)\frac{\nabla_\beta\phi\nabla_\gamma\phi}{1 + |\nabla\phi|^2}.$$

Under the assumption (4.17), we know $f \in L^\infty$. Noting that

$$\partial_t - a_{\beta\gamma}\frac{\partial^2}{\partial x^\beta \partial x^\gamma}$$

is a parabolic operator when $\alpha - 1$ is sufficiently small, by standard parabolic theory, for any $1 < p < \infty$, we have

$$\|U\|_{W_p^{2,1}(P_R^M(z_0))} \leq C(\|f\|_{L^p(P_R^M(z_0))} + \|\eta\varphi\|_{W_p^{2,1}(\partial P_R^M(z_0))}) \leq C(1 + \|\phi_0\|_{C^2(M)})$$

Then for any $0 < \beta = 1 - 4/p < 1$, Sobolev embedding tells us,

$$\begin{aligned} \|\nabla\phi\|_{C^{\beta,\beta/2}(P_{3R/4}^M(z_0))} &\leq \|\nabla U\|_{C^{\beta,\beta/2}(P_R^M(z_0))} \\ &\leq C\|U\|_{W_p^{2,1}(P_R^M(z_0))} \leq C(\beta)(1 + \|\phi_0\|_{C^2(M)}). \end{aligned} \tag{4.20}$$

Choose a cut-off function $\chi \in C_0^\infty(B_R^M(x_0))$ satisfying $0 \leq \chi \leq 1, \chi|_{B_{3R/4}^M(x_0)} \equiv 1$ and $|\nabla^j\chi| \leq \frac{C}{R^j}$, $j = 1, 2$. Set $V = \chi\psi$, then we have

$$\begin{cases} \partial\!\!\!/ V = h, & in \quad B_R^M(x_0); \\ \mathbf{B}V(x) = \chi\mathbf{B}\psi_0, & on \quad \partial B_R^M(x_0), \end{cases}$$

where

$$h = \chi\partial\!\!\!/\psi + \nabla\chi \cdot \psi = \chi\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) + \nabla\chi \cdot \psi \in L^\infty,$$

since the assumption (4.17) holds. By the standard theory of the usual Dirac operator and Sobolev embedding, we have

$$\begin{aligned} \|\psi\|_{C^{1-2/p}(B_{3R/4}^M(x_0))} &\leq C\|V\|_{W^{1,p}(B_R^M(x_0))} \\ &\leq C(\|h\|_{L^p(B_R^M(x_0))} + \|\mathbf{B}V\|_{W^{1-1/p,p}(\partial B_R^M(x_0))}) \\ &\leq C(1 + \|\mathbf{B}\psi_0\|_{C^1(\partial M)}) \end{aligned} \tag{4.21}$$

for any $2 < p < \infty$. Then (4.20) and (4.21) tell us $\partial\!\!\!/\psi \in C^\lambda(B_{3R/4}^M(z_0))$. By the Schauder estimates Theorem 4.6 in [12] and taking some suitable cut-off function as before, we have

$$\|\psi(t)\|_{C^{1+\lambda}(B_{R/2}^M(x_0))} \leq C(1 + \|\mathbf{B}\psi_0\|_{C^{1+\lambda}(\partial M)})(1 + \|\phi_0\|_{C^2(M)}) \tag{4.22}$$

for any $t_0 - \frac{R^2}{4} \leq t \leq t_0$. Then the inequality (4.18) follows from (4.20), (4.22) immediately.

For the estimate (4.19), we first rewrite the equation $\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi)$ as

$$\partial\!\!\!/\psi + \Omega \cdot \psi = 0$$

where

$$\Omega = \sum_{i=n+1}^{K} [\nu^i(\phi), d\nu^i(\phi)] = \sum_{i=n+1}^{K} \left((\nu^i)^A(\nabla_{e_\gamma}\nu^i)^B e_\gamma - (\nu^i)^B(\nabla_{e_\gamma}\nu^i)^A e_\gamma\right)$$

and $\{\nu^i\}_{i=n+1}^K$ is an orthonormal basis of the normal bundle $T^\perp N$ and $\nu^i = ((\nu^i)^1, ..., (\nu^i)^K)$ (see Remark 2.1 in [12]), then for any $t_0 - \frac{R^2}{4} < t, s < t_0$, we have

$$\begin{cases} \partial\!\!\!/(\psi(\cdot,t) - \psi(\cdot,s)) = -\Omega(\cdot,t)(\psi(\cdot,t) - \psi(\cdot,s)) + (\Omega(\cdot,s) - \Omega(\cdot,t))\psi(\cdot,s) & in \quad M; \\ \mathbf{B}(\psi(\cdot,t) - \psi(\cdot,s)) = 0 & on \quad \partial M. \end{cases}$$

Since $d\Omega = [d\nu(\phi), d\nu(\phi)]$, with (4.20) and (4.22), according to Theorem 4.1 in [12], for any $0 < \beta < 1$, by Sobolev embedding, we have

$$\|\psi(\cdot,t) - \psi(\cdot,s)\|_{C^\beta(M)} \leq C(\|\Omega(\cdot,t) - \Omega(\cdot,s)\|_{L^\infty(M)}) \leq C|s-t|^\beta.$$

So, we get $\|\psi\|_{C^{\beta,\frac{\beta}{2}}(M\times[t_0-\frac{R^2}{4},t_0])} \leq C$ and

$$\begin{cases} \partial_t\phi - a_{\beta\gamma}\frac{\partial^2\phi}{\partial x^\beta\partial x^\gamma} & \in C^{\beta,\beta/2}(M\times[t_0-\frac{R^2}{4},t_0]) \quad for \quad any \quad 0<\beta<1; \\ \phi|_{\partial M} = \varphi & \in C^{2+\lambda}(\partial M). \end{cases}$$

Taking some suitable cut-off function and by standard Schauder estimates for second order parabolic equations, when $\alpha - 1$ is sufficiently small, we have $\phi \in C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[t_0-\frac{R^2}{8},t_0])$ and

$$\begin{aligned} &\|\phi\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(M\times[t_0-\frac{R^2}{8},t_0])} \\ &\leq C(\|\partial_t\phi - a_{\beta\gamma}\frac{\partial^2\phi}{\partial x^\beta\partial x^\gamma}\|_{C^{\lambda,\lambda/2}(M\times[t_0-\frac{R^2}{4},t_0])} + \|\phi\|_{C^0(M\times[t_0-\frac{R^2}{4},t_0])} + \|\varphi\|_{C^{2+\lambda}(\partial M)}) \leq C. \end{aligned}$$

So we have proved (4.19).

Step 2: We prove (4.17).

We shall adapt the methods in [56, 61, 37]. Without loss of generality, we may assume $R = \frac{1}{2}$. Take $0 \leq \rho < 1$ such that

$$(1-\rho)^2 \sup_{P^M_\rho(z_0)} |\nabla\phi|^2 = \max_{0\leq\sigma\leq1}\{(1-\sigma)^2 \sup_{P^M_\sigma(z_0)} |\nabla\phi|^2\}$$

and then choose $z_1 = (x_1, t_1) \in P^M_\rho(z_0)$ such that

$$|\nabla\phi|^2(z_1) = \sup_{P^M_\rho(z_0)} |\nabla\phi|^2 := e.$$

We claim:

$$(1-\rho)^2 e \leq 4.$$

We proceed by contradiction. If $(1-\rho)^2 e > 4$, we set

$$u(x,t) := \phi(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \quad and \quad v(x) := e^{-\frac{1}{4}}\psi(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t).$$

Denoting $P_r(0) = D_r(0) \times [-r^2, 0] \subset \mathbb{R}^2 \times \mathbb{R}$ and

$$S_r := P_r(0) \cap \{(x,t)|(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \in P^M_1(0)\},$$

then (u, v) satisfy

$$\partial_t u = \Delta u + (\alpha - 1)\frac{\nabla|\nabla u|^2\nabla u}{e^{-1} + |\nabla u|^2} - A(\nabla u, \nabla u) - \frac{Re\left(P(\mathcal{A}(du(e_\gamma), e_\gamma \cdot v); v)\right)}{\alpha(1 + e|\nabla u|^2)^{\alpha-1}}, \tag{4.23}$$

$$\partial\!\!\!/ v = \mathcal{A}(du(e_\gamma), e_\gamma \cdot v) \tag{4.24}$$

with the boundary data

$$(u(x,t), \mathbf{B}v(x,t)) = (\varphi(x_1 + e^{-\frac{1}{2}}x), e^{-\frac{1}{4}}\mathbf{B}\psi_0(x_1 + e^{-\frac{1}{2}}x)), \quad if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M. \tag{4.25}$$

Moreover,

$$\sup_{S_1}|\nabla u|^2 = e^{-1}\sup_{P^M_{e^{-1/2}}(z_1)}|\nabla\phi|^2 \le e^{-1}\sup_{P^M_{\rho+e^{-1/2}}(z_0)}|\nabla\phi|^2 \le e^{-1}\sup_{P^M_{\frac{1+\rho}{2}}(z_0)}|\nabla\phi|^2 \le 4$$

and

$$|\nabla u|^2(0) = e^{-1}|\nabla\phi|^2(z_1) = 1.$$

Noting that v satisfies the equation $\partial\!\!\!/ v = \mathcal{A}(du(e_\gamma), e_\gamma \cdot v)$ and the facts

$$|du| \le 2, \quad \sup_{-1\le t\le 0}||v||_{L^4(D_1)} \le \sup_t ||\psi(\cdot, t)||_{L^4(M)} \le C,$$

where in the last step we have used Lemma 4.3 by taking $p = \frac{4}{3}$. By elliptic estimates of the usual Dirac operator and Sobolev embedding, we have

$$\sup_{-1\le t\le 0}||v||_{L^\infty(D_{3/4})} \le C\sup_{-1\le t\le 0}||v||_{W^{1,4}(D_{3/4})} \le C(1 + ||\mathbf{B}\psi_0||_{C^1(\partial M)}).$$

Next, we want to show that there exists a constant $C > 0$ such that

$$1 \le C\int_{S_{3/4}}|\nabla u|^2 dxdt. \tag{4.26}$$

In fact, if such a constant $C > 0$ does not exist, then there exists a sequence $\{(u_i, v_i)\}$ satisfying (4.23)-(4.24) with the boundary data (4.25) and

$$\sup_{S_{3/4}}(|\nabla u_i| + |v_i|) \le C, \tag{4.27}$$

$$|\nabla u_i|^2(0) = 1, \tag{4.28}$$

$$\int_{S_{3/4}}|\nabla u_i|^2 dxdt \le \frac{1}{i}. \tag{4.29}$$

Similarly to the argument in **Step 1** (since (u_i, v_i) satisfy (4.23)-(4.24), (4.25) and (4.27)), we obtain

$$||\nabla u_i||_{C^{\beta,\beta/2}(S_{1/2})} \le C(\beta)$$

for any $0 < \beta < 1$.

Therefore, there exists a subsequence of $\{u_i\}$ (we still denote it by $\{u_i\}$) and a function $\overline{u} \in C^{\delta,\frac{\delta}{2}}(S_{1/2})$ such that

$$\nabla u_i \to \nabla\overline{u} \quad in \quad C^{\delta,\delta/2}(S_{1/2})$$

where $0 < \delta < \beta$. By (4.29), we know

$$\int_{S_{1/2}} |\nabla \overline{u}|^2 dxdt = 0 \tag{4.30}$$

which implies $\nabla \overline{u} \equiv 0$ in $S_{1/2}$. But, (4.28) tells us $|\nabla \overline{u}|(0) = 1$. This is a contradiction and then (4.26) must be true. Thus, we have

$$\begin{aligned}
1 &\leq C \int_{S_{3/4}} |\nabla u|^2 dxdt \\
&\leq C \sup_{-1<t<0} \int_{B^M_{e^{\frac{1}{2}}}(x_1)} |\nabla \phi|^2 (t_1 + e^{-1} t) dx \\
&\leq C \sup_{-1<t<0} \int_{B^M_1(z_0)} |\nabla \phi|^2 (t) dx \leq C \epsilon_3.
\end{aligned}$$

Choosing $\epsilon_3 > 0$ sufficiently small leads to a contradiction, so we must have $(1-\rho)^2 e \leq 4$ and then

$$(1 - 3/4)^2 \sup_{P^M_{3/4}(z_0)} |\nabla \phi|^2 \leq (1-\rho)^2 e \leq 4.$$

Since ψ satisfies the equation $\partial\!\!\!/ \psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi)$ and $\|d\phi\|_{L^\infty(P^M_{3/4}(z_0))} \leq 8$, $\|\psi\|_{L^4(M)} \leq C$, by the elliptic theory for the Dirac operator and Sobolev embedding again, we shall easily obtain

$$\|\psi\|_{L^\infty(P^M_{1/2}(z_0))} \leq C.$$

Thus we get the inequality (4.17). This finishes the proof of the lemma. □

In the end of this section, we prove our first main result - Theorem 2.1.

Proof of Theorem 2.1. By the short-time existence result Theorem 4.1, there is a unique solution

$$\phi \in C^{2+\lambda, 1+\frac{\lambda}{2}}_{loc}(M \times [0,T), N)$$

and

$$\psi \in \cap_{0<S<T} L^\infty([0,S], \|\psi(\cdot,t)\|_{C^{1+\lambda}(M)}) \cap C^{\lambda,\frac{\lambda}{2}}_{loc}(M \times [0,T), \Sigma M \otimes \phi^* TN)$$

to the problem (2.10)-(2.11) with boundary data (2.12) for some $T > 0$.

Next, we will show that the solution (ϕ, ψ) can be extended to the time T. In fact, by Lemma 4.2, we have

$$\int_M (1 + |\nabla_g \phi|^2)^\alpha (\cdot, t) dM \leq E_\alpha(\phi_0) + 2\sqrt{2} \|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.$$

Then it is easy to see that, for any $0 < \epsilon < \epsilon_3$, there exists a positive constant r_0, depending only on ϵ, α, M, $E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$, such that for all $x \in M$ and $0 \leq t < T$, there holds

$$\int_{B^M_{r_0}(x)} |\nabla_g \phi|^2 (\cdot, t) dM \leq C E_\alpha(\phi)^{1/\alpha} r_0^{1-\frac{1}{\alpha}} \leq \epsilon.$$

By Lemma 4.4, we can extend the solution $(\phi(\cdot,t), \psi(\cdot,t))$ to the time T with $(\phi(\cdot,T), \psi(\cdot,T)) \in C^{2+\lambda}(M,N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi(\cdot,T)^* TN)$. Then the short-time existence result implies $T = \infty$.

For the limit behaviour as $t \to \infty$, by Lemma 4.2, we get

$$\int_0^\infty \int_M |\partial_t \phi|^2 dM dt \leq C,$$

which implies that there exists a time sequence $t_i \to \infty$, such that

$$\int_M |\partial_t \phi|^2 (\cdot, t_i) dM \to 0.$$

By Lemma 4.4, we have

$$\|\phi(t_i)\|_{C^{2+\lambda}(M)} + \|\psi(t_i)\|_{C^{1+\lambda}(M)} \leq C.$$

Thus, there exists a subsequence of $\{t_i\}$ (still denoted by $\{t_i\}$) and an α-Dirac-harmonic map $(\phi_\alpha, \psi_\alpha)$ with boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that $(\phi(\cdot, t_i), \psi(\cdot, t_i))$ converges to $(\phi_\alpha, \psi_\alpha)$ in $C^2(M) \times C^1(M)$. Since $(\varphi, \psi_0) \in C^{2+\lambda}(\partial M, N) \times C^{1+\lambda}(\partial M, \varphi^* TN)$, it is standard to obtain

$$(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi_\alpha^* TN)$$

from the Schauder theory for second order elliptic operators and Dirac operators. This completes the proof of theorem. □

5. Blow-up analysis and existence of Dirac-harmonic maps

By the results in the previous section, it is easy to see that there exists a sequence of α-Dirac-harmonic maps $\{(\phi_\alpha, \psi_\alpha)\}$ as $\alpha \searrow 1$ with Dirichlet-chiral boundary condition

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that

$$E_\alpha(\phi_\alpha) \leq E_\alpha(\phi_0) + 2\sqrt{2}\, \|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{5.1}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \leq C(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\, \|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{5.2}$$

for any $1 < p < 2$. In this section, we will study the limit behaviour of the sequence as $\alpha \searrow 1$ and show that the limit is just the Dirac-harmonic map we want to find.

First of all, we consider the blow-up sequence under the following more general assumption that

$$E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(M)} \leq \Lambda < \infty.$$

Note that the functional L_α and the equations of α-Dirac-harmonic maps are not conformally invariant in dimension two. Besides the concept of general α-Dirac-harmonic map, we need to introduce the following definition

$(u, v) : (D_1(0), g_\alpha) \to N \times (\Sigma M \otimes u^* TN)$ is called *a $\overline{\lambda}$-general α-Dirac-harmonic map* if it satisfies

$$\begin{cases} \Delta_{g_\alpha} u &= -(\alpha - 1) \frac{\nabla_{g_\alpha} |\nabla_{g_\alpha} u|^2 \nabla_{g_\alpha} u}{\sigma_\alpha + |\nabla_{g_\alpha} u|^2} + A(du, du) + \overline{\lambda} \frac{Re\left(P(\mathcal{A}(du(e_\gamma), e_\gamma \cdot v); v)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha} u|^2)^{\alpha - 1}}, \\ \not{\partial}_{g_\alpha} v &= \mathcal{A}(du(e_\gamma), e_\gamma \cdot v), \end{cases} \tag{5.3}$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$, $\varphi_\alpha \in C^\infty(D_1(0))$, $\varphi_\alpha(0) = 0$ is the Riemannian metric, $\overline{\lambda} \in \mathbb{R}$ and $\sigma_\alpha > 0$ are two constants. One can easily see that a $\overline{\lambda}$-general α-Dirac-harmonic map is a general α-Dirac-harmonic map if $\overline{\lambda} = 1$ and an α-Dirac-harmonic map if $\overline{\lambda} = 1$, $\sigma_\alpha = 1$.

For example, on an isothermal coordinate system around a point $p \in M$, if the metric is given by

$$g = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2)$$

with $\varphi_0(p) = 0$, setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) := \Big(\phi_\alpha(p + \lambda_\alpha x),\ \sqrt{\lambda_\alpha}\psi_\alpha(p + \lambda_\alpha x)\Big)$$

for some small positive number $\lambda_\alpha > 0$. By the conformal invariance of the spinor equation, it is easy to check that $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ is a $\sigma_\alpha^{\alpha-1}$-general α-Dirac-harmonic map which satisfies the following system

$$\begin{cases} \Delta_{g_\alpha} \widetilde{u}_\alpha &= -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2\nabla_{g_\alpha}\widetilde{u}_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\widetilde{u}_\alpha|^2} + A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\big(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha); \widetilde{v}_\alpha)\big)}{\alpha(1+\sigma_\alpha^{-1}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \not{\partial}_{g_\alpha} \widetilde{v}_\alpha &= \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha), \end{cases} \tag{5.4}$$

where $g_\alpha = e^{\varphi_0(p+\lambda_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and $\sigma_\alpha = \lambda_\alpha^2 > 0$.

Since α-Dirac-harmonic maps are not conformally invariant, in order to get unified bubbling equations, we need to add another factor $\lambda_\alpha^{\alpha-1}$ in the rescaling. Setting

$$(u_\alpha(x), v_\alpha(x)) := \Big(\widetilde{u}_\alpha(x), \lambda_\alpha^{\alpha-1}\widetilde{v}_\alpha(x)\Big) = \Big(\phi_\alpha(p + \lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(p + \lambda_\alpha x)\Big)$$

and noting that the equation for the spinor part is also invariant by multiplying a constant to the spinor, then one can verify that $(u_\alpha(x), v_\alpha(x))$ is a general α-Dirac-harmonic map, satisfying the following system:

$$\begin{cases} \Delta_{g_\alpha} u_\alpha &= -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}u_\alpha|^2\nabla_{g_\alpha}u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2} + A(du_\alpha, du_\alpha) + \frac{Re\big(P(\mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha); v_\alpha)\big)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}}, \\ \not{\partial}_{g_\alpha} v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha). \end{cases} \tag{5.5}$$

Since the spinor equation is conformally invariant, it is easy to check that the system (5.4) is equivalent to

$$\begin{cases} \Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) + \frac{Re\big(P(\mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha); v_\alpha)\big)}{\alpha(1+\sigma_\alpha^{-1}|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}} = 0, \\ \not{\partial} v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha), \end{cases} \tag{5.6}$$

and the system (5.5) is equivalent to

$$\begin{cases} \Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) + \frac{Re\big(P(\mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha); v_\alpha)\big)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}} = 0, \\ \not{\partial} v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha), \end{cases} \tag{5.7}$$

where $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$, the derivative ∇ and the Dirac operator $\not{\partial}$ are taken with respect to the standard Euclidean metric. The $\{e_\gamma\}$ in (5.6) and (5.7) is a local orthonormal basis with respect to the standard Euclidean metric and hence it is different from the one in (5.4) and (5.5), however, for simplicity, we shall use the same notation. More precisely, the above equivalences of the systems mean that $(u_\alpha \circ Id, e^{\frac{\varphi_\alpha}{2}} v_\alpha \circ Id)$ satisfies (5.6)-(5.7), where $Id : (D_1(0), (dx^1)^2 + (dx^2)^2) \to (D_1(0), g_\alpha)$ is a conformal map defined by $Id(x) = x$. In the sequel, for simplicity of notation, we shall identify $(u_\alpha \circ Id, e^{\frac{\varphi_\alpha}{2}} v_\alpha \circ Id)$ with (u_α, v_α) and use the appropriate forms of the systems.

Let $D_1(0) \subset \mathbb{R}^2$ be the unit ball centered at 0. Denote

$$D_1^+(0) := \left\{(x^1, x^2) \in D_1(0) | x^2 \geq 0\right\}, \ \partial^0 D_1^+(0) := \left\{(x^1, x^2) \in D_1(0) | x^2 = 0\right\}.$$

Next, we show a small energy regularity lemma for general α-Dirac-harmonic maps and $\overline{\lambda}$-general α-Dirac-harmonic maps. For the interior case, we have

Lemma 5.1. *Let $(\phi_\alpha, \psi_\alpha) : (D_1(0), g_\alpha) \to N$ be a general α-Dirac-harmonic map with*

$$0 < \beta_0 \leq \liminf_{\alpha \searrow 1} \sigma_\alpha^{\alpha-1} \leq 1$$

for some positive constant $\beta_0 > 0$ or a ρ_α-general α-Dirac-harmonic map with ρ_α satisfying

$$\sup_{\alpha \searrow 1} \frac{|\rho_\alpha|}{\sigma_\alpha^{\alpha-1}} \leq \Lambda_0$$

for some positive constant $\Lambda_0 > 0$. For any $1 < p < \infty$, there exist two positive constants $\epsilon_0 > 0$ and $\alpha_0 > 1$ depending only on g, N, such that if $E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(D_1(0))} \leq \Lambda$ and

$$E(\phi_\alpha) \leq \epsilon_0, \ 1 \leq \alpha \leq \alpha_0,$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\varphi_\alpha(0) = 0$, $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1(0))$ as $\alpha \to 1$, then there hold

$$\begin{aligned}
&\|\nabla\phi_\alpha\|_{W^{1,p}(D_{1/2}(0))} \leq C(p, g, \beta_0, \Lambda_0, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))},\\
&\|\psi_\alpha\|_{W^{1,p}(D_{1/2}(0))} \leq C(p, g, \beta_0, \Lambda_0, \Lambda, N)\|\psi_\alpha\|_{L^4(D_1(0))},\\
&\|\nabla\phi_\alpha\|_{C^1(D_{1/2}(0))} \leq C(g, \beta_0, \Lambda_0, \Lambda, N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))},\\
&\|\psi_\alpha\|_{C^0(D_{1/2}(0))} \leq C(g, \beta_0, \Lambda_0, \Lambda, N)\|\psi_\alpha\|_{L^4(D_1(0))}.
\end{aligned}$$

Near the boundary, we have

Lemma 5.2. *Let $(\phi_\alpha, \psi_\alpha) : (D_1^+(0), g_\alpha) \to N$ be a general α-Dirac-harmonic map with*

$$0 < \beta_0 \leq \liminf_{\alpha \searrow 1} \sigma_\alpha^{\alpha-1} \leq 1$$

for some positive constant $\beta_0 > 0$ or a ρ_α-general α-Dirac-harmonic map with ρ_α satisfying

$$\sup_{\alpha \searrow 1} \frac{|\rho_\alpha|}{\sigma_\alpha^{\alpha-1}} \leq \Lambda_0$$

for some positive constant $\Lambda_0 > 0$, satisfying Dirichlet-chiral boundary condition

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial^0 D_1^+(0)} = (\varphi, \mathbf{B}\psi_0).$$

For any $1 < p < \infty$, there exist two positive constants $\epsilon_0 > 0$ and $\alpha_0 > 1$ depending only on g, N, such that if $E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(D_1(0))} \leq \Lambda$ and

$$E(\phi_\alpha) \leq \epsilon_0, \ 1 \leq \alpha \leq \alpha_0,$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\varphi_\alpha(0) = 0$, $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1(0))$ as $\alpha \to 1$, then there hold

$$\begin{aligned}
&\|\nabla\phi_\alpha\|_{C^{1+\lambda}(D^+_{1/2}(0))} + \|\nabla\phi_\alpha\|_{W^{1,p}(D^+_{1/2}(0))} \leq C(\|\nabla\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^{1+\lambda}(\partial^0 D^+)}),\\
&\|\psi_\alpha\|_{C^0(D^+_{1/2}(0))} + \|\psi_\alpha\|_{W^{1,p}(D^+_{1/2}(0))} \leq C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_0\|_{C^1(\partial^0 D^+)}),
\end{aligned}$$

where C is a positive constant depending on $p, g, \beta_0, \Lambda_0, \lambda, \Lambda, N, \|\varphi\|_{C^2}, \|\mathbf{B}\psi_0\|_{C^1}$.

Since the proof for the interior case is similar to, but simpler than that of the boundary case, we only prove Lemma 5.2 here and omit the interior case.

Proof of Lemma 5.2. We prove the lemma for the case of ρ_α-general α-Dirac-harmonic map with ρ_α satisfying

$$\sup_{\alpha\searrow 1}\frac{|\rho_\alpha|}{\sigma_\alpha^{\alpha-1}}\leq\Lambda_0$$

for some positive constant $\Lambda_0>0$, i.e. $(\phi_\alpha,\psi_\alpha)$ satisfies (5.3). For the general α-Dirac-harmonic map case, i.e. $(\phi_\alpha,\psi_\alpha)$ satisfies (5.5) with $0<\beta_0\leq\liminf_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1}\leq 1$ for some positive constant $\beta_0>0$, the proof is almost the same.

Without loss of generality, we assume $\int_{\partial^0 D^+_{\frac{1}{2}}}\varphi=0$.

Choose a cut-off function $\eta\in C_0^\infty(D^+)$ satisfying $0\leq\eta\leq 1, \eta|_{D^+_{3/4}}\equiv 1, |\nabla\eta|+|\nabla^2\eta|\leq C$. Noting that ψ_α satisfies (5.6), by standard theory of first order elliptic equations, for any $1<q<2$, we have

$$\begin{aligned}
\|\eta\psi_\alpha\|_{W^{1,q}(D^+)}&\leq C(\|\partial\!\!\!/(\eta\psi_\alpha)\|_{L^q(D^+)}+\|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0D^+)})\\
&\leq C(\|\nabla\eta\cdot\psi_\alpha+\eta\partial\!\!\!/\psi_\alpha\|_{L^q(D^+)}+\|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0D^+)})\\
&\leq C\left(\|\psi_\alpha\|_{L^q(D^+)}+\||d\phi_\alpha||\eta\psi_\alpha|\|_{L^q(D^+)}+\|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0D^+)}\right)\\
&\leq C\|d\phi_\alpha\|_{L^2(D^+)}\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)}+C(\|\psi_\alpha\|_{L^q(D^+)}+\|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0D^+)})\\
&\leq C\epsilon_0\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)}+C(\|\psi_\alpha\|_{L^q(D^+)}+\|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0D^+)}).
\end{aligned}$$

Taking $\epsilon_0>0$ sufficiently small, by Sobolev embedding, we get

$$\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)}\leq\|\eta\psi_\alpha\|_{W^{1,q}(D^+)}\leq C(\|\psi_\alpha\|_{L^q(D^+)}+\|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0D^+)}). \tag{5.8}$$

In particular, taking $q=\frac{8}{5}$, we get

$$\|\eta\psi_\alpha\|_{L^8(D^+)}\leq\|\eta\psi_\alpha\|_{W^{1,\frac{8}{5}}(D^+)}\leq C(\|\psi_\alpha\|_{L^4(D^+)}+\|\mathbf{B}\psi_\alpha\|_{C^{3/8,8/5}(\partial^0D^+)}). \tag{5.9}$$

Noting that $\sup_{\alpha\searrow 1}\frac{|\rho_\alpha|}{\sigma_\alpha^{\alpha-1}}\leq\Lambda_0$ and

$$\Delta\phi_\alpha=-(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}+A(d\phi_\alpha,d\phi_\alpha)+\rho_\alpha\frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma),e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}},$$

where $\Delta=(\frac{\partial}{\partial x^1})^2+(\frac{\partial}{\partial x^2})^2$ is the Laplace operator of the standard Euclidean metric, computing directly, we obtain

$$\begin{aligned}
|\Delta(\eta\phi_\alpha)|&=|\eta\Delta\phi_\alpha+2\nabla\eta\nabla\phi_\alpha+\phi_\alpha\Delta\eta|\\
&\leq C\left(|\phi_\alpha|+|d\phi_\alpha|+(\alpha-1)|\eta\nabla^2\phi_\alpha|+|d\phi_\alpha||\eta d\phi_\alpha|+|\psi_\alpha|^2|\eta d\phi_\alpha|\right)\\
&\leq C(|d\phi_\alpha||d(\eta\phi_\alpha)|+(\alpha-1)|\nabla^2(\eta\phi_\alpha)|)+C\left(|\phi_\alpha|+|d\phi_\alpha|+|\psi_\alpha|^2|\eta d\phi_\alpha|\right).
\end{aligned} \tag{5.10}$$

By standard elliptic estimates and the Poincaré inequality, for any $1 < p < 2$, we have

$$\begin{aligned}\|\eta\phi_\alpha\|_{W^{2,p}(D^+)} &\le C(\||d\phi_\alpha||d(\eta\phi_\alpha)|\|_{L^p(D^+)} + (\alpha-1)\|\nabla^2(\eta\phi_\alpha)\|_{L^p(D^+)})\\ &\quad + C(\|d\phi_\alpha\|_{L^p(D^+)} + \|\varphi\|_{W^{2-1/p,p}(\partial^0 D^+)} + \||\psi_\alpha|^2|\eta d\phi_\alpha|\|_{L^p(D^+)})\\ &\le C\|d(\eta\phi_\alpha)\|_{L^{\frac{2p}{2-p}}(D^+)}\|d\phi_\alpha\|_{L^2(D^+)} + C(\alpha-1)\|\nabla^2(\eta\psi_\alpha)\|_{L^p(D^+)}\\ &\quad + C(\|d\phi_\alpha\|_{L^p(D^+)} + \|\,|\psi_\alpha|^2|\eta d\phi_\alpha|\,\|_{L^p(D^+)} + \|\varphi\|_{W^{2,p}(\partial^0 D^+)})\\ &\le C(\epsilon_0+\alpha-1)\|d(\eta\phi_\alpha)\|_{W^{1,p}(D^+)} + C(\|d\phi_\alpha\|_{L^p(D^+)}\\ &\quad + \|\,|\psi_\alpha|^2|\eta d\phi_\alpha|\,\|_{L^p(D^+)} + \|\nabla\varphi\|_{W^{1,p}(\partial^0 D^+)}).\end{aligned}$$

Choosing $\epsilon_0 > 0$ and $\alpha_0 - 1$ sufficiently small, we have

$$\|\nabla(\eta\phi_\alpha)\|_{W^{1,p}(D^+)} \le C(\|d\phi_\alpha\|_{L^p(D^+)} + \|\,|\psi_\alpha|^2|\eta d\phi_\alpha|\,\|_{L^p(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}). \tag{5.11}$$

In particular, we take $p = \frac{4}{3}$, then

$$\begin{aligned}\|\nabla\phi_\alpha\|_{L^4(D^+_{\frac{5}{8}})} &\le C\|\nabla\phi_\alpha\|_{W^{1,\frac{4}{3}}(D^+_{\frac{5}{8}})}\\ &\le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\psi_\alpha\|^2_{L^8(D^+_{3/4})}\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)})\\ &\le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}),\end{aligned} \tag{5.12}$$

where the last inequality follows from (5.9).

Applying the $W^{1,2}$-estimate for the usual Dirac operator, we have

$$\begin{aligned}\|\psi_\alpha\|_{W^{1,2}(D^+_{\frac{9}{16}})} &\le C(\|\partial\!\!\!/\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\mathbf{B}\psi_\alpha\|_{W^{1/2,2}(\partial^0 D^+)})\\ &\le C(\|d\phi_\alpha\|_{L^4(D^+_{\frac{5}{8}})}\|\psi_\alpha\|_{L^4(D^+_{\frac{5}{8}})} + \|\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\mathbf{B}\psi_\alpha\|_{W^{1/2,2}(\partial^0 D^+)})\\ &\le C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_\alpha\|_{C^1(\partial^0 D^+)}).\end{aligned}$$

By (5.10), we get

$$|\Delta(\eta\phi_\alpha)| \le C(\alpha-1)|\nabla^2(\eta\phi_\alpha)| + C\left(|\phi_\alpha| + |d\phi_\alpha| + |d\phi_\alpha|^2 + |\psi_\alpha|^2|d\phi_\alpha|\right). \tag{5.13}$$

Applying the $W^{2,2}$-estimate for the Laplace operator and choosing $\alpha_0 - 1$ small enough, by (5.9) and (5.12), we obtain

$$\|\nabla\phi_\alpha\|_{W^{1,2}(D^+_{\frac{9}{16}})} \le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(D^+)}). \tag{5.14}$$

By the Sobolev embedding theorem, we know $\nabla\phi_\alpha \in L^p(D^+_{9/16})$ and $\psi_\alpha \in L^p(D^+_{9/16})$ for any $1 < p < \infty$. Noting that ϕ_α satisfies the following elliptic equation that

$$\begin{aligned}\Delta\phi_\alpha + (\alpha-1)\frac{g_\alpha^{\kappa\beta}\frac{\partial\phi_\alpha}{\partial x^\kappa}\frac{\partial\phi_\alpha}{\partial x^\gamma}}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2}\frac{\partial^2\phi_\alpha}{\partial x^\beta\partial x^\gamma} &= A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) + \rho_\alpha\frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_g\phi_\alpha|^2)^{\alpha-1}}\\ &\quad - (\alpha-1)\frac{\frac{\partial g_\alpha^{\kappa\beta}}{\partial x^\gamma}\frac{\partial\phi_\alpha}{\partial x^\kappa}\frac{\partial\phi_\alpha}{\partial x^\beta}\frac{\partial\phi_\alpha}{\partial x^\gamma}}{\sigma_\alpha + |\nabla_g\phi_\alpha|^2}\\ &:= f(\phi_\alpha, \psi_\alpha, \nabla\phi_\alpha),\end{aligned}$$

when $\alpha - 1$ is sufficiently small, by $W^{2,p}$-estimate for elliptic operator, we have

$$\begin{aligned}\|\nabla\phi_\alpha\|_{W^{1,p}(D^+_{\frac{17}{32}}(0))} &\leq C(\|\nabla\phi_\alpha\|_{L^p(D^+_{\frac{9}{16}}(0))} + \|f\|_{L^p(D^+_{\frac{9}{16}}(0))} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}) \\ &\leq C(\|\nabla\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(D^+)}).\end{aligned}$$

Applying the $W^{1,p}$-estimate for the usual Dirac operator, we have

$$\|\psi_\alpha\|_{W^{1,p}(D^+_{\frac{17}{32}}(0))} \leq C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_\alpha\|_{C^1(\partial^0 D^+)}).$$

By Sobolev embedding, for any $0 < \delta < 1$, there hold

$$\begin{aligned}\|\nabla\phi_\alpha\|_{C^\delta(D^+_{17/32}(0))} &\leq C(\delta, g, \Lambda, N)(\|\nabla\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(D^+)}), \\ \|\psi_\alpha\|_{C^\delta(D^+_{17/32}(0))} &\leq C(\delta, g, \Lambda, N)(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_\alpha\|_{C^1(\partial^0 D^+)}).\end{aligned}$$

To estimate $\|\nabla\phi_\alpha\|_{C^1}$, using the standard schauder estimates of elliptic operator, we get

$$\begin{aligned}\|\nabla\phi_\alpha\|_{C^{1+\lambda}(D^+_{1/2}(0))} &\leq C(\|f\|_{C^\lambda(D^+_{17/32}(0))} + \|\nabla\phi_\alpha\|_{C^0(D^+_{17/32}(0))} + \|\nabla\varphi\|_{C^{1+\lambda}(D^+)}) \\ &\leq C(\lambda, g, \Lambda, N)(\|\nabla\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^{1+\lambda}(D^+)}).\end{aligned}$$

□

Applying the above small energy regularity results, we can now show Theorem 2.2, Theorem 2.4 and Theorem 2.5.

Proof of Theorem 2.2: Without loss of generality, let $\{x_1, ..., x_I\} \subset \mathbf{S}$ be any subset with finite points. Choosing $r > 0$ sufficiently small such that $B^M_r(x_i) \cap B^M_r(x_j) = \emptyset,\ i \neq j$, then

$$\Lambda \geq \liminf_{\alpha\to 1} E(\phi_\alpha; M) \geq \sum_{i=1}^{I} \liminf_{\alpha\to 1} E(\phi_\alpha; B^M_r(x_i)) \geq \frac{\epsilon_0}{2} I,$$

which implies $I \leq \frac{2\Lambda}{\epsilon_0}$. Therefore, $\mathbf{S}$ is a set with at most finitely many points..

For any $x_0 \in M \setminus \mathbf{S}$, there exist $r_0 > 0$ and a subsequence of $\alpha \searrow 1$, such that

$$E(\phi_\alpha; B^M_{r_0}(x_0)) < \frac{\epsilon_0}{2}.$$

If $x_0 \in M \setminus \partial M$, without loss of generality, we may assume $B^M_{r_0}(x_0) \cap \partial M = \emptyset$. By Lemma 5.1, we have

$$r_0\|\nabla\phi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} + \sqrt{r_0}\|\psi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} \leq C(\Lambda, M, N).$$

If $x_0 \in \partial M$, by Lemma 5.2, we have

$$r_0\|\nabla\phi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} + \sqrt{r_0}\|\psi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} \leq C(\Lambda, M, N, \|\varphi\|_{C^2}, \|\mathbf{B}\psi_0\|_{C^1}).$$

By standard theory of Dirac operators and second order elliptic operators, we obtain

$$\|\phi_\alpha\|_{C^k(B^M_{r_0/4}(x_0))} + \|\psi_\alpha\|_{C^k(B^M_{r_0/4}(x_0))} \leq C(k, r_0, \Lambda, M, N) \tag{5.15}$$

for $x_0 \in M \setminus \partial M$ and

$$\|\phi_\alpha\|_{C^{2+\lambda}(B^M_{r_0/4}(x_0))} + \|\psi_\alpha\|_{C^{1+\lambda}(B^M_{r_0/4}(x_0))} \leq C(r_0, \lambda, \Lambda, M, N, \|\varphi\|_{C^{2+\lambda}}, \|\mathbf{B}\psi_0\|_{C^{1+\lambda}}) \tag{5.16}$$

for $x_0 \in \partial M$.

Suppose (ϕ, ψ) is the weak limit of $(\phi_\alpha, \psi_\alpha)$ in $W^{1,2}(M) \times L^4(M)$, then by (5.15) and (5.16), we know there exists a subsequence of $(\phi_\alpha, \psi_\alpha)$ (not changing notation) such that

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \text{ in } C^2_{loc}(M \setminus \mathcal{S}) \times C^1_{loc}(M \setminus \mathcal{S}),$$

where

$$(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0).$$

By the removable singularity theorem of Dirac-harmonic maps (see Theorem 4.6 in [9] for the interior singularity case and see Theorem 1.4 and Theorem 1.5 in [26] for the boundary singularity case), we have $(\phi, \psi) \in C^2(M) \times C^1(M)$. Then, $(\phi, \psi) \in C^{2+\lambda}(M) \times C^{1+\lambda}(M)$ follows from the standard Schauder theory. □

Proof of Theorem 2.4: Take $r_0 > 0$ such that $x_0 \in \mathbf{S}$ is the only energy concentration point in $B^M_{r_0}(x_0)$. By standard blow-up analysis argument for harmonic map type problems, we can assume that, for the sequence $\alpha \searrow 1$, there exist sequences $x_\alpha \to x_0$ and $\lambda_\alpha \to 0$ such that

$$E(\phi_\alpha; B^M_{\lambda_\alpha}(x_\alpha)) = \sup_{\substack{x \in B^M_{r_0}(x_0), r \leq \lambda_\alpha \\ B^M_r(x) \subset B^M_{r_0}(x_0)}} E(\phi_\alpha; B^M_r(x)) = \frac{\epsilon_0}{4}, \tag{5.17}$$

where $\epsilon_0 > 0$ is the constant in Lemma 5.1 and Lemma 5.2.

Step 1: Let $x_0 \in \partial M$ and we prove the statement (2) under the assumption that

$$\limsup_{\alpha \to 1} \frac{dist(x_\alpha, \partial M)}{\lambda_\alpha} = \infty. \tag{5.18}$$

Without loss of generality, we may assume $x_0 = 0 \in D^+_1(0) \subset \mathbb{R}^2$ is the only energy concentration point in $D^+_1(0)$ and

$$g(x) = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2),$$

where $\varphi_0(x)$ is a smooth function satisfying $\varphi_0(0) = 0$.

Setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) = \left(\phi_\alpha(x_\alpha + \lambda_\alpha x),\ \sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right), \tag{5.19}$$

by (5.18) and (5.6), it is easy to see that, for any $R > 0$, $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ lives in $D_R(0) \subset \mathbb{R}^2$ for α close to 1 and satisfies

$$\begin{cases} \Delta \widetilde{u}_\alpha &= -(\alpha - 1)\frac{\nabla |\nabla_{g_\alpha} \widetilde{u}_\alpha|^2 \nabla \widetilde{u}_\alpha}{\lambda_\alpha^2 + |\nabla_{g_\alpha} \widetilde{u}_\alpha|^2} + A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\left(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha); \widetilde{v}_\alpha)\right)}{\alpha(1+\lambda_\alpha^{-2}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \partial\!\!\!/ \widetilde{v}_\alpha &= \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha), \end{cases} \tag{5.20}$$

where $g_\alpha(x) = e^{\varphi_0(x_\alpha + \lambda_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and we used the fact that the second equation, i.e. the equation for the spinor part, is conformally invariant.

Since $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ is a $\lambda_\alpha^{2(\alpha-1)}$-general α-Dirac-harmonic map (with $\sigma_\alpha = \lambda_\alpha^2$), by (5.17) and the small energy regularity Lemma 5.1 (noting that $\rho_\alpha = \lambda_\alpha^{2(\alpha-1)}$ which implies that $\frac{\rho_\alpha}{(\sigma_\alpha)^{\alpha-1}} \equiv 1$), we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and a limit $(\widetilde{\sigma},\widetilde{\xi}) \in W^{2,2}_{loc}(\mathbb{R}^2)\times W^{1,2}_{loc}(\mathbb{R}^2)$, such that $E(\widetilde{\sigma};D_1(0)) = \frac{\epsilon_0}{4}$ and

$$(\widetilde{u}_\alpha(x),\widetilde{v}_\alpha(x)) \to (\widetilde{\sigma},\widetilde{\xi}) \ in\ C^1_{loc}(\mathbb{R}^2)\times C^0_{loc}(\mathbb{R}^2). \tag{5.21}$$

Next, we make the following

Claim 1:

$$1 \le \liminf_{\alpha\searrow 1}\lambda_\alpha^{2(1-\alpha)} \le \limsup_{\alpha\searrow 1}\lambda_\alpha^{2(1-\alpha)} \le \mu_{max} < \infty. \tag{5.22}$$

To show this claim, we just need to prove that

$$\limsup_{\alpha\searrow 1}\lambda_\alpha^{2(1-\alpha)} < \infty.$$

In fact, if it does not hold, then there exists a subsequence $\alpha_j \to 1$ such that

$$\lim_{j\to\infty}\lambda_{\alpha_j}^{2(1-\alpha_j)} := \mu_1 = \infty.$$

By (5.20) and (5.21), it is easy to see that $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a harmonic map such that $\widetilde{u}_{\alpha_j}\to\widetilde{\sigma}$ in $C^1_{loc}(\mathbb{R}^2)$ as $j\to\infty$. Then we have

$$\begin{aligned}2\Lambda \ge \lim_{R\to\infty}\lim_{j\to\infty}\int_{D_{\lambda_{\alpha_j}R}(x_{\alpha_j})}|\nabla_{g_{\alpha_j}}\phi_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}} &= \lim_{R\to\infty}\lim_{j\to\infty}(\lambda_{\alpha_j})^{2-2\alpha_j}\int_{D_R(0)}|\nabla_{g_{\alpha_j}}\widetilde{u}_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}(x_{\alpha_j}+\lambda_{\alpha_j}x)}\\ &= \lim_{R\to\infty}\mu_1\int_{D_R(0)}|\nabla\widetilde{\sigma}|^2dx = 2\mu_1E(\widetilde{\sigma}).\end{aligned}$$

which is a contradiction to the fact that $E(\widetilde{\sigma})\ge\overline{\epsilon}>0$ which follows from the well known energy gap theorem for harmonic spheres, since $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a nontrivial harmonic map with finite energy and hence it can be conformally extended to a harmonic sphere. Thus, **Claim 1** holds true.

Now setting

$$(u_\alpha(x),v_\alpha(x)) := \left(\widetilde{u}_\alpha(x),\lambda_\alpha^{\alpha-1}\widetilde{v}_\alpha(x)\right) = \left(\phi_\alpha(x_\alpha+\lambda_\alpha x),\lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha+\lambda_\alpha x)\right), \tag{5.23}$$

since the equation for the spinor part is also invariant by multiplying a constant to the spinor, it is easy to see that (u_α,v_α) satisfies

$$\begin{cases}\Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\lambda_\alpha^2+|\nabla_{g_\alpha}u_\alpha|^2} + A(du_\alpha,du_\alpha) + \lambda_\alpha^{2(1-\alpha)}\frac{Re\left(P(\mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha);v_\alpha)\right)}{\alpha(1+\lambda_\alpha^{-2}|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}},\\ \partial\!\!\!/ v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha).\end{cases} \tag{5.24}$$

It is easy to see that (u_α,v_α) is a general α-Dirac-harmonic map with $\sigma_\alpha=\lambda_\alpha^2>0$. By (5.17),(5.22), (5.24), the small energy regularity result Lemma 5.1, we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and a nontrivial Dirac-harmonic map (σ,ξ) : $\mathbb{R}^2\to N$, such that

$$(u_\alpha(x),v_\alpha(x)) \to (\sigma,\xi)\ in\ C^1_{loc}(\mathbb{R}^2)\times C^0_{loc}(\mathbb{R}^2).$$

Next, we will show that (σ,ξ) has finite energy, i.e.

$$\|\nabla\sigma\|_{L^2(\mathbb{R}^2)} + \|\xi\|_{L^4(\mathbb{R}^2)} \le C < \infty.$$

In fact, for any $R > 0$,

$$\begin{aligned}\|\nabla\sigma\|_{L^2(D_R(0))} + \|\xi\|_{L^4(D_R(0))} &= \lim_{\alpha\searrow 1}(\|\nabla u_\alpha\|_{L^2(D_R(0))} + \|v_\alpha\|_{L^4(D_R(0))})\\ &= \lim_{\alpha\searrow 1}(\|\nabla\phi_\alpha\|_{L^2(D_{\lambda_\alpha R}(x_\alpha))} + \lambda_\alpha^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{\lambda_\alpha R}(x_\alpha))})\\ &\le \lim_{\alpha\searrow 1}(\|\nabla\phi_\alpha\|_{L^2(D_{\lambda_\alpha R}(x_\alpha))} + \|\psi_\alpha\|_{L^4(D_{\lambda_\alpha R}(x_\alpha))}) \le C(\Lambda) < \infty.\end{aligned}$$

Step 2: Let $x_0 \in \partial M$, then

$$\limsup_{\alpha\to 1}\frac{dist(x_\alpha,\partial M)}{\lambda_\alpha} = \infty. \tag{5.25}$$

If not, then there exists a converging subsequence of $\frac{dist(x_\alpha,\partial M)}{\lambda_\alpha}$. Without loss of generality, we may assume

$$\lim_{\alpha\to 1}\frac{dist(x_\alpha,\partial M)}{\lambda_\alpha} = a$$

where $a \ge 0$ is a constant.

Denoting

$$B_\alpha := \left\{x \in \mathbb{R}^2 |\ x_\alpha + \lambda_\alpha x \in D_1^+(0)\right\},$$

then

$$B_\alpha \to \mathbb{R}_a^2 := \left\{(x^1, x^2)|x^2 \ge -a\right\}.$$

Noting that $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ (see (5.19)) lives in B_α and satisfies (5.20) with the boundary data

$$(\widetilde{u}_\alpha(x), \mathbf{B}\widetilde{v}_\alpha(x)) = (\varphi(x_\alpha + \lambda_\alpha x),\ \sqrt{\lambda_\alpha}\mathbf{B}\psi_0(x_\alpha + \lambda_\alpha x)), \quad if\ x_\alpha + \lambda_\alpha x \in \partial^0 D_1^+(0),$$

by (5.17), Lemma 5.1 and Lemma 5.2, we have

$$\|\widetilde{u}_\alpha\|_{W^{2,p}(D_{4R}(0)\cap B_\alpha(0))} + \|\widetilde{v}_\alpha\|_{W^{1,p}(D_{4R}(0)\cap B_\alpha(0))} \le C(p, R, g, \Lambda, N, \|\varphi\|_{C^{2+\lambda}}, \|\mathbf{B}\psi_0\|_{C^{1+\lambda}}) \tag{5.26}$$

for any $D_R(0) \subset \mathbb{R}^2$ and $p > 1$, which implies

$$\|\widetilde{u}_\alpha(x - (0, \frac{d_\alpha}{r_\alpha}))\|_{W^{2,2}(D^+_{3R}(0))} + \|\widetilde{v}_\alpha(x - (0, \frac{d_\alpha}{r_\alpha}))\|_{W^{1,2}(D^+_{3R}(0))} \le C$$

when $\frac{1}{\alpha-1}$, R are large, where $d_\alpha := dist(x_\alpha, \partial^0 D^+)$.

Then there exist a subsequence of $(\widetilde{u}_\alpha, \widetilde{v}_\alpha)$ (still denoted by $(\widetilde{u}_\alpha, \widetilde{v}_\alpha)$) and

$$(\widetilde{u}, \widetilde{v}) \in W^{2,2}_{loc}(\mathbb{R}^{2+}_a) \times W^{1,2}_{loc}(\mathbb{R}^{2+}_a)$$

with the boundary data $(\widetilde{u}, \mathbf{B}\widetilde{v})|_{\partial\mathbb{R}^{2+}_a} = (\varphi(x_0), 0)$ where $\mathbb{R}^{2+}_a := \left\{(x^1, x^2)|x^2 > -a\right\}$, such that for any $R > 0$,

$$\lim_{\alpha\to 1}\|\widetilde{u}_\alpha(x - (0, \frac{d_\alpha}{r_\alpha})) - \widetilde{u}(x)\|_{W^{1,2}(D^+_{3R}(0))} = 0,\ \lim_{\alpha\to 1}\|\widetilde{v}_\alpha(x - (0, \frac{d_\alpha}{r_\alpha})) - \widetilde{v}(x)\|_{L^4(D^+_{3R}(0))} = 0.$$

We set $\widetilde{\sigma}(x) := \widetilde{u}(x + (0, a))$ and $\widetilde{\xi}(x) := \widetilde{v}(x + (0, a))$ and then conclude that, for any $R > 0$,

$$\lim_{\alpha\to 1}\|\widetilde{u}_\alpha(x) - \widetilde{\sigma}(x)\|_{W^{1,2}(D_{2R}(0)\cap B_\alpha\cap\mathbb{R}^2_a)} = 0,\ \lim_{\alpha\to 1}\|\widetilde{v}_\alpha(x) - \widetilde{\xi}(x)\|_{L^4(D_{2R}(0)\cap B_\alpha\cap\mathbb{R}^2_a)} = 0.$$

Combining this with (5.26) and noting that the measures of $D_{2R}(0) \cap B_\alpha \setminus \mathbb{R}^2_a$ and $D_{2R}(0) \cap \mathbb{R}^2_a \setminus B_\alpha$ go to zero, we have

$$\lim_{\alpha\to 1} \|\nabla\widetilde{u}_\alpha(x)\|_{L^2(D_R(0)\cap B_\alpha)} = \|\nabla\widetilde{\sigma}(x)\|_{L^2(D_R(0)\cap\mathbb{R}^2_a)}, \ \lim_{\alpha\to 1} \|\widetilde{v}_\alpha(x)\|_{L^4(D_R(0)\cap B_\alpha)} = \|\widetilde{\xi}(x)\|_{L^4(D_R(0)\cap\mathbb{R}^2_a)}. \tag{5.27}$$

By (5.17), we have $E(\widetilde{\sigma}; D_1(0) \cap \mathbb{R}^2_a) = \frac{\epsilon_0}{4}$.

Next, similarly to **Claim 1** in **Step 1**, we make the following

Claim 2:

$$1 \le \liminf_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \le \limsup_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \le \mu_{max} < \infty. \tag{5.28}$$

In fact, if it is not true, then there exists a subsequence $\alpha_j \to 1$ such that

$$\lim_{j\to\infty} \lambda_{\alpha_j}^{2(1-\alpha_j)} \to \infty.$$

In view of the equation (5.6), it follows from the above fact that $(\widetilde{u}_{\alpha_j}, \widetilde{v}_{\alpha_j}) \rightharpoonup (\widetilde{\sigma}, \widetilde{\xi})$ weakly in $W^{2,2}_{loc}(\mathbb{R}^{2+}_a) \times W^{1,2}_{loc}(\mathbb{R}^{2+}_a)$ as $j \to \infty$ and $\widetilde{\sigma} : \mathbb{R}^{2+}_a \to N$ is a harmonic map with boundary data $\widetilde{\sigma}|_{\partial\mathbb{R}^{2+}_a} = \varphi(x_0)$. By a well known result of Lemaire [31], we have that $\widetilde{\sigma}$ is a constant map, which is a contradiction to the fact that $E(\widetilde{\sigma}; D_1(0) \cap \mathbb{R}^2_a) = \frac{\epsilon_0}{4}$. Thus, **Claim 2** holds.

Then we know (u_α, v_α) (see (5.23)) is a general α-Dirac-harmonic map. By Lemma 5.2 and above arguments, there exist a subsequence of $\{\alpha\}$ (still denoted by itself) and a Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^{2+}_a \to N$ with the boundary data $(\sigma, \mathbf{B}\xi)|_{\partial\mathbb{R}^{2+}_a} = (\varphi(x_0), 0)$, such that

$$\lim_{\alpha\to 1} \|\nabla u_\alpha(x)\|_{L^2(D_R(0)\cap B_\alpha)} = \|\nabla\sigma(x)\|_{L^2(D_R(0)\cap\mathbb{R}^2_a)}, \ \lim_{\alpha\to 1} \|v_\alpha(x)\|_{L^4(D_R(0)\cap B_\alpha)} = \|\xi(x)\|_{L^4(D_R(0)\cap\mathbb{R}^2_a)}$$

for any $R > 0$, which implies $E(\sigma; D_1(0) \cap \mathbb{R}^2_a) = \frac{\epsilon_0}{4}$ according to (5.17). However, by Theorem 1.4 in [26], we know σ is a constant map and $\xi \equiv 0$. This is a contradiction and hence the statement (2) holds.

For the first statement (1), i.e., the case of $x_0 \in M \setminus \partial M$, the argument is almost the same as in **Step 1**, so we omit it. The proof of the theorem is finished. □

Proof of Theorem 2.5. By Theorem 2.1, we know there exists a sequence of α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^*_\alpha TN)$ for $\alpha \searrow 1$ with the Dirichlet-chiral boundary condition

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

satisfying

$$E_\alpha(\phi_\alpha) \le E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{5.29}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \le C(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{5.30}$$

for any $1 < p < 2$. All ϕ_α are in the homotopy class of ϕ_0.

Now, we claim that if the target manifold N does not admit any harmonic sphere, then the energy concentration set $\mathbf{S}$ defined in Theorem 2.2 is empty.

In fact, if not, taking a point $x_0 \in \mathbf{S}$, then by Theorem 2.4, there exist sequences $x_\alpha \to x_0$, $\lambda_\alpha \to 0$ and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that

$$(\phi_\alpha(x_\alpha + \lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)) \to (\sigma, \xi) \text{ in } C^2_{loc}(\mathbb{R}^2),$$

as $\alpha \to 1$. Choose any $4 < q < \infty$, taking $p = \frac{2q}{2+q} \in (\frac{4}{3}, 2)$ in (5.30), then we have

$$\|\psi_\alpha\|_{L^q(M)} \leq C(q, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{5.31}$$

and for any $R > 0$,

$$\|\xi\|_{L^4(D_R(0))} = \lim_{\alpha\to 1} \lambda_\alpha^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{R\lambda_\alpha}(x_\alpha))} \leq \lim_{\alpha\to 1} C\|\psi_\alpha\|_{L^q(M)}(R\lambda_\alpha)^{2(\frac{1}{4}-\frac{1}{q})} = 0.$$

Thus, $\xi \equiv 0$ and the Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$ is just a nontrivial harmonic map $\sigma : \mathbb{R}^2 \to N$ with finite energy, which can be extended to a nontrivial smooth harmonic sphere. This is a contradiction and hence $\mathbf{S}$ must be empty.

By Theorem 2.2, we have

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \text{ in } C^2(M) \times C^1(M), \quad \text{as } \alpha \to 1,$$

where $(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^* TN)$ is a Dirac-harmonic map with Dirichlet-chiral boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0).$$

Moreover, it is easy to see that (ϕ, ψ) is in the same homotopy class as ϕ_0. We have finished the proof. □

6. Pohozaev type identity and three-circle type method

In this section, we shall first prove several basic lemmas for general α-Dirac-harmonic maps, for instance, the energy gap theorem and a new Pohozaev type identity. Secondly, we shall establish a new three-circle type method for general α-Dirac-harmonic map system.

First of all, as an application of Lemma 5.1, we have

Lemma 6.1. *Let $D_1(0) \subset \mathbb{R}^2$ be the unit disk. Let $g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2)$ and $g = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2)$ be a family of metrics on $D_1(0)$, where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Let $(\phi_\alpha, \psi_\alpha) \in C^\infty(D_1(0), N)$ be a sequence of general α-Dirac-harmonic maps with uniformly bounded energy $E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha) \leq \Lambda$ and with $0 < \beta_0 \leq \lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \leq 1$, then there exists a positive number $\beta_1 > 0$ independent of α, such that*

$$\beta_0 \leq \liminf_{\alpha\searrow 1} \|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq \limsup_{\alpha\searrow 1} \|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq \beta_1.$$

Proof. Without loss of generality, we assume 0 is the only energy concentration point for the sequence $\{(\phi_\alpha, \psi_\alpha)\}$. Then by the blow-up process described in Section 2, we can get at most finitely many bubbles at this blow-up point, i.e. there exist a positive sequence $\lambda^i_\alpha \to 0$ and a sequence of points $x^i_\alpha \to 0$, $i = 1, ..., I$, as $\alpha \searrow 1$, which satisfy (A1) or (A2). Also, without loss of generality, we assume λ^1_α is the smallest one, i.e. $\limsup_{\alpha\searrow 1} \frac{\lambda^1_\alpha}{\lambda^j_\alpha} \leq C$ for $j = 2, ..., I$. By a standard blow-up argument (see Sec. 5), we know $(\phi_\alpha(x_0+\lambda_\alpha^1 x), (\lambda_\alpha^1)^{\alpha-1}\sqrt{\lambda_\alpha^1}\psi_\alpha(x_0+\lambda_\alpha^1 x))$ has no energy concentration points for all points $x_0 \in D_{\frac{1}{2}}$. By Lemma 5.1, we have

$$|\nabla\phi_\alpha|(x) \leq C, \quad \forall\, x \in D_1(0)\setminus D_{\frac{1}{2}}(0),$$

since 0 is the only energy concentration point and

$$|\nabla\phi_\alpha|(x) \leq \frac{C}{(\lambda_\alpha)^{\alpha-1}\lambda_\alpha^1} \leq \frac{C}{\lambda_\alpha^1}, \quad \forall\, x \in D_{\frac{1}{2}}(0),$$

where we used the fact that $1 \leq \liminf_{\alpha\searrow 1}(\lambda_\alpha)^{1-\alpha} \leq \limsup_{\alpha\searrow 1}(\lambda_\alpha)^{1-\alpha} \leq \sqrt{\frac{\Lambda}{\epsilon_4}}$ which follows from (2.24). Thus, we obtain

$$\|(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\|_{C^0(D)} \leq C(1+(\lambda_\alpha^1)^{2-2\alpha}) \leq C(1+\mu_{max}).$$

Then the conclusion of the lemma follows immediately from (2.24). □

The following energy gap result is a small improvement of the one given in Theorem 3.1. in [8].

Lemma 6.2 (Energy gap)**.** *There exists an $\epsilon_4 = \epsilon_4(N) > 0$ such that, if $(\phi,\psi): S^2 \to N$ is a smooth Dirac-harmonic map satisfying*

$$\int_{S^2} |d\phi|^2 < \epsilon_4.$$

Then both ϕ and ψ are trivial.

Proof. **Step1.** Claim: $\|\psi\|_{L^{4/3}(S^2)} \leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)}$, where ψ is a spinor on S^2 and $C>0$ is a universal constant.

In fact, if not, then there exists a sequence of spinors $\{\psi_k\}$ on S^2 such that

$$\|\psi_k\|_{L^{4/3}(S^2)} > k\,\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)}.$$

Without loss of generality, we assume $\|\psi_k\|_{L^{4/3}(S^2)} = 1$, then we have

$$\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)} < \frac{1}{k}. \tag{6.1}$$

By standard elliptic estimates, we get

$$\|\psi_k\|_{W^{1,4/3}(S^2)} \leq C.$$

Thus, there exists a subsequence of $\{\psi_k\}$ (we still denote it by $\{\psi_k\}$) and $\eta \in W^{1,4/3}(S^2)$ satisfying

$$\psi_k \to \eta \quad \textit{weakly in} \quad W^{1,4/3}(S^2) \quad \textit{and strongly in} \quad L^{4/3}(S^2). \tag{6.2}$$

Combining this fact with $\|\psi_k\|_{L^{4/3}(S^2)} = 1$ and the inequality (6.1), we get $\|\eta\|_{L^{4/3}(S^2)} = 1$ and

$$\|\partial\!\!\!/\eta\|_{L^{4/3}(S^2)} = 0. \tag{6.3}$$

It follows that $\eta \equiv 0$, since there is no nontrivial harmonic spinor on S^2. This is a contradiction.

Step2. By standard elliptic estimates, we have

$$\begin{aligned}\|\psi\|_{L^4(S^2)} &\leq C\|\psi\|_{W^{1,4/3}(S^2)}\\ &\leq C(\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} + \|\psi\|_{L^{4/3}(S^2)})\\ &\leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} \leq C\||d\phi||\psi|\|_{L^{4/3}(S^2)} \leq C\|d\phi\|_{L^2(S^2)}\|\psi\|_{L^4(S^2)} \leq C\sqrt{\epsilon_4}\|\psi\|_{L^4(S^2)}.\end{aligned}$$

Taking $\epsilon_4 > 0$ sufficiently small, then we have $\psi \equiv 0$. Therefore,

$$\begin{aligned}\|d\phi\|_{W^{1,4/3}(S^2)} &\leq C\|\Delta\phi\|_{L^{4/3}(S^2)}\\ &\leq C\||d\phi|^2\|_{L^{4/3}(S^2)}\\ &\leq C\|d\phi\|_{L^2(S^2)}\|d\phi\|_{L^4(S^2)} \leq C\|d\phi\|_{L^2(S^2)}\|d\phi\|_{W^{1,4/3}(S^2)} \leq C\sqrt{\epsilon_4}\|d\phi\|_{W^{1,4/3}(S^2)}.\end{aligned}$$

Thus ϕ has to be a constant map. □

Now we establish a new Pohozaev type identity for α-Dirac-harmonic maps.

Lemma 6.3. *Let (D, g_α) be the unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2 + (dx^2)^2\right)$, where $\varphi_\alpha \in C^\infty(D)$. If $(\phi_\alpha, \psi_\alpha)$ is a critical point of $L_{\alpha,\sigma_\alpha}(\phi, \psi)$, then for any $0 < t < 1$, the following Pohozaev type identity holds*

$$\begin{aligned}&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2 - \frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2\\ &= (1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx\\ &\quad + \frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha} + \frac{1}{2\alpha}\int_{\partial D_t}\langle\psi_\alpha, r^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle\\ &\quad + (1-\frac{1}{\alpha})\frac{1}{2t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 r\frac{\partial\varphi_\alpha}{\partial r}dx\\ &\quad - \frac{\sigma_\alpha}{\alpha t}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx.\end{aligned}\tag{6.4}$$

Here, $dx = dx^1dx^2$.

Proof. Multiplying (2.21) by $r\frac{\partial\phi_\alpha}{\partial r}$, we have

$$0 = \int_{D_t} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\} r\frac{\partial\phi_\alpha}{\partial r}dx - \int_{D_t}\left\langle\frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right), r\frac{\partial\phi_\alpha}{\partial r}\right\rangle dx.\tag{6.5}$$

On one hand, integrating by parts, by using the fact that $r\frac{\partial}{\partial r} = \sum_{\beta=1}^2 x^\beta\frac{\partial}{\partial x^\beta}$, we have

$$\begin{aligned}&\int_{D_t} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\} r\frac{\partial\phi_\alpha}{\partial r}dx\\ &= \int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}r|\frac{\partial\phi_\alpha}{\partial r}|^2 - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\nabla(x^\beta\frac{\partial\phi_\alpha}{\partial x^\beta})dx\\ &= \int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}r|\frac{\partial\phi_\alpha}{\partial r}|^2 - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx\\ &\quad - \int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla\phi_\alpha|^2 dx.\end{aligned}\tag{6.6}$$

Noting that

$$|\nabla\phi_\alpha|^2 = e^{\varphi_\alpha}|\nabla_{g_\alpha}\phi_\alpha|^2,$$

we get

$$
\begin{aligned}
&-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla\phi_\alpha|^2dx\\
&=-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla_{g_\alpha}\phi_\alpha|^2e^{\varphi_\alpha}dx-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&=-\frac{1}{2\alpha}\int_{D_t}r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx. \qquad (6.7)
\end{aligned}
$$

Integrating by parts yields that

$$
\begin{aligned}
&-\frac{1}{2\alpha}\int_{D_t}r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx\\
&=-\frac{1}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}+\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}div\{xe^{\varphi_\alpha}\}dx\\
&=-\frac{1}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2-\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}\\
&\quad+\frac{1}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx+\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx. \qquad (6.8)
\end{aligned}
$$

By (6.6), (6.7) and (6.8), we obtain

$$
\begin{aligned}
&\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}r|\frac{\partial\phi_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\\
&\quad-\int_{D_t}div\left\{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\}r\frac{\partial\phi_\alpha}{\partial r}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx-\frac{1}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx\\
&\quad+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}dx+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&\quad-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}r(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx
\end{aligned}
$$

which implies

$$\begin{aligned}
&\int_{D_t} div\left\{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}\nabla\phi_\alpha\right\} r\frac{\partial\phi_\alpha}{\partial r}dx\\
&= \int_{\partial D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} r|\frac{\partial\phi_\alpha}{\partial r}|^2 - \frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\\
&\quad - \frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha} - (1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx\\
&\quad + \frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx - (1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx.
\end{aligned}\tag{6.9}$$

On the other hand, according to Proposition 2.2 in [27] that

$$\langle\psi, \widetilde{\nabla}_X(\not{D}\psi) - \not{D}(\widetilde{\nabla}_X\psi)\rangle = 2\left\langle Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)\right), \nabla_X\phi\right\rangle$$

whenever $[X, e_\gamma] = 0,\ \gamma = 1, 2$, where $[\cdot,\cdot]$ is the Lie bracket and a well-known fact that

$$\int_M\langle\psi, \not{D}\omega\rangle = \int_M\langle\not{D}\psi, \omega\rangle - \int_{\partial M}\langle\overrightarrow{n}\cdot\psi, \omega\rangle$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, using the equation $\not{D}\psi_\alpha = 0$, we get

$$\begin{aligned}
&-\int_{D_t}\left\langle Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right), r\frac{\partial\phi_\alpha}{\partial r}\right\rangle dx\\
&= \frac{1}{2}\int_{D_t}\langle x^\beta\psi_\alpha, \not{D}\widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle dx\\
&= \frac{1}{2}\int_{D_t}\langle\not{D}(x^\beta\psi_\alpha), \widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle dx - \frac{1}{2}\int_{\partial D_t}\langle\overrightarrow{n}\cdot x^\beta\psi_\alpha, \widetilde{\nabla}_{\frac{\partial}{\partial\beta}}\psi_\alpha\rangle dx\\
&= -\frac{1}{2}\int_{D_t}\langle\psi_\alpha, \not{D}\psi_\alpha\rangle dx + \frac{1}{2}\int_{D_t}\langle\not{D}\psi_\alpha, r\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle dx + \frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r\frac{\partial}{\partial r}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle\\
&= \frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r\frac{\partial}{\partial r}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial r}}\psi_\alpha\rangle = -\frac{1}{2}\int_{\partial D_t}\langle\psi_\alpha, r^{-1}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle,
\end{aligned}\tag{6.10}$$

Then the conclusion of the lemma follows immediately from (6.9) and (6.10). □

As a direct corollary of Lemma 6.1 and Lemma 6.3, we have the following Pohozaev type estimate for α-Dirac-harmonic maps.

Corollary 6.4. *Let (D, g_α) be the unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2 + (dx^2)^2\right)$, where $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0 \in C^\infty(D)$ smoothly. If $(\phi_\alpha, \psi_\alpha)$ is a critical point of $L_{\alpha,\sigma_\alpha}(\phi, \psi)$, where $0 < \beta_0 < \lim_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1} \le 1$ and $E_{\alpha,\sigma_\alpha}(\phi_\alpha) \le \Lambda$, then for any $0 < t < 1$, we have the following*

Pohozaev type estimate

$$
\begin{aligned}
&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2\\
&=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\quad+\frac{1}{2\alpha}\int_{\partial D_t}\langle\psi_\alpha,r^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle+O(t)+O(\alpha-1).
\end{aligned}
\tag{6.11}
$$

In the end of this section, we shall develop the three-circle type method for general α-Dirac-harmonic type system. Let us first state the three-circle theorem for harmonic functions (see e.g. [59, 50]).

Theorem 6.5. *There exists a constant $L>0$, such that if u is a nontrivial smooth harmonic function defined in $[(i-1)L,(i+2)L]\times S^1$ that satisfies*

$$\int_{\{iL\}\times S^1}ud\theta=\int_{\{(i+1)L\}\times S^1}ud\theta=0,$$

then

$$\|u\|^2_{L^2([iL,(i+1)L]\times S^1)}<\frac{1}{2}\left(e^{-L}\|u\|^2_{L^2([(i-1)L,iL]\times S^1)}+e^{-L}\|u\|^2_{L^2([(i+1)L,(i+2)L]\times S^1)}\right).\tag{6.12}$$

Next, we shall establish the three-circle type method for a class of integro-differential equations. It is based on a series of lemmas. Such results can not be applied directly to general α-Dirac-harmonic map type systems to derive the decay estimates, in fact, as we will see in Section 8, it requires a lot of effort to do so.

We start with an L^2 interior estimate for a class of integro-differential equations.

Lemma 6.6. *Suppose $u\in W^{2,2}(D_4\setminus D_1)$ and $v\in W^{1,2}(D_4\setminus D_1)$ satisfy the following system of integro-differential equations*

$$\Delta u = A^1u+A^2\nabla u+A^3v+\frac{1}{2\pi}\int_0^{2\pi}A^4u+A^5\nabla u+A^6vd\theta+f_1,\tag{6.13}$$

$$\partial\!\!\!/ v = B^1u+B^2\nabla u+B^3v+\frac{1}{2\pi}\int_0^{2\pi}B^4u+B^5\nabla u+B^6vd\theta+f_2,\tag{6.14}$$

where

$$\sum_{i=1}^{6}(\|A^i\|_{L^\infty(D_4\setminus D_1)}+\|B^i\|_{L^\infty(D_4\setminus D_1)})\le\Lambda \text{ and } \sum_{i=1}^{2}\|f_i\|_{L^2(D_4\setminus D_1)}\le\Lambda.\tag{6.15}$$

Then there holds

$$\|u\|_{W^{2,2}(D_3\setminus D_2)}+\|v\|_{W^{1,2}(D_3\setminus D_2)}\le C(\Lambda)\left(\|u\|_{L^2(D_4\setminus D_1)}+\|v\|_{L^2(D_4\setminus D_1)}+\sum_{i=1}^{2}\|f_i\|_{L^2(D_4\setminus D_1)}\right).\tag{6.16}$$

Proof. The proof is similar to Lemma 3.2 in [27]. Here we present a sketch of proof and refer the details to [27].

Denote $B_\sigma = D_{3+\sigma} \setminus D_{2-\sigma}$, $0<\sigma<1$. Let $\sigma' = \frac{\sigma+1}{2}$. Take a cut-off function $\eta(x)=\eta(|x|)$ with compact support in $B_{\sigma'}$ satisfying $\eta(x)\equiv 1$ in B_σ and $|\nabla\eta| \le \frac{4}{(1-\sigma)}$ and $|\Delta\eta| \le \frac{16}{(1-\sigma)^2}$. Computing directly, we get

$$\begin{aligned}\Delta(\eta u) &= \eta\Delta u + 2\nabla\eta\nabla u + \Delta\eta u\\ &= (2\nabla\eta + \eta A^2)\nabla u + (\Delta\eta + \eta A^1)u + \eta A^3 v + \eta f_1 + \eta\cdot\frac{1}{2\pi}\int_0^{2\pi} A^4 u + A^5\nabla u + A^6 v d\theta.\end{aligned}$$

By applying standard elliptic estimate and following the calculations as in Lemma 3.2 in [27], we have

$$\|\eta u\|_{W^{2,2}(D_4)} \le C\left(\frac{\|\nabla u\|_{L^2(B_{\sigma'})}}{1-\sigma} + \frac{\|u\|_{L^2(B_{\sigma'})} + \|v\|_{L^2(B_{\sigma'})}}{(1-\sigma)^2} + \|\eta f_1\|_{L^2(D_4)}\right). \tag{6.17}$$

We now introduce the seminorms as in [27] and define for $j=0,1,2$

$$\Xi_j = \sup_{0\le\sigma\le1}(1-\sigma)^j\|D^j u\|_{L^2(B_\sigma)}.$$

Multiplying (6.17) by $(1-\sigma)^2$ and noting that $1-\sigma' = \frac{1-\sigma}{2}$, we have

$$\Xi_2 \le C\left(\Xi_1 + \Xi_0 + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right). \tag{6.18}$$

Since Ξ_j satisfy an interpolation inequality

$$\Xi_1 \le \epsilon\Xi_2 + \frac{C}{\epsilon}\Xi_0 \tag{6.19}$$

for any $\epsilon>0$, where $C>0$ is a universal constant. Using (6.19) in (6.18), we get

$$\Xi_2 \le C\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right),$$

this is

$$\|D^2u\|_{L^2(B_\sigma)} \le \frac{C}{(1-\sigma)^2}\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right).$$

Taking $\sigma=\frac12$, it follows

$$\|u\|_{W^{2,2}(B_{1/2})} \le C\left(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \|f_1\|_{L^2(D_4\setminus D_1)}\right). \tag{6.20}$$

Similarly, we can compute

$$\partial\!\!\!/(\eta v) = \eta B^1 u + \eta B^2\nabla u + (\eta B^3 + \nabla\eta)v + \eta\frac{1}{2\pi}\int_0^{2\pi} B^4 u + B^5\nabla u + B^6 v d\theta + \eta f_2.$$

By elliptic estimate for the Dirac operator and choosing a new cut-off function η suitably, we have

$$\begin{aligned}\|v\|_{W^{1,2}(B_{\frac14})} &\le C(\|\nabla u\|_{L^2(B_{\frac12})} + \|u\|_{L^2(B_{\frac12})} + \|v\|_{L^2(B_{\frac12})} + \|f_2\|_{L^2(B_{\frac12})})\\ &\le C(\|u\|_{L^2(D_4\setminus D_1)} + \|v\|_{L^2(D_4\setminus D_1)} + \sum_{i=1}^2\|f_i\|_{L^2(D_4\setminus D_1)}).\end{aligned}$$

Then it is easy to see that the conclusion of the lemma follows immediately. □

Denote $P_i := D_{e^{(i+1)L}r_2} \setminus D_{e^{iL}r_2}$ and

$$F_i(u,v) := \int_{P_i} \frac{1}{|x|^2}|u|^2 dx + \int_{P_i} \frac{1}{|x|}|v|^2 dx,$$

where $L > 0$ is the constant in Theorem 6.5.

Then, we have the following three-circle type theorem for the class of integro-differential equations considered in Lemma 6.6.

Theorem 6.7. *Suppose* $u \in W^{2,2}(P_{i-1} \cup P_i \cup P_{i+1})$, $v \in W^{1,2}(P_{i-1} \cup P_i \cup P_{i+1})$ *satisfy equations* (6.13) *and* (6.14)*. Then there exists a positive constant* $\delta_0 > 0$*, such that if*

$$\max_{i-1,i,i+1}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}) \le \delta_0 F_i(u,v), \tag{6.21}$$

and

$$\begin{aligned} &|x|^2(|A^1| + |A^4|) + |x|^{\frac{3}{2}}(|A^3| + |A^6| + |B^1| + |B^4|) \\ &+ |x|(|A^2| + |A^5| + |B^3| + |B^6|) + |x|^{\frac{1}{2}}(|B^2| + |B^5|) \le \delta_0, \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} &|\int_0^{2\pi} u(e^{iL}r_2,\theta)d\theta|^2 + |\int_0^{2\pi} u(e^{(i+1)L}r_2,\theta)d\theta|^2 \\ &+ e^{iL}r_2|\int_0^{2\pi} v(e^{iL}r_2,\theta)d\theta|^2 + e^{(i+1)L}r_2|\int_0^{2\pi} v(e^{(i+1)L}r_2,\theta)d\theta|^2 \le \delta_0 F_i(u,v), \end{aligned} \tag{6.23}$$

then, there hold

(a) $F_{i+1}(u,v) \le e^{-L}F_i(u,v)$ *implies* $F_i(u,v) \le e^{-L}F_{i-1}(u,v)$*;*

(b) $F_{i-1}(u,v) \le e^{-L}F_i(u,v)$ *implies* $F_i(u,v) \le e^{-L}F_{i+1}(u,v)$*;*

(c) *either* $F_i(u,v) \le e^{-L}F_{i-1}(u,v)$ *or* $F_i(u,v) \le e^{-L}F_{i+1}(u,v)$*.*

Proof. Note that the condition (6.22) is in fact stronger than (3.12) in [27], so the conclusions of theorem follow immediately from Theorem 3.3 in [27]. □

As a direct application of Theorem 6.7, we can get the following decay lemma

Lemma 6.8. *Let* $\delta_0 > 0$ *be the constant in Theorem 6.7. Let* $u \in W^{2,2}(D_{e^{(l+1)L}r_2} \setminus D_{r_2})$ *and* $v \in W^{1,2}(D_{e^{(l+1)L}r_2} \setminus D_{r_2})$*, for some integer* $l > 1$*, satisfy the system of integro-differential equations* (6.13) - (6.14) *and assume that the followings hold*

$$\begin{aligned} &|x|^2(|A^1| + |A^4|) + |x|^{\frac{3}{2}}(|A^3| + |A^6| + |B^1| + |B^4|) \\ &+ |x|(|A^2| + |A^5| + |B^3| + |B^6|) + |x|^{\frac{1}{2}}(|B^2| + |B^5|) \le \delta_0, \end{aligned} \tag{6.24}$$

and

$$\int_{\partial D_r} u = \int_{\partial D_r} v = 0.$$

Then for any $1 \le i \le l$, *there holds*

$$F_i(u,v) \le C \max_{j=1,\dots,l}\Big(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}\Big) + C\big(F_0(u,v) + F_l(u,v)\big)(e^{-(l-i)L} + e^{-iL}). \tag{6.25}$$

Proof. Denote the set

$$\mathbb{I} := \Big\{1 \le i \le l-1 \Big| \max_{i-1,i,i+1}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}) \ge \delta_0 F_i(u,v)\Big\}. \tag{6.26}$$

If $i \in \mathbb{I}$ then it is easy to see that (6.25) holds. If $i \notin \mathbb{I}$, by (c) of Theorem 6.7, we have

$$F_i(u,v) \le e^{-L}F_{i-1}(u,v) \text{ or } F_i(u,v) \le e^{-L}F_{i+1}(u,v).$$

If $F_i(u,v) \le e^{-L}F_{i-1}(u,v)$, we shall consider the following two cases:

Case 1-1: if there exists $1 \le i_1 < i$ such that $i_1 \in \mathbb{I}$ and for any $i_1 < j < i$, there holds $j \notin \mathbb{I}$, then by (a) in Theorem 6.7, we will get

$$F_i(u,v) \le e^{-(i-i_1)L}F_{i_1}(u,v) \le C \max_{i_1-1,i_1,i_1+1}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}).$$

Case 1-2: if such constant i_1 does not exist, i.e. $j \notin \mathbb{I},\ j = 2, ..., i$, then by (a) in Theorem 6.7, we will get

$$F_i(u,v) \le e^{-iL}F_0(u,v).$$

Similarly, if $F_i(u,v) \le e^{-L}F_{i+1}(u,v)$, then we consider the following two cases:

Case 2-1: if there exists $i < i_2 < l-1$ such that $i_2 \in \mathbb{I}$ and for any $i < j < i_2$, there holds $j \notin \mathbb{I}$, then by (b) in Theorem 6.7, we will get

$$F_i(u,v) \le e^{-(i_2-i)L}F_{i_2}(u,v) \le C \max_{i_2-1,i_2,i_2+1}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}).$$

Case 2-2: if such constant i_2 does not exist, i.e. $j \notin \mathbb{I},\ j = i, ..., l-1$, then by (b) in Theorem 6.7, we will get

$$F_i(u,v) \le e^{-(l-i)L}F_l(u,v).$$

Thus, we obtain

$$F_i(u,v) \le C \max_{j=1,\dots,l}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}) + Ce^{-iL}F_0(u,v)$$

or

$$F_i(u,v) \le C \max_{j=1,\dots,l}(\||x|f_1\|^2_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|^2_{L^2(P_j)}) + Ce^{-(l-i)L}F_l(u,v).$$

Then the conclusion of the lemma follows immediately. □

Finally, we have

Corollary 6.9. *Under the assumptions of Lemma 6.8, there holds*

$$\begin{aligned} &\|\nabla u\|_{L^2(P_i)} + \|\nabla v\|_{L^{\frac{4}{3}}(P_i)} \\ &\le C \max_{j=1,\dots,l}(\||x|f_1\|_{L^2(P_j)} + \||x|^{\frac{1}{2}} f_2\|_{L^2(P_j)}) + C(F_0^{1/2}(u,v) + F_l^{1/2}(u,v))(e^{-\frac{1}{2}(l-i)L} + e^{-\frac{1}{2}iL}). \end{aligned} \tag{6.27}$$

Proof. By the interior estimate in lemma 6.6 and the standard scaling argument, we have

$$\begin{aligned}
&\|\nabla u\|_{L^2(P_i)} + \|\nabla v\|_{L^{\frac{4}{3}}(P_i)} \\
&\leq C\left(F_i^{\frac{1}{2}}(u,v) + F_{i-1}^{\frac{1}{2}}(u,v) + F_{i+1}^{\frac{1}{2}}(u,v) + \||x|f_1\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})} + \||x|^{\frac{1}{2}}f_2\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})}\right) \\
&\leq C\max_{j=1,...,l}(\||x|f_1\|_{L^2(P_j)} + \||x|^{\frac{1}{2}}f_2\|_{L^2(P_j)}) + C(F_0^{1/2}(u,v) + F_l^{1/2}(u,v))(e^{-\frac{1}{2}(l-i)L} + e^{-\frac{1}{2}iL}).
\end{aligned}$$

□

7. Generalized energy identities for α-Driac-harmonic maps

In this section, we shall prove Theorem 2.6.

For simplicity, we first consider the local model case of a single interior blow-up point for a sequence of general α-Dirac harmonic maps.

Theorem 7.1. *Let $D = D_1(0) \subset \mathbb{R}^2$ be the unit disk. Assume that $g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2)$ and $g = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2)$ is a family of metrics on $D_1(0)$, where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Let $(\phi_\alpha, \psi_\alpha) \in C^\infty(D_1(0), N)$ be a sequence of general α-Dirac-harmonic maps satisfying the followings:*

(a) $\sup_\alpha(E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha)) \leq \Lambda$, *and* $0 < \beta_0 \leq \lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \leq 1$,

(b) $(\phi_\alpha, \psi_\alpha) \to (\phi, \psi)$, *strongly in* $C^\infty_{loc}(D \setminus \{0\})$ *as* $\alpha \searrow 1$.

Then there exist a subsequence of $(\phi_\alpha, \psi_\alpha)$ (still denoted by $(\phi_\alpha, \psi_\alpha)$) and a nonnegative integer L_1 such that, for any $i = 1, ..., L_1$, there exist sequences of points x^i_α and positive numbers λ^i_α, and a nonconstant Dirac-harmonic sphere (σ^i, ξ^i) such that:

(1) $x^i_\alpha \to 0$, $\lambda^i_\alpha \to 0$, *as* $\alpha \searrow 1$;

(2) $\lim_{\alpha\searrow 1}\left(\frac{\lambda^i_\alpha}{\lambda^j_\alpha} + \frac{\lambda^j_\alpha}{\lambda^i_\alpha} + \frac{|x^i_\alpha - x^j_\alpha|}{\lambda^i_\alpha + \lambda^j_\alpha}\right) = \infty$, *for any* $i \neq j$;

(3) (σ^i, ξ^i) *is the weak limit of* $\left(\phi_\alpha(x^i_\alpha + \lambda^i_\alpha x), (\lambda^i_\alpha)^{\alpha-1}\sqrt{\lambda^i_\alpha}\psi_\alpha(x^i_\alpha + \lambda^i_\alpha x)\right)$ *in* $W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$.

(4) ***Generalized energy identities:*** *we have*

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E_{\alpha,\sigma_\alpha}(\phi_\alpha, D_\delta(0)) = \sum_{i=1}^{L_1}\mu_i^2 E(\sigma^i), \tag{7.1}$$

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E(\psi_\alpha, D_\delta(0)) = \sum_{i=1}^{L_1}\mu_i^2 E(\xi^i), \tag{7.2}$$

where $\mu_i = \lim_{\alpha\searrow 1}(\lambda^i_\alpha)^{2-2\alpha}$.

Proof of Theorem 7.1. By our assumptions, we know that 0 is the only blow-up point in D for the sequence $\{(\phi_\alpha, \psi_\alpha)\}$, *i.e.*

$$\liminf_{\alpha\searrow 1} E(\phi_\alpha; D_r) \geq \frac{\epsilon_0}{2} \text{ for all } r > 0 \tag{7.3}$$

where $\epsilon_0 > 0$ is the constant in Lemma 5.1. By standard blow-up argument for harmonic map type problems, we can assume that there exist sequences $x_\alpha \to 0$ and $\lambda_\alpha \to 0$ such that

$$E(\phi_\alpha; D_{\lambda_\alpha}(x_\alpha)) = \sup_{\substack{x\in D, r\le\lambda_\alpha\\ D_r(x)\subset D}} E(\phi_\alpha; D_r(x)) = \frac{\epsilon_0}{4}. \tag{7.4}$$

Then, we shall construct the first bubble for the sequence of general α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha)$. The argument is similar to the proof of Theorem 2.4 for the critical points of L_α, since the objects considered in this part are critical points of L_{α,σ_α}. For reader's convenience, we recall this process once again.

Setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) := \left(\phi_\alpha(x_\alpha + \lambda_\alpha x),\ \sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right), \tag{7.5}$$

by (2.20) , it is easy to see that, for any $R > 0$, $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ lives in $D_R(0) \subset \mathbb{R}^2$ for α close to 1 and satisfies

$$\begin{cases} \Delta\widetilde{u}_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2\nabla\widetilde{u}_\alpha}{\sigma_\alpha\lambda_\alpha^2+|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2} + A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\left(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha); \widetilde{v}_\alpha)\right)}{\alpha(\sigma_\alpha+\lambda_\alpha^{-2}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \not{\partial}\widetilde{v}_\alpha &= \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha), \end{cases} \tag{7.6}$$

where $g_\alpha(x) = e^{\varphi_\alpha(x_\alpha+\lambda_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and we have used the fact that the second equation, i.e. the equation for the spinor part, is conformally invariant.

It is easy to see that $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ is a $\lambda_\alpha^{2(\alpha-1)}$-general α-Dirac-harmonic map (by replacing σ_α in the definition with $\sigma_\alpha\lambda_\alpha^2$). Noting that

$$\frac{\lambda_\alpha^{2(\alpha-1)}}{(\sigma_\alpha\lambda_\alpha^2)^{\alpha-1}} = \frac{1}{\sigma_\alpha^{\alpha-1}},$$

by (7.4), the fact $0 < \beta_0 \le \lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \le 1$ and Lemma 5.1, we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and a limit $(\widetilde{\sigma}, \widetilde{\xi}) \in W^{2,2}_{loc}(\mathbb{R}^2) \times W^{1,2}_{loc}(\mathbb{R}^2)$, such that

$$E(\widetilde{\sigma}; D_1(0)) = \frac{\epsilon_0}{4}$$

and

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) \rightharpoonup (\widetilde{\sigma}, \widetilde{\xi}), \quad \textit{weakly in } W^{2,2}_{loc}(\mathbb{R}^2) \times W^{1,2}_{loc}(\mathbb{R}^2), \tag{7.7}$$

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) \to (\widetilde{\sigma}, \widetilde{\xi}), \quad \textit{in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2). \tag{7.8}$$

Next, we prove the following

Claim 1:

$$1 \le \liminf_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \le \limsup_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} \le \mu_{max} < \infty. \tag{7.9}$$

To show this claim, we just need to prove that

$$\limsup_{\alpha\searrow 1} \lambda_\alpha^{2(1-\alpha)} < \infty.$$

In fact, if it does not hold, then there exists a subsequence $\alpha_j \to 1$ such that

$$\lim_{j\to\infty} \lambda_{\alpha_j}^{2(1-\alpha_j)} := \mu = \infty.$$

By (7.6) and (7.7), it is easy to see that $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a harmonic map such that $\widetilde{u}_{\alpha_j}\to\widetilde{\sigma}$ in $C^1_{loc}(\mathbb{R}^2)$ as $j\to\infty$. Then we have

$$\Lambda \geq \lim_{R\to\infty}\lim_{j\to\infty}\int_{D_{\lambda_{\alpha_j}R}(x_{\alpha_j})}|\nabla_{g_{\alpha_j}}\phi_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}} = \lim_{R\to\infty}\lim_{j\to\infty}(\lambda_{\alpha_j})^{2-2\alpha_j}\int_{D_R(0)}|\nabla_{g_{\alpha_j}}\widetilde{u}_{\alpha_j}|^{2\alpha_j}dvol_{g_{\alpha_j}(x_{\alpha_j}+\lambda_{\alpha_j}x)}$$
$$= \lim_{R\to\infty}\mu\int_{D_R(0)}|\nabla\widetilde{\sigma}|^2dx = \mu E(\widetilde{\sigma}).$$

which is a contradiction to the fact that $E(\widetilde{\sigma})\geq\overline{\epsilon}>0$, which follows from the well known energy gap theorem for harmonic spheres, since $\widetilde{\sigma}:\mathbb{R}^2\to N$ is a nontrivial harmonic map with finite energy and hence it can be conformally extended to a harmonic sphere. Thus, **Claim 1**, i.e. (7.9) holds true.

Now setting

$$u_\alpha(x) := \phi_\alpha(x_\alpha+\lambda_\alpha x),\quad v_\alpha(x) := \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha+\lambda_\alpha x),$$

since the equation for the spinor part is also invariant by multiplying a constant to the spinor, it is easy to see that (u_α, v_α) satisfies

$$\begin{cases}\Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha\lambda_\alpha^2+|\nabla_{g_\alpha}u_\alpha|^2} + A(du_\alpha,du_\alpha) + \lambda_\alpha^{2(1-\alpha)}\frac{Re\big(P(\mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha);v_\alpha)\big)}{\alpha(\sigma_\alpha+\lambda_\alpha^{-2}|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}},\\ \not{\partial}v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha).\end{cases}\tag{7.10}$$

It is easy to see that $(u_\alpha(x), v_\alpha(x))$ is a general α-Dirac-harmonic map (by replacing σ_α with $\sigma_\alpha\lambda_\alpha^2$) living in some region which exhausts $\mathbb{R}^2$ as $\alpha\searrow 1$. Moreover, for any $x\in\mathbb{R}^2$, when α is sufficiently close to 1, by (7.4), we have

$$E(u_\alpha;D_1(x))\leq\frac{\epsilon_0}{4}.\tag{7.11}$$

From (7.9), we have

$$\frac{\beta_0}{\mu_{max}}\leq\liminf_{\alpha\searrow1}(\sigma_\alpha\lambda_\alpha^2)^{(\alpha-1)}\leq\limsup_{\alpha\searrow1}(\sigma_\alpha\lambda_\alpha^2)^{(\alpha-1)}\leq 1.$$

According to Lemma 5.1, there exist a subsequence of (u_α, v_α) (we still denote it by (u_α, v_α)) and a finite energy Dirac-harmonic map $(\sigma^1,\xi^1)\in W^{2,2}(\mathbb{R}^2,N)\times W^{1,2}(\mathbb{R}^2,N)$ such that

$$\lim_{\alpha\searrow1}u_\alpha(x)=\sigma^1(x)\quad\text{in } W^{1,2}_{loc}(\mathbb{R}^2)\quad \textit{and}\quad \lim_{\alpha\searrow1}v_\alpha(x)=\xi^1(x)\quad\text{in } L^4_{loc}(\mathbb{R}^2).$$

Besides, by (7.4), there holds $E(\sigma^1;D_1(0))=\frac{\epsilon_0}{4}$. By classical theory of Dirac-harmonic maps, (σ^1,ξ^1) can be extended to a nontrivial finite energy Dirac-harmonic sphere, which is called the first bubble.

Similarly to the blow-up theory of approximate harmonic maps with L^2-uniformly bounded tension fields [17], to prove our theorem, without loss of generality, it is sufficient to consider the case where there is only one bubble occurring at the blow-up point. The case of multiple bubbles occurring can be handled by a standard induction argument as in [17]. See [35] for a more detailed discussion on such a induction argument for the case of α-harmonic maps.

Now, under the “one bubble” assumption, we first make the following:

Claim 2: For any $\epsilon > 0$, there exist $\delta > 0$ and $R > 0$ such that

$$\int_{D_{8t}(x_\alpha)\setminus D_t(x_\alpha)} |\nabla\phi_\alpha|^2 dx + \int_{D_{8t}(x_\alpha)\setminus D_t(x_\alpha)} |\psi_\alpha|^4 dx \le \epsilon^4, \quad \text{for any } t \in (\frac{1}{4}\lambda_\alpha R, 2\delta) \tag{7.12}$$

when $\alpha - 1$ is small enough.

The proof of this claim is now standard and follows from a contradiction argument. In fact, if (7.12) is not true, then we can find $\overline{\epsilon} > 0$, $t_\alpha \to 0$, such that $\lim_{\alpha\searrow 1} \frac{t_\alpha}{\lambda_\alpha} = \infty$ and

$$E(\phi_\alpha, \psi_\alpha; D_{8t_\alpha}(x_\alpha) \setminus D_{t_\alpha}(x_\alpha)) \ge \overline{\epsilon} > 0. \tag{7.13}$$

Setting

$$u'_\alpha(x) := \phi_\alpha(x_\alpha + t_\alpha x), \quad v'_\alpha(x) := t_\alpha^{\alpha-1}\sqrt{t_\alpha}\psi_\alpha(x_\alpha + t_\alpha x),$$

then from the above arguments, we know that (u'_α, v'_α) is also a general α-Dirac-harmonic map. Furthermore, it is easy to see that 0 is an energy concentration point for (u'_α, v'_α) and (σ^1, ξ^1) is also a bubble for (u'_α, v'_α). To construct the second bubble, we have to consider the following two cases:

(a): (u'_α, v'_α) has no other energy concentration points except 0.

By Lemma 5.1, passing to a subsequence, we may assume that (u'_α, v'_α) converges to a Dirac-harmonic map $(\sigma^2, \xi^2) : \mathbb{R}^2 \to N$ strongly in $W^{1,2}_{loc}(\mathbb{R}^2 \setminus \{0\}) \times L^4_{loc}(\mathbb{R}^2 \setminus \{0\})$ as $\alpha \searrow 1$. In particular, we have

$$\begin{aligned} E(\sigma^2, \xi^2; D_8(0) \setminus D_1(0)) &= \lim_{\alpha\searrow 1} E(\phi_\alpha; D_{8t_\alpha}(x_\alpha) \setminus D_{8t_\alpha}(x_\alpha)) + \lim_{\alpha\searrow 1} t_\alpha^{4(\alpha-1)} E(\psi_\alpha; D_{8t_\alpha}(x_\alpha) \setminus D_{8t_\alpha}(x_\alpha)) \\ &\ge \lim_{\alpha\searrow 1} (\lambda_\alpha R)^{4(\alpha-1)} E(\phi_\alpha, \psi_\alpha; D_{8t_\alpha}(x_\alpha) \setminus D_{8t_\alpha}(x_\alpha)) \ge \frac{1}{\mu^2_{max}} \frac{\epsilon_0}{4}. \end{aligned}$$

Here, the last inequality follows from (2.24) where $\mu_{max} := \frac{\Lambda}{\epsilon_4}$.

By classical theory of Dirac-harmonic maps, we know that (σ^2, ξ^2) is a nontrivial finite energy Dirac-harmonic sphere. This is the second bubble, which is a contradiction to the "one bubble" assumption.

(b): (u'_α, v'_α) has another energy concentration point $p \ne 0$.

Without loss of generality, we may assume that p is the only blow-up point in $D_r(p)$ for some small $r > 0$. By standard blow-up argument for harmonic map type problems, there exist $x'_\alpha \to p$ and $\lambda'_\alpha \to 0$ such that

$$E(u'_\alpha; D_{\lambda'_\alpha}(x'_\alpha)) = \sup_{\substack{x\in D_r(p), s\le r_n \\ D_s(x)\subset D_r(p)}} E(u'_\alpha; D_s(x)) = \frac{\epsilon_0}{4}. \tag{7.14}$$

From the process of constructing the first bubble, we know that there exists a nontrivial Dirac-harmonic sphere (σ^2, ξ^2) such that

$$(u'_\alpha(x'_\alpha + \lambda'_\alpha x), (\lambda'_\alpha)^{\alpha-1}\sqrt{\lambda'_\alpha} v'_\alpha(x'_\alpha + \lambda'_\alpha x)) \to (\sigma^2, \xi^2) \text{ in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$$

as $\alpha \searrow 1$. This is

$$\Big(\phi_\alpha(x_\alpha + t_\alpha x'_\alpha + t_\alpha\lambda'_\alpha x), (t_\alpha\lambda'_\alpha)^{\alpha-1}\sqrt{t_\alpha\lambda'_\alpha}\psi_\alpha(x_\alpha + t_\alpha x'_\alpha + t_\alpha\lambda'_\alpha x)\Big) \to (\sigma^2, \xi^2) \text{ in } W^{1,2}_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2)$$

as $\alpha \searrow 1$. By (7.14), (σ^2, ξ^2) is nontrivial. Therefore, we get the second bubble, contradicting the "one bubble" assumption.

So, we proved **Claim 2** and (7.12) holds.

To proceed, we need to establish a series of auxiliary lemmas and we leave the rest of the proof of Theorem 7.1 to the end of this section. □

Without loss of generality, we always assume $\delta = e^{k_\alpha L}\lambda_\alpha R$ where $L > 0$ is the constant in Theorem 6.5 and k_α is an integer which goes to infinity as $\alpha \searrow 1$. For simplicity of notation, we still denote $P_i = D_{e^{(i+1)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{iL}\lambda_\alpha R}(x_\alpha)$.

Firstly, we show the generalized energy identity for the spinor part.

Lemma 7.2. *Under the assumption of Theorem 7.1 and the one bubble assumption, if there is no energy concentration for the sequence $(\phi_\alpha, \psi_\alpha)$ in the region $D_\delta(x_\alpha) \setminus D_{\lambda R}(x_\alpha)$, i.e.* (7.12) *holds* [9], *then we have*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\left(\|\psi_\alpha\|_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))} + \|\nabla\psi_\alpha\|_{L^{\frac{4}{3}}(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\right) = 0.$$

Proof. Firstly we use a finite decomposition argument that is similar to those in [64, 65] to decompose the region $\Sigma := D_\delta(x_\alpha) \setminus D_{\lambda_\alpha R}(x_\alpha)$ into finite parts

$$\Sigma = \cup_{j=1}^{s_\alpha} Q_j, \quad Q_j := \cup_{i=m_{j-1}}^{m_j-1} P_i, \quad 0 = m_0 < m_1 <, ..., < m_{s_\alpha} = k_\alpha$$

such that $s_\alpha \le S$ and

$$E(\phi_\alpha, \psi_\alpha; Q_j) \le \frac{1}{C_1(N)}, \quad j = 1, ..., s_\alpha, \tag{7.15}$$

where $C_1(N) > 0$ is a constant depending only on N to be determined later and S is a uniform integer for all $\alpha - 1$ small enough.

From (7.12), for any $\epsilon << \frac{1}{2C_1(N)}$, we have

$$E(\phi_\alpha, \psi_\alpha; P_i) < C(L)\epsilon < \frac{1}{2C_1(N)}, \quad i = 1, ..., k_\alpha$$

when $\alpha - 1$ is small.

If

$$E(\phi_\alpha, \psi_\alpha; \Sigma) \le \frac{1}{C_1(N)},$$

let $m_1 = k_\alpha$ and then $Q_1 = \Sigma$. Otherwise, we can choose an integer $1 \le m_1 < k_\alpha$ such that

$$\frac{1}{2C_1(N)} < E(\phi_\alpha, \psi_\alpha; Q_1) \le \frac{1}{C_1(N)} \quad \text{and} \quad E(\phi_\alpha, \psi_\alpha; Q_1 \cup P_{m_1}) > \frac{1}{C_1(N)}.$$

This is the first step of the division. Inductively, suppose that m_j is chosen such that

$$E(\phi_\alpha, \psi_\alpha; Q_j) \le \frac{1}{C_1(N)}.$$

[9] We remark that the region $D_\delta(x_\alpha) \setminus D_{\lambda R}(x_\alpha)$ satisfying the property (7.12) is usually called the neck domain.

If

$$E(\phi_\alpha, \psi_\alpha; \cup_{i=m_j}^{k_\alpha} P_i) \leq \frac{1}{C_1(N)},$$

let $m_{j+1} = k_\alpha$, thus $Q_{j+1} = \cup_{i=m_j}^{k_\alpha - 1} P_i$. If not, then similarly to the first step, we can find $m_j < m_{j+1} < k_\alpha$ such that

$$\frac{1}{2C_1(N)} < E(\phi_\alpha, \psi_\alpha; Q_{j+1}) \leq \frac{1}{C_1(N)} \quad \textit{and} \quad E(\phi_\alpha, \psi_\alpha; Q_{j+1} \cup P_{m_{j+1}}) > \frac{1}{C_1(N)}.$$

Since $E(\phi_\alpha, \psi_\alpha)$ is uniformly bounded by Λ, we will finish our division after at most

$$S = [2C_1(N)\Lambda] + 1$$

steps. This gives the finite decomposition.

Take a cut-off function $\eta \in C_0^\infty(D_{e^{m_j L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{(m_{j-1}-1)L}\lambda_\alpha R}(x_\alpha))$ such that $0 \leq \eta \leq 1$ and

$$\eta\Big|_{D_{e^{(m_j-1)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{m_{j-1}L}\lambda_\alpha R}(x_\alpha)} \equiv 1$$

and

$$|\nabla\eta| \leq \frac{C}{e^{m_j L}\lambda_\alpha R} \quad \textit{on} \quad D_{e^{m_j L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{(m_j-1)L}\lambda_\alpha R}(x_\alpha) \quad \textit{and}$$

$$|\nabla\eta| \leq \frac{C}{e^{m_{j-1} L}\lambda_\alpha R} \quad \textit{on} \quad D_{e^{m_{j-1} L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{(m_{j-1}-1)L}\lambda_\alpha R}(x_\alpha).$$

By standard elliptic estimates, we have

$$\begin{aligned}
\|\eta\psi_\alpha\|_{W^{1,4/3}(D_1)} &\leq C\|\eta \partial\!\!\!/ \psi_\alpha + \nabla\eta \cdot \psi_\alpha\|_{L^{\frac{4}{3}}(D_1)} \\
&\leq \frac{1}{4}C(N)\||d\phi_\alpha||\eta\psi_\alpha|\|_{L^{\frac{4}{3}}(D_1)} + C\||\nabla\eta||\psi_\alpha|\|_{L^{\frac{4}{3}}(D_1)} \\
&\leq \frac{1}{4}C(N)\|d\phi_\alpha\|_{L^2(D_{e^{(m_j+2)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{m_{j-1}L}\lambda_\alpha R}(x_\alpha))}\|\eta\psi_\alpha\|_{L^4(D_1)} + C\|\nabla\eta\psi_\alpha\|_{L^{\frac{4}{3}}(P_{m_{j-1}-1} \cup P_{m_j - 1})} \\
&\leq \frac{1}{4}C(N)\frac{2}{\sqrt{C_1(N)}}\|\eta\psi_\alpha\|_{L^4(D_1)} + C\|\nabla\eta\|_{L^2(P_{m_{j-1}-1} \cup P_{m_j-1})}\|\psi_\alpha\|_{L^4(P_{m_{j-1}-1} \cup P_{m_j-1})},
\end{aligned}$$

where the last inequality is from (7.15) and (7.12). Then, taking $C_1(N) = C^2(N) + 1$, by (7.12) and Sobolev embedding, for any $\epsilon > 0$, we have

$$\|\psi_\alpha\|_{L^4(Q_j)} + \|\nabla\psi_\alpha\|_{L^{4/3}(Q_j)} \leq C\|\psi_\alpha\|_{L^4(P_{m_{j-1}-1} \cup P_{m_j-1})} \leq C\epsilon,$$

when $\alpha - 1,\ \delta,\ \frac{1}{R}$ are small enough.

So,

$$\|\psi_\alpha\|_{L^4(\Sigma)} + \|\nabla\psi_\alpha\|_{L^{4/3}(\Sigma)} \leq \sum_{j=1}^{s_\alpha}(\|\psi_\alpha\|_{L^4(Q_j)} + \|\nabla\psi_\alpha\|_{L^{4/3}(Q_j)}) \leq CS\epsilon,$$

which implies the conclusion of the lemma immediately. □

As a corollary of Lemma 7.2, we get the following generalized energy identity for the spinor.

Corollary 7.3. *Under the assumptions of Lemma 7.2, we have*

$$\lim_{\delta \to 0} \lim_{\alpha \searrow 1} E(\psi_\alpha, D_\delta(0)) = \mu^2 E(\xi^1).$$

Proof. By Lemma 7.2, we have

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E(\psi_\alpha, D_\delta(0))\\
&= \lim_{\delta\to 0}\lim_{\alpha\searrow 1}\Big(E(\psi_\alpha, D_\delta(0)\setminus D_{\frac{\delta}{2}}(x_\alpha)) + E(\psi_\alpha, D_{\frac{\delta}{2}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)) + E(\psi_\alpha, D_{\lambda_\alpha R}(x_\alpha))\Big)\\
&= \lim_{\alpha\searrow 1} E(\psi_\alpha, D_{\lambda_\alpha R}(x_\alpha)) = \lim_{\alpha\searrow 1}\lambda_\alpha^{4-4\alpha}\int_{D_R(0)}|v_\alpha|^4 dx = \mu^2 E(\xi^1),
\end{aligned}$$

where $\mu := \lim_{\alpha\searrow 1}(\lambda_\alpha)^{2-2\alpha}$. □

Next, we will show that there is no concentration for some kind of stronger weighted energy of the spinor part on the neck domain. The proof is based on some Hardy-type inequality on $\mathbb{R}^2$.

Lemma 7.4. *Under the assumption of Lemma 7.2, there holds*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\to 1}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0. \tag{7.16}$$

Proof. Without loss of generality, we may assume $x_\alpha = 0$.

The key of the proof is the following Hardy-type inequality on $\mathbb{R}^2$: for any $f\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, there holds

$$\|\frac{f}{|x|}\|_{L^1(\mathbb{R}^2)} \le \|\nabla f\|_{L^1(\mathbb{R}^2)} \tag{7.17}$$

where the constant 1 on the right hand side is the best possible constant (see [3] for a simple proof).

We choose a cut-off function $\eta\in C_0^\infty(D_\delta\setminus D_{\lambda_\alpha R})$ such that $0\le\eta\le 1$ and $\eta\equiv 1$ on $D_{\frac12\delta}\setminus D_{2\lambda_\alpha R}$ and

$$|\nabla\eta|\le\frac{C}{\delta}\quad on\quad D_\delta\setminus D_{\frac12\delta}\quad and\quad |\nabla\eta|\le\frac{C}{\lambda_n R}\quad on\quad D_{2\lambda_\alpha R}\setminus D_{\lambda_\alpha R}.$$

Taking $f=\eta|\psi_\alpha|^2$ in the inequality (7.17), we have

$$\begin{aligned}
\|\eta\frac{|\psi_\alpha|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\le \|\nabla(\eta|\psi_\alpha|^2)\|_{L^1(\mathbb{R}^2)}\\
&\le \|2\eta\psi_\alpha\nabla\psi_\alpha\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(D_\delta\setminus D_{\lambda_\alpha R})}\\
&\le \|\psi_\alpha\|_{L^4(D_\delta\setminus D_{\lambda_\alpha R})}\|\nabla\psi_\alpha\|_{L^{\frac43}(D_\delta\setminus D_{\lambda_\alpha R})} + C\frac{1}{\delta}\||\psi_\alpha|^2\|_{L^1(D_\delta\setminus D_{\frac12\delta})} + C\frac{1}{\lambda_\alpha R}\||\psi_\alpha|^2\|_{L^1(D_{2\lambda_\alpha R}\setminus D_{\lambda_\alpha R})}\\
&\le \|\psi_\alpha\|_{L^4(D_\delta\setminus D_{\lambda_\alpha R})}\|\nabla\psi_\alpha\|_{L^{\frac43}(D_\delta\setminus D_{\lambda_\alpha R})} + C\|\psi_\alpha\|^2_{L^4(D_\delta\setminus D_{\frac12\delta})} + C\|\psi_\alpha\|^2_{L^4(D_{2\lambda_\alpha R}\setminus D_{\lambda_\alpha R})}\\
&\le C\epsilon
\end{aligned}$$

where the last inequality is from (7.12) and Lemma 7.2. Thus,

$$\int_{D_{\frac12\delta}\setminus D_{2\lambda_\alpha R}}\frac{|\psi_\alpha|^2}{|x|}dx\le C\epsilon.$$

Combining this with (7.12) again, we get the conclusion of the lemma. □

As an application of Lemma 5.1, we have the following lemma.

Lemma 7.5. *Under the assumption of Lemma 7.2, for any $\lambda_\alpha R \le t_1 \le t_2 \le \delta$, there hold*

$$|x-x_\alpha||\nabla\phi_\alpha| + |x-x_\alpha|^2|\nabla^2\phi_\alpha| + \sqrt{|x-x_\alpha|}|\psi_\alpha| \le C\epsilon,\ \forall\ x \in D_{t_2}(x_\alpha)\setminus D_{t_1}(x_\alpha) \tag{7.18}$$

and

$$\int_{D_{t_2}(x_\alpha)\setminus D_{t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha - \phi^*_\alpha| dx \le C\int_{D_{4t_2}(x_\alpha)\setminus D_{\frac{1}{2}t_1}(x_\alpha)} |\nabla\phi_\alpha|^2 dx, \tag{7.19}$$

where $\phi^*_\alpha(r) = \frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta$ and $C > 0$ is independent of α.

Proof. For any $t_1 \le t \le t_2$, it is easy to see that $\left(\phi_\alpha(x_\alpha + tx), t^{\alpha-1}\sqrt{t}\psi_\alpha(x_\alpha+tx)\right)$ is a α-Dirac-harmonic map which is defined on $D_2(0)\setminus D_{\frac{1}{4}}(0)$. Under the assumption of Lemma 7.2, (7.12) holds. Then by Lemma 5.1, we have

$$t\|\nabla\phi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))} + t^2\|\nabla^2\phi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))} \le C\|\nabla\phi_\alpha\|_{L^2(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))},$$
$$t^{\alpha-1}\sqrt{t}\|\psi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))} \le Ct^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))}.$$

Noting that

$$\frac{C}{\sqrt{\mu_{max}}} \le (\lambda_\alpha R)^{\alpha-1} \le t^{\alpha-1} \le 1$$

where $\mu_{max} = \frac{\Lambda}{\epsilon_1}$ (see (2.24)), we obtain

$$t\|\nabla\phi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))} + t^2\|\nabla^2\phi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))} + \sqrt{t}\|\psi_\alpha(x_\alpha+tx)\|_{L^\infty(D_1(0)\setminus D_{\frac12}(0))}$$
$$\le C\left(\|\nabla\phi_\alpha\|_{L^2(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))} + \|\psi_\alpha\|_{L^4(D_{2t}(x_\alpha)\setminus D_{\frac{t}{4}}(x_\alpha))}\right),$$

which implies (7.18).

For (7.19), denoting by I_α a positive integer such that

$$2^{I_\alpha-1}t_1 \le t_2 \le 2^{I_\alpha}t_1,$$

then according to Lemma 5.1, we have

$$\begin{aligned}\int_{D_{t_2}(x_\alpha)\setminus D_{t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha-\phi^*_\alpha|dx &\le \sum_{i=1}^{I_\alpha}\int_{D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha)} |\nabla^2\phi_\alpha||\phi_\alpha-\phi^*_\alpha|dx\\ &\le C\sum_{i=1}^{I_\alpha}(2^it_1)^2\|\nabla^2\phi_\alpha\|_{L^\infty(D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha))}2^it_1\|\nabla\phi_\alpha\|_{L^\infty(D_{2^it_1}(x_\alpha)\setminus D_{2^{i-1}t_1}(x_\alpha))}\\ &\le C\sum_{i=1}^{I_\alpha}\int_{D_{2^{i+1}t_1}(x_\alpha)\setminus D_{2^{i-2}t_1}(x_\alpha)}|\nabla\phi_\alpha|^2dx \le C\int_{D_{4t_2}(x_\alpha)\setminus D_{\frac{t_1}{2}}(x_\alpha)}|\nabla\phi_\alpha|^2dx.\end{aligned}$$

□

To estimate the energy of the map part, we first prove that the tangential energy of the map part on the neck domain is converging to zero.

Lemma 7.6. *Under the assumptions of Lemma 7.2, there holds:*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx=0.$$

Proof. Denote

$$\phi_\alpha^*(r)=\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta.$$

Then by Lemma 5.1, we have

$$\begin{aligned}
\|\phi_\alpha-\phi_\alpha^*\|_{L^\infty(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))} &= \sup_{\lambda_\alpha R\le t\le\delta}\|\phi_\alpha-\phi_\alpha^*\|_{D_{2t}(x_\alpha)\setminus D_t(x_\alpha)}\\
&\le \sup_{\lambda_\alpha R\le t\le\delta}\|\phi_\alpha\|_{Osc(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))}\\
&\le C\sup_{\lambda_\alpha R\le t\le\delta}\|\nabla\phi_\alpha\|_{L^2(D_{4t}(x_\alpha)\setminus D_{\frac{1}{2}t}(x_\alpha))}\le C\epsilon,
\end{aligned}$$

where the last inequality follows from (7.12).

Using equation (2.20), we have

$$\begin{aligned}
\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}-\Delta\phi_\alpha(\phi_\alpha-\phi_\alpha^*)dx &= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}\Big\{(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}-A(\phi_\alpha)(d\phi_\alpha,d\phi_\alpha)\\
&\quad-\frac{Re\Big(P(\mathcal{A}(d\phi_\alpha(e_\gamma),e_\gamma\cdot\psi_\alpha);\psi_\alpha)\Big)}{\alpha(\sigma_\alpha+|\nabla_g\phi_\alpha|^2)^{\alpha-1}}\Big\}(\phi_\alpha-\phi_\alpha^*)dx
\end{aligned}$$

On one hand, by integrating by parts, we get

$$\begin{aligned}
left &= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}-\Delta\phi_\alpha(\phi_\alpha-\phi_\alpha^*)dx\\
&= \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(|\nabla\phi_\alpha|^2-\frac{\partial\phi_\alpha}{\partial r}\frac{\partial\phi_\alpha^*}{\partial r})dx-\int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi_\alpha^*)dx\\
&\ge \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(|\nabla\phi_\alpha|^2-|\frac{\partial\phi_\alpha}{\partial r}|^2)dx-\int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))}\frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi_\alpha^*).
\end{aligned}$$

On the other hand, by Lemma 6.1 and Lemma 7.5, there holds

$$\begin{aligned}
right &\le C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(\alpha-1)|\nabla^2\phi_\alpha||\phi_\alpha-\phi_\alpha^*|dx+C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|\nabla\phi_\alpha|^2|\phi_\alpha-\phi_\alpha^*|dx\\
&\quad+C\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|\nabla\phi_\alpha||\psi_\alpha|^2|\phi_\alpha-\phi_\alpha^*|dx\\
&\le C(\alpha-1)\int_{D_{2\delta}(x_\alpha)}|\nabla\phi_\alpha|^2dx+C\epsilon\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}(|\nabla\phi_\alpha|^2+|\psi_\alpha|^4)dx\le C((\alpha-1)+\epsilon).
\end{aligned}$$

Combining these together, we conclude that

$$\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} |x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx = \int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (|\nabla\phi_\alpha|^2 - |\frac{\partial\phi_\alpha}{\partial r}|^2)dx$$
$$\leq \int_{\partial(D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha))} \frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha - \phi_\alpha^*) + C((\alpha-1)+\epsilon).$$

As for the boundary term, by trace theory, we have

$$\int_{\partial D_\delta(x_\alpha)} \frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi_\alpha^*) \leq C\epsilon\int_{\partial D_\delta(x_\alpha)}|\frac{\partial\phi_\alpha}{\partial r}|$$
$$\leq C\epsilon\,(\|\nabla\phi_\alpha\|_{L^2(D_\delta(x_\alpha)\setminus D_{\frac{1}{2}\delta}(x_\alpha))} + \delta\|\nabla^2\phi_\alpha\|_{L^2(D_\delta(x_\alpha)\setminus D_{\frac{1}{2}\delta}(x_\alpha))})$$
$$\leq C\epsilon\,\|\nabla\phi_\alpha\|_{L^2(D_{2\delta}(x_\alpha)\setminus D_{\frac{1}{4}\delta}(x_\alpha))} \leq C\epsilon,$$

where the third inequality follows from Lemma 5.1.

Similarly, there holds

$$\int_{\partial D_{\lambda_\alpha R}(x_\alpha)} \frac{\partial\phi_\alpha}{\partial r}(\phi_\alpha-\phi_\alpha^*) \leq C\epsilon.$$

Then the conclusion of the lemma follows immediately. □

Combining Lemma 7.6 with Lemma 6.1, we get

Lemma 7.7. *Under the assumptions of Lemma 7.2, there holds*

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}\int_{D_\delta(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx = 0.$$

Denote

$$F_\alpha(t) := \int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx,$$

$$F_{r,\alpha}(t) := \int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\frac{\partial\phi_\alpha}{\partial r}|^2dx$$

and

$$F_{\theta,\alpha}(t) := \int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx,\ 0 < t \leq t_0 < 1.$$

By Corollary 6.4, for $t\in[\epsilon, t_0]$, we have

$$(1-\frac{1}{2\alpha})\frac{d}{dt}F_{r,\alpha}(t) - \frac{1}{2\alpha}\frac{d}{dt}F_{\theta,\alpha}$$
$$= \frac{\alpha-1}{\alpha}\log\lambda_\alpha F_\alpha(t) + \frac{1}{2\alpha}\lambda_\alpha^t\log\lambda_\alpha\int_{\partial D_{\lambda_\alpha^t}(x_\alpha)}\langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle + O(\lambda_\alpha^t\log\lambda_\alpha).$$

Then

$$(1-\frac{1}{2\alpha})F_{r,\alpha}(t)-\frac{1}{2\alpha}F_{\theta,\alpha}=\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{1}{\alpha}\log\lambda_\alpha^{2(1-\alpha)}F_\alpha(t)+O(\lambda_\alpha^t\log\lambda_\alpha)\right\}dt \tag{7.20}$$
$$+\frac{1}{2\alpha}\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle.$$

It is easy to check that the function defined by the following integration

$$\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{1}{\alpha}\log\lambda_\alpha^{2(1-\alpha)}F_\alpha(t)+O(\lambda_\alpha^t\log\lambda_\alpha)\right\}dt$$

is compact in $C^0([\epsilon,t_0])$. By Lemma 7.2,

$$\frac{1}{2\alpha}\int_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle\to 0 \ in \ C^0([\epsilon,t_0]).$$

Combining these with the fact that $F_{\theta,\alpha}(t)\to 0$, by Lemma 7.7, we know that the sequences

$$\left\{(1-\frac{1}{2\alpha})F_{r,\alpha}(t)-\frac{1}{2\alpha}F_{\theta,\alpha}\right\},\{F_\alpha(t)\},\{F_{r,\alpha}(t)\}\text{ and }\{F_{\theta,\alpha}(t)\}$$

are compact in the $C^0([\epsilon,t_0])$ topology for any $\epsilon>0$. Thus, there exist two functions F and F_r, belonging to $C^0([\epsilon,t_0])$, such that

$$F_\alpha\to F\quad and\quad F_{r,\alpha}\to F_r,\quad in\ C^0([\epsilon,t_0])$$

as $\alpha\searrow 1$.

Lemma 7.8. *The functionals $F(t)$ and $F_r(t)$ satisfy the following relation:*

$$F_r(t)=\mu^{t_0-t}F(t_0)-F(t_0),\quad\forall\ 0<t\le t_0<1. \tag{7.21}$$

Moreover, we have

$$\lim_{t_0\to 1^-}F(t_0)=\mu E(\sigma^1(x)), \tag{7.22}$$

where $\mu:=\lim_{\alpha\searrow 1}(\lambda_\alpha)^{2-2\alpha}$.

Proof. With the help of Lemma 7.6 and equality (7.20), the proof of this lemma is similar to the case of α-harmonic maps in Lemma 3.4 in [35]. Noting that

$$|\widetilde{\nabla}_{\partial\theta}\psi_\alpha|\le C(|\frac{\partial\psi_\alpha}{\partial\theta}|+|\frac{\partial\phi_\alpha}{\partial\theta}||\psi_\alpha|),$$

we have

$$\begin{aligned}
&\left|\frac{1}{2\alpha}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| \\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\left(|\psi_\alpha||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}| + |\psi_\alpha|^2|x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\right)dx \\
&\le C\|\psi_\alpha\|_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\|\,|x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\,\|_{L^{\frac{4}{3}}(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))} \\
&\quad + C\|\psi_\alpha\|^2_{L^4(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\|\,|x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\,\|_{L^2(D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha))}\to 0
\end{aligned}\tag{7.23}$$

as $\alpha\searrow 1$, $R\to\infty$, $\delta\to 0$, where we used Lemma 7.2.

Letting $\alpha\searrow 1$ in (7.20) and using (7.23), we get

$$F_r(t) = -\log\mu\int_{t_0}^t F(s)ds,\quad \forall 0<t<t_0.\tag{7.24}$$

Since

$$F_\alpha(t) = F_{r,\alpha}(t) + F_{\theta,\alpha}(t) + F_\alpha(t_0),$$

letting $\alpha\searrow 1$, Lemma 7.7 yields

$$F(t) = F_r(t) + F(t_0).$$

Then,

$$F_r(t) = -\log\mu\int_{t_0}^t F(s)ds = -\log\mu\int_{t_0}^t\left(F_r(s)+F(t_0)\right)ds,$$

which implies that $F_r(t)\in C^1$ and

$$\frac{d}{dt}F_r(t) = -\log\mu(F_r(t)+F(t_0)).$$

Thus,

$$F_r(t) = \mu^{t_0-t}F(t_0) - F(t_0).$$

For (7.22), by Corollary 6.4, integrating the Pohozaev type estimate (6.11) from $\lambda_\alpha R$ to $\lambda_\alpha^{t_0}$, we have

$$\begin{aligned}
&F_\alpha(t_0) - \int_{D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&\leq C \int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| + C\int_{\lambda_\alpha R}^{\lambda_\alpha^{t_0}} \frac{\alpha-1}{r}dr + C(\lambda_\alpha^{t_0} - \lambda_\alpha R) \\
&\leq C \int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| \\
&\quad + C\left((t_0-1)(\alpha-1)\log\lambda_\alpha - (\alpha-1)\log R\right) + C(\lambda_\alpha^{t_0} - \lambda_\alpha R).
\end{aligned} \tag{7.25}$$

Combining this with Lemma 7.7 and (7.23), we get

$$\begin{aligned}
\lim_{t_0\to 1^-} F(t_0) &= \lim_{R\to\infty}\lim_{\alpha\searrow 1} \int_{D_{\lambda_\alpha R}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&= \lim_{R\to\infty}\lim_{\alpha\searrow 1} \int_{D_R(0)} (\sigma_\alpha\lambda_\alpha^2 + |\nabla_{g_\alpha(x_\alpha+\lambda_\alpha x)}u_\alpha|^2)^{\alpha-1}(\lambda_\alpha)^{2-2\alpha}|\nabla u_\alpha|^2 dx \\
&= \mu \lim_{R\to\infty}\int_{D_R(0)} |\nabla\sigma^1|^2 dx = \mu E(\sigma^1(x)).
\end{aligned}$$

□

Lemma 7.9. *Under the assumptions of Lemma 7.2, for any $t \in (0,1)$, there holds*

$$\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx = \mu^{2-t}E(\sigma^1).$$

Proof. By Lemma 7.7 and Lemma 7.8, we have

$$\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx = F_r(t) + F(t_0) = \mu^{t_0-t}F(t_0).$$

Then the conclusion of the lemma follows from letting $t_0 \nearrow 1$ and Lemma 7.8. □

In the end of this section, we complete the proof of Theorem 7.1.

Proof of Theorem 7.1. By the Pohozaev type estimate in Corollary 6.4, we have

$$
\begin{aligned}
&\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle dx\right| + C\int_{\lambda_\alpha^t}^{\delta}\frac{\alpha-1}{r}dr + C(\delta-\lambda_\alpha^t) \\
&\le C\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\
&\quad + C\left|\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} \langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\partial_\theta}\psi_\alpha\rangle dx\right| \\
&\quad + C\left((\alpha-1)\log\delta - t(\alpha-1)\log\lambda_\alpha\right) + C(\delta-\lambda_\alpha^t). \qquad (7.26)
\end{aligned}
$$

Combining (7.26) with (7.23), we have

$$\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx = 0.$$

By Lemma 7.9, we get

$$
\begin{aligned}
&\lim_{\delta\to0}\lim_{\alpha\searrow1}\int_{D_\delta(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx \\
&= \lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\left(\int_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx + \int_{D_{\lambda_\alpha^t}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2 dx\right) \\
&= \mu^2 E(\sigma^1).
\end{aligned}
$$

Combining the above equality with Corollary 7.3, we finished the proof of Theorem 7.1. □

Proof of Theorem 2.6. It is easy to see that Theorem 2.6 is a consequence of Theorem 7.1. □

8. Decay estimates and refined asymptotic neck analysis

In this section, we shall study the refined asymptotic behaviour of the α-Dirac-harmonic necks and prove Theorem 2.8.

For simplicity, we first consider the local model case as in Theorem 7.1.

Theorem 8.1. *Under the assumptions of Theorem 7.1, if we assume that there is only one bubble* (σ^1, ξ^1)*, which is a Dirac-harmonic sphere, let*

$$\nu = \liminf_{\alpha\searrow1}(\lambda_\alpha)^{-\sqrt{\alpha-1}}.$$

Then we have the following alternatives:

(1) *when* $\nu = 1$, *the set* $\phi(D_1(0)) \cup \sigma^1(S^2)$ *is a connected set in the target* N;

(2) *when* $\nu \in (1, \infty)$, *the set* $\phi(D_1(0))$ *and* $\sigma^1(S^2)$ *are connected by a geodesic* γ *in the target* N *of length*

$$L(\gamma) = \sqrt{\frac{E(\sigma^1)}{\pi}} \log \nu;$$

(3) *when* $\nu = \infty$, *the neck contains at least an infinite length geodesic curve in* N.

There are two main crucial steps in proving Theorem 8.1. The first one is to apply the three-circle method developed for the class of integro-differential equations considered in Section 6 to the general α-Dirac-harmonic map system to get some decay of tangential neck energies of both the map part and the spinor part. Here, we get a decay at the speed of $\alpha - 1$ rather than the exponential decay as in the (approximate) Dirac-harmonic map case [38, 27]. The key point is to write the general α-Dirac-harmonic system into the special forms given in our three-circle result - Theorem 6.7. The treatment of the two bad error terms mentioned in Section 2, namely, the second derivative term (2.26) and the curvature term (2.27), is complicated and subtle, see Lemma 8.2. The second step is to derive the decay of some weighted neck energy of the spinor part. The key point here is to apply some Hardy-type inequality to derive a differential inequality on the neck domain, see Lemma 8.4 and Lemma 8.5. This step is tricky.

Aa a first step towards Theorem 8.1, by applying the three-circle theorem in Section 6, we shall establish the following decay estimate for the tangential energies on the neck domain of both the map part and the spinor part.

Lemma 8.2. (Decay of tangential energies.) *Under the assumption of Theorem 7.1 and the one bubble assumption. If there is no energy concentration for the sequence* $(\phi_\alpha, \psi_\alpha)$ *in the region* $D_\delta(x_\alpha) \setminus D_{\lambda_\alpha R}(x_\alpha)$, *i.e.* (7.12) *holds, then we have the following decay estimates*

$$\left(\int_{P_i} |x - x_\alpha|^{-2} |\frac{\partial \phi_\alpha}{\partial \theta}|^2 dx\right)^{\frac{1}{2}} + \left(\int_{P_i} |x - x_\alpha|^{-\frac{4}{3}} |\frac{\partial \psi_\alpha}{\partial \theta}|^{\frac{4}{3}} dx\right)^{\frac{3}{4}} \leq \left((\alpha - 1) + e^{-\frac{iL}{2}} + e^{-\frac{(k_\alpha - i)L}{2}}\right) o(\alpha, \delta, R),$$

where $P_i = D_{e^{(i+1)L}\lambda_\alpha R}(x_\alpha) \setminus D_{e^{iL}\lambda_\alpha R}(x_\alpha)$, $i = 1, ..., k_\alpha$ *and* $\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{\alpha \searrow 1} o(\alpha, \delta, R) = 0$.

Proof. Since (7.12) holds, by (7.18), we have

$$|x - x_\alpha| |\nabla \phi_\alpha| + |x - x_\alpha|^2 |\nabla^2 \phi_\alpha| + \sqrt{|x - x_\alpha|} |\psi_\alpha| \leq C\epsilon, \ \forall \ x \in D_\delta(x_\alpha) \setminus D_{\lambda_\alpha R}(x_\alpha). \tag{8.1}$$

Denote

$$\phi^*_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi_\alpha(r, \theta) d\theta, \quad \psi^*_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} \psi_\alpha(r, \theta) d\theta,$$

and

$$u = \phi_\alpha - \phi^*_\alpha, \ v = \psi_\alpha - \psi^*_\alpha.$$

Next, we compute the equation for $(\phi_\alpha - \phi^*_\alpha, \psi_\alpha - \psi^*_\alpha)$. By equation (2.20), we have

$$\begin{aligned}\Delta\phi^*_\alpha &= \frac{1}{2\pi}\int_0^{2\pi}\Delta\phi_\alpha d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2} + A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) + \frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

Computing directly, we have

$$\begin{aligned}\mathrm{II} &= \frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) - A(\phi^*_\alpha)(d\phi_\alpha, d\phi_\alpha) + A(\phi^*_\alpha)(d\phi_\alpha, d\phi_\alpha) - A(\phi^*_\alpha)(d\phi^*_\alpha, d\phi^*_\alpha) \\ &\quad + A(\phi^*_\alpha)(d\phi^*_\alpha, d\phi^*_\alpha) d\theta \\ &= A(\phi^*_\alpha)(d\phi^*_\alpha, d\phi^*_\alpha) + \frac{1}{2\pi}\int_0^{2\pi} A^4(\phi_\alpha - \phi^*_\alpha) + A^5\nabla(\phi_\alpha - \phi^*_\alpha) d\theta,\end{aligned}$$

where A^i may change from line to line and it just stands for some expression satisfying

$$|A^4| \le C(N)(|d\phi_\alpha|^2 + |d\phi_\alpha||\psi_\alpha|^2),\ \ |A^5| \le C(N)(|d\phi_\alpha| + |\psi_\alpha|^2).$$

Similarly,

$$\begin{aligned}\mathrm{III} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi_\alpha)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha}\phi^*_\alpha|^2)^{\alpha-1}}\right) d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} Re\left(P(\phi^*_\alpha)(\mathcal{A}(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x)}\phi^*_\alpha|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi^*_\alpha|^2)^{\alpha-1}}\right) d\theta \\ &\quad + \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi^*_\alpha|^2)^{\alpha-1}} \\ &= \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi^*_\alpha(e_\gamma), e_\gamma\cdot\psi^*_\alpha);\psi^*_\alpha)\right)}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi^*_\alpha|^2)^{\alpha-1}} + \frac{1}{2\pi}\int_0^{2\pi} h_1 d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} A^4(\phi_\alpha - \phi^*_\alpha) + A^5\nabla(\phi_\alpha - \phi^*_\alpha) + \frac{1}{2\pi}Re\int_0^{2\pi} A^6(\psi_\alpha - \psi^*_\alpha) d\theta,\end{aligned}$$

where

$$|A^6| \leq C(N)|d\phi_\alpha||\psi_\alpha|,$$

$$h_1 = Re\Big(P(\phi_\alpha^*)(\mathcal{A}(d\phi_\alpha^*(e_\gamma), e_\gamma \cdot \psi_\alpha^*); \psi_\alpha^*)\Big)\left(\frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x)}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{\alpha(\sigma_\alpha + |\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}}\right)$$

and we have used the estimate

(8.2)

$$Re\Big(P(\phi_\alpha^*)(\mathcal{A}(\phi_\alpha^*)(d\phi_\alpha^*(e_\gamma), e_\gamma \cdot \psi_\alpha^*); \psi_\alpha^*)\Big)\left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}}\right) = A^5\nabla(\phi_\alpha - \phi_\alpha^*).$$

To show (8.2), in fact, if $|\nabla_{g_\alpha}\phi_\alpha| = |\nabla_{g_\alpha}\phi_\alpha^*|$, then the estimate holds immediately by taking $A^5 = 0$. If $|\nabla_{g_\alpha}\phi_\alpha| \neq |\nabla_{g_\alpha}\phi_\alpha^*|$, without loss of generality, we assume $|\nabla_{g_\alpha}\phi_\alpha^*| \leq |\nabla_{g_\alpha}\phi_\alpha|$. Then

$$\begin{aligned}
0 &\leq \left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right)\\
&= \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\left([(1 + \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha^*|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2})^\alpha - 1](\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2) + |\nabla_{g_\alpha}\phi_\alpha^*|^2 - |\nabla_{g_\alpha}\phi_\alpha|^2\right)\\
&\leq \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha^*|^2}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}, \qquad (8.3)
\end{aligned}$$

where in the last estimate we have used the inequality

$$(1 + x)^\alpha - 1 \leq 2x, \quad \forall x \in [0, b_\alpha]$$

when $\alpha - 1$ is small enough, where b_α satisfies

$$\limsup_{\alpha \searrow 1} (1 + b_\alpha)^{\alpha-1} \leq C < \infty.$$

Taking

$$b_\alpha = \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha^*|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2}$$

and noting that, by Lemma 6.1, there holds

$$\begin{aligned}
\limsup_{\alpha \searrow 1}(1 + \frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha^*|^2}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2})^{\alpha-1} &\leq \limsup_{\alpha \searrow 1}(1 + \frac{|\nabla_{g_\alpha}\phi_\alpha|^2}{\sigma_\alpha})^{\alpha-1}\\
&= \limsup_{\alpha \searrow 1} \sigma_\alpha^{1-\alpha}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1} \leq C < \infty,
\end{aligned}$$

it is easy to see that (8.3) follows immediately.

According to (8.3), we can write that

$$\left(\frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right) = \widetilde{A}^5\frac{|\nabla_{g_\alpha}\phi_\alpha|^2 - |\nabla_{g_\alpha}\phi_\alpha^*|^2}{(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}$$

where $0 < \widetilde{A}^5 \le 1$ and then

$$
\begin{aligned}
& Re\left(P(\phi_\alpha^*)(\mathcal{A}(\phi_\alpha^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*);\psi_\alpha^*)\right)\left(\frac{1}{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha^*|^2)^{\alpha-1}}-\frac{1}{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}}\right)\\
&= Re\left(P(\phi_\alpha^*)(\mathcal{A}(\phi_\alpha^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*);\psi_\alpha^*)\right)\widetilde{A}^5\frac{|\nabla_{g_\alpha}\phi_\alpha|^2-|\nabla_{g_\alpha}\phi_\alpha^*|^2}{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\\
&= Re\left(P(\phi_\alpha^*)(\mathcal{A}(\phi_\alpha^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*);\psi_\alpha^*)\right)\widetilde{A}^5\frac{\langle\nabla_{g_\alpha}\phi_\alpha,\nabla_{g_\alpha}(\phi_\alpha-\phi_\alpha^*)\rangle+\langle\nabla_{g_\alpha}\phi_\alpha^*,\nabla_{g_\alpha}(\phi_\alpha-\phi_\alpha^*)\rangle}{(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha}}\\
&= A^5\nabla(\phi_\alpha-\phi_\alpha^*),
\end{aligned}
$$

which implies that (8.2) holds.

Therefore, we get

$$
\begin{aligned}
\Delta\phi_\alpha^* &= A(\phi_\alpha^*)(d\phi_\alpha^*, d\phi_\alpha^*) + \frac{Re\left(P(\phi^*)(\mathcal{A}(\phi^*)(d\phi_\alpha^*(e_\gamma), e_\gamma\cdot\psi_\alpha^*);\psi_\alpha^*)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2)^{\alpha-1}} - \frac{1}{2\pi}\int_0^{2\pi}(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}d\theta\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}h_1 d\theta + \frac{1}{2\pi}\int_0^{2\pi}A^4(\phi_\alpha-\phi_\alpha^*) + A^5\nabla(\phi_\alpha-\phi_\alpha^*) + \frac{1}{2\pi}Re\int_0^{2\pi}A^6(\psi_\alpha-\psi_\alpha^*)d\theta.
\end{aligned}
$$

Using the same method, we get

$$
\begin{aligned}
\Delta(\phi_\alpha-\phi_\alpha^*) &= A^1(\phi_\alpha-\phi_\alpha^*) + A^2\nabla(\phi_\alpha-\phi_\alpha^*) + Re(A^3(\psi_\alpha-\psi_\alpha^*))\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}A^4(\phi_\alpha-\phi_\alpha^*) + A^5\nabla(\phi_\alpha-\phi_\alpha^*) + Re(A^6(\psi_\alpha-\psi_\alpha^*))d\theta\\
&\quad + h_1 - \frac{1}{2\pi}\int_0^{2\pi}h_1 d\theta - (\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2} + \frac{1}{2\pi}\int_0^{2\pi}(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}d\theta,
\end{aligned}
\tag{8.4}
$$

and

$$
\begin{aligned}
\partial\!\!\!/(\psi_\alpha-\psi_\alpha^*) &= B^1(\phi_\alpha-\phi_\alpha^*) + B^2\nabla(\phi_\alpha-\phi_\alpha^*) + B^3(\psi_\alpha-\psi_\alpha^*)\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}B^4(\phi_\alpha-\phi_\alpha^*) + B^5\nabla(\phi_\alpha-\phi_\alpha^*) + B^6(\psi_\alpha-\psi_\alpha^*)d\theta,
\end{aligned}
\tag{8.5}
$$

where $A^i,\ B^i,\ i = 1, ..., 6$ satisfy

$$
\begin{aligned}
&|A^1| + |A^4| \le C(N)(|d\phi_\alpha|^2 + |d\phi_\alpha||\psi_\alpha|^2), \quad |A^3| + |A^6| + |B^1| + |B^4| \le C(N)|d\phi_\alpha||\psi_\alpha|\\
&|A^2| + |A^5| + |B^3| + |B|^6 \le C(N)(|d\phi_\alpha| + |\psi_\alpha|^2), \quad |B^2| + |B^5| \le C(N)|\psi_\alpha|.
\end{aligned}
$$

By Lemma 6.1, similarly to the derivation of (8.3), we have

$$
\begin{aligned}
|h_1| &\le C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{|\nabla_{g_\alpha(x)}\phi_\alpha^*|^2-|\nabla_{g_\alpha(x_\alpha)}\phi_\alpha^*|^2}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right|\\
&= C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{\left(g_\alpha^{\beta\gamma}(x)-g_\alpha^{\beta\gamma}(x_\alpha)\right)\frac{\partial\phi_\alpha^*}{\partial x^\beta}\frac{\partial\phi_\alpha^*}{\partial x^\gamma}}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right|\\
&\le C|\nabla\phi_\alpha^*||\psi_\alpha^*|^2\left|\frac{|x-x_\alpha||\nabla\phi_\alpha^*|^2}{(\sigma_\alpha+|\nabla\phi_\alpha^*|^2)^\alpha}\right| \le C|x-x_\alpha||\nabla\phi_\alpha^*||\psi_\alpha^*|^2,
\end{aligned}
$$

which implies

$$\begin{aligned}\| |x-x_\alpha|h_1 \|_{L^2(P_i)} &\leq \| |x-x_\alpha|^2|\nabla\phi_\alpha^*||\psi_\alpha^*|^2 \|_{L^2(P_i)}\\ &\leq e^{iL}\lambda_\alpha R\, o(\alpha,\delta,R) = e^{-(k_\alpha-i)L}\delta\, o(\alpha,\delta,R) = e^{-(k_\alpha-i)L}o(\alpha,\delta,R),\end{aligned} \tag{8.6}$$

where we used the fact that

$$|x-x_\alpha||\nabla\phi_\alpha^*| + \sqrt{|x-x_\alpha|}|\psi_\alpha^*| \leq \frac{1}{2\pi}\int_0^{2\pi}\Big(|x-x_\alpha||\nabla\phi_\alpha| + \sqrt{|x-x_\alpha|}|\psi_\alpha|\Big)d\theta = o(\alpha,\delta,R),$$

which follows from (8.1).

Note that

$$\begin{aligned}&|x|^2(|A^1|+|A^4|) + |x|^{\frac{3}{2}}(|A^3|+|A^6|+|B^1|+|B^4|)\\ &+|x|(|A^2|+|A^5|+|B^3|+|B^6|) + |x|^{\frac{1}{2}}(|B^2|+|B^5|) \leq C\epsilon.\end{aligned} \tag{8.7}$$

Substituting $u=\phi_\alpha-\phi_\alpha^*$, $v=\psi_\alpha-\psi_\alpha^*$, $f_2=0$ and

$$f_1 = h_1 - \frac{1}{2\pi}\int_0^{2\pi}h_1 d\theta - (\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2} + \frac{1}{2\pi}\int_0^{2\pi}2(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2}d\theta$$

in Corollary 6.9 and noting that

$$\begin{aligned}\||x-x_\alpha|f_1\|_{L^2(P_i)} &\leq C(\alpha-1)\||x-x_\alpha||\nabla^2\phi_\alpha|\|_{L^2(P_i)} + C\||x-x_\alpha||h_1|\|_{L^2(P_i)}\\ &\leq C(\alpha-1)\|\nabla\phi_\alpha\|_{L^2(P_{i-1}\cup P_i\cup P_{i+1})} + C\||x-x_\alpha||h_1|\|_{L^2(P_i)}\\ &\leq \Big((\alpha-1)+e^{-(k_\alpha-i)L}\Big)o(\alpha,\delta,R),\quad i=1,...,k_\alpha,\end{aligned}$$

where we used (8.6) and Lemma 5.1, we obtain the energy decay in the θ-direction,

$$\begin{aligned}&\|r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\|_{L^2(P_i)} + \|r^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(P_i)}\\ &\leq \|\nabla u\|_{L^2(P_i)} + \|\nabla v\|_{L^{\frac{4}{3}}(P_i)}\\ &\leq C\Big((\alpha-1)+e^{-(k_\alpha-i)L}\Big)o(\alpha,\delta,R) + C\big(F_0^{1/2}(u,v)+F_{k_\alpha}^{1/2}(u,v)\big)\big(e^{-\frac{1}{2}(k_\alpha-i)L}+e^{-\frac{1}{2}iL}\big)\\ &\leq C\Big((\alpha-1)+e^{-\frac{1}{2}(k_\alpha-i)L}+e^{-\frac{1}{2}iL}\Big)o(\alpha,\delta,R),\end{aligned} \tag{8.8}$$

where we used the fact that

$$F_0^{1/2}(u,v)+F_{k_\alpha}^{1/2}(u,v) = o(\alpha,\delta,R),$$

which follows from the Poincaré inequality and (7.12). We finished the proof of this lemma. □

As a direct corollary of Lemma 8.2, we have

Corollary 8.3. *Under the assumption of Lemma 8.2, assume $\nu>1$, then for $t_\alpha\in[t_1,t_2]$ where $0<t_1\leq t_2<1$, we have*

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\left(\left\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\right\|_{L^2(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))} + \left\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\right\|_{L^{\frac{4}{3}}(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\right)=0,$$

where $K>1$ is any fixed positive constant.

Proof. Let $i_1, i_2 \in [0, k_\alpha]$ be two integers such that

$$e^{i_1 L}\lambda_\alpha R \le \frac{1}{K}\lambda_\alpha^{t_\alpha} < e^{(i_1+1)L}\lambda_\alpha R \ \ and \ \ e^{i_2 L}\lambda_\alpha R \le K\lambda_\alpha^{t_\alpha} < e^{(i_2+1)L}\lambda_\alpha R.$$

Then we have

$$e^{-i_1 L} \le KRe^L\lambda_\alpha^{1-t_\alpha},\ e^{-(k_\alpha - i_2)L} \le \frac{K}{\delta}\lambda_\alpha^{t_\alpha},\ i_2 - i_1 \le \frac{2\log K}{L}.$$

By Lemma 8.2, we have

$$\begin{aligned}
&\frac{1}{\alpha-1}\left(\Big\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\Big\|_{L^2(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))} + \Big\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\Big\|_{L^{\frac{4}{3}}(D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\right)\\
&\le \frac{1}{\alpha-1}\sum_{i=i_1}^{i_2}\left(\Big\||x-x_\alpha|^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|\Big\|_{L^2(P_i)} + \Big\||x-x_\alpha|^{-1}|\frac{\partial\psi_\alpha}{\partial\theta}|\Big\|_{L^{\frac{4}{3}}(P_i)}\right)\\
&\le \frac{1}{\alpha-1}\left((\alpha-1)(i_2-i_1) + e^{-\frac{i_1 L}{2}} + e^{-\frac{(k_\alpha - i_2)L}{2}}\right)o(\alpha,\delta,R)\\
&\le \frac{1}{\alpha-1}\left((\alpha-1)\frac{2\log K}{L} + \sqrt{KRe^L}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \sqrt{\frac{K}{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R).
\end{aligned}$$

Since $\nu > 1$, then $\lambda_\alpha \le e^{-\frac{C}{\sqrt{\alpha-1}}}$ and the conclusion of the corollary follows immediately from the above inequality. □

Next, by applying the Hardy type inequality (7.17) and Lemma 8.2, we shall derive the following new decay estimate of the weighted energy of spinor part on the neck region, which is crucial in the proofs of two main results - Theorem 2.8 (e.g. Proposition 8.9) and Theorem 2.12 (e.g. Lemma 9.5).

Lemma 8.4. (Decay of spinor.) *Under the assumption of Lemma 8.2, assume $\nu > 1$, then for $t_\alpha \in [t_1, t_2]$ where $0 < t_1 \le t_2 < 1$, we have*

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0, \tag{8.9}$$

which implies

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\psi_\alpha|^4 dx = 0, \tag{8.10}$$

where $K > 1$ is any fixed positive constant.

Proof. We divide the proof into two steps.

Step 1 We prove

$$\lim_{\alpha\searrow 1}\frac{1}{\sqrt{\alpha-1}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx = 0. \tag{8.11}$$

For simplicity of notation, we also assume $x_\alpha = 0$. Define

$$f(t) := \int_{D_{e^t\lambda_\alpha^{t_\alpha}} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}} \frac{|\psi_\alpha|^2}{|x|} dx,\ t \in [0, 8\log\frac{1}{\alpha-1}].$$

Since $\nu > 1$, then $\lambda_\alpha \leq e^{-\frac{C}{\sqrt{\alpha-1}}}$ and it is easy to see that

$$e^t \leq (\frac{1}{\alpha-1})^8 \leq \lambda_\alpha^{-\epsilon},$$

where $\epsilon < \min\{t_1, 1-t_2\}$.

For any $\rho > 0$, taking the cut-off function $\eta \in C_0^\infty(D_{e^t\lambda_\alpha^{t_\alpha}+\rho} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $D_{e^t\lambda_\alpha^{t_\alpha}} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}$ and $|\nabla\eta| \leq \frac{2}{\rho}$. Taking $f = \eta|\psi_\alpha|^2$ in the Hardy inequality (7.17), we get

$$\begin{aligned}
\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)} &\leq \|\nabla(\eta|\psi_\alpha|^2)\|_{L^1(\mathbb{R}^2)} \\
&\leq \|2\eta\psi_\alpha\nabla\psi_\alpha\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_\alpha|^2\|_{L^1(\mathbb{R}^2)} \\
&\leq \left\|2\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^1(\mathbb{R}^2)} + \left\|2\eta\psi_\alpha\frac{\partial\psi_\alpha}{\partial r}\right\|_{L^1(\mathbb{R}^2)} + \left\|\nabla\eta|\psi_\alpha|^2\right\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

On one hand, we know

$$\frac{\partial\psi_\alpha}{\partial r} = \frac{\partial}{\partial r}\cdot\frac{1}{|x|}\frac{\partial}{\partial\theta}\cdot\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta} + \frac{\partial}{\partial r}\cdot\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha) \tag{8.12}$$

from equation (2.19). So, we have

$$|2\eta\psi_\alpha\frac{\partial\psi_\alpha}{\partial r}| \leq |2\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}| + C|\eta||d\phi_\alpha||\psi_\alpha|^2.$$

On the other hand, by inequality (7.18), we have

$$|x||d\phi_\alpha| + \sqrt{|x|}|\psi_\alpha| \leq C\epsilon \quad on \quad D_\delta \setminus D_{\lambda_\alpha R}. \tag{8.13}$$

Combining these, we get

$$\begin{aligned}
\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)} &\leq 4\left\|\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^1(\mathbb{R}^2)} + C\left\|\eta|d\phi_\alpha||\psi_\alpha|^2\right\|_{L^1(\mathbb{R}^2)} + \left\|\nabla\eta|\psi_\alpha|^2\right\|_{L^1(\mathbb{R}^2)} \\
&\leq 4\left\|\eta\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^1(\mathbb{R}^2)} + C\epsilon\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)} + \left\|\nabla\eta|\psi_\alpha|^2\right\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

Taking $\epsilon > 0$ sufficiently small such that $C\epsilon \le \frac{1}{2}$, we have

$$\begin{aligned}
\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)} &\le 8\left\|\psi_\alpha\frac{1}{|x|}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})} + 2\left\|\nabla\eta|\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le 8\|\psi_\alpha\|_{L^4(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\left\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\quad + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\sigma})}\\
&\le C\left\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}. \qquad (8.14)
\end{aligned}$$

where $\rho > 0$ is small enough such that $D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho}\subset D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}}$.

Let $i_1, i_2 \in [0, k_\alpha]$ be two integers such that

$$e^{i_1L}\lambda_\alpha R \le e^{-(t+1)}\lambda_\alpha^{t_\alpha} < e^{(i_1+1)L}\lambda_\alpha R \ \ and\ \ e^{i_2L}\lambda_\alpha R \le e^{(t+1)}\lambda_\alpha^{t_\alpha} < e^{(i_2+1)L}\lambda_\alpha R.$$

Then we have

$$e^{-i_1L} \le Re^{(t+1+L)}\lambda_\alpha^{1-t_\alpha} \le CR(\alpha-1)^{-8}\lambda_\alpha^{1-t_\alpha},\ e^{-(k_\alpha-i_2)L} \le \frac{e^{t+1}}{\delta}\lambda_\alpha^{t_\alpha} \le C\frac{(\alpha-1)^{-8}}{\delta}\lambda_\alpha^{t_\alpha},$$

and

$$i_2 - i_1 \le \frac{2(1+t)}{L} + 1 \le C\log\frac{1}{\alpha-1}.$$

Combining this with (8.14) and Lemma 8.2, we get

$$\begin{aligned}
&\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)}\\
&\le \sum_{i=i_1}^{i_2}\left((\alpha-1) + e^{-\frac{iL}{2}} + e^{-\frac{(k_\alpha-i)L}{2}}\right)o(\alpha,\delta,R) + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \left((\alpha-1)(i_2-i_1) + e^{-\frac{i_1L}{2}} + e^{-\frac{(k_\alpha-i_2)L}{2}}\right)o(\alpha,\delta,R) + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \left((\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)\\
&\quad + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})} + \frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}
\end{aligned}$$

Letting $\rho \to 0$, we get

$$
\begin{aligned}
f(t) &= \int_{D_{e^t\lambda_\alpha^{t_\alpha}} \setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}}} \frac{|\psi_\alpha|^2}{|x|} dx \\
&\leq \left((\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}} \right) o(\alpha,\delta,R) + 4e^t\lambda_\alpha^{t_\alpha}\int_{\partial D_{e^t\lambda_\alpha^{t_\alpha}}} \frac{|\psi_\alpha|^2}{|x|} \\
&\quad + 4e^{-t}\lambda_\alpha^{t_\alpha}\int_{\partial D_{e^{-t}\lambda_\alpha^{t_\alpha}}} \frac{|\psi_\alpha|^2}{|x|} \\
&\leq \left((\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}} \right) o(\alpha,\delta,R) + 4f'(t).
\end{aligned}
$$

This is

$$
(e^{-\frac{1}{4}t}f(t))' \geq -e^{-\frac{1}{4}t}\left((\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}} \right) o(\alpha,\delta,R). \tag{8.15}
$$

Integrating the above ODE from K to $8\log\frac{1}{\alpha-1}$, we get

$$
\begin{aligned}
f(K) &\leq C(\alpha-1)^2 f(8\log\frac{1}{\alpha-1}) + \left((\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}} \right) o(\alpha,\delta,R) \\
&\leq \left((\alpha-1)^2 + (\alpha-1)\log\frac{1}{\alpha-1} + \sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}} + \frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}} \right) o(\alpha,\delta,R).
\end{aligned}
$$

where the last inequality follows from Lemma 7.4 since

$$
f(8\log\frac{1}{\alpha-1}) \leq \int_{D_\delta \setminus D_{\lambda_\alpha R}} \frac{|\psi_\alpha|^2}{|x|}.
$$

Then (8.11) follows immediately, which implies

$$
\lim_{\alpha\searrow 1} \frac{1}{\sqrt{\alpha-1}} \int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha) \setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)} |\psi_\alpha|^4 dx = 0. \tag{8.16}
$$

Step 2 We prove the conclusions of the lemma.

Using the decay estimate (8.16), i.e. $\|\psi_\alpha\|_{L^4(P_i)} = o\Big((\alpha-1)^{\frac{1}{8}}\Big)$ in (8.14) and repeat the above process again, we have

$$\begin{aligned}\left\|\eta\frac{|\psi_\alpha|^2}{|x|}\right\|_{L^1(\mathbb{R}^2)} &\le 8\|\psi_\alpha\|_{L^4(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}-\rho})}\left\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(D_{e^{t+1}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-(t+1)}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\quad+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \sum_{i=i_1}^{i_2}8\|\psi_\alpha\|_{L^4(P_i)}\sum_{i=i_1}^{i_2}\left\||x|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(P_i)}\\
&\quad+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le (\alpha-1)^{\frac{1}{8}}(i_2-i_1)\Big((\alpha-1)(i_2-i_1)+e^{-\frac{i_1L}{2}}+e^{-\frac{(k_\alpha-i_2)L}{2}}\Big)o(\alpha,\delta,R)\\
&\quad+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}\\
&\le \left((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R)\\
&\quad+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^t\lambda_\alpha^{t_\alpha}+\rho}\setminus D_{e^t\lambda_\alpha^{t_\alpha}})}+\frac{4}{\rho}\left\||\psi_\alpha|^2\right\|_{L^1(D_{e^{-t}\lambda_\alpha^{t_\alpha}}\setminus D_{e^{-t}\lambda_\alpha^{t_\alpha}-\rho})}.\end{aligned}$$

Letting $\rho\to 0$ and similar to deriving (8.15), we have

$$(8.17)\quad (e^{-\frac{1}{4}t}f(t))'\ge -e^{-\frac{1}{4}t}\left((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R).$$

Integrating the above ODE from K to $8\log\frac{1}{\alpha-1}$, we get

$$f(K)\le C(\alpha-1)^2f(8\log\frac{1}{\alpha-1})+\left((\alpha-1)^{1+\frac{1}{8}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R),$$

which implies

$$(8.18)\qquad \lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{1+\frac{1}{16}}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0.$$

Thus we proved (8.9). The inequality (8.10) is a consequence of (8.9) and (7.18). □

From the proof of Lemma 8.4, we can actually get a better decay estimate.

Lemma 8.5. (Improved decay of spinor.) *Under the assumption of Lemma 8.4, we have*

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\frac{4}{3}}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0,$$

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\frac{4}{3}}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\psi_\alpha|^4dx=0.$$

Proof. Applying the new decay estimate (8.18) and repeating the process of **Step 2** in the proof of Lemma 8.4 again and again, we will get the best decay estimate, i.e. there exists a best constant[10] $\beta > 1$ such that

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\beta}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0, \tag{8.19}$$

and for any $\beta' > \beta$, the estimate (8.19) not holds.

We make the following

Claim: $\beta = \frac{4}{3}$.

In fact, if not, then $\beta < \frac{4}{3}$ and there exists a small constant $\epsilon' > 0$ such that $\beta < \frac{4}{3}(1-\epsilon')$. Using the new decay estimate (8.19) and repeating the process of **Step 2** in the proof of Lemma 8.4 again, we have

$$f(K)\le C(\alpha-1)^2 f(8\log\frac{1}{\alpha-1})+\left((\alpha-1)^{1+\frac{\beta}{4}}(\log\frac{1}{\alpha-1})^2+\sqrt{R}(\alpha-1)^{-4}\lambda_\alpha^{\frac{1-t_\alpha}{2}}+\frac{(\alpha-1)^{-4}}{\sqrt{\delta}}\lambda_\alpha^{\frac{t_\alpha}{2}}\right)o(\alpha,\delta,R),$$

which implies

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{1+\frac{\beta}{4}-\epsilon'}}\int_{D_{K\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\frac{1}{K}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\frac{|\psi_\alpha|^2}{|x-x_\alpha|}dx=0. \tag{8.20}$$

Since $\beta < \frac{4}{3}(1-\epsilon')$, it is easy to see that $1+\frac{\beta}{4}-\epsilon' > \beta$. This is a contradiction, since β is the best constant. Thus the **Claim** holds and the lemma is proved. □

It is interesting to ask whether the constant $\frac{4}{3}$ in the decay estimate in Lemma 8.5 is the best constant to characterise the decay of the spinor or not.

As a corollary of Lemma 8.4 and Lemma 5.1, we get

Corollary 8.6. *Assume $\nu > 1$. Let $0 < t_1 \le t_2 < 1$, by passing to a subsequence, there holds*

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)^{\frac{1}{4}}}\left\|\sqrt{|x-x_\alpha|}\,|\psi_\alpha|\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_2}}(x_\alpha))}=0.$$

Proof. For any $t \in [\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}]$, by Lemma 5.1 (or see the proof of Lemma 7.5), we have

$$\left\|\sqrt{|x-x_\alpha|}\,|\psi_\alpha|\right\|_{C^0(D_{2t}(x_\alpha)\setminus D_t(x_\alpha))}\le C\|\psi_\alpha\|_{L^4(D_{4t}(x_\alpha)\setminus D_{\frac{1}{2}t}(x_\alpha))}.$$

Then the conclusion of the corollary follows immediately from Lemma 8.4. □

By Lemma 8.2 (or Corollary 8.3), we have

Lemma 8.7. *Under the assumption of Lemma 8.2, suppose that $\nu > 1$ and t_α is a positive number such that $0 < t_1 \le t_\alpha \le t_2 < 1$, then we have:*

[10] Here, the best constant is in the sense that the best decay estimate can be derived by using our approach.

(1)

$$\frac{1}{\sqrt{\alpha-1}}\left(\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x)-\phi_\alpha(x_\alpha+(\lambda_\alpha^{t_\alpha},0))\right)\to\overrightarrow{a}\log|x|$$

strongly in $C^j_{loc}(\mathbb{R}^2\setminus\{0\},\mathbb{R}^K)$ *for any integer* j*, where* $\overrightarrow{a}\in T_yN$ *is a vector in* $\mathbb{R}^K$ *with*

$$|\overrightarrow{a}|=\mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{\frac{E(\sigma^1)}{\pi}}.$$

(2)

$$\left\|\frac{1}{\sqrt{\alpha-1}}\left(\phi_\alpha-\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta\right)\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_2}}(x_\alpha))}+\left\|\frac{|x-x_\alpha|}{\sqrt{\alpha-1}}\nabla\left(\phi_\alpha-\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta\right)\right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_2}}(x_\alpha))}\to 0.$$

Proof. Set

$$u_\alpha(x):=\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x),\quad w_\alpha(x):=(\lambda_\alpha^{t_\alpha})^{\alpha-1}\sqrt{\lambda_\alpha^{t_\alpha}}\,\psi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x)$$

and

$$v_\alpha(x):=\frac{1}{\sqrt{\alpha-1}}\left\{\phi_\alpha(x_\alpha+\lambda_\alpha^{t_\alpha}x)-\phi_\alpha(x_\alpha+(\lambda_\alpha^{t_\alpha},0))\right\}.$$

By (7.26), (7.23), Lemma 6.1 and Lemma 8.2, we have

$$\begin{aligned}
&\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\nabla\phi_\alpha|^2dx\\
&\le C\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&\le C\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\\
&\quad+C\left|\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}\langle\psi_\alpha,|x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\partial_\theta}\psi_\alpha\rangle dx\right|+C\int_{2^{-k}\lambda_\alpha^{t_\alpha}}^{2^k\lambda_\alpha^{t_\alpha}}\frac{\alpha-1}{r}dr+C(k)\lambda_\alpha^{t_\alpha}\\
&\le C\left\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\right\|^2_{L^2(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+\|\psi_\alpha\|_{L^4(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\left\||x-x_\alpha|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\\
&\quad+C\|\psi_\alpha\|^2_{L^4(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\left\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\right\|_{L^2(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}+C(k)\left(\alpha-1+\lambda_\alpha^{t_\alpha}\right)
\end{aligned}$$

(8.21)

$$\le C(k)(\alpha-1),$$

where we used the fact that

$$\lambda_\alpha=o((\alpha-1)^m)$$

for any $m>0$, since $\nu>1$.

By Lemma 5.1, we have

$$\|\nabla u_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} + \|\nabla^2 u_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} \leq C(k)\sqrt{\alpha-1}. \tag{8.22}$$

Then,

$$\|\nabla v_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} + \|\nabla^2 v_\alpha\|_{L^\infty(D_{2^k}(0)\setminus D_{2^{-k}}(0))} \leq C(k).$$

Noting that $v_\alpha(1,0)=0$, thus

$$\|v_\alpha\|_{C^2(D_{2^k}(0)\setminus D_{2^{-k}}(0))} \leq C(k).$$

Since v_α satisfies the following equation

$$\Delta v_\alpha + \sqrt{\alpha-1}O(|\nabla v_\alpha|^2) + (\alpha-1)O(|\nabla^2 v_\alpha|) + O(|w_\alpha|^2)O(|\nabla v_\alpha|) = 0$$

and by (7.18), there holds

$$\lim_{\alpha\searrow 1}\|w_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} = 0,$$

then there exists a subsequence of v_α (without changing the notation) such that

$$v_\alpha \to v_0 \quad in \quad C^1_{loc}(\mathbb{R}^2\setminus\{0\})$$

where v_0 satisfies $v_0(1,0)=0$ and

$$\Delta v_0 = 0.$$

Moreover, Corollary 8.3 tells us that $v_0(x) = v_0(|x|)$. Set

$$v_0 = \overrightarrow{a}\log r = (a_1,...,a_K)\log r.$$

By Corollary 6.4 and Corollary 8.3, we have

$$\begin{aligned}
&\frac{1}{\alpha-1}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx\\
&= \frac{2}{2\alpha-1}\int_{\lambda_\alpha^{t_\alpha}}^{2\lambda_\alpha^{t_\alpha}}\frac{1}{t}F_\alpha(\log_{\lambda_\alpha}t)dt + \frac{1}{(2\alpha-1)(\alpha-1)}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}\langle\psi_\alpha, |x-x_\alpha|^{-2}\frac{\partial}{\partial\theta}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial\theta}}\psi_\alpha\rangle + o(1)\\
&= \frac{2}{2\alpha-1}\log 2F_\alpha(t_\alpha) + o(1) \to 2\log 2F(\lim_{\alpha\searrow 1}t_\alpha),
\end{aligned}$$

where we used the fact that

$$\lambda_\alpha^{t_\alpha} = o((\alpha-1)^m)$$

for any $m>0$, since $\nu>1$.

On the other hand, there also holds

$$\begin{aligned}
\frac{1}{\alpha-1}\int_{D_{2\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}(\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}|\nabla\phi_\alpha|^2dx &= \int_{D_2(0)\setminus D_1(0)}\left(\sigma_\alpha + |\nabla_{g_\alpha}v_\alpha|^2\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\right)^{\alpha-1}|\nabla v_\alpha|^2dx\\
&\to 2\pi\log 2|\overrightarrow{a}|^2\mu^{\lim_{\alpha\searrow 1}t_\alpha}.
\end{aligned}$$

Therefore,

$$|\overrightarrow{a}|^2 = \frac{1}{\pi}\mu^{-\lim_{\alpha\searrow 1}t_\alpha}F(\lim_{\alpha\searrow 1}t_\alpha) = \frac{E(\sigma^1)}{\pi}\mu^{2-2\lim_{\alpha\searrow 1}t_\alpha},$$

where the last equality follows from Lemma 7.9 and we proved the statement (1).

For statement (2), if it was false, then there would exist $t_\alpha \in [t_1, t_2]$ and $\theta_\alpha \in [0, 2\pi]$ such that

$$\frac{1}{\sqrt{\alpha-1}}\left(\lambda_\alpha^{t_\alpha}|\nabla\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha) - \nabla\phi_\alpha^*(\lambda_\alpha^{t_\alpha})| + |\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha) - \phi_\alpha^*(\lambda_\alpha^{t_\alpha})|\right) \to b \neq 0, \tag{8.23}$$

where

$$\phi_\alpha^* := \frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha d\theta.$$

Let $\lim_{\alpha\searrow 1}\theta_\alpha = \theta_0$, then it is obvious that

$$|\nabla v_0(1,\theta_0) - \nabla v_0^*(1)| + |v_0(1,\theta_0) - v_0^*(1)| = 0,$$

where v_0^* is the average of v_0, defined in a similar way as ϕ_α^*.

However, by (8.23) and the fact that $v_\alpha \to v_0$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$, we have

$$\begin{aligned} b &= \lim_{\alpha\searrow 1}\frac{1}{\sqrt{\alpha-1}}\left(\lambda_\alpha^{t_\alpha}|\nabla\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha) - \nabla\phi_\alpha^*(\lambda_\alpha^{t_\alpha})| + |\phi_\alpha(\lambda_\alpha^{t_\alpha},\theta_\alpha) - \phi_\alpha^*(\lambda_\alpha^{t_\alpha})|\right) \\ &= \lim_{\alpha\searrow 1}|\nabla v_\alpha(1,\theta_\alpha) - \nabla v_\alpha^*(1)| + \lim_{\alpha\searrow 1}|v_\alpha(1,\theta_\alpha) - v_\alpha^*(1)| \\ &= |\nabla v_0(1,\theta_0) - \nabla v_0^*(1)| + |v_0(1,\theta_0) - v_0^*(1)| = 0, \end{aligned}$$

which is a contradiction. So, the statement (2) holds and we finished the proof of the lemma. □

Corollary 8.8. *Under the assumption of Lemma 8.7, we have:*

$$\int_{\lambda_\alpha^t}^{2\lambda_\alpha^t}\frac{1}{\sqrt{\alpha-1}}|\frac{\partial\phi_\alpha}{\partial r}|dr \to \log 2\mu^{1-t}\sqrt{\frac{E(\sigma^1)}{\pi}} \quad in\ C^0([t_1,t_2])$$

and

$$\begin{aligned} &\frac{1}{\sqrt{\alpha-1}}(r|\frac{\partial\phi_\alpha}{\partial r}|)(\lambda_\alpha^t,\theta) \to \mu^{1-t}\sqrt{\frac{E(\sigma^1)}{\pi}} \quad in\ C^0([t_1,t_2]), \\ &\frac{1}{\sqrt{\alpha-1}}(r^{-1}|\frac{\partial\phi_\alpha}{\partial\theta}|)(\lambda_\alpha^t,\theta) \to 0 \quad in\ C^0([t_1,t_2]). \end{aligned}$$

Proof. We shall only prove the second conclusion, since the other two conclusions can be proved in a similar way.

In fact, if it was false, then there would exist $t_\alpha \in [t_1, t_2]$ and $\theta_\alpha \in [0, 2\pi]$ such that

$$\left|\frac{1}{\sqrt{\alpha-1}}(r|\frac{\partial\phi_\alpha}{\partial r}|)(\lambda_\alpha^{t_\alpha},\theta_\alpha) - \mu^{1-t_\alpha}\sqrt{\frac{E(\sigma^1)}{\pi}}\right| \geq b > 0. \tag{8.24}$$

However, by Lemma 8.7, we have

$$\frac{\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha-1}}\frac{\partial\phi_\alpha}{\partial r}(\lambda_\alpha^{t_\alpha},\theta_\alpha) \to \vec{a},$$

where $|\vec{a}| = \mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{\frac{E(\sigma^1)}{\pi}}$. This yields the following

$$\frac{1}{\sqrt{\alpha-1}}(r|\frac{\partial\phi_\alpha}{\partial r}|)(\lambda_\alpha^{t_\alpha},\theta_\alpha) \to \mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{\frac{E(\sigma^1)}{\pi}},$$

which contradicts to (8.24). We finished the proof of this corollary. □

For $0 < t_1 < t_2 < 1$, we define the following curve:

$$\omega_\alpha(r) := \frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta : \quad [\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}] \to \mathbb{R}^K,$$

which is denoted by γ_α, where (r,θ) is the polar coordinate around the point x_α.

Denote

$$\ddot{\omega}_\alpha := (\frac{d}{dr})^2\omega_\alpha, \quad \dot{\omega}_\alpha := \frac{d}{dr}\omega_\alpha.$$

By a direct computation, we have

$$\begin{aligned}
\ddot{\omega}_\alpha &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2}{\partial r^2}\phi_\alpha(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Delta\phi_\alpha(r,\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r}\frac{\partial}{\partial r}\phi_\alpha(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}A(\phi_\alpha)(d\phi_\alpha,d\phi_\alpha)d\theta - \frac{1}{2\pi}\int_0^{2\pi}O(|\nabla\phi_\alpha||\psi_\alpha|^2)d\theta + \frac{\alpha-1}{2\pi}\int_0^{2\pi}O(|\nabla^2\phi_\alpha|)d\theta - \frac{1}{r}\dot{\omega}_\alpha.
\end{aligned} \tag{8.25}$$

Proposition 8.9. *After passing to a subsequence, the sequence of the curves $\{\gamma_\alpha\}$ in $\mathbb{R}^N$, which is defined by ω_α, and parameterized by its arc length, converges to a curve $\omega \subset N$ as $\alpha \searrow 1$, where ω is a geodesic on N, i.e. it satisfies*

$$\frac{d^2\omega}{ds^2} + A(\omega)(\frac{d\omega}{ds},\frac{d\omega}{ds}) = 0.$$

Proof. Denote the induced second fundamental form of γ_α in $\mathbb{R}^K$ by A_{γ_α}. Set

$$G_\alpha := -\ddot{\omega}_\alpha - \frac{\dot{\omega}_\alpha}{r}.$$

By Lemma 5.1, it is easy to see that γ_α converges to some curve on N, denoted by γ.

Next, we will show that γ is just a geodesic on N.

Let s be the arc length parameter of γ_α, i.e.

$$s(r) = \int_{\lambda_\alpha^{t_\alpha}}^r |\dot{\omega}_\alpha(\tau)|d\tau,$$

where $t_\alpha \in [t_1,t_2]$.

By Lemma 8.7 and Corollary 8.8, for any $t_1 \in (0,1)$, we have

$$\begin{aligned}
\int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}}|\dot{\omega}_\alpha|dr &\le \int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}}\frac{\sqrt{\alpha-1}}{r}\mu\sqrt{\frac{E(\sigma^1)}{\pi}}dr \\
&= (t_1-t_\alpha)\sqrt{\alpha-1}\log\lambda_\alpha\mu\sqrt{\frac{E(\sigma^1)}{\pi}} \to (\lim_{\alpha\searrow 1}t_\alpha - t_1)\log\nu\mu\sqrt{\frac{E(\sigma^1)}{\pi}}
\end{aligned}$$

as $\alpha \searrow 1$. So, if $\nu = \infty$, then for any $s \in (0,\infty)$,

$$\omega_\alpha|_{[0,s]} \subset \omega_\alpha|_{[\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}]}.$$

If $\nu \in (1, \infty)$, then $\mu = 1$ and noting that

$$\int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} |\dot{\omega}_\alpha| dr \geq \int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t_1}} \frac{\sqrt{\alpha - 1}}{r} \sqrt{\frac{E(\sigma^1)}{\pi}} dr$$
$$= (t_1 - t_\alpha) \sqrt{\alpha - 1} \log \lambda_\alpha \sqrt{\frac{E(\sigma^1)}{\pi}} \to (\lim_{\alpha \searrow 1} t_\alpha - t_1) \log \nu \sqrt{\frac{E(\sigma^1)}{\pi}}$$

as $\alpha \searrow 1$, there exists $t_1' \in (0, t_1)$ such that

$$\omega_\alpha|_{[0,s]} \subset \omega_\alpha|_{[\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1'}]}$$

whenever $s \in \left(0, (\lim_{\alpha \searrow 1} t_\alpha - t_1) \log \nu \sqrt{\frac{E(\sigma^1)}{\pi}}\right)$.

Computing directly, we obtain

$$\frac{d^2 \omega_\alpha}{ds^2} = A_{\gamma_\alpha}(\omega_\alpha(s))(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds}) = \frac{1}{|\dot{\omega}_\alpha|^2} A_{\gamma_\alpha}(\omega_\alpha(t))(\frac{d\omega_\alpha}{dt}, \frac{d\omega_\alpha}{dt})$$
$$= \frac{1}{|\dot{\omega}_\alpha|^2}(\ddot{\omega}_\alpha - \frac{\langle \ddot{\omega}_\alpha, \dot{\omega}_\alpha \rangle}{|\dot{\omega}_\alpha|^2} \dot{\omega}_\alpha) = \frac{1}{|\dot{\omega}_\alpha|^2}(-G_\alpha + \frac{\langle G_\alpha, \dot{\omega}_\alpha \rangle}{|\dot{\omega}_\alpha|^2} \dot{\omega}_\alpha)$$
$$= \frac{1}{|\dot{\omega}_\alpha|^2} \frac{1}{2\pi} \int_0^{2\pi} A(\phi_\alpha)(\nabla \phi_\alpha, \nabla \phi_\alpha) d\theta + \frac{1}{|\dot{\omega}_\alpha|^2} \frac{1}{2\pi} \left(\int_0^{2\pi} O(|\nabla \phi_\alpha||\psi_\alpha|^2) + (\alpha - 1) O(|\nabla^2 \phi_\alpha|) d\theta \right)$$
$$- \frac{\dot{\omega}_\alpha}{|\dot{\omega}_\alpha|^4} \frac{1}{2\pi} \int_0^{2\pi} \langle A(\phi_\alpha)(\nabla \phi_\alpha, \nabla \phi_\alpha), \dot{\omega}_\alpha \rangle d\theta$$
$$- \frac{\dot{\omega}_\alpha}{|\dot{\omega}_\alpha|^4} \frac{1}{2\pi} \int_0^{2\pi} \langle O(|\nabla \phi_\alpha||\psi_\alpha|^2) + (\alpha - 1) O(|\nabla^2 \phi_\alpha|), \dot{\omega}_\alpha \rangle d\theta := \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}. \tag{8.26}$$

On one hand, by Lemma 8.7 and using the fact that

$$\langle A(\phi_\alpha)(\nabla \phi_\alpha, \nabla \phi_\alpha), \frac{\partial \phi_\alpha}{\partial r} \rangle = 0,$$

we have

$$\mathbf{III} = \frac{\dot{\omega}_\alpha}{|\dot{\omega}_\alpha|^4} \frac{1}{2\pi} \int_0^{2\pi} \langle A(\phi_\alpha)(\nabla \phi_\alpha, \nabla \phi_\alpha), \dot{\omega}_\alpha - \frac{\partial \phi_\alpha}{\partial r} \rangle d\theta$$
$$\leq \left\| \frac{|x - x_\alpha|}{\sqrt{\alpha - 1}} \nabla(\phi_\alpha - \phi_\alpha^*) \right\|_{C^0(D_{\lambda_\alpha^{t_1}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha))} \to 0 \tag{8.27}$$

as $\alpha \searrow 1$, where we used the fact derived from Lemma 8.7 and Corollary 8.8 that

$$|\dot{\omega}_\alpha| \geq C \frac{\sqrt{\alpha - 1}}{|x - x_\alpha|} \; and \; |\nabla \phi_\alpha| \leq C \frac{\sqrt{\alpha - 1}}{|x - x_\alpha|}.$$

On the other hand, by (8.22) and Corollary 8.6, Corollary 8.8, we have

$$\mathbf{II} + \mathbf{IV} \leq C \left(\frac{|x - x_\alpha||\nabla \phi_\alpha|}{\sqrt{\alpha - 1}} \frac{|x - x_\alpha||\psi_\alpha|^2}{\sqrt{\alpha - 1}} + \sqrt{\alpha - 1} \frac{|x - x_\alpha|^2 |\nabla^2 \phi_\alpha|}{\sqrt{\alpha - 1}} \right) \to 0 \tag{8.28}$$

as $\alpha \searrow 1$.

Therefore, it is easy to see that

$$|\frac{d^2\omega_\alpha}{ds^2}| \leq C(N).$$

Thus, $\phi_\alpha(s)$ will converge locally to a smooth vector valued function from $[0, s]$ into $\mathbb{R}^K$, denoted by $\omega(s)$, in the sense of C^1, i.e. $\gamma_\alpha|_{[\lambda_\alpha^{t_2}, \lambda_\alpha^{t_1}]}$ converges locally to the curve γ.

Now, we will show that γ is a geodesic.

By Lemma 8.7 and Corollary 8.8, we obtain

$$\begin{aligned}
\mathbf{I} &= \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\phi_\alpha)(\nabla\phi_\alpha, \nabla\phi_\alpha)d\theta \\
&= \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\left(\int_0^{2\pi}(A(\phi_\alpha) - A(\omega_\alpha))(\nabla\phi_\alpha, \nabla\phi_\alpha) + A(\omega_\alpha)(\nabla(\phi_\alpha - \omega_\alpha), \nabla\phi_\alpha)d\theta\right) \\
&\quad + \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\omega_\alpha)(\nabla\omega_\alpha, \nabla(\phi_\alpha - \omega_\alpha))d\theta + \frac{1}{|\dot{\omega}_\alpha|^2}\frac{1}{2\pi}\int_0^{2\pi} A(\omega_\alpha)(\nabla\omega_\alpha, \nabla\omega_\alpha)d\theta \\
&= \frac{1}{|\dot{\omega}_\alpha|^2}A(\omega_\alpha)(\nabla\omega_\alpha, \nabla\omega_\alpha) + O\left(\frac{|x - x_\alpha||\nabla(\phi_\alpha - \omega_\alpha)|}{\sqrt{\alpha - 1}}\right) + O(|\phi_\alpha - \omega_\alpha|). \qquad (8.29)
\end{aligned}$$

Combining (8.27), (8.28), (8.29) with Lemma 8.7, letting $\alpha \searrow 1$, we have

$$\frac{d\omega}{ds}(s) - \frac{d\omega}{ds}(0) = \int_0^s A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds})$$

which implies

$$\frac{d^2\omega}{ds^2} = A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}).$$

We finished the proof of this proposition. □

Now we can prove our main result in this section.

Proof of Theorem 8.1. By Proposition 8.9, we just need to compute the length of the geodesic. Without loss of generality, we assume $\lambda_\alpha^{t_1} = 2^{m_\alpha}\lambda_\alpha^{t_2}$ for some integer

$$m_\alpha = \frac{t_1 - t_2}{\log 2}\log\lambda_\alpha,$$

which tends to infinity as $\alpha \searrow 1$. Then it is sufficient to consider the three cases listed in the statement of the theorem.

Case (3): $\nu = \infty$.

By Corollary 8.8, there holds

$$L(\gamma_\alpha|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)}) \geq \sqrt{\alpha - 1}\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2 + o(1)\right).$$

Then

$$L(\gamma_\alpha) \geq Cm_\alpha\sqrt{\alpha - 1} \geq -C\sqrt{\alpha - 1}\log\lambda_\alpha \to \infty.$$

Case (2): $\nu \in (1, \infty)$.

First, we prove the following equalities

$$\lim_{\delta\to 0}\lim_{t\to 0}\lim_{\alpha\searrow 1} Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\phi_\alpha = 0 \tag{8.30}$$

and

$$\lim_{R\to\infty}\lim_{t\to 1}\lim_{\alpha\searrow 1} Osc_{D_{\lambda_\alpha^t}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha = 0. \tag{8.31}$$

In fact, by Lemma 8.2, there holds

$$\left(\int_{P_i}|x-x_\alpha|^{-2}|\frac{\partial\phi_\alpha}{\partial\theta}|^2dx\right)^{\frac{1}{2}} + \left(\int_{P_i}|x-x_\alpha|^{-\frac{4}{3}}|\frac{\partial\psi_\alpha}{\partial\theta}|^{\frac{4}{3}}dx\right)^{\frac{3}{4}} \le \left((\alpha-1)+e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}}\right)o(\alpha,\delta,R).$$

where $P_i = D_{e^{(i+1)L}\lambda_\alpha R}\setminus D_{e^{iL}\lambda_\alpha R}$, $0\le i\le k_\alpha := [\frac{\delta}{\lambda_\alpha R}]-1$.

By (8.21), we obtain

$$\begin{aligned}\int_{P_i}|\nabla\phi_\alpha|^2dx &\le C\left\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\right\|^2_{L^2(P_i)} + \|\psi_\alpha\|_{L^4(P_i)}\left\||x-x_\alpha|^{-1}\frac{\partial\psi_\alpha}{\partial\theta}\right\|_{L^{\frac{4}{3}}(P_i)}\\ &\quad + C\|\psi_\alpha\|^2_{L^4(P_i)}\left\||x-x_\alpha|^{-1}\frac{\partial\phi_\alpha}{\partial\theta}\right\|_{L^2(P_i)} + C(\alpha-1+\lambda_\alpha^{t_\alpha})\\ &\le (e^{-\frac{iL}{2}}+e^{-\frac{(k_\alpha-i)L}{2}})o(\alpha,\delta,R) + C(\alpha-1+e^{iL}\lambda_\alpha R).\end{aligned} \tag{8.32}$$

According to Lemma 5.1, we get

$$Osc_{P_i}\phi_\alpha \le (e^{-\frac{iL}{4}}+e^{-\frac{(k_\alpha-i)L}{4}})o(\alpha,\delta,R) + C(\sqrt{\alpha-1}+e^{\frac{iL}{2}}\sqrt{\lambda_\alpha R}). \tag{8.33}$$

Let $i_t\in[1,k_\alpha]$ be the positive integer such that

$$e^{i_tL}\lambda_\alpha R \le \lambda_\alpha^t < e^{(i_t+1)L}\lambda_\alpha R.$$

Then $k_\alpha - i_t \le C\log\frac{\delta}{\lambda_\alpha^t}$ and

$$Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}\phi_\alpha \le \sum_{i=i_t}^{k_\alpha} Osc_{P_i}\phi_\alpha \le o(\alpha,\delta,R) + C(\sqrt{\alpha-1}\log\frac{\delta}{\lambda_\alpha^t}+\sqrt{\delta})$$

and (8.30) follows immediately. The proof of (8.31) is similar.

Secondly, since $\nu\in(1,\infty)$ implies $\mu=1$, by Corollary 8.8, there holds

$$L(\gamma_\alpha|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)}) = \sqrt{\alpha-1}\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2 + o(1)\right).$$

Then

$$L(\gamma) = \lim_{\alpha\searrow 1}\sqrt{\alpha-1}m_\alpha\left(\sqrt{\frac{E(\sigma^1)}{\pi}}\log 2+o(1)\right) = (t_2-t_1)\sqrt{\frac{E(\sigma^1)}{\pi}}\log\nu.$$

Case (1): $\nu = 1$.

By (8.33), we have

$$Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha \le \sum_{i=1}^{k_\alpha} Osc_{P_i}\phi_\alpha \le o(\alpha,\delta,R) + C(\sqrt{\alpha-1}\log\frac{\delta}{\lambda_\alpha R} + \sqrt{\delta})$$

which implies that

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1} Osc_{D_\delta(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)}\phi_\alpha = 0.$$

We finished the proof of Theorem 8.1. □

Proof of Theorem 2.8. It is easy to see that Theorem 2.8 is a consequence of Theorem 8.1. □

9. Compactness with bounded Morse index

In this section, we shall first calculate the second variation formulas for the functionals L_α and L and define the notion of the Morse index of α-Dirac-harmonic maps and Dirac-harmonic maps. Then, we shall prove Theorem 2.12.

Let $(\phi,\psi): M\to N$ be an α-Dirac-harmonic map or a Dirac-harmonic map. Let ϕ^*TN be the pull-back bundle over M. Let V be a section of ϕ^*TN. We vary (ϕ,ψ) via

$$\phi_\tau(x) = exp_{\phi(x)}(\tau V),\ \psi_\tau(x) = \psi^i(x)\otimes\frac{\partial}{\partial y^i}(\phi_\tau(x)). \tag{9.1}$$

The second variation formula for the energy of the map is standard (see e.g. [55]),

$$\begin{aligned}
&\delta^2 E_\alpha(\phi)(V,V)\\
=\ & 2\alpha\int_M (1+|\nabla_g\phi|^2)^{\alpha-1}\Big(\langle\nabla_g V,\nabla_g V\rangle - R(V,\nabla_g\phi,\nabla_g\phi,V) - \langle div_g\{(1+|\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\},\nabla_V V\rangle\Big)dM\\
&+4\alpha(\alpha-1)\int_M (1+|\nabla_g\phi|^2)^{\alpha-2}\langle\nabla_g\phi,\nabla_g V\rangle^2 dM + 2\alpha\int_{\partial M}\langle(1+|\nabla_g\phi|^2)^{\alpha-1}\frac{\partial\phi}{\partial\overrightarrow{n}},\nabla_V V\rangle,
\end{aligned}$$

where $\overrightarrow{n}$ is the unit outside normal vector field of ∂M.

Next, we compute

$$\frac{d^2}{d\tau^2}|_{\tau=0}\int_M\langle\psi_\tau, \not{D}\psi_\tau\rangle dM = 2\int_M\langle\frac{d}{d\tau}|_{\tau=0}\psi_\tau, \frac{d}{d\tau}|_{\tau=0}\not{D}\psi_\tau\rangle dM + \int_M\langle\psi_\tau,\frac{d^2}{d\tau^2}|_{\tau=0}\not{D}\psi_\tau\rangle dM.$$

Choosing a local orthonormal basis $\{e_\beta\}$ on M such that $[e_\beta,\frac{\partial}{\partial\tau}]=0$, $\nabla_{e_\gamma}e_\beta = 0$ at a considered point, we have

$$\begin{aligned}
\frac{d}{d\tau}\not{D}\psi_\tau &= \frac{d}{d\tau}\left(e_\beta\cdot\widetilde{\nabla}_{e_\beta}(\psi^i\otimes\frac{\partial}{\partial y^i}(\phi_\tau))\right)\\
&= \frac{d}{d\tau}\left(e_\beta\cdot\nabla_{e_\beta}\psi^i\otimes\frac{\partial}{\partial y^i}(\phi_\tau) + e_\beta\cdot\psi^i\otimes\nabla_{e_\beta}\frac{\partial}{\partial y^i}(\phi_\tau)\right)\\
&= e_\beta\cdot\nabla_{e_\beta}\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau) + e_\beta\cdot\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{e_\beta}\frac{\partial}{\partial y^i}(\phi_\tau)\\
&= e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right) + e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{d\tau^2}|_{\tau=0}\not{D}\psi_\tau &= e_\beta\cdot\nabla_{e_\beta}\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau) + e_\beta\cdot\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{e_\beta}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\\
&\quad + e_\beta\cdot\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\left(R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right)\\
&= e_\beta\cdot\nabla_{e_\beta}\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau) + e_\beta\cdot\psi^i\otimes\nabla_{e_\beta}\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\\
&\quad + e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\\
&\quad + e_\beta\cdot\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\left(R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right)\\
&= e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right) + e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\\
&\quad + e_\beta\cdot\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\left(R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right).
\end{aligned}$$

Noting that

$$R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau) = R^j_{ikl}V^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j},$$

we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial\tau}}\left(R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right)\\
&= R^j_{ikl;p}V^pV^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j} + R(\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau) + R(\frac{\partial}{\partial\tau},\nabla_{\frac{\partial}{\partial\tau}}e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau).
\end{aligned}$$

Combining these with the fact that

$$\int_M\langle\psi,\not{D}\omega\rangle = \int_M\langle\not{D}\psi,\omega\rangle - \int_{\partial M}\langle\overrightarrow{n}\cdot\psi,\omega\rangle, \tag{9.2}$$

where $\psi,\omega\in C^1(M,\Sigma M\otimes\phi^\star TN)$, using the equation $\not{D}\psi$=0, we get

$$
\begin{aligned}
&\frac{d^2}{d\tau^2}|_{\tau=0}\int_M\langle\psi_\tau,\not{D}\psi_\tau\rangle dM\\
&=2\int_M\left\langle\psi^j\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^j}(\phi_\tau),e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right)+e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right\rangle dM\\
&\quad+\int_M\left\langle\psi,e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right)+e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right.\\
&\quad\left.+e_\beta\cdot\psi^i\otimes\left(R^j_{ikl;p}V^pV^kd\phi^l(e_\beta)\frac{\partial}{\partial y^j}+R(\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)+R(\frac{\partial}{\partial\tau},\nabla_{\frac{\partial}{\partial\tau}}e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right)\right\rangle dM\\
&=2\int_M\left\langle\psi^j\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^j}(\phi_\tau),e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\right)+e_\beta\cdot\psi^i\otimes R(\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right\rangle dM\\
&\quad-\int_{\partial M}\langle\overrightarrow{n}\cdot\psi,\psi^i\otimes\nabla_{\frac{\partial}{\partial\tau}}\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)\rangle+\int_M\left\langle\psi,e_\beta\cdot\psi^i\otimes R(\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right.\\
&\quad\left.+e_\beta\cdot\psi^i\otimes\left(R^j_{ikl;p}V^pV^kd\phi^l(e_\beta)\frac{\partial}{\partial y^j}+R(\frac{\partial}{\partial\tau},e_\beta)\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial y^i}(\phi_\tau)+R(\frac{\partial}{\partial\tau},\nabla_{\frac{\partial}{\partial\tau}}e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\right)\right\rangle dM.
\end{aligned}
$$

By the fact that

$$
\begin{aligned}
\langle\psi,e_\beta\cdot\psi^i\otimes R(\nabla_{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\tau},e_\beta)\frac{\partial}{\partial y^i}(\phi_\tau)\rangle&=2\langle R(\phi,\psi),\nabla_VV\rangle\\
&=2\langle Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right),\nabla_VV\rangle\\
&=2\alpha\left\langle\mathrm{div_g}\left\{(1+|\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right\},\nabla_VV\right\rangle,
\end{aligned}
$$

where the last equality follows from the following equivalent equation of (2.8) that

$$
\begin{aligned}
div_g\left\{(1+|\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right\}&-(1+|\nabla_g\phi|^2)^{\alpha-1}A(\phi)(\nabla_g\phi,\nabla_g\phi)\\
&-\frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right)=0,
\end{aligned}
$$

we obtain the following

Proposition 9.1. *Let $(\phi,\psi):M\to N$ be a α-Dirac-harmonic map and V be a smooth section of ϕ^*TN. Then the second variational formula of the functional $L_\alpha(\phi,\psi)$ with respect to the variations*

(9.1) *is*

$$
\begin{aligned}
\delta^2 L_\alpha(\phi,\psi)(V,V) &= \frac{d^2}{d\tau^2}|_{\tau=0} L_\alpha(\phi_\tau,\psi_\tau)\\
&= 2\alpha\int_M (1+|\nabla_g\phi|^2)^{\alpha-1}\Big(\langle\nabla_g V,\nabla_g V\rangle - R(V,\nabla_g\phi,\nabla_g\phi,V)\Big)dM\\
&\quad + 4\alpha(\alpha-1)\int_M (1+|\nabla_g\phi|^2)^{\alpha-2}\langle\nabla_g\phi,\nabla_g V\rangle^2 dM\\
&\quad + 2\int_M \Big\langle \psi^j\otimes\nabla_V\frac{\partial}{\partial y^j}, e_\beta\cdot\widetilde{\nabla}_{e_\beta}\Big(\psi^i\otimes\nabla_V\frac{\partial}{\partial y^i}\Big) + e_\beta\cdot\psi^i\otimes R(V,e_\beta)\frac{\partial}{\partial y^i}\Big\rangle dM\\
&\quad + \int_M \Big\langle \psi, e_\beta\cdot\psi^i\otimes\Big(R^j_{ikl;p}V^pV^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j} + R(V,e_\beta)\nabla_V\frac{\partial}{\partial y^i} + R(V,\nabla_{e_\beta}V)\frac{\partial}{\partial y^i}\Big)\Big\rangle dM\\
&\quad + 2\alpha\int_{\partial M}\langle(1+|\nabla_g\phi|^2)^{\alpha-1}\frac{\partial\phi}{\partial\vec{n}},\nabla_V V\rangle - \int_{\partial M}\langle\vec{n}\cdot\psi,\psi^i\otimes\nabla_V\nabla_V\frac{\partial}{\partial y^i}\rangle.
\end{aligned}
$$

By a similar computation, we have

Proposition 9.2. *Let $(\phi,\psi): M\to N$ be a Dirac-harmonic map and V be a smooth section of ϕ^*TN. Then the second variational formula of the functional $L(\phi,\psi)$ with respect to the variations* (9.1) *is*

$$
\begin{aligned}
\delta^2 L(\phi,\psi)(V,V) &= \frac{d^2}{d\tau^2}|_{\tau=0} L(\phi_\tau,\psi_\tau)\\
&= 2\int_M \Big(\langle\nabla_g V,\nabla_g V\rangle - R(V,\nabla_g\phi,\nabla_g\phi,V)\Big)dM\\
&\quad + 2\int_M \Big\langle \psi^j\otimes\nabla_V\frac{\partial}{\partial y^j}, e_\beta\cdot\widetilde{\nabla}_{e_\beta}\Big(\psi^i\otimes\nabla_V\frac{\partial}{\partial y^i}\Big) + e_\beta\cdot\psi^i\otimes R(V,e_\beta)\frac{\partial}{\partial y^i}\Big\rangle dM\\
&\quad + \int_M \Big\langle \psi, e_\beta\cdot\psi^i\otimes\Big(R^j_{ikl;p}V^pV^k d\phi^l(e_\beta)\frac{\partial}{\partial y^j} + R(V,e_\beta)\nabla_V\frac{\partial}{\partial y^i} + R(V,\nabla_{e_\beta}V)\frac{\partial}{\partial y^i}\Big)\Big\rangle dM\\
&\quad + 2\int_{\partial M}\langle\frac{\partial\phi}{\partial\vec{n}},\nabla_V V\rangle - \int_{\partial M}\langle\vec{n}\cdot\psi,\psi^i\otimes\nabla_V\nabla_V\frac{\partial}{\partial y^i}\rangle.
\end{aligned}
$$

Before giving the proof of our last Theorem 2.12, we first prove the following theorem.

Theorem 9.3. *Let N be a compact Riemannian manifold with $Ric_N \geq \lambda_0 > 0$. Let $(\phi_\alpha,\psi_\alpha)$: $(D_1(0),g)\to(N,h)$ be a sequence of α-Dirac-harmonic maps with uniformly bounded energy*

$$E_\alpha(\phi_\alpha)+E(\psi_\alpha)\leq\Lambda.$$

Suppose there is only one energy concentration point 0 in $D_1(0)$ for the sequence $(\phi_\alpha,\psi_\alpha)$ and there is only one bubble occurring at this point, if $\nu=\infty$, then the Morse index of $(\phi_\alpha,\psi_\alpha)$ tends to infinity.

Since $\mathrm{Ric}_N\geq\lambda_0>0$, by Myers' theorem, we know that, if

$$b\geq\frac{\pi}{\sqrt{\lambda_0(n-1)^{-1}}}+2\epsilon,$$

then there exists a tangent vector field $V_0(s)$ on N, which is smooth on γ, and is vanishing on $\gamma|_{[0,\epsilon]}$ and $\gamma|_{[b-\epsilon,b]}$, such that the second variation of the length of γ satisfies

$$I_\gamma(V_0,V_0)=\int_0^b(\langle\nabla_{\dot\gamma}V_0,\nabla_{\dot\gamma}V_0\rangle-R(V_0,\dot\gamma,\dot\gamma,V_0))ds<-\delta<0. \tag{9.3}$$

Let $s=s(r)$ be the arc-length parametrization of the curve

$$\omega_\alpha(r)=\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha(r,\theta)d\theta,$$

with $s(\lambda_\alpha^{t_0})=0$. Set $s(\lambda_\alpha^{t_\alpha^b})=b$, where $0<t_\alpha^b<t_0<1$. By Proposition 8.9, $\omega_\alpha(s)$ converges to γ in $C^1([0,b])$.

Lemma 9.4. *Suppose the limiting neck of $\{(\phi_\alpha,\psi_\alpha)\}$ is a geodesic of infinite length, then, for any given $b>0$ and fixed θ, $\phi_\alpha(se^{\sqrt{-1}\theta})$ converges to the geodesic γ in $C^1([0,b])$. Moreover, we have*

$$\left\|\frac{r(s)}{\sqrt{\alpha-1}}\left|\frac{\partial s}{\partial r}\right|-\mu^{1-t_0}\sqrt{\frac{E(\sigma^1)}{\pi}}\right\|_{C^0([0,b])}\to 0. \tag{9.4}$$

Proof. Since

$$s(\lambda_\alpha^{t_\alpha^b})=b,$$

by Lemma 8.7 and Corollary 8.8, we have

$$a=\int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}}\left|\frac{d\phi_\alpha^*(r)}{dr}\right|dr\ge C\int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}}\frac{\sqrt{\alpha-1}}{r}dr=C(t_\alpha^b-t_0)\sqrt{\alpha-1}\log\lambda_\alpha. \tag{9.5}$$

On the other hand, Theorem 2.8 (or Theorem 8.1) tells us

$$\sqrt{\alpha-1}\log\lambda_\alpha\to-\infty,$$

since the limiting neck (geodesic) is of infinite length. Hence, from above facts, we have

$$t_\alpha^b-t_0\to 0\quad\text{as}\quad\alpha\searrow 1. \tag{9.6}$$

We assume that $\phi_\alpha(se^{\sqrt{-1}\theta})$ does not converge to γ in $C^1[0,b]$. Then there exists $s_\alpha\in[0,b]$, such that

$$\sup_\theta\left|\frac{\partial\phi_\alpha}{\partial s}(s_\alpha e^{\sqrt{-1}\theta})-\frac{d\phi_\alpha^*}{ds}(s_\alpha)\right|>\epsilon>0.$$

Let $s_\alpha=s(\lambda_\alpha^{t_\alpha})$. Obviously, $t_\alpha\in[t_\alpha^b,t_0]$. Thus $t_\alpha\to t_0$. By Lemma 8.7, after passing to a subsequence, we have

$$\lim_{\alpha\searrow 1}\frac{\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha-1}}\left|\frac{\partial\phi_\alpha}{\partial r}(\lambda_\alpha^{t_\alpha}e^{\sqrt{-1}\theta})-\frac{d\phi_\alpha^*}{dr}(\lambda_\alpha^{t_\alpha})\right|\to 0.$$

Therefore, noting

$$\left|\frac{\partial\phi_\alpha(s_\alpha e^{\sqrt{-1}\theta})}{\partial s}-\frac{d\phi_\alpha^*(s_\alpha)}{ds}\right|=\left|\frac{dr}{ds}\right|\cdot\left|\frac{\partial\phi_\alpha}{\partial r}-\frac{d\phi_\alpha^*(r)}{dr}\right|_{r=\lambda_\alpha^{t_\alpha}}$$

and

$$\left|\frac{dr}{ds}\right|_{r=\lambda_\alpha^{t_\alpha}} \leq \frac{C\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha-1}},$$

we have

$$\left|\frac{\partial\phi_\alpha(s_\alpha e^{\sqrt{-1}\theta})}{\partial s} - \frac{d\phi_\alpha^*(s_\alpha)}{ds}\right| \leq \frac{C\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha-1}}\left|\frac{\partial\phi_\alpha}{\partial r} - \frac{d\phi_\alpha^*(r)}{dr}\right|_{r=\lambda_\alpha^{t_\alpha}} \to 0.$$

Thus, we get a contradiction and it follows

$$\|\frac{\partial\phi_\alpha}{\partial s}(se^{\sqrt{-1}\theta}) - \frac{d\phi_\alpha^*}{ds}(s)\|_{C^0([0,b])} \to 0.$$

Combining this with Lemma 8.7, we obtain that for any fixed θ,

$$\|\phi_\alpha(se^{\sqrt{-1}\theta}) - \phi_\alpha^*(s)\|_{C^1([0,b])} \to 0.$$

By the same way, we can prove (9.4). □

Lemma 9.5. *Under the assumption of Theorem 9.3, if the limiting neck of $\{(\phi_\alpha, \psi_\alpha)\}$ is an unstable geodesic which is parameterized on $[0, b]$ by arc length, then, for $\alpha - 1$ sufficiently small, there exists a section V_α of $\phi_\alpha^*(TN)$, which is supported in $D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)$, such that*

$$\delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha, V_\alpha) < 0.$$

Proof. Since the limiting neck of $\{(\phi_\alpha, \psi_\alpha)\}$, denoted by $\gamma : [0, b] \to N$, is not a stable geodesic, there exists a vector field V_0 on γ with $V_0|_{\gamma(0)} = 0$ and $V_0|_{\gamma(b)} = 0$ such that

$$I_\gamma(V_0, V_0) < 0.$$

Let P be the projection from $T\mathbb{R}^K$ to TN. We define

$$V_\alpha\left(r(s)e^{\sqrt{-1}\theta} + x_\alpha\right) = P_{\phi_\alpha\left(se^{\sqrt{-1}\theta}\right)}(V_0(s)),$$

where s is the arc-length parametrization of $\omega_\alpha(r)$ with $s(\lambda_\alpha^{t_0}) = 0$. Then, V_α is a smooth section of $\phi_\alpha^*(TN)$ which is supported in $D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)$. By Lemma 9.4, for any fixed θ, $V_\alpha(\phi_\alpha(se^{\sqrt{-1}\theta}))$

converges to $V_0(\gamma(s))$ in $C^1[0,b]$. Then[11]

$$\begin{aligned}
&\delta^2 L_\alpha(\phi_\alpha,\psi_\alpha)(V_\alpha,V_\alpha)\\
=\ & 2\alpha\int_{D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)}\left(\langle\nabla V_\alpha,\nabla V_\alpha\rangle - R(V_\alpha,\nabla\phi_\alpha,\nabla\phi_\alpha,V_\alpha)\right)dx\\
&+4\alpha(\alpha-1)\int_{D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha}}(x_0)}(1+|\nabla_g\phi_\alpha|^2)^{\alpha-2}\langle\nabla_g\phi_\alpha,\nabla_g V_\alpha\rangle^2 dvol_g\\
&+2\int_{D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\left\langle\psi_\alpha^j\otimes\nabla_{V_\alpha}\frac{\partial}{\partial y^j}, e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi_\alpha^i\otimes\nabla_{V_\alpha}\frac{\partial}{\partial y^i}\right)+e_\beta\cdot\psi_\alpha^i\otimes R(V_\alpha,e_\beta)\frac{\partial}{\partial y^i}\right\rangle dvol_g\\
&+\int_{D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_0}}(x_\alpha)}\left\langle\psi_\alpha, e_\beta\cdot\psi_\alpha^i\otimes\left(R^j_{ikl;p}V_\alpha^p V_\alpha^k d\phi_\alpha^l(e_\beta)\frac{\partial}{\partial y^j}+R(V_\alpha,e_\beta)\nabla_{V_\alpha}\frac{\partial}{\partial y^i}+R(V_\alpha,\nabla_{e_\beta}V_\alpha)\frac{\partial}{\partial y^i}\right)\right\rangle dvol_g\\
=\ & \mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}.
\end{aligned}$$

Next, we will show that

$$\lim_{\alpha\searrow 1}\frac{1}{\sqrt{\alpha-1}}\delta^2 L_\alpha(V_\alpha,V_\alpha) = 4\pi\mu\sqrt{\frac{E(\sigma^1)}{\pi}}I_\gamma(V_0,V_0). \tag{9.7}$$

First, we calculate $\mathbf{I}$:

$$\begin{aligned}
&\frac{\mathbf{I}}{\sqrt{\alpha-1}}\\
=\ & \frac{2\alpha}{\sqrt{\alpha-1}}\int_{D_{\lambda_\alpha^{t_\alpha^b}}\setminus D_{\lambda_\alpha^{t_\alpha}}(x_\alpha)}(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)}\left(\langle\nabla V_\alpha,\nabla V_\alpha\rangle - R(V_\alpha,\nabla\phi_\alpha,\nabla\phi_\alpha,V_\alpha)\right)dx\\
=\ & 2\alpha\int_0^{2\pi}\int_0^b(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)}\left(\langle\nabla_{\frac{\partial\phi_\alpha}{\partial s}}V_\alpha,\nabla_{\frac{\partial\phi_\alpha}{\partial s}}V_\alpha\rangle - R(V_\alpha,\frac{\partial\phi_\alpha}{\partial s},\frac{\partial\phi_\alpha}{\partial s},V_\alpha)\right)\frac{\left|\frac{\partial s}{\partial r}\right|}{\sqrt{\alpha-1}}rdsd\theta\\
&+\frac{2\alpha}{\sqrt{\alpha-1}}\int_0^{2\pi}\int_0^b(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)}\left(\langle\nabla_{r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}}V_\alpha,\nabla_{r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}}V_\alpha\rangle - R(V_\alpha,r^{-1}\frac{\partial\phi_\alpha}{\partial\theta},r^{-1}\frac{\partial\phi_\alpha}{\partial\theta},V_\alpha)\right)rdrd\theta\\
:=\ & \mathbf{I}_1+\mathbf{I}_2.
\end{aligned}$$

We claim that

$$\left(\left|\frac{\lambda_\alpha^{t(s)}}{\sqrt{\alpha-1}}\nabla_g\phi_\alpha\right|^2\frac{\alpha-1}{\lambda_\alpha^{2t(s)}}\right)^{(\alpha-1)}\longrightarrow\mu^{t_0}. \tag{9.8}$$

In fact, by Corollary 8.8, we have

$$\lim_{\alpha\searrow 1}\left(\left|\frac{\lambda_\alpha^{t(s)}}{\sqrt{\alpha-1}}\nabla\phi_\alpha\right|^2\frac{\alpha-1}{\lambda_\alpha^{2t(s)}}\right)^{(\alpha-1)} = \lim_{\alpha\searrow 1}\left(\frac{1}{\lambda_\alpha^{2t(s)}}\right)^{(\alpha-1)}.$$

[11] Noting that we do not have the boundary term in this case.

Since $\nu = \infty$, by Corollary 8.8, we have

$$b = \int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}} |\phi_\alpha^*| dr \geq C \int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}} \frac{\sqrt{\alpha-1}}{r} dr = -C(t_0 - t_\alpha^b)\sqrt{\alpha-1}\log\lambda_\alpha. \tag{9.9}$$

Thus, we obtain that

$$t_\alpha^b \to t_o.$$

This implies that (9.8) holds.

It follows from the fact that $\mu \geq 1$ and (9.8) that

$$(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)} = \left(1 + \left|\frac{\lambda_\alpha^{t(s)}}{\sqrt{\alpha-1}}\nabla_g\phi_\alpha\right|^2 \frac{\alpha-1}{\lambda_\alpha^{2t(s)}}\right)^{(\alpha-1)} \longrightarrow \mu^{t_0}.$$

Hence, we infer from the above and Lemma 9.4

$$\lim_{\alpha\searrow 1} \frac{\mathbf{I}_1}{\sqrt{\alpha-1}} = 4\pi\mu\sqrt{\frac{E(\nu)}{\pi}} I_\gamma(V_0, V_0).$$

For the term $\mathbf{I}_2$, by definition, we have

$$\nabla_{\frac{\partial\phi_\alpha}{\partial\theta}} V_\alpha = P_{\phi_\alpha(se^{\sqrt{-1}\theta})}\left(\frac{\partial V_\alpha}{\partial\theta}\right) = P_{\phi_\alpha(se^{\sqrt{-1}\theta})}\left(\frac{\partial}{\partial\theta}(P_{\phi_\alpha(se^{\sqrt{-1}\theta})})(V_0)\right),$$

where $\frac{\partial V_\alpha}{\partial\theta}$ is the derivative in $\mathbb{R}^K$. This leads to

$$|\nabla_{\frac{\partial\phi_\alpha}{\partial\theta}} V_\alpha| \leq C(a)\left|\frac{\partial\phi_\alpha}{\partial\theta}\right|.$$

Hence, we have

$$\begin{aligned}
\mathbf{I}_2 &= 2\alpha\int_0^{2\pi}\int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}} \frac{(1+|\nabla_g\phi_\alpha|^2)^{(\alpha-1)}}{\sqrt{\alpha-1}}\left(\langle\nabla_{r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}} V_\alpha, \nabla_{r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}} V_\alpha\rangle - R(V_\alpha, r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}, r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}, V_\alpha)\right) rdrd\theta \\
&\leq \frac{C}{\sqrt{\alpha-1}}\int_0^{2\pi}\int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}} |r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}|^2 rdrd\theta.
\end{aligned}$$

For a given $K > 1$, set

$$m_\alpha = \left[\frac{\log\lambda_\alpha^{t_\alpha^b - t_0}}{\log K}\right] + 1.$$

It is easy to see that

$$D_{\lambda_\alpha^{t_\alpha^b}} \setminus D_{\lambda_\alpha^{t_0}} \subset \cup_{i=1}^{m_\alpha}(D_{K^i\lambda_\alpha^{t_0}} \setminus D_{K^{i-1}\lambda_\alpha^{t_0}}).$$

By (9.9), there holds

$$\sqrt{\alpha-1}\, m_\alpha \leq C(K).$$

Then according to Corollary 8.3, we obtain

$$\begin{aligned}\mathbf{I}_2 &\leq \frac{C}{\sqrt{\alpha-1}}\int_0^{2\pi}\int_{\lambda_\alpha^{t_0}}^{\lambda_\alpha^{t_\alpha^b}} |r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}|^2 rdrd\theta \\ &\leq \frac{C\sqrt{\alpha-1}m_\alpha}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)} |r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \\ &\leq \frac{C(K)}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)} |r^{-1}\frac{\partial\phi_\alpha}{\partial\theta}|^2 dx \to 0. \qquad (9.10)\end{aligned}$$

Now, we turn to the term **II**. It is easy to check that

$$|\langle\nabla_g\phi_\alpha, \nabla_g V_\alpha\rangle| \leq C|\nabla_g\phi_\alpha|^2.$$

So, there exists a constant C such that

$$(1+|\nabla_g\phi_\alpha|^2)^{\alpha-2}\langle\nabla_g\phi_\alpha, \nabla_g V_\alpha\rangle^2 \leq C(1+|\nabla_g\phi_\alpha|^2)^\alpha.$$

This leads to

$$\frac{\mathbf{II}}{\sqrt{\alpha-1}} \leq C\sqrt{\alpha-1}\int_{D_{\lambda_\alpha^{t_0}}}(1+|\nabla_g\phi_\alpha|^2)^\alpha dvol_g \to 0.$$

Finally, we estimate the terms **III** and **IV**. In fact, by Young's inequality, we have

$$\begin{aligned}\frac{\mathbf{III}+\mathbf{IV}}{\sqrt{\alpha-1}} \leq \frac{C(N,\|V_0\|_{C^1})}{\sqrt{\alpha-1}}&\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}\left(|\psi_\alpha|^2+|\psi_\alpha||\nabla\psi_\alpha|+|\psi_\alpha|^2|\nabla\phi_\alpha|\right)dvol_g \\ \leq C(N,\|V_0\|_{C^1})\frac{\lambda_\alpha^{t_0}}{\sqrt{\alpha-1}}&\left(\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}|\psi_\alpha|^4 dvol_g\right)^{\frac12} \\ + C(N,\|V_0\|_{C^1})\frac{1}{\sqrt{\alpha-1}}&\left(\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}|\nabla\psi_\alpha|^{\frac43}dvol_g\right)^{\frac34}\left(\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}|\psi_\alpha|^4 dvol_g\right)^{\frac14} \\ + C(N,\|V_0\|_{C^1})\frac{1}{\sqrt{\alpha-1}}&\left(\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}|\nabla\phi_\alpha|^2 dvol_g\right)^{\frac12}\left(\int_{D_{\lambda_\alpha^{t_0}}(x_\alpha)\setminus D_{\lambda_\alpha^{t_\alpha^b}}(x_\alpha)}|\psi_\alpha|^4 dvol_g\right)^{\frac12}.\end{aligned}$$

On one hand, similarly to deriving (9.10), by (8.21) and Lemma 8.4, we have

$$\begin{aligned}\frac{1}{\sqrt{\alpha-1}}\int_{D_{\lambda_\alpha^{t_0}}\setminus D_{\lambda_\alpha^{t_\alpha^b}}}|\nabla\phi_\alpha|^2 dx &\leq \frac{\sqrt{\alpha-1}m_\alpha}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\nabla\phi_\alpha|^2 dx \\ &\leq \frac{C(K)}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\nabla\phi_\alpha|^2 dx \leq C(K)\end{aligned}$$

and

$$\frac{1}{\sqrt{\alpha-1}}\int_{D_{\lambda_\alpha^{t_0}}\setminus D_{\lambda_\alpha^{t_\alpha^b}}}|\psi_\alpha|^4dx \le \frac{\sqrt{\alpha-1}m_\alpha}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\psi_\alpha|^4dx$$
$$\le \frac{C(K)}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\psi_\alpha|^4dx \to 0.$$

On the other hand, by (8.21), (8.12), Corollary 8.3 and Lemma 8.4, for any fixed $k>1$, we have

$$\|\frac{\partial\psi_\alpha}{\partial r}\|_{L^{\frac{4}{3}}(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}$$
$$\le C\||x-x_\alpha|\frac{\partial\psi_\alpha}{\partial\theta}\|_{L^{\frac{4}{3}}(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))} + C\|\nabla\phi_\alpha\|_{L^2(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}\|\psi_\alpha\|_{L^4(D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha))}$$
$$= o((\alpha-1)^{\frac{3}{4}}),$$

which implies

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\int_{D_{2^k\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D_{2^{-k}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|\nabla\psi_\alpha|^{\frac{4}{3}}dx = 0. \tag{9.11}$$

Then, by (9.11) and Lemma 8.4, we get

$$\frac{1}{\sqrt{\alpha-1}}\int_{D_{\lambda_\alpha^{t_0}}\setminus D_{\lambda_\alpha^{t_\alpha^b}}}|\psi_\alpha|^4dx \le \frac{\sqrt{\alpha-1}m_\alpha}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\psi_\alpha|^4dx$$
$$\le \frac{C(K)}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\psi_\alpha|^4dx \to 0.$$

and

$$\frac{1}{\sqrt{\alpha-1}}\int_{D_{\lambda_\alpha^{t_0}}\setminus D_{\lambda_\alpha^{t_\alpha^b}}}|\nabla\psi_\alpha|^{\frac{4}{3}}dx \le \frac{\sqrt{\alpha-1}m_\alpha}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\nabla\psi_\alpha|^{\frac{4}{3}}dx$$
$$\le \frac{C(K)}{\alpha-1}\frac{1}{m_\alpha}\int_{\cup_{i=1}^{m_\alpha}\left(D_{K^i\lambda_\alpha^{t_0}}\setminus D_{K^{i-1}\lambda_\alpha^{t_0}}\right)}|\nabla\psi_\alpha|^{\frac{4}{3}}dx \to 0.$$

Combining these with the fact that

$$\lambda_\alpha = o((\alpha-1)^m),\ \textit{since}\ \nu>1,$$

we get

$$\frac{\mathbf{III}+\mathbf{IV}}{\sqrt{\alpha-1}} \to 0.$$

Thus, we finished the proof. □

Proof of Theorem 9.3. Since $\nu=\infty$, Theorem 8.1 tells us that the limiting neck of $\{(\phi_\alpha,\psi_\alpha)\}$ is a geodesic of infinite length. Therefore, for any given t_1, by Corollary 8.8, we know that there exists a suitable positive constant $\epsilon_1 > 0$ such that, as $\alpha - 1$ is small enough, the arc length b of $\omega_\alpha(s)$ on $D_{\lambda_\alpha^{t_1-\epsilon}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_1}}(x_\alpha)$ satisfies

$$b > l_N = \frac{\pi}{\sqrt{\lambda(n-1)^{-1}}}.$$

According to Lemma 9.5, there exists a section V_α^1 of $\phi_\alpha^*(TN)$, which is 0 outside the region $D_{\lambda_\alpha^{t_1-\epsilon}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_1}}(x_\alpha)$, satisfying

$$\delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha^1, V_\alpha^1) < 0.$$

By the same method, for $t_2 = t_1 - 2\epsilon_1$, we can also pick $\epsilon_2 > 0$ and construct a section V_α^2, which is 0 outside $D_{\lambda_\alpha^{t_2-\epsilon_2}}(x_\alpha) \setminus D_{\lambda_\alpha^{t_2}}(x_\alpha)$, such that

$$\delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha^2, V_\alpha^2) < 0.$$

Since the limiting neck is a geodesic of infinite length, then, when $\alpha - 1$ is sufficiently small, there exists a series of sections $\{V_\alpha^3, V_\alpha^4, \cdots, V_\alpha^k\}$ satisfying that for any $1 \le i \le k$ there holds

$$\delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha^i, V_\alpha^i) < 0.$$

Obviously, $V_\alpha^1, V_\alpha^2, \cdots, V_\alpha^k$ are linearly independent. This means that

$$\text{Index }(\phi_\alpha, \psi_\alpha; L_\alpha) \ge k.$$

Therefore, we get

$$\text{Index }(\phi_\alpha, \psi_\alpha; L_\alpha) \to +\infty, \quad \text{as } \alpha \searrow 1.$$

Thus, we complete the proof of the theorem. □

From the proof of Theorem 9.3, we obtain the following theorem

Theorem 9.6. *Let N be a compact Riemannian manifold with finite fundamental group. Let $(\phi_\alpha, \psi_\alpha) : (D_1(0), g) \to (N, h)$ be a sequence of α-Dirac-harmonic maps with uniformly bounded energy*

$$E_\alpha(\phi_\alpha) + E(\psi_\alpha) \le \Lambda.$$

Suppose there is only one energy concentration point 0 in $D_1(0)$ for the sequence of $(\phi_\alpha, \psi_\alpha)$ and there is only one bubble occurring at this point, if $\nu = \infty$, then the Morse index of $(\phi_\alpha, \psi_\alpha)$ tends to infinity.

Proof. We prove this theorem by a contradiction. If it is false, then there exists an uniform integer $\overline{m}$ such that

$$\text{Index }(\phi_\alpha, \psi_\alpha; L_\alpha) \le \overline{m} < \infty.$$

Let $s = s(r)$ be the arc-length parameter as before with $s(\lambda_\alpha^{t_0}) = 0$, $s(\lambda_\alpha^{t_\alpha^b}) = b$ and $s(\lambda_\alpha^{s_\alpha^b}) = -b$ where $0 < t_\alpha^b < t_0 < s_\alpha^b < 1$. By Proposition 8.9, $\omega_\alpha(s)$ converges to γ in $C^1([-b, b])$.

Next, we will show that the Morse index of γ on any subintervals $[-b, b]$ is uniformly bounded.

In fact, let $V_1(s), ..., V_m(s)$ be the m linearly independent tangent vector fields on N, which are smooth on γ, and are vanishing on $\gamma|_{[-b,-b+\epsilon]}$ and $\gamma|_{[b-\epsilon,b]}$ for any big $b > 0$ and small $\epsilon > 0$, such that for any $q \in \{1, ..., m\}$, the second variation of the length of γ satisfies

$$I_\gamma(V_q, V_q) = \int_{-b}^{b} \Big(\langle \nabla_{\dot\gamma} V_q, \nabla_{\dot\gamma} V_q\rangle - R(V_q, \dot\gamma, \dot\gamma, V_q)\Big)\, ds < -\delta < 0. \tag{9.12}$$

Let P be the projection from $T\mathbb{R}^K$ to TN. We define

$$V_\alpha^q\left(r(s)e^{\sqrt{-1}\theta}+x_\alpha\right)=P_{\phi_\alpha(se^{\sqrt{-1}\theta})}(V_q(s)).$$

Similarly to the proof of Lemma 9.5, we have

$$\delta^2 L_\alpha(\phi_\alpha,\psi_\alpha)(V_\alpha^q,V_\alpha^q)<0,$$

when $\alpha-1$ is small enough.

Thus,

$$m\leq \text{ Index }(\phi_\alpha,\psi_\alpha;L_\alpha)\leq\overline{m}<\infty,$$

which implies that the Morse index of geodesic γ is uniformly bounded by $\overline{m}$, since b is arbitrary.

Since the target manifold N has a finite fundamental group, by Gromov's estimate (see Corollary 3.3.5 in [47]), the length of the geodesic is controlled by the bound of the Morse index, i.e.

$$length(\gamma)\leq C(Index(\gamma)+1)\leq C(\overline{m}+1)<\infty,$$

which is a contradiction to $\nu=\infty$. Thus, we finished the proof of this theorem. □

It is well known that one can remove finitely many points from a non-trivial harmonic map $f:(M,g)\to(N,h)$ without affecting its Morse index (see Sect. 4 in [44] and Prop. 1.9 in [20]). Now, we will extend this property to the case of Dirac-harmonic maps.

Lemma 9.7. *Let m be the Morse index of a non-trivial Dirac-harmonic map $(\phi,\psi):M\to N$. Given any finite collections of interior points $\{z_1,...,z_l\}\subset M\setminus\partial M$, there are a linear subspace Ξ of $\Gamma(\phi^*TN)$ of dimension m and m linearly independent sections $V_1,...,V_m$ of Ξ, such that*

(1) *the second variation of L on Ξ with respect to the variations* (9.1) *is negative, i.e. for any $V\in\Xi$, there holds*

$$\delta^2 L(\phi,\psi)(V,V)<0;$$

(2) *$V_1,...,V_m$ vanish in neighborhoods of $\{z_1,...,z_l\}$.*

Proof. By assumptions of the lemma, there is a linear subspace $\Xi_0=span\{V_1^0,...,V_m^0\}$ of $\Gamma(\phi^*TN)$ of dimension m such that the second variation $\delta^2L(\phi,\psi)$ is negative on Ξ_0.

Denote $B_{r_0}^M(z_i)$ the geodesic disk of radius r at center point z_i on M. Choose

$$r_0<\min\left\{\min_{i=1,...,l}dist(z_i,\partial M),1\right\}$$

small enough such that $B_{r_0}^M(z_i)\cap B_{r_0}^M(z_j)=\emptyset$ if $i\neq j$ and the radial distance functions $r_i:B_{r_0}^M(z_i)\setminus\{z_i\}\to\mathbb{R}$ is smooth. For each $\tau\in(0,r_0)$, define the Lipschitz cut-off function as follows:

$$\eta_\tau(p)=\begin{cases}0, & if\ 0\leq r_i(p)<\tau^2 \text{ for some } i,\\ \frac{2\log\tau-\log r_i(p)}{\log\tau}, & if\ \tau^2\leq r_i(p)<\tau \text{ for some } i,\\ 1, & \text{otherwise.}\end{cases}$$

It is easy to see that there is a constant $C>0$, independent of τ, such that

$$\int_M|\nabla_g\eta_\tau|^2dM\leq\frac{C}{|\log\tau|}.$$

By mollifying, in fact, for any $\tau > 0$, we can construct smooth cut-off function η_τ (still use this notation for simplicity) which is zero near the points $z_1, ..., z_l$, such that $\eta_\tau \in [0, 1+\tau]$ and

$$\int_M |\nabla_g \eta_\tau|^2 dM \leq \frac{C}{|\log \tau|},$$

where $C > 0$ is independent of τ.

By Young's inequality, we have that for any $\epsilon > 0$, there holds

$$\begin{aligned}\int_M |\nabla_g(\eta_\tau V_q^0)|^2 dM - \int_M \eta_\tau^2 |\nabla_g V_q^0|^2 dM &\leq \int_M |\nabla_g \eta_\tau V_q^0|^2 dM + 2\int_M |\eta_\tau \nabla_g V_q^0||\nabla_g \eta_\tau V_q^0| dM \\ &\leq \|V_q^0\|^2_{C^1(M)} \left(\int_M |\nabla_g \eta_\tau V_q^0|^2 dM + \int_M |\nabla_g \eta_\tau||\nabla_g \phi| dM \right) \\ &\leq \frac{C}{\sqrt{|\log \tau|}} \|V_q^0\|^2_{C^1(M)} (1 + \|\nabla_g \phi\|_{L^2(M)})\end{aligned}$$

and

$$\begin{aligned}&\left| 2\int_M \left\langle \psi^j \otimes \nabla_{\eta_\tau V_q^0} \frac{\partial}{\partial y^j}, e_\beta \cdot \nabla_{e_\beta} \eta_\tau (\psi^i \otimes \nabla_{V_q^0} \frac{\partial}{\partial y^i}) \right\rangle \right| + \left| \int_M \left\langle \psi, e_\beta \cdot \psi^i \otimes R(\eta_\tau V_q^0, \nabla_{e_\beta} \eta_\tau V_q^0) \frac{\partial}{\partial y^i} \right\rangle dM \right| \\ &\leq C \|\nabla \eta_\tau\|_{L^2(M)} \|\psi\|^2_{L^4(M)} \|V_q^0\|^2_{C^1(M)} \leq \frac{C}{\sqrt{|\log \tau|}} \|\psi\|^2_{L^4(M)} \|V_q^0\|^2_{C^1(M)}.\end{aligned}$$

Combining these with the fact that $\eta_\tau|_{\partial M} \equiv 1$, we have

$$\begin{aligned}&\delta^2 L(\phi, \psi)(\eta_\tau V_q^0, \eta_\tau V_q^0) \\ &\leq 2\int_M \eta_\tau^2 \left(\langle \nabla_g V_q^0, \nabla_g V_q^0 \rangle - R(V_q^0, \nabla_g \phi, \nabla_g \phi, V_q^0) \right) dM \\ &\quad + 2\int_M \eta_\tau^2 \left\langle \psi^j \otimes \nabla_{V_q^0} \frac{\partial}{\partial y^j}, e_\beta \cdot \widetilde{\nabla}_{e_\beta} \left(\psi^i \otimes \nabla_{V_q^0} \frac{\partial}{\partial y^i} \right) + e_\beta \cdot \psi^i \otimes R(V_q^0, e_\beta) \frac{\partial}{\partial y^i} \right\rangle dM \\ &\quad + \int_M \eta_\tau^2 \left\langle \psi, e_\beta \cdot \psi^i \otimes \left(R^j_{ikl;p} (V_q^0)^p (V_q^0)^k d\phi^l(e_\beta) \frac{\partial}{\partial y^j} + R(V_q^0, e_\beta) \nabla_{V_q^0} \frac{\partial}{\partial y^i} + R(V_q^0, \nabla_{e_\beta} V_q^0) \frac{\partial}{\partial y^i} \right) \right\rangle dM \\ &\quad + 2\int_{\partial M} \eta_\tau^2 \langle \frac{\partial \phi}{\partial \vec{n}}, \nabla_{V_q^0} V_q^0 \rangle - \int_{\partial M} \eta_\tau^2 \langle \vec{n} \cdot \psi, \psi^i \otimes \nabla_{V_q^0} \nabla_{V_q^0} \frac{\partial}{\partial y^i} \rangle \\ &\quad + \frac{C}{\sqrt{|\log \tau|}} \|V_q^0\|^2_{C^1(M)} (1 + \|\nabla_g \phi\|_{L^2(M)} + \|\psi\|^2_{L^4(M)}) \\ &\to \delta^2 L(\phi, \psi)(V_q^0, V_q^0) < 0,\end{aligned}$$

as $\tau \to 0$.

Therefore, when τ is small enough, we take $V_i = \eta_\tau V_q^0,\ q = 1, ..., m$ and $\Xi = span\{V_1, ..., V_m\}$ which satisfy (1) and (2). We finished the proof of this lemma. □

With the help of Theorem 9.3 and Theorem 9.6, we now give the proof of Theorem 2.12.

Proof of Theorem 2.12. For the conclusion (1), we prove it by contradiction. In fact, if it false, by Lemma 7.5, there exist a linear subspace Ξ of $\Gamma(\phi^* TN)$ of dimension $m \geq \Lambda_{index} + 1$ and m linearly independent sections $V_1, ..., V_m$ of Ξ, such that

(1) the second variation of L on Ξ with respect to the variations (9.1) is negative, i.e. for any $V \in \Xi$, there holds

$$\delta^2 L(\phi, \psi)(V, V) < 0;$$

(2) $V_1, ..., V_m$ vanish in neighborhoods of $\mathbf{S} = \{p_1, ..., p_I\}$.

Let P be the projection from $T\mathbb{R}^K$ to TN. We define

$$V_\alpha^q(x) = P_{\phi_\alpha(x)} V_q(x),\ q = 1, ..., m.$$

Since $V_1, ..., V_m$ vanish in neighborhoods of $\mathbf{S} = \{p_1, ..., p_I\}$ and $\phi_\alpha \to \phi$ strongly in $C^2_{loc}(M \setminus \mathbf{S})$, it is easy to see that $V_\alpha^q \to V_q$ in $C^1(M)$ as $\alpha \searrow 1$.

By Proposition 9.1, we have

$$\begin{aligned}
&\delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha^q, V_\alpha^q)\\
&= 2\alpha \int_M (1 + |\nabla_g \phi_\alpha|^2)^{\alpha-1} \left(\langle \nabla_g V_\alpha^q, \nabla_g V_\alpha^q \rangle - R(V_\alpha^q, \nabla_g \phi_\alpha, \nabla_g \phi_\alpha, V_\alpha^q) \right) dM\\
&\quad + 4\alpha(\alpha - 1) \int_M (1 + |\nabla_g \phi_\alpha|^2)^{\alpha-2} \langle \nabla_g \phi_\alpha, \nabla_g V_\alpha^q \rangle^2 dM\\
&\quad + 2 \int_M \left\langle \psi_\alpha^j \otimes \nabla_{V_\alpha^q} \frac{\partial}{\partial y^j}, e_\beta \cdot \widetilde{\nabla}_{e_\beta} \left(\psi_\alpha^i \otimes \nabla_{V_\alpha^q} \frac{\partial}{\partial y^i} \right) + e_\beta \cdot \psi_\alpha^i \otimes R(V_\alpha^q, e_\beta) \frac{\partial}{\partial y^i} \right\rangle dM\\
&\quad + \int_M \left\langle \psi_\alpha, e_\beta \cdot \psi_\alpha^i \otimes \left(R^j_{ikl;p} (V_\alpha^q)^p (V_\alpha^q)^k d\phi_\alpha^l(e_\beta) \frac{\partial}{\partial y^j} + R(V_\alpha^q, e_\beta) \nabla_{V_\alpha^q} \frac{\partial}{\partial y^i} + R(V_\alpha^q, \nabla_{e_\beta} V_\alpha^q) \frac{\partial}{\partial y^i} \right) \right\rangle dM\\
&\quad + 2\alpha \int_{\partial M} \langle (1 + |\nabla_g \phi_\alpha|^2)^{\alpha-1} \frac{\partial \phi_\alpha}{\partial \overrightarrow{n}}, \nabla_{V_\alpha^q} V_\alpha^q \rangle - \int_{\partial M} \langle \overrightarrow{n} \cdot \psi_\alpha, \psi_\alpha^i \otimes \nabla_{V_\alpha^q} \nabla_{V_\alpha^q} \frac{\partial}{\partial y^i} \rangle.
\end{aligned}$$

First, we can easily see that

$$4\alpha(\alpha - 1) \int_M (1 + |\nabla_g \phi_\alpha|^2)^{\alpha-2} \langle \nabla_g \phi_\alpha, \nabla_g V_\alpha^q \rangle^2 dM \le C(\alpha - 1) \|V_\alpha^q\|_{C^1(M)} \int_M (1 + |\nabla_g \phi_\alpha|^2)^\alpha dM \to 0,$$

as $\alpha \searrow 1$.

Combining this with the facts that $V_\alpha^1, ..., V_\alpha^m$ vanish in neighborhoods of $\mathbf{S} = \{p_1, ..., p_I\}$, $(\phi_\alpha, \psi_\alpha) \to (\phi, \psi)$ strongly in $C^2_{loc}(M \setminus \mathbf{S}) \times C^1_{loc}(M \setminus \mathbf{S})$ and $V_\alpha^i \to V_i$ in $C^1(M)$ as $\alpha \searrow 1$, we get

$$\lim_{\alpha \searrow 1} \delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V_\alpha^q, V_\alpha^q) = \delta^2 L(\phi, \psi)(V_q, V_q) < 0,\ q = 1, ..., m.$$

Thus, when α is sufficiently close to 1, $(\phi_\alpha, \psi_\alpha)$ must have Morse index at least $m \ge \Lambda_{index} + 1$ which is a contradiction to the assumption.

For the conclusion (2), if it is false, then there exists a bubble, i.e. a nontrivial Dirac-harmonic sphere $(\sigma_i^l, \xi_i^l) : S^2 (or\, \mathbb{R}^2) \to N$, $1 \le l \le l_i$, $1 \le i \le I$, such that

$$\text{Index } (\sigma_i^l, \xi_i^l; L) = m \ge \Lambda_{index} + 1.$$

By classical blow-up theory (see Page 8), we know that there exist a sequence of points x_α^{il} and a sequence of positive numbers λ_α^{il}, such that $x_\alpha^{il} \to x_i$, $\lambda_\alpha^{il} \to 0$ and the two rescaled fields

$$\sigma_\alpha^{il} = \phi_\alpha(x_\alpha^{il} + \lambda_\alpha^{il} x), \quad \xi_\alpha^{il} = (\lambda_\alpha^{il})^{\alpha-1} \sqrt{\lambda_\alpha^{il}} \psi_\alpha(x_\alpha^{il} + \lambda_\alpha^{il} x)$$

converge in $C^k_{loc}(\mathbb{R}^2\setminus\widetilde{\mathbf{S}})$ to (σ^l_i,ξ^l_i), where $\widetilde{\mathbf{S}}$ is a finite set. By Lemma 7.5, there exist a linear subspace Ξ of $\Gamma((\sigma^l_i)^*TN)$ of dimension $m\geq\Lambda_{index}+1$ and m linearly independent sections $V_1,...,V_m$ of Ξ, such that

(1) the second variation of L on Ξ with respect to the variations (9.1) is negative, i.e. for any $V\in\Xi$, there holds
$$\delta^2L(\sigma^l_i,\xi^l_i)(V,V)<0;$$
(2) $V_1,...,V_m$ vanish in neighborhoods of $\widetilde{\mathbf{S}}\cup\{\infty\}$.

Here, V_q vanishes in a neighborhood of point $\{\infty\}$ means that there exists a big constant $R_0>0$ such that $V_q(x)\equiv 0$ when $|x|\geq\frac{R_0}{2}$.

Choosing $R>R_0$ big enough such that $\widetilde{\mathbf{S}}\subset D_{\frac{R}{2}}(0)$, for $q=1,...,m$, define
$$V^q_\alpha(x):=\begin{cases}P_{\phi_\alpha(x)}V_q(\frac{x-x^{il}_\alpha}{\lambda^{il}_\alpha}), & x\in B^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha),\\ 0, & otherwise,\end{cases}$$
where P is the projection from $T\mathbb{R}^K$ to TN. Since $V_1,...,V_m$ vanish in neighborhoods of $\widetilde{\mathbf{S}}\cup\{\infty\}$ and $\phi_\alpha(x^{il}_\alpha+\lambda^{il}_\alpha x)\to\sigma^l_i$ strongly in $C^2_{loc}(\mathbb{R}^2\setminus\mathbf{S})$, it is easy to see that $\widetilde{V}^q_\alpha(x):=V^q_\alpha(x^{il}_\alpha+\lambda^{il}_\alpha x)\to V_q(x)$ in $C^1(\mathbb{R}^2)$ as $\alpha\searrow 1$.

Noting that V^q_α vanishes near the boundary ∂M, by Proposition 9.1, we have
$$\begin{aligned}
&\delta^2L_\alpha(\phi_\alpha,\psi_\alpha)(V^q_\alpha,V^q_\alpha)\\
&=2\alpha\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)}(1+|\nabla_g\phi_\alpha|^2)^{\alpha-1}\left(\langle\nabla_gV^q_\alpha,\nabla_gV^q_\alpha\rangle-R(V^q_\alpha,\nabla_g\phi_\alpha,\nabla_g\phi_\alpha,V^q_\alpha)\right)dM\\
&\quad+4\alpha(\alpha-1)\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)}(1+|\nabla_g\phi_\alpha|^2)^{\alpha-2}\langle\nabla_g\phi_\alpha,\nabla_gV^q_\alpha\rangle^2dM\\
&\quad+2\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)}\left\langle\psi^j_\alpha\otimes\nabla_{V^q_\alpha}\frac{\partial}{\partial y^j},e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left(\psi^i_\alpha\otimes\nabla_{V^q_\alpha}\frac{\partial}{\partial y^i}\right)+e_\beta\cdot\psi^i_\alpha\otimes R(V^q_\alpha,e_\beta)\frac{\partial}{\partial y^i}\right\rangle dM\\
&\quad+\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)}\left\langle\psi_\alpha,e_\beta\cdot\psi^i_\alpha\otimes\left(R^j_{ikl;p}(V^q_\alpha)^p(V^q_\alpha)^kd\phi^l_\alpha(e_\beta)\frac{\partial}{\partial y^j}+R(V^q_\alpha,e_\beta)\nabla_{V^q_\alpha}\frac{\partial}{\partial y^i}+R(V^q_\alpha,\nabla_{e_\beta}V^q_\alpha)\frac{\partial}{\partial y^i}\right)\right\rangle dM\\
&=2\alpha(\lambda^{il}_\alpha)^{2(1-\alpha)}\int_{D_R(0)}\left((\lambda^{il}_\alpha)^2+|\nabla_{g(x^{il}_\alpha+\lambda^{il}_\alpha x)}\sigma^{il}_\alpha|^2\right)^{\alpha-1}\left(\langle\nabla\widetilde{V}^q_\alpha,\nabla\widetilde{V}^q_\alpha\rangle-R(\widetilde{V}^q_\alpha,\nabla\sigma^{il}_\alpha,\nabla\sigma^{il}_\alpha,\widetilde{V}^q_\alpha)\right)dx\\
&\quad+4\alpha(\alpha-1)\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)}(1+|\nabla_g\phi_\alpha|^2)^{\alpha-2}\langle\nabla_g\phi_\alpha,\nabla_gV^q_\alpha\rangle^2dM\\
&\quad+2(\lambda^{il}_\alpha)^{2(1-\alpha)}\int_{D_R(0)}\left\langle(\xi^{il}_\alpha)^j\otimes\nabla_{\widetilde{V}^q_\alpha}\frac{\partial}{\partial y^j},e_\beta\cdot\widetilde{\nabla}_{e_\beta}\left((\xi^{il}_\alpha)^k\otimes\nabla_{\widetilde{V}^q_\alpha}\frac{\partial}{\partial y^k}\right)+e_\beta\cdot(\xi^{il}_\alpha)^k\otimes R(\widetilde{V}^q_\alpha,e_\beta)\frac{\partial}{\partial y^k}\right\rangle dx\\
&\quad+(\lambda^{il}_\alpha)^{2(1-\alpha)}\int_{D_R(0)}\\
&\qquad\left\langle(\xi^{il}_\alpha),e_\beta\cdot(\xi^{il}_\alpha)^m\otimes\left(R^j_{mkn;p}(\widetilde{V}^q_\alpha)^p(\widetilde{V}^q_\alpha)^kd(\sigma^{il}_\alpha)^n(e_\beta)\frac{\partial}{\partial y^j}+R(\widetilde{V}^q_\alpha,e_\beta)\nabla_{\widetilde{V}^q_\alpha}\frac{\partial}{\partial y^m}+R(\widetilde{V}^q_\alpha,\nabla_{e_\beta}\widetilde{V}^q_\alpha)\frac{\partial}{\partial y^m}\right)\right\rangle dx.
\end{aligned}$$

Noting the facts that $\widetilde{V}^1_\alpha, ..., \widetilde{V}^m_\alpha$ vanish in neighborhoods of $\widetilde{\mathbf{S}}$, $(\sigma^{il}_\alpha, \xi^{il}_\alpha) \to (\sigma^{il}, \xi^{il})$ strongly in $C^k_{loc}(\mathbb{R}^2 \setminus \widetilde{\mathbf{S}})$, $\widetilde{V}^q_\alpha \to V_q$ in $C^1(\mathbb{R}^2)$ as $\alpha \searrow 1$ and

$$4\alpha(\alpha-1)\int_{D^M_{\lambda^{il}_\alpha R}(x^{il}_\alpha)} (1+|\nabla_g \phi_\alpha|^2)^{\alpha-2}\langle \nabla_g \phi_\alpha, \nabla_g V^q_\alpha\rangle^2 dM \to 0,$$

as $\alpha \searrow 1$, we get

$$\lim_{\alpha \searrow 1} \delta^2 L_\alpha(\phi_\alpha, \psi_\alpha)(V^q_\alpha, V^q_\alpha) = \mu_{il}\delta^2 L(\sigma^l_i, \xi^l_i)(V_q, V_q) < 0, \ q = 1, ..., m.$$

Thus, when α is sufficiently close to 1, $(\phi_\alpha, \psi_\alpha)$ must have Morse index at least $m \geq \Lambda_{index} + 1$ which is a contradiction to the assumption.

For the conclusions (3) and (4), it is easy to see that they are two consequences of Theorem 9.3 and Theorem 9.6. We finished the proof. □

References

[1] B. Ammann and N. Ginoux, *Dirac-harmonic maps from index theory*, Calc. Var. Partial Differential Equations **47** (2013), no. 3-4, 739–762.

[2] B. Ammann and N. Ginoux, *Some examples of Dirac-harmonic maps*, Lett. Math. Phys. (2018), 1–14.

[3] A. Balinsky and W. Evans, *Some recent results on Hardy-type inequalities*, Applied Mathematics and Information Sciences, **4** (2010), no. 2, 191–208.

[4] C. Bär and W. Ballmann, *Boundary value problems for elliptic differential operators of first order*, Surveys in differential geometry. Vol. XVII, Surv. Differ. Geom., vol. 17, Int. Press, Boston, MA, 2012, pp. 1–78.

[5] V. Branding, *The evolution equations for regularized Dirac-geodesics*. J. Geom. Phys. **100** (2016), 1–19.

[6] K. C. Chang, *Heat flow and boundary value problem for harmonic maps*, Ann. Inst. H. Poincare Anal. Non Lineaire 6 (1989), no. 5, 363-395.

[7] J. Chen and Y. Li, *Homotopy classes of harmonic maps of the stratified 2-spheres and applications to geometric flows*, Adv. Math. 263 (2014), 357-388.

[8] Q. Chen, J. Jost, J. Li, and G. Wang, *Regularity theorems and energy identities for Dirac-harmonic maps*, Math. Z. **251** (2005), no. 1, 61–84.

[9] Q. Chen, J. Jost, J. Li, and G. Wang, *Dirac-harmonic maps*, Math. Z. **254** (2006), no. 2, 409–432.

[10] Q. Chen, J. Jost and G. Wang, *The maximum principle and the Dirichlet problem for Dirac-harmonic maps*, Calc. Var. Partial Differential Equations, **47** (2013), no.12, 87–116.

[11] Q. Chen, J. Jost, G. Wang and M. Zhu, *The boundary value problem for Dirac-harmonic maps*, J. Eur. Math. Soc. (JEMS). Volume 15, Issue 3, 2013, 997-1031.

[12] Q. Chen, J. Jost, L. Sun and M. Zhu, *Estimates for solutions of Dirac equations and an application to a geometric elliptic-parabolic problem*, J. Eur. Math. Soc. (JEMS), Volume 21, Issue 3, 2019, 665-707.

[13] Q. Chen, J. Jost, L. Sun and M. Zhu, *Dirac-harmonic maps between Riemann surfaces*, Asian J. Math., Volume 23 (2019), Number 1, 107-126.

[14] P. Deligne, *Quantum fields and strings: a course for mathematicians*, vol. 2, Amer Mathematical Society, 1999.

[15] W. Ding, *Lusternik-Schnirelman theory for harmonic maps*, Acta Math. Sinica, **2** (1986), 105–122

[16] W. Ding, *Lectures on heat flow of harmonic maps*, Lecture notes at CTS, NTHU, Taiwan (1998).

[17] W. Ding and G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom. 3 (1995), no. 3-4, 543–554.

[18] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964) 109-160.

[19] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, *Positive mass theorems for black holes*, Comm. Math. Phys. **88** (1983), no. 3, 295–308.

[20] R. Gulliver and H. Lawson, *The structure of stable minimal hypersurfaces near a singularity*, Geometric measure theory and the calculus of variations (Arcata, 1984), 213–237. Proc. Sympos. Pure Math., 44. American Mathematical Society, Providence, R.I., 1986.
[21] R. Hamilton, *Harmonic maps of manifolds with boundary*, Springer LNM 471, 1975
[22] O. Hijazi, S. Montiel, and A. Roldán, *Eigenvalue boundary problems for the Dirac operator*, Comm. Math. Phys. **231** (2002), no. 3, 375–390.
[23] M. Hong and H. Yin, *On the Sacks-Uhlenbeck flow of Riemannian surfaces*, Comm. Anal. Geom. 21 (2013), no. 5, 917-955.
[24] J. Jost, *Two-dimensional geometric variational problems*, New York, Wiley, 1991.
[25] J. Jost, *Geometry and physics*, Springer, 2009
[26] J. Jost, L. Liu and M. Zhu, *A global weak solution of the Dirac-harmonic map flow*, Ann. Inst. H. Poincare Anal. Non Lineaire, 34 (2017), no. 7, 1851-1882.
[27] J. Jost, L. Liu and M. Zhu, *Blow-up analysis for approximate Dirac-harmonic maps in dimension 2 with applications to Dirac-harmonic heat flow*, Calc. Var. Partial Differential Equations 56 (2017), no. 4, 56:108.
[28] J. Jost and M. Struwe, *Morse-Conley theory for minimal surfaces of varying topological type*, Invent. Math., **102** (1990), 465–499
[29] J. Jost, C. Zhou and M. Zhu, *Vanishing Pohozaev constant and removability of singularities*, J. Differential Geom., Vol. 111, No. 1 (2019), pp. 91-144.
[30] H. Lawson and M. Michelsohn, *Spin geometry*, vol. 38, Princeton University Press, 1989.
[31] L. Lemaire, *Applications harmoniques de surfaces riemanniennes*, J. Differential Geom. 13 (1978) 51-78.
[32] J. Li and X. Zhu, *Energy identity and necklessness for a sequence of Sacks-Uhlenbeck maps to a sphere*, Ann. Inst. H. Poincare Anal. Non Lineaire, 36 (2019), no. 1, 103–118.
[33] P. Li and L. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. 105 (1991), no. 1, 1-46.
[34] Y. Li, L. Liu and Y. Wang, *Blow-up behavior of harmonic maps with finite index*, Calc. Var. Partial Differential Equations 56 (2017), no. 5, Art. 146, 16 pp.
[35] Y. Li and Y. Wang, *A weak energy identity and the length of necks for a sequence of Sacks-Uhlenbeck α-harmonic maps*, Adv. Math. 225 (2010), no. 3, 1134-1184.
[36] Gary M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439 pp.
[37] F. Lin and C. Wang, *The Analysis of Harmonic Maps and Their Heat Flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008, xii+267 pp.
[38] L. Liu, *No neck for Dirac-harmonic maps*, Calc. Var. Partial Differ. Equ. **52** (2015), no. 1-2, 1–15.
[39] F. C. Marques, *Minimal surfaces: variational theory and applications*, Proceedings of the International Congress of Mathematicians–Seoul 2014. Vol. 1, 283–310, Kyung Moon Sa, Seoul, 2014.
[40] F. C. Marques and A. Neves, *Topology of the space of cycles and existence of minimal varieties*, Advances in geometry and mathematical physics, 165–177, Surv. Differ. Geom., 21, Int. Press, Somerville, MA, 2016.
[41] F. C. Marques and A. Neves, *Morse index and multiplicity one min-max minimal hypersurfaces*, Camb. J. Math. 4 (2016), no. 4, 463–511.
[42] F. C. Marques and A. Neves, *Morse index of multiplicity one min-max minimal hypersurfaces*, arXiv:1803.04273.
[43] A. Neves, *New applications of min-max theory*, Proceedings of the International Congress of Mathematicians–Seoul 2014. Vol. II, 939–957, Kyung Moon Sa, Seoul, 2014.
[44] M. Micallef and J. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math. (2) 127 (1988), no. 1, 199–227
[45] R. Moser, *An L^p regularity theory for harmonic maps*, Trans. Amer. Math. Soc. 367 (2015), no. 1, 1-30
[46] J. Moore, *Energy growth in minimal surface bubbles*, http://www.math.ucsb.edu/ moore/growthrev.pdf
[47] J. Moore, *Introduction to global analysis. Minimal surfaces in Riemannian manifolds*, Graduate Studies in Mathematics, 187. American Mathematical Society, Providence, RI, 2017. xiv+368 pp. ISBN: 978-1-4704-2950-8
[48] T. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom. 44(3), 595–633 (1996)
[49] T. Parker and J. Wolfson, *A compactness theorem for Gromov's moduli space*, J. Geom. Analysis **3** (1993), 63–98

[50] J. Qing and G. Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math **50** (1997), no. 4, 295–310.
[51] T. Rivière, *A viscosity method in the min-max theory of minimal surfaces*. Publ. Math. Inst. Hautes Etudes Sci. 126 (2017), 177-246.
[52] T. Rivière, *Minmax Hierarchies and Minimal Surfaces in Manifolds*, arXiv:1705.09848.
[53] T. Rivière, *Lower Semi-Continuity of the Index in the Visosity Method for Minimal Surfaces*, arXiv:1808.00426.
[54] A. Pigati and T. Rivière, *A proof of the multiplicity one conjecture for min-max minimal surfaces in arbitrary codimension*, arXiv:1807.04205.
[55] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*. Ann. of Math. 113 (1981), 1-24.
[56] R. Schoen, *Analytic aspects of the harmonic map problem*, Math. Sci. Res. Inst. PUBL.2 (1984), Springer-Verlag, New York, 321-358.
[57] B. Sharp, *Compactness of minimal hypersurfaces with bounded index*, J. Differential Geom. 106 (2017), no. 2, 317-339.
[58] B. Sharp and M. Zhu, *Regularity at the free boundary for Dirac-harmonic maps from surfaces*, Calc. Var. Partial Differ. Equ. 55 (2016), no. 2, 55:27.
[59] L. Simon, *Asympotics for a class of nolinear evolution equations with applications to geometric problems*, Ann. of Math. 118 (1983), 525-571.
[60] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Commun. Math. Helv. **60** (1985), 558-581.
[61] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. 28 (1988), 485-502.
[62] M. Struwe, *On a critical point theory for minimal surfaces spanning a wire in* $\mathbb{R}^n$, J. Reine Angew. Math. **349** (1984), 1–23
[63] J. Wittmann, *Short time existence of the heat flow for Dirac-harmonic maps on closed manifolds*, Calc. Var. Partial Differential Equations 56 (2017), no. 6, Art. 169, 32 pp.
[64] R. Ye, *Gromov's compactness theorem for pseudo-holomorphic curves*, Trans. Am. Math. Soc. 342(2), 671-694 (1994).
[65] L. Zhao, *Energy identities for Dirac-harmonic maps*, Calc. Var. Partial Differential Equations **28** (2007), no. 1, 121–138.
[66] M. Zhu, *Regularity of weakly Dirac-harmonic maps to hypersurfaces*, Ann. Global Anal. Geom. **35** (2009), no.4 405-412.

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany
Email address: jost@mis.mpg.de

Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Ernst-Zermelo-Strasse 1, D-79104, Freiburg im Breisgau, Germany
Email address: lei.liu@math.uni-freiburg.de

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, P. R. China
Email address: mizhu@sjtu.edu.cn