Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Constructions of Unextendible Maximally Entangled Bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$

by

Gui-Jun Zhang, Yuan-Hong Tao, Yi-Fan Han,
Xin-Lei Yong, and Shao-Ming Fei

Preprint no.: 51 2018

Constructions of Unextendible Maximally Entangled Bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$

Gui-Jun Zhang[1], Yuan-Hong Tao[1] *, Yi-Fan Han[1], Xin-Lei Yong[1], Shao-Ming Fei[2,3]

1 Department of Mathematics College of Sciences, Yanbian University, Yanji 133002, China
2 School of Mathematics Sciences, Capital Normal University, Beijing 100048, China
3 Max-Planck-Institute for Mathematics in the Science, 04103 Leipzig, Germany

* Correspondence to taoyuanhong12@126.com

Abstract

We study unextendible maximally entangled bases (UMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d < d')$. An operational method to construct $d(d'-1)$-number UMEBs is established, and two 25-member UMEBs in $\mathbb{C}^5 \otimes \mathbb{C}^6$ and $\mathbb{C}^5 \otimes \mathbb{C}^{12}$ are given as examples. Furthermore, a systematic way of constructing $d(d'-r)$-number UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ is presented for $r = 1, 2, \cdots, d-1$. Correspondingly, two UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$ are obtained.

1 Introduction

Quantum entanglement lies in the heart of the quantum information processing. It plays important roles in many fields such as quantum teleportation, quantum coding, quantum key distribution protocol, quantum non-locality [1, 2, 3, 4]. Maximally entangled states attract much attention due to their importance in ensuring the highest fidelity and efficiency in quantum teleportation [5]. A pure state $|\psi\rangle$ is said to be a $d \otimes d'$ $(d < d')$ maximally entangled state if and only if for an arbitrary given orthonormal basis $\{|i_A\rangle\}$ of subsystem A, there exists an orthonormal basis $\{|i_B\rangle\}$ of subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$ [6].

Since the unextendible product basis (UPB) has been studied, many interesting conclusions have been gained towards its applications [7]. Bravyi and Smolin [8] generalized the notion of UPB to unextendible maximally entangled bases (UMEB). Chen and Fei [9] provided a way to construct d^2-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(\frac{d'}{2} < d < d')$. Later, Nan et al. [10] and Li et al. [11] constructed two sets of UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d < d')$ independently. Wang et al. [12] put forward a method of constructing UMEBs in $\mathbb{C}^{qd} \otimes \mathbb{C}^{qd}$ from that in $\mathbb{C}^d \otimes \mathbb{C}^d$, and gave a 30-member UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^6$. They proved that there exist UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$ except for $d = p$ or $2p$, where p is a prime and $p = 3 \bmod 4$. They also presented a 23-member UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^5$ and a 45-member UMEB in $\mathbb{C}^7 \otimes \mathbb{C}^7$ [13]. Then Guo [14, 15, 16] proposed a scenario of constructing UMEBs via the space decomposition, which improves the previous work about UMEBs.

In this paper, we give two methods of constructing UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}(d \leq d')$. In Sec. 2 we first recall some basic notions and lemmas about UMEB and space decomposition. In Sec. 3 we give an operational method to construct $d(d'-1)$-number UMEB and then present explicit constructions of UMEBs in $\mathbb{C}^5 \otimes \mathbb{C}^6$ and $\mathbb{C}^5 \otimes \mathbb{C}^{12}$. In Sec. 4 we present an approach of systematically constructing $d(d'-r)$-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $r = 1, 2, \cdots, d-1$, and give two examples in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$. We summarize in Sec. 5.

2 Preliminaries

Throughout this paper, we assume that $d < d'$. Let us first recall some basic notions and lemmas [8, 9, 14]. Let $\{|k\rangle\}$ and $\{|\ell'\rangle\}$ be the standard computational bases of $\mathbb{C}^d$ and $\mathbb{C}^{d'}$, respectively, and $\{|\phi_i\rangle\}_{i=1}^{dd'}$ an orthonormal basis of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Let $\mathcal{M}_{d\times d'}$ be the Hilbert space of all $d \times d'$ complex matrices equipped with the inner product defined by $\langle A|B\rangle = Tr(A^\dagger B)$ for any $A, B \in \mathcal{M}_{d\times d'}$. If $\{A_i\}_{i=1}^{dd'}$ constitutes a Hilbert-Schmidt basis of $\mathcal{M}_{d\times d'}$, where $\langle A_i|A_j\rangle = d\delta_{ij}$, then there is a one-to-one correspondence between $\{|\phi_i\rangle\}$ and $\{A_i\}$ as follows [15, 16]:

$$|\phi_i\rangle = \sum_{k,\ell} a_{kl}^{(i)}|k\rangle|\ell'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \quad \Leftrightarrow \quad A_i = [\sqrt{d}a_{k\ell}^{(i)}] \in \mathcal{M}_{d\times d'},$$

$$Sr(|\phi_i\rangle) = rank(A_i), \qquad \langle\phi_i|\phi_j\rangle = \frac{1}{d}Tr(A_i^\dagger A_j), \tag{1}$$

where $Sr(|\phi_i\rangle)$ denotes the Schmidt number of $|\phi_i\rangle$. Obviously, $|\phi_i\rangle$ is a maximally entangled pure state in $C^d \otimes C^{d'}$ iff $\sqrt{d}A_i$ is a $d \times d'$ singular-value-1 matrix (a matrix whose singular values all equal to 1).

A basis $\{|\phi_i\rangle\}_{i=1}^{dd'}$ constituted by maximally entangled states in $C^d \otimes C^{d'}$ is called a maximally entangled basis (MEB) of $C^d \otimes C^{d'}$. A set of pure states $\{|\phi_i\rangle\}_{i=1}^n \in C^d \otimes C^{d'}$ with the following conditions is called an unextendible maximally entangled basis (UMEB) [8, 9]:

(i) $|\phi_i\rangle$, $i = 1, 2, 3...n$ are all maximally entangled states.

(ii) $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, $i, j = 1, 2, 3...n$.

(iii) $n < dd'$, and if a pure state $|\psi\rangle$ satisfies that $\langle\phi_i|\psi\rangle = 0$, $i = 1, 2, 3...n$, then $|\psi\rangle$ can not be maximally entangled.

A Hilbert-Schmidt basis $\{A_i\}_{i=1}^{dd'}$ constituted by single-value-1 matrices in $\mathcal{M}_{d\times d'}$ is called single-value-1 Hilbert-Schmidt basis (SV1B) of $\mathcal{M}_{d\times d'}$. A set of $d\times d'$ matrices $\{A_i\}_{i=1}^{n}$ with the following conditions is called unextendible singular-value-1 Hilbert-Schmidt basis (USV1B) of $\mathcal{M}_{d\times d'}$ [14]:

(i) A_i, $i=1,2,3...n$ are all single-value-1 matrices.

(ii) $Tr(A_i^{\dagger}A_j)=d\delta_{ij}$, $i,j=1,2,3...n$.

(iii) $n<dd'$, and if a matrix X satisfies that $Tr(X^{\dagger}A_i)=0$, $i=1,2,3...n$,then X can not be a single-value-1 matrix.

It is obvious that $\{A_i\}_{i=1}^{dd'}$ is an SV1B of $\mathcal{M}_{d\times d'}$ iff $\{|\phi_i\rangle\}_{i=1}^{dd'}$ is a MEB of $C^d\otimes C^{d'}$, and $\{A_i\}_{i=1}^{n}$ is a USV1B of $\mathcal{M}_{d\times d'}$ iff $\{|\phi_i\rangle\}_{i=1}^{n}$ is a UMEB of $C^d\otimes C^{d'}$. Therefore, for convenience, we may just call an SV1B $\{A_i\}_{i=1}^{dd'}$ of $\mathcal{M}_{d\times d'}$ an MEB $\{|\phi_i\rangle\}_{i=1}^{dd'}$ of $\mathbb{C}^d\otimes\mathbb{C}^{d'}$, and call a USV1B $\{A_i\}_{i=1}^{n}$ of $\mathcal{M}_{d\times d'}$ a UMEB $\{|\phi_i\rangle\}_{i=1}^{n}$ of $\mathbb{C}^d\otimes\mathbb{C}^{d'}$.

In deriving our main results, we need the following lemma in Ref [14].

Lemma 1. [14] Let $\mathcal{M}_{d\times d'}=\mathcal{M}_1\bigoplus\mathcal{M}_1^{\perp}$. If $\{|\phi_i\rangle\}$ is a MEB in $\mathcal{M}_1$ and $\{|\psi_i\rangle\}$ is a UMEB in $\mathcal{M}_1^{\perp}$, then $\{|\phi_i\rangle\}\cup\{|\psi_i\rangle\}$ is a UMEB in $\mathcal{M}$. If $\{|\phi_i\rangle\}$ is a MEB in $\mathcal{M}_1$ and $\mathcal{M}_1^{\perp}$ contains no single-value-1 matrix (maximally entangled state), then $\{|\phi_i\rangle\}$ is a UMEB in $\mathcal{M}$.

3 $d(d'-1)$-member UMEBs in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$

In this section, we will establish a flexible method to construct $d(d'-1)$-member UMEBs in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$.

Theorem 1. Let $\mathcal{M}_{d\times d'}$ be the Hilbert space of all $d\times d'$ complex matrices. If V is a subspace of $\mathcal{M}_{d\times d'}$ such that each matrix in V is a $d\times d'$ matrix ignoring d entries which occupy different rows and N columns with $N<d$, then there exists a $d(d'-1)$-member MEB in V, as well as a $d(d'-1)$-member UMEB in $\mathcal{M}_{d\times d'}$.

Proof. Without loss of generality, we can always assume the ignored d entries in V only occupy the former N columns. Let b_i, $i=0,1,...,d-1$, denote the column number of the ignored element in the i-th row. Obviously, $b_{i+1}-b_i=0$ or 1.

Denote

$$C(k,l)=\begin{cases}1, & l=b_k;\\ 0, & otherwise.\end{cases}\tag{2}$$

We can construct $d(d'-1)$ pure states in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$ as follows,

$$|\phi'_{j,n}\rangle=\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\omega_d^{nm}|m\rangle|t_{mj}\rangle,\ j=0,1...,d'-2;\ n=0,1...,d-1, \tag{3}$$

where $\omega_d=e^{\frac{2\pi\sqrt{-1}}{d}}$, and

$$t_{mj}=\begin{cases} j+1, & m=0;\\ t_{m-1,j}+1\oplus_{d'}C(m,t_{m-1,j}+1), & m=1,2...d-1,\end{cases} \tag{4}$$

$p\oplus_{d'}m$ denotes $(p+m) \mod d'$.

Next, we prove that all the states in (3) constitute an MEB in V.

(i) Maximally entangled.

If $C(m,t_{m-1,j}\oplus_{d'}1)=0$ for any m, it is obvious that $t_{mj}\neq t_{m'j}$ for $m\neq m'$.

If $C(m,t_{m-1,j}\oplus_{d'}1)=1$ for some $m\neq 0$, from the definition of t_{mj} one has $t_{m-1,j}\neq b_{m-1}$. Note that b_m-b_{m-1} =0 or 1, then $t_{m-1,j}=b_{m-1}\oplus_{d'}1$.

From the definition of t_{mj}, we also have $C(k+1,t_{kj}+1)=0$ for $k\neq m-1$. Hence

$$t_{kj}=\begin{cases} t_{m-1,j}\ominus_{d'}(m-1-k), & 0\leqslant k<m-1;\\ t_{mj}\oplus_{d'}(k-m), & k\geqslant m.\end{cases} \tag{5}$$

where $p\ominus_{d'}m$ denotes $(p-m) \mod d'$. In particular,

$$t_{0j}=t_{m-1,j}\ominus_{d'}(m-1),\qquad t_{d-1,j}=t_{mj}\oplus_{d'}(d-1-m). \tag{6}$$

Then

$$\begin{aligned} t_{d-1,j}-t_{0j} &= t_{mj}\oplus_{d'}(d-1-m)-t_{m-1,j}\ominus_{d'}(m-1)\\ &= d-2+(t_{mj}-t_{m-1,j})\\ &= d<d'. \end{aligned} \tag{7}$$

Hence $t_{mj}\neq t_{m'j}$ for $m\neq m'$. Namely, the states $|\phi'_{j,n}\rangle$ in (3) are all maximally entangled.

(ii) Orthogonality.

We first show that $|t_{mj}\rangle=|t_{mj'}\rangle$ if and only if $j=j'$.

Obviously, $t_{mj}=t_{mj'}$ for $j=j'$. If $j\neq j'$, without loss of generality, let $j'>j$. It is easy to show that $t_{mj}\neq t_{mj'}$ when $t_{m-1,j}\neq t_{m-1,j'}$. Otherwise, from the definition of t_{mj} we have $t_{mj}=t_{m-1,j'}\ominus_{d'}C(m,t_{m-1,j}\oplus 1)$ when $C(m,t_{m-1,j}\oplus_{d'}1)=1$. Note that $t_{m-1,j'}=b_{m-1}\ominus_{d'}1$

when $C(m, t_{m-1,j} \oplus_{d'} 1) = 1$, as proved in (i). Therefore, $t_{m-1,j'} = b_{m-1}$, which contradicts to the definition of t_{mj}. Furthermore, $t_{mj} \neq t_{mj'}$ when $t_{0j} \neq t_{0j'}$. Therefore,

$$\begin{aligned}\langle \phi'_{j,n} | \phi'_{j',n'} \rangle &= \frac{1}{d} \sum_{m=0}^{d-1} \overline{\omega^{n'm}} \omega^{nm} \langle t_{mj} | t_{mj'} \rangle \\ &= \frac{1}{d} \sum_{m=0}^{d-1} \omega^{(n-n')m} \delta_{jj'} \\ &= \delta_{nn'} \delta_{jj'}. \end{aligned} \tag{8}$$

Thus, the $d(d'-1)$ states $\{|\phi_{j,n}\rangle\}$ in (3) constitutes an MEB in V. Furthermore, there exist no MEBs in $V^{\perp}$ because $N < d$. Hence $\{|\phi_{j,n}\rangle\}$ is a UMEB in $\mathcal{M}_{d\times d'}$, as well as in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Example 1. Constructing two UMEBs in $C^5 \otimes C^6$, whereas $V = \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 & 0 \end{pmatrix}$.

We can get the following matrix V' by using suitable unitary transformation on V,

$$V' = PVQ = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to Theorem 1, we first construct an MEB $\{|\phi'_j\rangle\}_{j=1}^{25}$ in V', i.e. a UMEB in $C^5 \otimes C^6$ as follows:

$$\begin{cases} |\phi'_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle + \alpha^3|34'\rangle + \alpha^4|45'\rangle), \\ |\phi'_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|02'\rangle + \alpha|13'\rangle + \alpha^2|24'\rangle + \alpha^3|35'\rangle + \alpha^4|40'\rangle), \\ |\phi'_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|03'\rangle + \alpha|14'\rangle + \alpha^2|25'\rangle + \alpha^3|30'\rangle + \alpha^4|41'\rangle), \\ |\phi'_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|04'\rangle + \alpha|15'\rangle + \alpha^2|21'\rangle + \alpha^3|32'\rangle + \alpha^4|43'\rangle), \\ |\phi'_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|05'\rangle + \alpha|11'\rangle + \alpha^2|22'\rangle + \alpha^3|33'\rangle + \alpha^4|44'\rangle), \end{cases} \tag{9}$$

where $\alpha = 1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$.

By inverse unitary transformation $|\phi_j\rangle = (P^{-1} \otimes Q^{-1})|\phi'_j\rangle$, we get the following MEB $\{|\phi_j\rangle\}_{j=1}^{25}$ in V, i.e., another UMEB in $C^5 \otimes C^6$:

$$\begin{cases} |\phi_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|01'\rangle + \alpha|25'\rangle + \alpha^2|40'\rangle + \alpha^3|12'\rangle + \alpha^4|34'\rangle), \\ |\phi_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|05'\rangle + \alpha|20'\rangle + \alpha^2|42'\rangle + \alpha^3|14'\rangle + \alpha^4|33'\rangle), \\ |\phi_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|00'\rangle + \alpha|22'\rangle + \alpha^2|44'\rangle + \alpha^3|13'\rangle + \alpha^4|31'\rangle), \\ |\phi_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|02'\rangle + \alpha|24'\rangle + \alpha^2|41'\rangle + \alpha^3|15'\rangle + \alpha^4|30'\rangle), \\ |\phi_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|04'\rangle + \alpha|21'\rangle + \alpha^2|45'\rangle + \alpha^3|10'\rangle + \alpha^4|32'\rangle), \end{cases} \tag{10}$$

where $\alpha = 1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$.

Remark 1. Actually both (9) and (10) are UMEBs in $\mathbb{C}^5 \otimes \mathbb{C}^6$. However, they are different although they can be unitarily transformed to each other. We will reveal the difference in the following example.

Example 2. Constructing a UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{12}$, whereas

$$V = (V_1|V_2) = \left(\begin{array}{cccccc|cccccc} 0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{11}$$

One can easily get the following simple formations V_1' and V_2' from V_1 and V_2 by elementary transformation respectively:

$$V_1' = P_1 V_1 Q_1 = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_2' = P_2 V_2 Q_2 = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

Then following Theorem 1 we can construct the following UMEBs $\{|\phi'_j\rangle\}_{j=1}^{25}$ and $\{|\psi'_j\rangle\}_{j=1}^{25}$

in V_1' and V_2' respectively:

$$\begin{cases} |\phi'_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle + \alpha^3|34'\rangle + \alpha^4|45'\rangle), \\ |\phi'_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|02'\rangle + \alpha|13'\rangle + \alpha^2|24'\rangle + \alpha^3|35'\rangle + \alpha^4|40'\rangle), \\ |\phi'_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|03'\rangle + \alpha|14'\rangle + \alpha^2|25'\rangle + \alpha^3|30'\rangle + \alpha^4|42'\rangle), \\ |\phi'_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|04'\rangle + \alpha|15'\rangle + \alpha^2|20'\rangle + \alpha^3|32'\rangle + \alpha^4|43'\rangle), \\ |\phi'_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|05'\rangle + \alpha|10'\rangle + \alpha^2|22'\rangle + \alpha^3|33'\rangle + \alpha^4|44'\rangle), \end{cases} \tag{13}$$

$$\begin{cases} |\psi'_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|07'\rangle + \alpha|18'\rangle + \alpha^2|29'\rangle + \alpha^3|3,10'\rangle + \alpha^4|4,11'\rangle), \\ |\psi'_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|08'\rangle + \alpha|19'\rangle + \alpha^2|2,10'\rangle + \alpha^3|3,11'\rangle + \alpha^4|46'\rangle), \\ |\psi'_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|09'\rangle + \alpha|1,10'\rangle + \alpha^2|2,11'\rangle + \alpha^3|3,6'\rangle + \alpha^4|48'\rangle), \\ |\psi'_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|0,10'\rangle + \alpha|1,11'\rangle + \alpha^2|27'\rangle + \alpha^3|38'\rangle + \alpha^4|49'\rangle), \\ |\psi'_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|0,11'\rangle + \alpha|17'\rangle + \alpha^2|28'\rangle + \alpha^3|39'\rangle + \alpha^4|4,10'\rangle) \end{cases} \tag{14}$$

where $\alpha = 1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$.

By inverse transformation $|\phi_j\rangle = (P_1^{-1} \otimes Q_1^{-1})|\phi'_j\rangle$ and $|\psi_j\rangle = (P_2^{-1} \otimes Q_2^{-1})|\psi'_j\rangle$, we can obtain the following UMEBs $\{|\phi_j\rangle\}_{j=1}^{25}$ and $\{|\psi_j\rangle\}_{j=1}^{25}$ in V_1 and V_2, respectively,

$$\begin{cases} |\phi_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle + \alpha^3|34'\rangle + \alpha^4|45'\rangle), \\ |\phi_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|02'\rangle + \alpha|13'\rangle + \alpha^2|24'\rangle + \alpha^3|35'\rangle + \alpha^4|40'\rangle), \\ |\phi_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|03'\rangle + \alpha|14'\rangle + \alpha^2|25'\rangle + \alpha^3|30'\rangle + \alpha^4|42'\rangle), \\ |\phi_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|04'\rangle + \alpha|15'\rangle + \alpha^2|20'\rangle + \alpha^3|32'\rangle + \alpha^4|43'\rangle), \\ |\phi_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|05'\rangle + \alpha|10'\rangle + \alpha^2|22'\rangle + \alpha^3|33'\rangle + \alpha^4|44'\rangle), \end{cases} \tag{15}$$

$$
\begin{cases}
|\psi_{1,2,3,4,5}\rangle = \frac{1}{\sqrt{5}}(|07'\rangle + \alpha|18'\rangle + \alpha^2|29'\rangle + \alpha^3|3,10'\rangle + \alpha^4|4,11'\rangle), \\
|\psi_{6,7,8,9,10}\rangle = \frac{1}{\sqrt{5}}(|08'\rangle + \alpha|19'\rangle + \alpha^2|2,10'\rangle + \alpha^3|3,11'\rangle + \alpha^4|46'\rangle), \\
|\psi_{11,12,13,14,15}\rangle = \frac{1}{\sqrt{5}}(|09'\rangle + \alpha|1,10'\rangle + \alpha^2|2,11'\rangle + \alpha^3|36'\rangle + \alpha^4|48'\rangle), \\
|\psi_{16,17,18,19,20}\rangle = \frac{1}{\sqrt{5}}(|0,10'\rangle + \alpha|1,11'\rangle + \alpha^2|27'\rangle + \alpha^3|38'\rangle + \alpha^4|49'\rangle), \\
|\psi_{21,22,23,24,25}\rangle = \frac{1}{\sqrt{5}}(|0,11'\rangle + \alpha|17'\rangle + \alpha^2|28'\rangle + \alpha^3|39'\rangle + \alpha^4|4,10'\rangle).
\end{cases}
\tag{16}
$$

Thus, $\{|\phi_j\rangle\} \cup \{|\psi_j\rangle\}$ constitutes a UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{12}$ with V in (8). However, neither $(P_1^{-1} \otimes Q_1^{-1})$ nor $(P_2^{-1} \otimes Q_2^{-1})$ can transform $\{|\psi'_j\rangle\} \cup \{|\phi'_j\rangle\}$ to $\{|\phi_j\rangle\} \cup \{|\psi_j\rangle\}$, which shows the difference between (9) and (10).

4 $d(d'-r)$-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$

In this section, we construct UMEBs consisting of fewer elements in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. The following theorem provides a systematic way of constructing $d(d'-r)$-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, $r = 1, 2, \cdots, d-1$, that is to say, it presents $d-1$ constructions of UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Theorem 2. Let $d' = \sum_{i=1}^{s} a_i + r$, where $s \geqslant 1,\ a_i \geqslant d,\ 0 < r < d$. Then the following vectors constitute a $d(d'-r)$-member UMEB in $C^d \otimes C^{d'}$:

$$
|\phi_{l,j,n}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega_d^{nm} |m\rangle |b_j + (l_{j+i} \oplus_{a_{j+1}} m)\rangle, \quad l_{j+1} = 0, 1, \cdots, a_{j+1} - 1, \tag{17}
$$

where $b_j = \sum_{k=1}^{j} a_k$; $j = 0, 1, \cdots, s-1$; $n = 0, 1, \cdots, d-1$.

Proof. (i) It is obvious that $|\phi_{l,j,n}\rangle$ in (16) are all maximally entangled.

(ii) Orthogonality,

$$
\begin{aligned}
\langle \phi_{l,j,n} | \phi_{l',j',n'} \rangle =& \frac{1}{d} \sum_{m=0}^{d-1} \overline{\omega_d^{n'm}} \omega_d^{nm} \langle b_j + (l_{j+1} \oplus_{a_{j+1}} m) | b_{j'} + (l'_{j'+1} \oplus_{a_{j'+1}} m) \rangle \\
=& \frac{1}{d} \sum_{m=0}^{d-1} \omega_d^{(n-n')m} \delta_{jj'} \langle l_{j+1} \oplus_{a_{j+1}} m | l'_{j'+1} \oplus_{a_{j'+1}} m \rangle \\
=& \frac{1}{d} \sum_{m=0}^{d-1} \omega_d^{(n-n')m} \delta_{jj'} \delta_{ll'} \\
=& \delta_{nn'} \delta_{jj'} \delta_{ll'}.
\end{aligned}
\tag{18}
$$

(iii) Denote $\mathcal{M}_1$ the $d \otimes (d'-n)$ matrix space, a subspace of $\mathcal{M}_{d\times d'}$. Since the number of $\{|\phi_{l,j,n}\rangle\}$ in (17) equals to the dimension of $\mathcal{M}_1$, $\{|\phi_{l,j,n}\rangle\}$ is an MEB of $\mathcal{M}_1$. Moreover, since $\mathcal{M}_1^{\perp}$ is a $d \times r$ matrix space and $r < d$, there contains no UMEB in $\mathcal{M}_1^{\perp}$. From Lemma 1, $\{|\phi_{l,j,n}\rangle\}$ is a UMEB of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Example 3. UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$.

Obviously, $10 = 4+5+1$ or $10 = 4+4+2$. According to Theorem 2, we can construct the following 27-number UMEB (19) and 24-number UMEB (20) in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$ respectively.

$$\begin{cases}
|\phi_{1,2,3}\rangle = \frac{1}{\sqrt{3}}(|00'\rangle + \alpha|11'\rangle + \alpha^2|22'\rangle), \\
|\phi_{4,5,6}\rangle = \frac{1}{\sqrt{3}}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle), \\
|\phi_{7,8,9}\rangle = \frac{1}{\sqrt{3}}(|02'\rangle + \alpha|13'\rangle + \alpha^2|20'\rangle), \\
|\phi_{10,11,12}\rangle = \frac{1}{\sqrt{3}}(|03'\rangle + \alpha|10'\rangle + \alpha^2|21'\rangle), \\
|\phi_{13,14,15}\rangle = \frac{1}{\sqrt{3}}(|04'\rangle + \alpha|15'\rangle + \alpha^2|26'\rangle), \\
|\phi_{16,17,18}\rangle = \frac{1}{\sqrt{3}}(|05'\rangle + \alpha|16'\rangle + \alpha^2|27'\rangle), \\
|\phi_{19,20,21}\rangle = \frac{1}{\sqrt{3}}(|06'\rangle + \alpha|17'\rangle + \alpha^2|28'\rangle), \\
|\phi_{22,23,24}\rangle = \frac{1}{\sqrt{3}}(|07'\rangle + \alpha|18'\rangle + \alpha^2|24'\rangle), \\
|\phi_{25,26,27}\rangle = \frac{1}{\sqrt{3}}(|08'\rangle + \alpha|14'\rangle + \alpha^2|25'\rangle),
\end{cases} \tag{19}$$

and

$$\begin{cases} |\phi_{1,2,3}\rangle = \frac{1}{\sqrt{3}}(|00'\rangle + \alpha|11'\rangle + \alpha^2|22'\rangle), \\ |\phi_{4,5,6}\rangle = \frac{1}{\sqrt{3}}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle), \\ |\phi_{7,8,9}\rangle = \frac{1}{\sqrt{3}}(|02'\rangle + \alpha|13'\rangle + \alpha^2|20'\rangle), \\ |\phi_{10,11,12}\rangle = \frac{1}{\sqrt{3}}(|03'\rangle + \alpha|10'\rangle + \alpha^2|21'\rangle), \\ |\phi_{13,14,15}\rangle = \frac{1}{\sqrt{3}}(|04'\rangle + \alpha|15'\rangle + \alpha^2|26'\rangle), \\ |\phi_{16,17,18}\rangle = \frac{1}{\sqrt{3}}(|05'\rangle + \alpha|16'\rangle + \alpha^2|27'\rangle), \\ |\phi_{19,20,21}\rangle = \frac{1}{\sqrt{3}}(|06'\rangle + \alpha|17'\rangle + \alpha^2|24'\rangle), \\ |\phi_{22,23,24}\rangle = \frac{1}{\sqrt{3}}(|07'\rangle + \alpha|14'\rangle + \alpha^2|25'\rangle), \end{cases} \tag{20}$$

where $\alpha = 1, \omega_3, \omega_3^2$.

Remark 2. Theorem 2 gives much more UMEBs in $C^d \otimes C^{d'}$ than that from previous results. For example, the 27-number UMEB (19) and 24-number UMEB (20) in Example 3 are only two kinds of UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$. Actually, according to Theorem 2, there are five more kinds of UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$, since $10 = 3 + 5 + 2$, $10 = 3 + 6 + 1$, $10 = 3 + 3 + 3 + 1$, $10 = 8 + 2$ and $10 = 9 + 1$.

Remark 3. Theorem 2 in Ref. [11] is a special case of the above Theorem 2 at $d' = a_1 + r$. Theorem 1 in Ref. [10] and Theorem 1 in Ref. [11] are both special cases of our Theorem 1, where all the a_i are equal.

5 Conclusion

We have provided new constructions of unextendible maximally entangled bases in arbitrary bipartite spaces $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. We have presented a systematic way of constructing $d(d'-1)$-member UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, and constructed two different UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^6$ and $\mathbb{C}^3 \otimes \mathbb{C}^{12}$ respectively. We have established a flexible method to construct $d(d-r)$-number UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, $r = 1, 2, \cdots, d-1$. Namely, we have presented more than $d-1$ constructions of UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Such generalized the main results in Ref.[11] and Ref.[10]. We have also shown 27-number UMEB and 24-number UMEB in $\mathbb{C}^3 \otimes \mathbb{C}^{10}$, respectively.

References

[1] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki. Quantum entanglement. Reviews of Modern Physics, 2009, 81: 865-942

[2] Adriano Barenco, Artur K. Ekert. Dense Coding Based on Quantum Entanglement. Journal of Modern Optics, 1995, 42(6): 1253-1259

[3] M. Curty, M. Lewenstein, N. L$\ddot{u}$ tkenhaus. Entanglement as a precondition for secure quantum key distribution. Physical Review Letters, 2004, 92(21): 217903

[4] Zheng S.B.. Quantum nonlocality for a three-particle nonmaximally entangled state without inequalities. Physical Review A, 2002, 66(1): 90-95

[5] S. Adhikari, A. S. Majumdar, S. Roy, B. Ghosh, N. Nayak. Teleportation via maximally and non-maximally entangled mixed states. Quantum Information & Computation, 2010, 10(5) : 398-419

[6] A. Peres. Quantum Theory: Concepts and Methods. Kluwer Academic Publishers , 1995, 28(1): 131-135

[7] P. Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. Physics Letters A, 1997, 232(5): 333-339

[8] S. Bravyi, J. A. Smolin. Unextendible maximally entangled bases. Physical Review A, 2011, 84: 042306

[9] B. Chen, S.M. Fei. Unextendible maximally entangled bases and mutually unbiased bases. Physical Review A, 2013, 88: 034301

[10] H. Nan, Y. H. Tao, L. S. Li, J. Zhang. Unextendible Maximally Entangled Bases and Mutually Unbiased Bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. International Journal of Theoretical Physics, 2015, 54: 927

[11] M. S. Li, Y. L. Wang, S. M. Fei, Z.J. Zheng. Unextendible maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Physical Review A, 2014, 89: 062313

[12] Y. L. Wang, M. S. Li, S. M. Fei. Unextendible maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Physical Review A, 2014, 90: 034301

[13] Y. L. Wang, M. S. Li, S. M. Fei. Connecting the UMEB in $\mathbb{C}^d \otimes \mathbb{C}^d$ with partial Hadamard matrices. Quantum Information Processing, 2017, 16(3): 84

[14] Y. Guo. Constructing the unextendible maximally entangled basis from the maximally entangled basis. Physical Review A, 2016, 94: 052302

[15] Y. Guo, S. P. Du, X. L. Li, S. J. Wu. Entangled bases with fixed Schmidt number. Journal of Physics A: Mathematical and Theoretical, 2015, 48: 245301

[16] Y. Guo, Y. P. Jia, X. L. Li. Multipartite unextendible entangled basis. Quantum Inf Process. 2015, 14: 3553

Acknowledgements

The work is supported by the NSFC under number 11361065, 11675113, 11761073.

Author Contributions

G.-J.Z and Y.-H.Y. wrote the main manuscript text. All of the authors reviewed the manuscript.