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measures**

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Ordering states with various coherence measures

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Abstract

Quantum coherence is one of the most significant theories in quantum physics. Ordering states with various coherence measures is an intriguing task in quantification theory of coherence. In this paper, we study this problem by use of four important coherence measures – the l_1 norm of coherence, the relative entropy of coherence, the geometric measure of coherence and the modified trace distance measure of coherence. We show that each pair of these measures give a different ordering of qudit states when $d \geq 3$. However, for single-qubit states, the l_1 norm of coherence and the geometric coherence provide the same ordering. We also show that the relative entropy of coherence and the geometric coherence give a different ordering for single-qubit states. Then we partially answer the open question proposed in [Quantum Inf. Process. 15, 4189 (2016)] whether all the coherence measures give a different ordering of states.

Key words: l_1 -norm of coherence, relative entropy of coherence, geometric measure of coherence, modified trace distance of coherence, ordering states.

1 Introduction

Quantum coherence is one of the most outstanding features in quantum mechanics. It is very essential in various research fields such as low-temperature thermodynamics [1, 2, 3, 4, 5], quantum biology [6, 7, 8, 9, 10, 11], nanoscale physics [12, 13], etc. Although formulating resource theory of quantum coherence is a long-standing open problem, it has only been proposed by Baumgratz *et al.* recently [14]. In their seminal work, conditions that a suitable measure of coherence should satisfy have been put forward. After that, many efforts have been made in quantification of coherence. Up to now, various proper quantifiers have been given, such as the l_1 norm of coherence, the relative entropy of coherence [14], the geometric measure of coherence [15] and the modified trace distance measure of coherence [16, 17], etc.

Based on various physical contexts, different values of coherence may reflect different properties of quantum states. Generally, one cannot say that the coherence of a state ρ_1 is smaller than that of ρ_2 , since different coherence measures may provide a different ordering for these two states.

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Similar to the case of quantum entanglement, different measures of quantum coherence characterize different aspects of the state, and play different roles in quantum information processing. Hence, a given quantum state may behavior better in one information processing, but worse in another information task. On the other hand, one can identify two measures of coherence to some extent if they give the same ordering for all quantum states. Therefore, it is worthy of a study on the ordering of quantum states under different measures of quantum coherence. It should be noted that this issue has been extensively investigated in the theory of quantum entanglement. In [18], Virmani *et al.* showed that any two entanglement measures placing the same ordering on states must be identical, as long as they coincide on pure states. However, less has been known for quantum coherence. In this paper, we focus on ordering states based on various coherence measures.

In [19], Liu *et al.* consider the l_1 norm of coherence and the relative entropy of coherence, and show that these two measures do not give the same ordering of states. Then they propose an open question: whether all the coherence measures give a different ordering of states? That is to say, whether there exist quantum states ρ_1 and ρ_2 , such that the following relation fails for any two coherence measures C_A and C_B : $C_A(\rho_1) \leq C_A(\rho_2)$ iff $C_B(\rho_1) \leq C_B(\rho_2)$. In this paper, we investigate this problem. We mainly focus on four coherence measures – the l_1 norm of coherence, the relative entropy of coherence, the geometric measure of coherence and the modified trace distance measure of coherence. We show that each pair of these measures do not give the same ordering of high-dimensional states in general. However, the l_1 norm of coherence and the geometric coherence provide the same ordering for single-qubit states, while the relative entropy of coherence and the geometric coherence still give rise to a different ordering in this case. Thus we partially answer the open question proposed in [19]. Additionally, we provide some special sets of quantum states such that each pairs of the four coherence measures give the same ordering.

This paper is organized as follows. In Sect. 2, we review some basic concepts about quantification theory of coherence and some coherence measures we will use in this paper. In Sect. 3, we present our main results via detailed examples by using pairwise coherence measures. We also extend our discussion to the coherence of formation and the Tsallis relative α -entropy of coherence for single-qubit states. We conclude our results in Sect. 4.

2 The quantification of coherence

In this section, we first review some basic concepts about quantification of coherence.

For a given d -dimensional Hilbert space \mathcal{H} , let us fix an orthonormal basis $\{|i\rangle\}_{i=1}^d$. Then the incoherent states are defined as:

$$\sigma = \sum_{i=1}^d p_i |i\rangle\langle i|, \quad (1)$$

where $p_i \geq 0$, $\sum_{i=1}^d p_i = 1$. The set of all the incoherent states is denoted as \mathcal{I} . Let Λ be a completely positive trace preserving (CPTP) map:

$$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger, \quad (2)$$

where $\{K_n\}$ is a set of Kraus operators satisfying $\sum_n K_n^\dagger K_n = \mathbb{I}_d$, with \mathbb{I}_d the identity operator. If $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$ for all n , then we call $\{K_n\}$ a set of incoherent Kraus operators, and the corresponding Λ an incoherent operation.

In Ref. [14], Baumgratz *et al.* proposed a resource-theoretic framework for quantifying quantum coherence. Any function C defined on a space of quantum states can be employed as a proper measure of coherence, if it satisfies the following four conditions :

$$(B1) C(\rho) \geq 0, C(\rho) = 0 \text{ if and only if } \rho \in \mathcal{I};$$

$$(B2) C(\Lambda(\rho)) \leq C(\rho) \text{ for any incoherent operation } \Lambda;$$

(B3) $\sum_n p_n C(\rho_n) \leq C(\rho)$, where $p_n = \text{Tr}(K_n \rho K_n^\dagger)$, $\rho_n = K_n \rho K_n^\dagger / p_n$, $\{K_n\}$ is a set of incoherent Kraus operators;

(B4) $C(\sum_i p_i \rho_i) \leq \sum_i p_i C(\rho_i)$ for any set of quantum states $\{\rho_i\}$ and any probability distribution $\{p_i\}$.

Recently, Yu *et al.* put forward an alternative framework for quantifying coherence [16]. This framework is equivalent to the previous one proposed by Baumgratz *et al.*[14]. A nonnegative function C can be used as a measure of coherence, if it satisfies:

$$(C1) C(\rho) \geq 0, C(\rho) = 0 \text{ if and only if } \rho \in \mathcal{I};$$

$$(C2) C(\Lambda(\rho)) \leq C(\rho) \text{ for any incoherent operation } \Lambda;$$

$$(C3) C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2) \text{ for block diagonal states } \rho \text{ in the incoherent basis.}$$

In accordance with the above frameworks, several legitimate coherence measures have been provided so far. In this paper, we mainly consider four coherence measures – the l_1 norm of coherence, the relative entropy of coherence, the geometric measure of coherence and the modified trace distance measure of coherence.

The l_1 norm of coherence is defined as

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|, \quad (3)$$

where $\rho_{ij} = \langle i | \rho | j \rangle$.

The relative entropy of coherence is defined as

$$C_r(\rho) = \min_{\sigma \in \mathcal{I}} \mathcal{S}(\rho \| \sigma) = \mathcal{S}(\rho_{diag}) - \mathcal{S}(\rho), \quad (4)$$

where $\mathcal{S}(\rho \| \sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$ is the quantum relative entropy, $\mathcal{S}(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy, and $\rho_{diag} = \sum_i \rho_{ii} |i\rangle\langle i|$.

The geometric measure of coherence is defined as

$$C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma), \quad (5)$$

where $F(\rho, \sigma) = \left(\text{Tr} \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right)^2$ is the fidelity of two density operators ρ and σ . When ρ is a pure state, $C_g(\rho) = 1 - \max_i \{\rho_{ii}\}$, where $\rho_{ii} = \langle i | \rho | i \rangle$ [20].

The modified trace distance measure of coherence is defined as

$$C'_{tr}(\rho) = \min_{\lambda \geq 0, \delta \in \mathcal{I}} \|\rho - \lambda \delta\|_{tr}. \quad (6)$$

It has been shown that $C'_{tr}(\rho) = C_{l_1}(\rho)$ if ρ is a single-qubit state [17].

It should be noted that the last two coherence measures have no analytical expressions in general. However, for some special classes of coherent states, explicit formulae have been presented in [20, 17]. For example, for the maximally coherent mixed states (MCMS) [21],

$$\rho_m = p|\phi_d\rangle\langle\phi_d| + \frac{1-p}{d}\mathbb{I}_d, \quad (7)$$

where $0 < p \leq 1$, and $|\phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle$ is the maximally coherent state, one has $C_g(\rho_m) = 1 - [\sqrt{1-p} + \frac{1}{d}(\sqrt{1-p+dp} - \sqrt{1-p})]^2$ [20], and $C'_{tr}(\rho_m) = p$ [17].

3 Ordering states with coherence measures

Let us recall the concept of ordering states with coherence measures. We first note that all the states can be ordered under a coherence measure C , since $C(\rho)$ is always a nonnegative real number. Then a natural question is raised: are there any two coherence measures C_A and C_B which give rise to the same ordering of all states? Here the same ordering means that the following relation holds for any two states ρ_1 and ρ_2 :

$$C_A(\rho_1) \leq C_A(\rho_2) \Leftrightarrow C_B(\rho_1) \leq C_B(\rho_2). \quad (8)$$

Otherwise, we say that the two measures give a different ordering.

In this section, we discuss ordering states with pairs of coherence measures among C_{l_1} , C_r , C_g and C'_{tr} via detailed examples. We will show that C_{l_1} , C_r , C_g and C'_{tr} generate a different ordering of qudit ($d \geq 3$) states. For single-qubit states, we show that C_{l_1} and C_g give the same ordering, while C_r and C_g provide a different ordering.

3.1 Ordering states with C_{l_1} and C_g

We first consider two-dimensional quantum systems. Any density operator acting on a two-dimensional quantum system can be generally written as

$$\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}, \quad (9)$$

where $|a|^2 + |b|^2 \leq 1$. Then we have $C_{l_1}(\rho) = 2|b|$ and $C_g(\rho) = \frac{1-\sqrt{1-4|b|^2}}{2}$ [20]. It can be seen that $C_{l_1}(\rho)$ and $C_g(\rho)$ are both increasing functions with respect to $|b|$. Thus, for all single-qubit states, the coherence measures C_{l_1} and C_g give the same ordering, since $C_{l_1}(\rho_1) \leq C_{l_1}(\rho_2) \Leftrightarrow |b_1| \leq |b_2| \Leftrightarrow C_g(\rho_1) \leq C_g(\rho_2)$, where $\rho_1 = \begin{pmatrix} a_1 & b_1 \\ b_1^* & 1-a_1 \end{pmatrix}$ and $\rho_2 = \begin{pmatrix} a_2 & b_2 \\ b_2^* & 1-a_2 \end{pmatrix}$ are arbitrary single-qubit states.

We now discuss the case of high-dimensional quantum systems. Let $|\psi\rangle = \sum_{i=1}^d \sqrt{\lambda_i}|i\rangle$ and $|\phi\rangle = \sum_{i=1}^d \sqrt{\mu_i}|i\rangle$ be two pure states, where $\lambda_i \geq 0$, $\sum_{i=1}^d \lambda_i = 1$, and $\mu_i \geq 0$, $\sum_{i=1}^d \mu_i = 1$. Then we have $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\phi\rangle) \Leftrightarrow \sum_{i=1}^d \sqrt{\lambda_i} \leq \sum_{i=1}^d \sqrt{\mu_i}$, and $C_g(|\psi\rangle) \leq C_g(|\phi\rangle) \Leftrightarrow \max_i\{\lambda_i\} \geq \max_i\{\mu_i\}$.

Thus $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\phi\rangle) \Leftrightarrow C_g(|\psi\rangle) \leq C_g(|\phi\rangle)$ if the two conditions $\sum_{i=1}^d \sqrt{\lambda_i} \leq \sum_{i=1}^d \sqrt{\mu_i}$ and $\max_i\{\lambda_i\} \geq \max_i\{\mu_i\}$ hold at the same time.

Let us consider a special case where $\lambda_1 = \lambda \geq 0$, $\lambda_2 = \lambda_3 = \dots = \lambda_d$, i.e., $|\psi\rangle = \sqrt{\lambda}|1\rangle + \sqrt{\frac{1-\lambda}{d-1}} \sum_{i=2}^d |i\rangle$. Then we have $C_{l_1}(|\psi\rangle) = (\sqrt{\lambda} + \sqrt{(d-1)(1-\lambda)})^2 - 1$. Note that $C_{l_1}(|\psi\rangle)$ is an increasing function with respect to λ when $\lambda \leq \frac{1}{d}$, while a decreasing function when $\lambda \geq \frac{1}{d}$. Let $|\phi\rangle = \sqrt{\mu}|1\rangle + \sqrt{\frac{1-\mu}{d-1}} \sum_{i=2}^d |i\rangle$. We consider the following cases:

(i) If $\lambda \leq \frac{1}{d}$, $\mu \leq \frac{1}{d}$, then we have $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\phi\rangle) \Leftrightarrow \lambda \leq \mu \Leftrightarrow C_g(|\psi\rangle) = 1 - \frac{1-\lambda}{d-1} \leq C_g(|\phi\rangle) = 1 - \frac{1-\mu}{d-1}$. Thus the coherence measures C_{l_1} and C_g generate the same ordering in this case.

(ii) If $\lambda \geq \frac{1}{d}$, $\mu \geq \frac{1}{d}$, then we have $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\phi\rangle) \Leftrightarrow \lambda \geq \mu \Leftrightarrow C_g(|\psi\rangle) = 1 - \lambda \leq C_g(|\phi\rangle) = 1 - \mu$. Thus C_{l_1} and C_g also generate the same ordering in this case.

(iii) If $\lambda \geq \frac{1}{d}$, $\mu < \frac{1}{d}$, then we have $C_g(|\psi\rangle) \leq C_g(|\phi\rangle) \Leftrightarrow (d-1)\lambda \geq 1 - \mu$. To find different ordering pairs, one may choose λ and μ that satisfy $C_{l_1}(|\psi\rangle) > C_{l_1}(|\phi\rangle) \Leftrightarrow \sqrt{\lambda} + \sqrt{(d-1)(1-\lambda)} > \sqrt{\mu} + \sqrt{(d-1)(1-\mu)}$. This implies that $\sqrt{1-\lambda} + \sqrt{1-\mu} > \sqrt{(d-1)\lambda} + \sqrt{(d-1)\mu} \geq \sqrt{1-\mu} + \sqrt{(d-1)\mu}$. Hence $\frac{1-\mu}{d-1} \leq \lambda < 1 - (d-1)\mu$, $d \geq 3$, and in this case C_{l_1} and C_g generate a different ordering. Therefore we conclude that the coherence measures C_{l_1} and C_g do not give the same ordering in d -dimensional quantum systems when $d \geq 3$. They can only provide the same ordering for families of quantum states.

As another example, let us consider ρ_m defined in (7). We have that C_{l_1} and C_g provide the same ordering for this class of states, since $C_{l_1}(\rho_m) = (d-1)p$ and $C_g(\rho_m) = 1 - [\sqrt{1-p} + \frac{1}{d}(\sqrt{1-p} + \sqrt{dp - \sqrt{1-p}})]^2$ are both increasing functions with respect to p , thus $C_{l_1}(\rho_m) \leq C_{l_1}(\widetilde{\rho}_m) \Leftrightarrow p \leq \widetilde{p} \Leftrightarrow C_g(\rho_m) \leq C_g(\widetilde{\rho}_m)$, where $\widetilde{\rho}_m = \widetilde{p}|\phi_d\rangle\langle\phi_d| + \frac{1-\widetilde{p}}{d}\mathbb{I}_d$.

3.2 Ordering states with C_r and C_g

In Ref. [19], the authors have shown that C_r and C_{l_1} give rise to a different ordering of single-qubit states. Taking into account the previous result that C_{l_1} and C_g provide the same ordering of single-qubit states, we have that C_r and C_g must provide a different ordering in this case. Just like the discussion in [22], to find σ_1 and σ_2 that satisfy both $C_r(\sigma_1) > C_r(\sigma_2)$ and $C_g(\sigma_1) < C_g(\sigma_2)$, one can choose t_1 and t_2 ($t_1 < t_2$) such that $H\left(\frac{1-\sqrt{1-t_1^2}}{2}\right) > 1 - H\left(\frac{1-t_2}{2}\right)$, and then find z_1 and z_2 ($0 \leq z_1, z_2 \leq 1$) by using $H\left(\frac{1}{2} - \frac{z_1}{2}\right) - H\left(\frac{1}{2} - \frac{\sqrt{z_1^2+t_1^2}}{2}\right) > H\left(\frac{1}{2} - \frac{z_2}{2}\right) - H\left(\frac{1}{2} - \frac{\sqrt{z_2^2+t_2^2}}{2}\right)$, where $\sigma_1 = \frac{1}{2} \begin{pmatrix} 1+z_1 & t_1 \\ t_1 & 1-z_1 \end{pmatrix}$, $\sigma_2 = \frac{1}{2} \begin{pmatrix} 1+z_2 & t_2 \\ t_2 & 1-z_2 \end{pmatrix}$, and $H(x) = -x \log x - (1-x) \log(1-x)$. For instance, assume

$$\sigma_1 = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}. \quad (10)$$

we have that C_r and C_g must provide a different ordering. This can be seen from the fact that $C_r(\sigma_1) = 0.7219 > C_r(\sigma_2) = 0.5576$, and $C_g(\sigma_1) = \frac{1}{5} < C_g(\sigma_2) = \frac{3-\sqrt{3}}{6}$.

For high-dimensional quantum systems, let us define two d -dimensional states ($d \geq 3$) as follows:

$$\sigma_1^{(d)} = p\sigma_1 \oplus (1-p)\delta_1^{(d-2)}, \quad \sigma_2^{(d)} = p\sigma_2 \oplus (1-p)\delta_2^{(d-2)}, \quad (11)$$

where $0 < p \leq 1$, and $\delta_1^{(d-2)}, \delta_2^{(d-2)}$ are $(d-2)$ -dimensional incoherent states. Then $C_r(\sigma_1^{(d)}) = pC_r(\sigma_1) > pC_r(\sigma_2) = C_r(\sigma_2^{(d)})$, and $C_r(\sigma_1^{(d)}) = pC_g(\sigma_1) < pC_g(\sigma_2) = C_g(\sigma_2^{(d)})$. Thus, the coherence measures C_r and C_g give rise to a different ordering of arbitrary dimensional states.

However, for some special classes of states, C_r and C_g could generate the same ordering. For instance, it has been shown that for all single-qubit states with a fixed mixedness, the coherence measures C_{l_1} and C_r have the same ordering [23], thus C_r and C_g also have the same ordering in this case, since C_{l_1} and C_g provide the same ordering for all single-qubit states. Let us consider again a class of MCMS ρ_m , one has $C_r(\rho_m) = \log d + \frac{1+(d-1)p}{d} \log \frac{1+(d-1)p}{d} + \frac{(d-1)(1-p)}{d} \log \frac{1-p}{d}$. It can be seen that $C_r(\rho_m)$ and $C_g(\rho_m)$ are both increasing functions with respect to p , hence give rise to the same ordering.

3.3 Ordering states with C'_{tr} and C_g

It is obvious that C'_{tr} and C_g give the same ordering of single-qubit states, since in this case $C'_{tr}(\rho) = C_{l_1}(\rho)$, C_{l_1} and C_g provide the same ordering.

For high-dimensional quantum systems, since the two coherence measures C'_{tr} and C_g have no analytical expressions in general, we can only take into account special examples to show that they do not provide the same ordering. To this end, let us consider two qutrit states $\rho_1 = |\phi_2\rangle\langle\phi_2| \oplus 0$, and $\rho_2 = p|\phi_3\rangle\langle\phi_3| + \frac{1-p}{3}\mathbb{I}_3$, where $|\phi_2\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^2 |i\rangle$, and $|\phi_3\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |i\rangle$. Then we have $C'_{tr}(\rho_1) = C'_{tr}(|\phi_2\rangle\langle\phi_2|) = 1$, $C'_{tr}(\rho_2) = p$, $C_g(\rho_1) = C_g(|\phi_2\rangle\langle\phi_2|) = \frac{1}{2}$, $C_g(\rho_2) = 1 - (\frac{2}{3}\sqrt{1-p} + \frac{1}{3}\sqrt{1+2p})^2$. It can be seen that $C_g(\rho_2) \leq \frac{2}{3}$, since $C_g(\rho_2)$ is an increasing function with respect to p . Thus there exists a $p < 1$ such that $C_g(\rho_2) > C_g(\rho_1)$. For instance, let $\rho'_2 = \frac{99}{100}|\phi_3\rangle\langle\phi_3| + \frac{1}{300}\mathbb{I}_3$, then we get $C_g(\rho'_2) > C_g(\rho_1)$, and $C'_{tr}(\rho'_2) = \frac{99}{100} < C'_{tr}(\rho_1) = 1$. That is to say, C'_{tr} and C_g provide a different ordering of qutrit states. Using similar approach which transforms a 3×3 density matrix to a $d \times d$ density matrix ($d \geq 3$) by direct sum of an incoherent state, we have that the coherence measures C'_{tr} and C_g generate a different ordering of qudit ($d \geq 3$) states.

Similar to the above discussion, C'_{tr} and C_g provide the same ordering for ρ_m , since $C'_{tr}(\rho_m)$ and $C_g(\rho_m)$ are both increasing functions with respect to p .

3.4 Ordering states with C'_{tr} and C_{l_1}

Before discussing ordering states with C'_{tr} and C_{l_1} , let us first provide an upper bound of C'_{tr} . Let σ be a pure qudit state and $\delta_i = |i\rangle\langle i|$, $1 \leq i \leq d$. Then we have

$$\begin{aligned} C'_{tr}(\sigma) &\leq \min_{\lambda \geq 0, 1 \leq i \leq d} \|\sigma - \lambda \delta_i\|_{tr} \\ &= \min_{\lambda \geq 0} \sqrt{\lambda^2 + (2 - 4\max_i \{\sigma_{ii}\})\lambda + 1} \\ &= \begin{cases} \sqrt{1 - (2\max_i \{\sigma_{ii}\} - 1)^2} & \text{if } \max_i \{\sigma_{ii}\} \geq \frac{1}{2}, \\ 1 & \text{if } \max_i \{\sigma_{ii}\} < \frac{1}{2}. \end{cases} \end{aligned}$$

Thus, for any qudit state ρ , $C'_{tr}(\rho) \leq \sum_i p_i C'_{tr}(\rho_i) \leq 1$, where $\rho = \sum_i p_i \rho_i$ is any pure state decomposition of ρ with $p_i \geq 0$, $\sum_i p_i = 1$. For single-qubit states, since $C_{l_1}(\rho) = C'_{tr}(\rho)$, C_{l_1} and C'_{tr} of course

provide the same ordering in this case.

Now consider the following pure qutrit states, $|\psi\rangle = \sqrt{\lambda_1}|1\rangle + \sqrt{\lambda_2}|2\rangle$ and $|\phi\rangle = \sqrt{\mu_1}|1\rangle + \sqrt{\mu_2}(|2\rangle + |3\rangle)$, where $\lambda_1 + \lambda_2 = 1$, $\mu_1 + 2\mu_2 = 1$. Assume that $\frac{1}{2} \leq \lambda_1 < \mu_1 < \frac{8}{9}$. Then we find $C'_{tr}(|\phi\rangle) \leq \sqrt{1 - (2\mu_1 - 1)^2} < 2\sqrt{\lambda_1(1 - \lambda_1)} = C'_{tr}(|\psi\rangle)$, and $C_{l_1}(|\phi\rangle) = (\sqrt{\mu_1} + \sqrt{2(1 - \mu_1)})^2 - 1 > 1 \geq C_{l_1}(|\psi\rangle)$. For instance, let ρ_1 and ρ_2 be two pure qutrit states,

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{6}}{8} & \frac{\sqrt{6}}{8} \\ \frac{\sqrt{6}}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{\sqrt{6}}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

Using the previous results, one can easily get $C'_{tr}(\rho_2) \leq \sqrt{1 - (2 \cdot \frac{3}{4} - 1)^2} = \frac{\sqrt{3}}{2} < C'_{tr}(\rho_1) = 1$, and $C_{l_1}(\rho_2) = \frac{2\sqrt{6}+1}{4} > C_{l_1}(\rho_1) = 1$. Thus the coherence measures C'_{tr} and C_{l_1} give a different ordering in this case, hence also give a different ordering of states in d -dimensional quantum systems when $d \geq 3$.

Noting that $C_{l_1}(\rho_m) = (d - 1)p$ and $C'_{tr}(\rho_m) = p$ are both increasing functions with respect to p , they provide the same ordering for ρ_m . If we consider the density matrices which have the block-diagonal form under the incoherent basis, and the dimension of each block is at most 2, then the coherence measures C_{l_1} and C'_{tr} also give the same ordering.

3.5 Ordering states with C'_{tr} and C_r

Consider the coherence measures C'_{tr} and C_r for arbitrary d -dimensional quantum systems. When $d = 2$, $C'_{tr}(\rho) = C_{l_1}(\rho)$, it has been proved in [19] that C'_{tr} and C_r give rise to a different ordering. When $d \geq 3$, similar to the discussion in Sect. 3.2, one can demonstrate that C'_{tr} and C_r also give rise to a different ordering.

There also have been sets of quantum states such that C'_{tr} and C_r provide the same ordering. Note that the coherence measures C_{l_1} and C_r have the same ordering for all single-qubit states with a fixed mixedness. Thus C'_{tr} and C_r also provide the same ordering in this case, since $C'_{tr}(\rho) = C_{l_1}(\rho)$ for all single-qubit states. Besides, similarly to the previous discussion, C'_{tr} and C_r give the same ordering for ρ_m .

We now extend our discussion to other coherence measures – the coherence of formation and the Tsallis relative α -entropies of coherence. The coherence of formation is defined as $C_f(\rho) = \min_{\{p_i, \varphi_i\}} \sum_i p_i \mathcal{S}(|\varphi_i\rangle\langle\varphi_i|_{diag})$, where $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ is any pure state decomposition of ρ . The Tsallis relative α -entropies of coherence is defined as $C_\alpha(\rho) = \min_{\delta \in I} \mathcal{D}_\alpha(\rho || \delta) = \frac{r^\alpha - 1}{\alpha - 1}$, where $r = \sum_i \langle i | \rho^\alpha | i \rangle^{\frac{1}{\alpha}}$, $\alpha \in (0, 1) \cup (1, 2]$.

It has been shown that for single-qubit states, C_{l_1} , C_f give the same ordering [19]. Thus the four measures C_{l_1} , C'_{tr} , C_g and C_f provide the same ordering of single-qubit states. Moreover, we claim that C_α and C_g , C_α and C'_{tr} , as well as C_α and C_f do not generate the same ordering of single-qubit states when $\alpha = \frac{1}{2}$ and 2, since in this case, C_α and C_{l_1} generate a different ordering [22]. The fact that C_α and C_r give rise to a different ordering of single-qubit states has also been proposed in Ref. [22].

In Ref. [23], the authors studied the coherent-induced state ordering with fixed mixedness. They proved that C_{l_1} , C_r and C_α give the same ordering of single-qubit states with a fixed mixedness. Thus, we get that with a fixed mixedness, the coherence measures C_{l_1} , C_r , C_g , C'_{tr} , C_f and C_α give the same ordering of single-qubit states. Therefore the problem of ordering single-qubit states with these six coherence measures is completely solved.

4 Conclusion

We have investigated the issue of ordering states with the l_1 norm of coherence, the relative entropy of coherence, the geometric measure of coherence and the modified trace distance measure of coherence. For single-qubit states, the l_1 norm of coherence, the modified trace distance measure of coherence and the geometric coherence give the same ordering. We also have shown that the relative entropy of coherence and the geometric measure of coherence do not give the same ordering of single-qubit states. Furthermore, for high-dimensional quantum systems, each pair of the four measures C_{l_1} , C_r , C_g and C'_{tr} give a different ordering. However, for some special classes of quantum states, each pair of these measures may provide the same ordering. For instance, we have shown that they give the same ordering for a class of maximally coherent mixed states ρ_m . We also have completely solved the problem of ordering single-qubit states with the above four measures, coherence of formation and the Tsallis relative α -entropies of coherence. For each pair of the four measures C_{l_1} , C_r , C_g and C'_{tr} , we also give some sets under which they give the same ordering. It should be noted that, as it was shown in [24], the "non-equivalence" between the relative entropy and l_1 -norm coherence is due to the dependence of the first on the density matrix populations, in contrast to the last. However, we cannot follow the idea in this paper since the two coherence measures C_g and C'_{tr} have no analytical expressions in general. In other words, we do not know whether or not these two coherence measures are dependence of density matrix populations. Further efforts can be made towards whether or not there exist other coherence measures which generate the same ordering of qudit states.

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