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Notes on modified trace distance measure of coherence

by

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Abstract

We investigate the modified trace distance measure of coherence recently introduced in [Phys. Rev. A 94, 060302(R) (2016)]. We show that for any single-qubit state, the modified trace norm of coherence is equal to the l_1-norm of coherence. For any d-dimensional quantum system, an analytical formula of this measure for a class of maximally coherent mixed states is provided. The trade-off relation between the coherence quantified by the new measure and the mixedness quantified by the trace norm is also discussed. Furthermore, we explore the relation between the modified trace distance measure of coherence and other measures such as the l_1-norm of coherence and the geometric measure of coherence.

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1 Introduction

Quantum coherence, as one of the most fundamental and striking features in quantum mechanics, plays a key role in various research fields such as low-temperature thermodynamics [1, 2, 3, 4, 5], quantum biology [6, 7, 8, 9, 10, 11], nanoscale physics [12, 13], ect. Although characterizing coherence in a rigorous framework is a desirable and intriguing task, considerable progresses have only been made in quantum optics [14, 15]. Recently, Baumgratz *et al.* proposed a resource-theoretic framework for quantifying quantum coherence [16] by following the quantitative theory of quantum entanglement. This framework includes two important concepts – incoherent states and incoherent operations, which are analogous to the separable states and local operations and classical communication (LOCC) respectively in the quantum entanglement theory. A nonnegative function C defined on a space of quantum states can be used as a measure of coherence, if it satisfies the following four conditions [16]:

(B1) $C(\rho) = 0$ if and only if ρ is an incoherent state, i.e., ρ can be written as $\rho = \sum_{i=1}^{d} p_i |i\rangle\langle i|,\ p_i \geq 0,\ \sum_{i=1}^{d} p_i = 1$, for a fixed basis $\{|i\rangle\}_{i=1}^{d}$ in a d-dimensional Hilbert space;

(B2) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ, i.e., Λ is a completely positive trace preserving (CPTP) map, $\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger$, where $\{K_n\}$ is a set of incoherent Kraus operators satisfying $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$ for all n, $\mathcal{I}$ is the set of incoherent states;

(B3) $\sum_n p_n C(\rho_n) \leq C(\rho)$, where $p_n = \mathrm{Tr}(K_n \rho K_n^\dagger), \rho_n = K_n \rho K_n^\dagger / p_n, \{K_n\}$ is a set of incoherent Kraus operators;

(B4) C is a convex function, i.e., $C(\sum_i p_i \rho_i) \leq \sum_i p_i C(\rho_i)$ for any set of quantum states $\{\rho_i\}$ and any probability distribution $\{p_i\}$.

Conditions (B2) and (B3) are often referred as monotonicity and strong monotonicity under incoherent channel, respectively. Similar to the entanglement quantifiers, very few functions satisfy the strong monotonicity condition and can be used as proper measures of coherence. In their seminal paper, Baumgratz *et al.* presented two coherence measures satisfying conditions (B1)–(B4) for all states, i.e., the l_1-norm-based measure C_{l_1} defined as $C_{l_1}(\rho) = \min_{\sigma \in \mathcal{I}} \|\rho - \sigma\|_{l_1} = \sum_{i \neq j} |\rho_{ij}|$, and the relative entropy based measure C_r defined as $C_r(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho \| \sigma)$. Then a natural question is raised that whether l_p-norm and Schatten-p-norm can also be employed to be proper coherence measures for all p.

However, this is not the case. In Ref. [17] it has been shown that both l_p-norm-based and Schatten-p-norm-based measures violate the strong monotonicity for $p > 1$, leaving the problem whether the trace norm (Schatten-1-norm) of coherence is a legitimate coherence measure still open.

Recently, Yu *et al.* put forward an alternative framework for quantifying coherence [18]. This framework consists of the following three conditions that a function C should satisfy as a measure of coherence:

(C1) $C(\rho) \geq 0$, and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$;

(C2) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ;

(C3) $C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block diagonal states ρ in the incoherent basis.

It has been shown that this framework is equivalent to the previous one proposed by Baumgratz *et al.*. That is to say, the three conditions (C1)–(C3) can be derived from conditions (B1)–(B4), and vice visa. However, the new framework is more convenient for various applications. By using condition (C3), the authors in Ref. [18] proved that the trace norm of coherence is not a legitimate coherence measure, thus it must violate the strong monotonicity. Furthermore, they introduced a new measure, called the modified trace norm of coherence,

$$C'_{\mathrm{tr}}(\rho) = \min_{\lambda \geq 0, \delta \in \mathcal{I}} \|\rho - \lambda\delta\|_{\mathrm{tr}}, \tag{1}$$

which can be shown to satisfy (C1)–(C3): Like the l_1-norm-based measure C_{l_1} and the relative entropy based measure C_r, this new measure C'_{tr} is also worthy of being further investigated.

In this paper, we compute the modified trace norm of coherence for any single-qubit state and a class of maximally coherent mixed states for any qudit system. We also discuss the trade-off relations between the coherence quantified by the modified trace distance measure and the mixedness quantified by the trace norm. Moreover, we study the relations among the modified trace distance measure, the l_1-norm-based measure and the geometric measure of coherence.

2 Modified trace norm of coherence

In this paper, we fix a set of basis $\{|i\rangle\}_{i=0}^{d-1}$ in a d-dimensional Hilbert space. To find the analytic form of modified trace norm of coherence defined in (1) for any single-qubit state, let us consider a 2×2 Hermitian matrix A with eigenvalues λ_1 and λ_2. We have that $\|A\|_{\mathrm{tr}}^2 = (|\lambda_1| + |\lambda_2|)^2 = \mathrm{Tr}(A^2) + 2|\det(A)|$. Using this fact, we have the following result.

Proposition 1. Let $\rho = \frac{1}{2}(I + \overrightarrow{r} \cdot \overrightarrow{\sigma})$ be a qubit state, where $\overrightarrow{r} = (r_1, r_2, r_3)$, $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Then the modified trace norm of coherence for ρ is equal to the l_1-norm of coherence, i.e., $C'_{\mathrm{tr}}(\rho) = C_{l_1}(\rho) = \sqrt{r_1^2 + r_2^2}$.

Proof. We need to minimize $\|\rho - \lambda\delta\|_{\mathrm{tr}}$, where $\lambda \geq 0$, $\delta = \mathrm{diag}\{x_1, x_2\}$, $x_1, x_2 \geq 0$, $x_1 + x_2 = 1$. Under the fixed basis, one has

$$\rho - \lambda\delta = \begin{pmatrix} \frac{1}{2}(1 + r_3) - \lambda x_1 & \frac{1}{2}(r_1 - \mathrm{i}r_2) \\ \frac{1}{2}(r_1 + \mathrm{i}r_2) & \frac{1}{2}(1 - r_3) - \lambda x_2 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \|\rho - \lambda\delta\|_{\mathrm{tr}}^2 = & \left[\frac{1}{2}(1 + r_3) - \lambda x_1\right]^2 + \left[\frac{1}{2}(1 - r_3) - \lambda x_2\right]^2 + \frac{1}{2}(r_1^2 + r_2^2) \\ & + 2\left|\left[\frac{1}{2}(1 + r_3) - \lambda x_1\right]\left[\frac{1}{2}(1 - r_3) - \lambda x_2\right] - \frac{1}{4}(r_1^2 + r_2^2)\right|. \end{aligned} \tag{2}$$

Set $a = \frac{1}{2}(1 + r_3) - \lambda x_1$, $b = \frac{1}{2}(1 - r_3) - \lambda x_2$, $c = \frac{1}{4}(r_1^2 + r_2^2)$. Using the inequality [17]: $a^2 + b^2 + 2|ab - c| \geq 2|c|$, we get

$$\|\rho - \lambda\delta\|_{\mathrm{tr}} \geq \sqrt{r_1^2 + r_2^2}, \tag{3}$$

and the equality holds when $\lambda = 1$, $\delta = \mathrm{diag}\{\frac{1}{2}(1 + r_3), \frac{1}{2}(1 - r_3)\}$. Thus we have $C'_{\mathrm{tr}}(\rho) = \sqrt{r_1^2 + r_2^2}$. On the other hand, $C_{l_1}(\rho) = \frac{1}{2}|r_1 - \mathrm{i}r_2| + \frac{1}{2}|r_1 + \mathrm{i}r_2| = \sqrt{r_1^2 + r_2^2}$. Hence $C'_{\mathrm{tr}}(\rho) = C_{l_1}(\rho)$ holds for any qubit state. □

For high-dimensional quantum states, the computation of $C'_{\mathrm{tr}}(\rho)$ becomes more difficult, since it is not easy to find the closest incoherent state with multiplier λ. However, we can calculate the modified trace norm of coherence for a class of important coherent states – the maximally coherent mixed states (MCMS) [19], which are defined as

$$\rho_m = p|\phi_d\rangle\langle\phi_d| + \frac{1-p}{d}\mathbb{I}_d, \tag{4}$$

where $0 < p \leq 1$, and $|\phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$ is the maximally coherent state. To this end, let us consider a class of unitary operators $U_n = \sum_{k=0}^{d-1} |k \oplus n\rangle\langle k|,\ n = 0, \ldots d-1$, where $k \oplus n$ denotes $(k+n)$ mod d. Taking into account the unitary invariance and the subadditivity of the trace norm, we have

$$\begin{aligned} \|\rho_m - \lambda\delta\|_{\mathrm{tr}} &= \frac{1}{d} \sum_{n=0}^{d-1} \left\| U_n \left(p|\phi_d\rangle\langle\phi_d| + \frac{1-p}{d}\mathbb{I}_d - \lambda\delta \right) U_n^\dagger \right\|_{\mathrm{tr}} \\ &\geq \frac{1}{d} \left\| \sum_{n=0}^{d-1} U_n \left(p|\phi_d\rangle\langle\phi_d| + \frac{1-p}{d}\mathbb{I}_d - \lambda\delta \right) U_n^\dagger \right\|_{\mathrm{tr}} . \end{aligned} \tag{5}$$

By using the facts that $U_n|\phi_d\rangle = |\phi_d\rangle,\ 0 \leq n \leq d-1$, and $\sum_{n=0}^{d-1} U_n \delta U_n^\dagger = \mathbb{I}_d$ [18], we get

$$\begin{aligned} \|\rho_m - \lambda\delta\|_{\mathrm{tr}} &\geq \left\| p|\phi_d\rangle\langle\phi_d| + \frac{1-p-\lambda}{d}\mathbb{I}_d \right\|_{\mathrm{tr}} \\ &= \left| p + \frac{1-p-\lambda}{d} \right| + (d-1) \left| \frac{1-p-\lambda}{d} \right| \\ &= |p+x| + (d-1)|x|, \end{aligned} \tag{6}$$

where $x = \frac{1-p-\lambda}{d}$, and $x \leq \frac{1-p}{d}$ since $\lambda \geq 0$. To find the minimum value of $|p+x|+(d-1)|x|$, we consider the following cases:

(i) If $x \leq -p$, then $|p+x| + (d-1)|x| = -p - dx \geq (d-1)p \geq p$;

(ii) If $-p \leq x \leq 0$, then $|p+x| + (d-1)|x| = p - (d-2)x \geq p$;

(iii) If $0 \leq x \leq \frac{1-p}{d}$, then $|p+x| + (d-1)|x| = p + dx \geq p$.

Thus we have $\|\rho_m - \lambda\delta\|_{\mathrm{tr}} \geq p$. Setting $\lambda = 1-p$ and $\delta = \frac{1}{d}\mathbb{I}_d$, one can easily get $\|\rho_m - \lambda\delta\|_{\mathrm{tr}} = p$. Therefore we obtain $C'_{\mathrm{tr}}(\rho_m) = p$.

3 Trade-off relations between coherence and mixedness

In Ref. [19], Singh *et al.* discussed the relations between quantum coherence and mixedness in any d-dimensional quantum system. They claimed that the amount of coherence that a quantum system can possess must be limited, if the mixedness of the system is fixed. A trade-off relation between coherence quantified by the l_1-norm $C_{l_1}(\rho)$ and the mixedness quantified by the normalized linear entropy $M_l(\rho) = \frac{d}{d-1}(1 - \mathrm{Tr}(\rho^2))$ is given [19]:

$$\frac{C_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \leq 1. \tag{7}$$

Quantum states with maximal coherence for a fixed mixedness are called maximally coherent mixed states (MCMS). It has been shown that ρ_m defined in (4) is the only form of MCMS with respect to the above inequality [19]. However, for other measures of coherence and mixedness, few results have been known so far for such trade-off relations [19, 20].

In this section, we discuss trade-off relations between coherence quantified by the modified trace distance measure and the mixedness quantified by the trace norm. To define a new measure of quantum mixedness, we first determine the range of $\|\rho - \frac{1}{d}\mathbb{I}_d\|_{\rm tr}$ for arbitrary qudit state ρ. Let $\lambda_i,\ i = 1, \ldots, d$, be the eigenvalues of ρ, and assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \frac{1}{d} \geq \lambda_{k+1} \geq \cdots \geq \lambda_d,\ 1 \leq k \leq d$. Then we have

$$\begin{aligned} \left\|\rho - \frac{1}{d}\mathbb{I}_d\right\|_{\rm tr} &= \sum_{i=1}^{d} \left|\lambda_i - \frac{1}{d}\right| \\ &= \sum_{i=1}^{k} \lambda_i - \frac{k}{d} - \left(\sum_{i=k+1}^{d} \lambda_i - \frac{d-k}{d}\right) \\ &= 2\left(\sum_{i=1}^{k} \lambda_i - \frac{k}{d}\right) \\ &\leq 2\left(1 - \frac{1}{d}\right), \end{aligned} \tag{8}$$

where the equality holds if ρ is a pure state. Thus we can define a new measure of mixedness based on the trace norm

$$M_{\rm tr}(\rho) = 1 - \frac{d}{2(d-1)} \left\|\rho - \frac{1}{d}\mathbb{I}_d\right\|_{\rm tr}. \tag{9}$$

It can easily be seen that $0 \leq M_{\rm tr}(\rho) \leq 1$, and $M_{\rm tr}(\rho) = 0$ if ρ is pure.

For general cases, the trade-off relations between $C'_{\rm tr}(\rho)$ and $M_{\rm tr}(\rho)$ are very difficult to derive, since we have no analytical form of $C'_{\rm tr}(\rho)$ for arbitrary quantum states. However, in the next, we will present such trade-off relation for single-qubit state and a complementarity relation for ρ_m.

Consider again the single-qubit state $\rho = \frac{1}{2}(I + \overrightarrow{r} \cdot \overrightarrow{\sigma})$ with eigenvalues $\lambda_1 = \frac{1}{2}(1 + |\overrightarrow{r}|)$ and $\lambda_2 = \frac{1}{2}(1 - |\overrightarrow{r}|)$. Then we have $M_{\rm tr}(\rho) = 1 - (|\lambda_1 - \frac{1}{2}| + |\lambda_2 - \frac{1}{2}|) = 1 - |\overrightarrow{r}|$. Using the result derived in Proposition 1, $C'_{\rm tr}(\rho) = \sqrt{r_1^2 + r_2^2}$, we have the following trade-off relation,

$$C'_{\rm tr}(\rho) + M_{\rm tr}(\rho) = \sqrt{r_1^2 + r_2^2} + (1 - |\overrightarrow{r}|) \leq 1. \tag{10}$$

The equality holds if and only if $r_3 = 0$. Thus the maximally coherent mixed states in this case are those states with Bloch vectors in the closed unit disk on the r_1-r_2 plane.

For a class of MCMS ρ_m given in (4), one can easily get $\|\rho_m - \frac{1}{d}\mathbb{I}_d\|_{\rm tr} = \frac{2p(d-1)}{d}$ by simple algebra, thus $M_{\rm tr}(\rho_m) = 1 - p$. Taking into account that $C'_{\rm tr}(\rho_m) = p$, we obtain the following complementarity relation between coherence and mixedness

$$C'_{\rm tr}(\rho_m) + M_{\rm tr}(\rho_m) = 1. \tag{11}$$

4 Relation among $C'_{\rm tr}$, C_{l_1} and the geometric measure of coherence

The l_1 norm of coherence has been deeply investigated and the analytical expression of C_{l_1} is given in [16]. It is interesting to find the relationa among C_{l_1} and other measures. The interrelations between C_{l_1} and the relative entropy of coherence C_r have been derived in [17]. Here we will prove that C_{l_1} is an upper bound for the modified trace norm of coherence $C'_{\rm tr}$.

In Ref. [21], the authors introduced the geometric measure of coherence defined by $C_g(\rho) = 1 - \max_{\sigma\in\mathcal{I}} F(\rho,\sigma)$, where $F(\rho,\sigma) = ({\rm Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}})^2$ is the fidelity of two density operators ρ and σ. The computation of C_g for any qudit state is formidably difficult. Recently Zhang *et al.* provided the lower and upper bounds of C_g [20]. It is also worth studying the relations between C_g and $C'_{\rm tr}$.

We have the following Theorem.

Theorem 1 *Let ρ be a qudit state. Then we have*

$$C_g(\rho) \le C'_{\rm tr}(\rho) \le C_{l_1}(\rho). \tag{12}$$

Proof. To see that $C_{l_1}(\rho)$ is an upper bound of $C'_{\rm tr}(\rho)$, we only need to use the fact that $\|A\|_{\rm tr} \le \|A\|_{l_1}$ for any Hermitian matrix A [22]. Then we have

$$\begin{aligned} C'_{\rm tr}(\rho) &= \min_{\lambda\ge 0,\delta\in\mathcal{I}} \|\rho - \lambda\delta\|_{\rm tr} \\ &\le \min_{\delta\in\mathcal{I}} \|\rho - \delta\|_{\rm tr} \\ &\le \min_{\delta\in\mathcal{I}} \|\rho - \delta\|_{l_1} \\ &= C_{l_1}(\rho). \end{aligned} \tag{13}$$

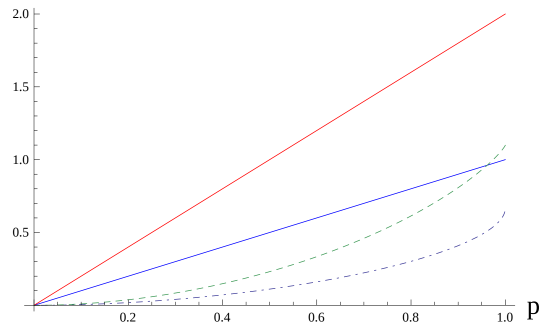

Figure 1: The red and blue solid lines are the values of $C_{l_1}(\rho_m)$ and $C'_{\rm tr}(\rho_m)$, respectively. The dot-dashed line is $C_g(\rho_m)$, and the dashed line is $C_r(\rho_m)$.

Now we prove that $C'_{\rm tr}(\rho) \geq C_g(\rho)$. For two positive semidefinite matrices P and Q, it holds that $\|P-Q\|_{\rm tr} \geq \|\sqrt{P}-\sqrt{Q}\|^2_{\rm HS}$, where $\|A\|_{\rm HS} = \sqrt{{\rm Tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm [23]. Then we have $C'_{\rm tr}(\rho) \geq \min_{\lambda\geq 0,\delta\in\mathcal{I}} \|\sqrt{\rho}-\sqrt{\lambda\delta}\|^2_{\rm HS}$. Thus we only need to minimize $\|\sqrt{\rho}-\sqrt{\lambda\delta}\|^2_{\rm HS}$. Let $\sqrt{\rho} = \sum_{i,j} b_{ij}|i\rangle\langle j|$, $\delta = \sum_i x_i|i\rangle\langle i|$, $x_i \geq 0$, $\sum_i x_i = 1$. Note that

$$\begin{aligned}
\|\sqrt{\rho}-\sqrt{\lambda\delta}\|^2_{\rm HS} &= {\rm Tr}(\sqrt{\rho}-\sqrt{\lambda\delta})^2 \\
&= {\rm Tr}(\rho+\lambda\delta) - 2{\rm Tr}(\sqrt{\rho}\sqrt{\lambda\delta}) \\
&= 1+\lambda-2\sum_i b_{ii}\sqrt{\lambda x_i} \\
&= 1-\sum_i b_{ii}^2 + \sum_i (b_{ii}-\sqrt{\lambda x_i})^2 \\
&\geq 1-\sum_i b_{ii}^2,
\end{aligned} \tag{14}$$

and the equality holds if and only if $\lambda x_i = b_{ii}$, $\forall i$, which yields that $\lambda = \sum_i b_{ii}^2$, $x_i = b_{ii}^2/\sum_i b_{ii}^2$. Then we obtain $C'_{\rm tr}(\rho) \geq 1-\sum_i b_{ii}^2$. On the other hand, it has been shown that $C_g(\rho) \leq 1-\sum_i b_{ii}^2$ [20]. Therefore we have $C'_{\rm tr}(\rho) \geq C_g(\rho)$. This completes the proof. □

As an example, FIG 1 shows the relations among $C'_{\rm tr}(\rho_m)$, $C_g(\rho_m)$, $C_{l_1}(\rho_m)$ and $C_r(\rho_m)$ for $d=3$. It can be seen that $C'_{\rm tr}(\rho_m) > C_r(\rho_m)$ when $0<p<0.96151$, and $C'_{\rm tr}(\rho_m) \leq C_r(\rho_m)$ when $0.96151 \leq p \leq 1$. Thus there is no order relation between the quantities $C'_{\rm tr}(\rho)$ and $C_r(\rho)$ in general.

5 Conclusion

We have investigated the modified trace norm of coherence $C'_{\rm tr}(\rho)$. The analytical formulae of this new measure are provided for any single-qubit state and for a class of maximally coherent mixed states. For qubit states, the modified trace norm of coherence is equal to the l_1-norm of coherence. However, the calculation of $C'_{\rm tr}(\rho)$ for arbitrary quantum states is more complicated, since it is not easy to find the optimal incoherent state δ and the multiplier λ simultaneously in general.

We have also discussed the trade-off relation between coherence quantified by this new measure and the mixedness quantified by the trace norm. A new class of maximally coherent mixed states for qubit system has been obtained by providing a trade-off relation between these two quantities. For a special class of coherent states given in (4), a complementarity relation has also been presented.

As a new measure of coherence, its relations to other measures are worth studying. We have shown that the l_1-norm coherence provides an upper bound for $C'_{\rm tr}(\rho)$, while the geometric measure of coherence $C_g(\rho)$ is the lower bound of $C'_{\rm tr}(\rho)$. Further efforts should be made toward analytical formulae of $C'_{\rm tr}(\rho)$ for arbitrary or any other special classes of states ρ, new trade-off relations between the coherence and the mixedness, as well as the relations to other coherence quantifiers.

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