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by

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Abstract

The entanglement detection via local measurements can be experimentally implemented. Based on mutually unbiased measurements and general symmetric informationally complete positive-operator-valued measures, we present separability criteria for bipartite quantum states, which, by theoretical analysis, are stronger than the related existing criteria via these measurements. A detailed example is presented to show that the separability criteria can be more efficient than some well-known criteria such as positive partial transpose (Peres-Horodecki) criteria, realignment (computable cross norm) criteria, the criteria based on the realignment of $\rho - \rho^A \otimes \rho^B$, the criteria based on Bloch representations and some evaluated covariance matrix criteria.

1 Introduction

Entanglement is one of the main differences between quantum mechanics and classical physics. As an essence resource, it has been widely used to implement some quantum tasks from quantum cryptography to quantum teleportation; see, e.g., [1]. Thus, the detection or determination of entanglement of any quantum state becomes extremely important and necessary. Although many efforts have been devoted to the study of this problem, it is still open except for the case of $m \times n$ quantum states with $mn \leq 6$ [2, 3, 4]. Nevertheless, in the last years, a variety of sufficient conditions for entanglement have been proposed; see [5] for a survey.

Among them, the separability criteria based on quantum measurements are attractive in the last years, since they are easily experimentally implemented. In [6], the mutually unbiased bases (MUBs) [7] were connected with the entanglement detection for bipartite, mulitpartite and continuous-variable quantum systems. The detection ability of the presented criteria partly depends on the maximum number of MUBs.

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For $\mathbb{C}^d$, if the dimension d is an integer power of a prime number, then extremal sets containing $d+1$ MUBs are known. However, for arbitrary d the maximum number of MUBs is still not known [8].

In [9], MUBs were generalized to mutually unbiased measurements (MUMs). Unlike MUBs, a complete set of MUMs, i.e., $d+1$ MUMs, can always be constructed with certain efficiency parameters for any d-dimensional space. Based on MUMs, Chen and Fei [10] obtained a new separability criterion including the corresponding criterion in [6] as a special case. In [11], by applying MUMs to $\rho-\rho^A\otimes\rho^B$, where ρ^A and ρ^B are the reduced density matrices of ρ, the authors obtained a new separability criterion, which is more powerful than the corresponding ones in [6] and [10]. Later, these criteria were further generalized to the multipartite case [12, 13]. It should be mentioned that the criteria in [12] can be used to the quantum systems with subsystems having different dimensions. Recently, Rastegin [14] derived separability criteria in terms of local fine-grained uncertainty relations under MUBs and MUMs.

Another kind of quantum measurements, known as symmetric informationally complete positive-operator-valued measures (SIC-POVMs) [15], has a close connection with MUBs. In [16], a family of general SIC-POVMs (GSIC-POVMs) has been constructed. Unlike SCI-POVMs, the measurement operators of GSIC-POVMs are not necessarily rank 1. By analogous arguments to MUMs, the separability criteria based on GSIC-POVMs were also investigated in [11, 13, 17, 18].

The aim of this paper is to achieve some new separability criteria via MUMs and GSIC-POVMs. On one hand, they are strictly stronger than the corresponding ones in [6, 10, 11, 12, 17, 18]. On the other hand, by an example, the proposed criteria can be more efficient than the well-known criteria such as the positive partial transpose (PPT) criterion or Peres-Horodecki criterion [2, 3], the computable cross norm or realignment (CCNR) criterion [19], the criterion based on Bloch representations (for simplicity, we call it the BR criterion) [20], the realignment criterion based on $\rho-\rho^A\otimes\rho^B$ (for simplicity, we call it the RM criterion) [21], and some evaluated covariance matrix (CM) criteria [22, 23]. Meanwhile, the connection between the presented criteria and the CM criteria is discussed.

The remainder of the paper is organized as follows. In Section 2, some preliminaries about MUMs and GSIC-POVMs are briefly reviewed. In Section 3, the separability criteria based on MUMs are obtained for bipartite quantum states. Its connection with the CM criteria is investigated in detail. In Section 4, by similar methods used in Section 3, the separability criteria via GSIC-POVMs for bipartite quantum states are derived. Some conclusions and future work are drawn in Section 5.

2 Some classes of measurements

The concept of mutually unbiased bases (MUBs) was first introduced by Schwinger [7]. The two orthonormal bases $\{|a_i\rangle\}_{i=1}^d$ and $\{|b_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$ are said to be mutually unbiased, if and only if

$$|\langle a_i|b_j\rangle|^2=\frac{1}{d}, i,j=1,\cdots,d. \tag{1}$$

A set of othonormal bases are MUBs if each two of them are mutually unbiased. The MUBs have key applications in quantum information processing such as quantum state tomography, mean kings problems, quantum cryptography and so on; see, e.g., [8]. When d is an integer power of a prime number, a complete set of MUBs, i.e., $d+1$ MUBs, can be constructed [24]. However, whether there exists a complete set of MUBs for any d is still unknown.

Mutually unbiased measurements (MUMs) [9] can be seen as a generalization of MUBs under weaker requirements. Two measurements on $\mathbb{C}^d$

$$\mathcal{P}^{(b)} = \left\{ P_n^{(b)} | P_n^{(b)} \geq 0, \sum_{n=1}^{d} P_n^{(b)} = I_d \right\}, b = 1, 2$$

are said to be mutually unbiased if and only if

(i) $\mathrm{Tr}(P_n^{(b)}) = 1$;

(ii) $\mathrm{Tr}(P_n^{(b)} P_{n'}^{(b')}) = \delta_{nn'}\delta_{bb'}\kappa + (1-\delta_{nn'})\delta_{bb'}\frac{1-\kappa}{d-1} + (1-\delta_{bb'})\frac{1}{d}$,

where the efficiency parameter κ satisfies $\frac{1}{d} < \kappa \leq 1$. A complete set of MUMs, i.e., $d+1$ MUMs, can be constructed in any d-dimensional space; see [9] for a detail. If κ can be chosen to be $\kappa = 1$, a complete set of MUMs reduces to a complete set of MUBs.

A set of d^2 operators

$$\left\{ M_i | M_i \geq 0, \sum_{i=1}^{d^2} M_i = I_d \right\}$$

is said to be a general symmetric informationally complete positive-operator-valued measure (GSIC-POVM) [16] if and only if

(i) $\mathrm{Tr}(M_i^2) = \alpha, i = 1, \cdots, d^2$;

(ii) $\mathrm{Tr}(M_i M_j) = \frac{1-d\alpha}{d(d^2-1)}, 1 \leq i \neq j \leq d^2$,

where the parameter α satisfies $\frac{1}{d^3} < \alpha \leq \frac{1}{d^2}$. The complete set of GSIC-POVMs can be constructed explicitly in all finite dimensions for some choices of α [16]. If α can be chosen to be $\alpha = \frac{1}{d^2}$, then the complete set of GSIC-POVMs becomes the complete set of SIC-POVMs [25].

3 Entanglement detection via MUMs

In this section, based on MUMs, we give separability criteria for bipartite states. In the follows, for any matrix A, we denote by $\mathrm{Tr}(A)$, A^T, $||A||_{\mathrm{tr}}$ and $||A||_2$ the trace, the transpose, the trace norm (i.e., the sum of singular values) and the spectral norm (i.e., the maximum singular value) of A, respectively.

Let $\{\mathcal{P}^{(b)}\}_{b=1}^{m_1}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{m_2}$ be two sets of MUMs on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively. Then we define $X = \{X_i\}_{i=1}^{d_1 m_1}$ and $Y = \{Y_j\}_{j=1}^{d_2 m_2}$ with

$$X_i = P_{n_1}^{(b_1)}, i = (b_1 - 1)d_1 + n_1, b_1 = 1, \cdots, m_1, n_1 = 1, \cdots, d_1, \tag{2}$$

$$Y_j = Q_{n_2}^{(b_2)}, j = (b_2 - 1)d_2 + n_2, b_2 = 1, \cdots, m_2, n_2 = 1, \cdots, d_2. \tag{3}$$

For any state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we further define

$$\mathcal{M}^{(X,Y)}(\rho) = (w_{ij}) \in \mathbb{C}^{d_1 m_1 \times d_2 m_2} \tag{4}$$

with

$$w_{ij} = \mathrm{Tr}\left(X_i \otimes Y_j(\rho - \rho^A \otimes \rho^B)\right),$$

where ρ^A and ρ^B are the reduced density matrices acting on the first and second subsystems, respectively.

3.1 Separability criteria via MUMs

The following theorem gives the new separability criterion based on $\mathcal{M}^{(X,Y)}(\rho)$.

Theorem 3.1. *Let $\{\mathcal{P}^{(b)}\}_{b=1}^{m_1}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{m_2}$ be two sets of MUMs on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$ with efficiency parameters κ_1 and κ_2, respectively, and let $X = \{X_i\}_{i=1}^{d_1 m_1}$ and $Y = \{Y_j\}_{j=1}^{d_2 m_2}$ be defined as in* (2) *and* (3)*. If the quantum state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is separable, then*

$$\left\|\mathcal{M}^{(X,Y)}(\rho)\right\|_{\mathrm{tr}} \leq \sqrt{\frac{m_1 - 1}{d_1} + \kappa_1 - \sum_{i=1}^{d_1 m_1} (\mathrm{Tr}(X_i \rho^A))^2} \sqrt{\frac{m_2 - 1}{d_2} + \kappa_2 - \sum_{i=1}^{d_2 m_2} (\mathrm{Tr}(Y_i \rho^B))^2}. \tag{5}$$

Proof. Since ρ is separable, it can be decomposed into

$$\rho = \sum_{k=1}^{r} p_k \rho_k^A \otimes \rho_k^B, \tag{6}$$

where $p_k > 0, \sum_k p_k = 1$, ρ_k^A and ρ_k^B are pure states on the first and second subsystems, respectively. From the equality [21]

$$\rho - \rho^A \otimes \rho^B = \frac{1}{2} \sum_{s,t=1}^{r} p_s p_t (\rho_s^A - \rho_t^A) \otimes (\rho_s^B - \rho_t^B),$$

we can derive

$$\mathcal{M}^{(X,Y)}(\rho) = \frac{1}{2} \sum_{s,t=1}^{r} p_s p_t \mathcal{M}\left((\rho_s^A - \rho_t^A) \otimes (\rho_s^B - \rho_t^B)\right), \tag{7}$$

where

$$\begin{aligned}
&\mathcal{M}\left((\rho_s^A - \rho_t^A) \otimes (\rho_s^B - \rho_t^B)\right) = \left(\mathrm{Tr}\left((X_i \otimes Y_j)(\rho_s^A - \rho_t^A) \otimes (\rho_s^B - \rho_t^B)\right)\right)_{d_1 m_1 \times d_2 m_2} \\
&= \left(\mathrm{Tr}(X_i(\rho_s^A - \rho_t^A))\mathrm{Tr}(Y_j(\rho_s^B - \rho_t^B))\right)_{d_1 m_1 \times d_2 m_2} \\
&= \begin{pmatrix} \mathrm{Tr}(X_1(\rho_s^A - \rho_t^A)) \\ \vdots \\ \mathrm{Tr}(X_{d_1 m_1}(\rho_s^A - \rho_t^A)) \end{pmatrix} \begin{pmatrix} \mathrm{Tr}(Y_1(\rho_s^B - \rho_t^B)) & \cdots & \mathrm{Tr}(Y_{d_2 m_2}(\rho_s^B - \rho_t^B)) \end{pmatrix} \\
&:= \beta_{s,t} \eta_{s,t}^T.
\end{aligned} \tag{8}$$

From (7) and (8), it is easy to deduce

$$\begin{aligned}||\mathcal{M}^{(X,Y)}(\rho)||_{\rm tr} &\le \frac{1}{2}\sum_{s,t=1}^{r} p_s p_t ||\mathcal{M}\left((\rho_s^A-\rho_t^A)\otimes(\rho_s^B-\rho_t^B)\right)||_{\rm tr}\\ &= \frac{1}{2}\sum_{s,t=1}^{r} p_s p_t ||\beta_{s,t}\eta_{s,t}^T||_{\rm tr} = \frac{1}{2}\sum_{s,t=1}^{r} p_s p_t ||\beta_{s,t}||_2||\eta_{s,t}||_2\\ &\le \frac{1}{2}\sqrt{\sum_{s,t=1}^{r} p_s p_t ||\beta_{s,t}||_2^2}\sqrt{\sum_{s,t=1}^{r} p_s p_t ||\beta_{s,t}||_2^2}\\ &\le \sqrt{\frac{m_1-1}{d_1}+\kappa_1-\sum_{i=1}^{d_1 m_1}(\mathrm{Tr}(X_i\rho^A))^2}\sqrt{\frac{m_2-1}{d_2}+\kappa_2-\sum_{i=1}^{d_2 m_2}(\mathrm{Tr}(Y_i\rho^B))^2}.\end{aligned}$$

where, in the second equality, the second inequality and the third inequality, we have used, respectively, the equality

$$|||a\rangle\langle b|||_{\rm tr} = |||a\rangle||_2|||b\rangle||_2 \quad \text{for any vectors } |a\rangle \text{ and } |b\rangle,$$

the well-known Cauchy-Schwarz inequality, and the inequalities [26]

$$\sum_{i=1}^{d_1 m_1}\mathrm{Tr}(X_i\rho_1)^2 \le \frac{m_1-1}{d_1}+\kappa_1, \quad \sum_{i=1}^{d_2 m_2}\mathrm{Tr}(Y_i\rho_2)^2 \le \frac{m_2-1}{d_2}+\kappa_2$$

for any pure states ρ_1 and ρ_2 in $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively. □

Consider the case $d_1=d_2=d$, $m_1=m_2=d+1$, and $\kappa_1=\kappa_2=\kappa$. (5) in Theorem 3.1 reduces to

$$||\mathcal{M}^{(X,Y)}(\rho)||_{\rm tr} \le \sqrt{1+\kappa-\sum_{i=1}^{d(d+1)}(\mathrm{Tr}(X_i\rho^A))^2}\sqrt{1+\kappa-\sum_{i=1}^{d(d+1)}(\mathrm{Tr}(Y_i\rho^B))^2}. \qquad (10)$$

In this case, the criterion [11, Theorem 2] states that any separable state ρ in $\mathbb{C}^d\otimes\mathbb{C}^d$ satisfies

$$\sum_{i=1}^{d(d+1)}\left|\mathrm{Tr}\left(X_i\otimes Y_i(\rho-\rho^A\otimes\rho^B)\right)\right| \le \sqrt{1+\kappa-\sum_{i=1}^{d(d+1)}(\mathrm{Tr}(X_i\rho^A))^2}\sqrt{1+\kappa-\sum_{i=1}^{d(d+1)}(\mathrm{Tr}(Y_i\rho^B))^2},$$

which is weaker than (10) by the inequality [20]

$$\sum_{i=1}^{n}|g_{ii}| \le ||G||_{\rm tr} \text{ for any matrix } G=(g_{ij})\in\mathbb{C}^{n\times n}.$$

Since the criteria given in [6, 10] are weaker than (11), the criterion (10) from Theorem 3.1 is the strongest one among these criteria.

Recently, Liu et al. [12] presented separable criteria for quantum states with different dimensions of subsystems. Let $d = \min\{d_1, d_2\}$. Consider $m_1 = m_2 = m$. We define two subsets

$$\{p_1, p_2, \cdots, p_{md}\} \subseteq \{1, 2, \cdots, md_1\}, \quad \{q_1, q_2, \cdots, q_{md}\} \subseteq \{1, 2, \cdots, md_2\}$$

where

$$id_1 + 1 \leq p_{id+l} \leq (i+1)d_1, \quad jd_2 + 1 \leq q_{jd+l} \leq (j+1)d_2$$

for $l = 1, \cdots, d$ and $i, j = 0, 1, \cdots, m-1$. Set $G = (g_{ij}) \in \mathbb{C}^{md \times md}$ with $g_{ij} = \mathrm{Tr}(X_{p_i} \otimes Y_{q_j} \rho)$. The criterion [12, Theorem 3] shows that any separable state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ satisfies

$$\mathrm{Tr}(G) = \sum_{i=1}^{md} \mathrm{Tr}(X_{p_i} \otimes Y_{q_i} \rho) \leq \sqrt{\frac{m-1}{d_1} + \kappa_1} \sqrt{\frac{m-1}{d_2} + \kappa_2}. \tag{13}$$

If W is a sub-matrix of the matrix V, then from [27] we have $||W||_{\mathrm{tr}} \leq ||V||_{\mathrm{tr}}$. By this conclusion and similar comparison between [10] and [11], it can be found that Theorem 3.1, with $m_1 = m_2 = m$, is stronger than the criterion (13).

3.2 The connection between Theorem 3.1 and CM criteria

We first give the definition of the covariance matrix.

Definition 3.1 [23]. Let $\{M_k\}_{k=1}^{n}$ be some observables on $\mathbb{C}^d$, and let ρ in $\mathbb{C}^d$ be a given quantum state. Then the covariance matrix $\gamma = \gamma(\rho, \{M_k\})$ is defined with entries

$$\gamma_{ij} = \langle M_i M_j \rangle - \langle M_i \rangle \langle M_j \rangle.$$

We now consider the observables $\{N_k\} = \{X_k \otimes I_{d_2}, I_{d_1} \otimes Y_k\}$, where $\{X_k\}_{k=1}^{d_1 m_1}$ and $\{Y_k\}_{k=1}^{d_2 m_2}$ are defined as in (2) and (3). The covariance matrix (CM) criterion says that, for any separable state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, there must exist pure states $|\phi_k\rangle\langle\phi_k|$, $|\xi_k\rangle\langle\xi_k|$, and convex weights p_k such that

$$\gamma(\rho, \{N_k\}) = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \geq \begin{pmatrix} \chi_A & 0 \\ 0 & \chi_B \end{pmatrix},$$

where

$$\begin{aligned} &A = \gamma(\rho^A, \{X_k\}), \quad C = \gamma(\rho^B, \{Y_k\}), \\ &B = (B_{ij}) \text{ with } B_{ij} = \langle X_i \otimes Y_j \rangle - \langle X_i \rangle \langle Y_j \rangle, \\ &\chi_A = \sum_k p_k \gamma(|\phi_k\rangle\langle\phi_k|, \{X_k\}), \quad \chi_B = \sum_k p_k \gamma(|\xi_k\rangle\langle\xi_k|, \{Y_k\}). \end{aligned}$$

The proof of (15) is completely similar to that of the CM criterion given in [23]. Nevertheless, it is different with the CM criterion given in [23], where the observables are orthogonal, not mutually unbiased.

Clearly, the CM criterion (15) cannot be used directly to detect entanglement. Thus, an "evaluation of the CM criterion" is necessary [22, 23]. In fact, Theorem 3.1 can be seen as an evaluation of the criterion (15) via singular values. Thus, another proof of Theorem 3.1 can be obtained as follows.

An alternative proof of Theorem 3.1 via the CM criterion. Since ρ is separable, ρ can be expressed as the form in (6). From the CM criterion (15) and [23, Lemma IV.1], one can get

$$\begin{aligned}||\mathcal{M}^{(X,Y)}(\rho)||_{\mathrm{tr}}^2 &= ||B||_{\mathrm{tr}}^2 \leq ||A-\chi_A||_{\mathrm{tr}}||B-\chi_B||_{\mathrm{tr}} \\ &= \left(\mathrm{Tr}(A)-\mathrm{Tr}(\chi_A)\right)\left(\mathrm{Tr}(B)-\mathrm{Tr}(\chi_B)\right). \end{aligned} \tag{16}$$

Simple computations yield

$$\mathrm{Tr}(A) = \sum_{i=1}^{d_1m_1} A_{ii} = \sum_{i=1}^{d_1m_1}\left(\langle X_i^2\rangle - \langle X_i\rangle^2\right) = \sum_{i=1}^{d_1m_1}\mathrm{Tr}(X_i^2\rho^A) - \sum_{i=1}^{d_1m_1}\left(\mathrm{Tr}(X_i\rho^A)\right)^2,$$

$$\mathrm{Tr}(\chi_A) = \sum_{i=1}^{d_1m_1}\sum_{j=1}^{r} p_j\gamma(\rho_j^A,\{X_k\})_{ii} = \sum_{i=1}^{d_1m_1}\sum_{j=1}^{r} p_j\left(\mathrm{Tr}(X_i^2\rho_j^A) - (\mathrm{Tr}(X_i\rho_j^A))^2\right) \tag{17}$$

$$\leq \sum_{i=1}^{d_1m_1}\mathrm{Tr}(X_i^2\rho^A) - \left(\frac{m_1-1}{d_1}+\kappa_1\right), \tag{18}$$

where, in the last inequality, we have used the inequality (3.1). Thus, from (16)-(18) we can derive

$$||\mathcal{M}^{(X,Y)}(\rho)||_{\mathrm{tr}}^2 \leq \left(\frac{m_1-1}{d_1}+\kappa_1-\sum_{i=1}^{d_1m_1}\left(\mathrm{Tr}(X_i\rho^A)\right)^2\right)\left(\frac{m_2-1}{d_2}+\kappa_2-\sum_{i=1}^{d_2m_2}\left(\mathrm{Tr}(Y_i\rho^B)\right)^2\right).$$

The proof is completed. □

3.3 Example

In this subsection, we compare Theorem 3.1 with the PPT [2, 3], CCNR [19], BR [20], RM [21], and CM [22, 23] criteria by an example. It was shown in [23] that the CCNR, BR, and RM criteria can be seen as corollaries of the CM criterion. In what follows, the complete set of MUMs is always constructed by using generalized Gell-Mann operators; see [9] for a detail.

The following 3×3 bound entangled state is due to [28]:

$$\rho = \frac{1}{4}\left(I_9 - \sum_{i=0}^{4}|\eta_i\rangle\langle\eta_i|\right), \tag{20}$$

where

$$|\eta_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle-|1\rangle),\ |\eta_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|2\rangle,\ |\eta_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle-|2\rangle),$$
$$|\eta_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle-|2\rangle)|0\rangle,\ |\eta_4\rangle = \frac{1}{3}(|0\rangle+|1\rangle+|2\rangle)(|0\rangle+|1\rangle+|2\rangle).$$

Consider the mixture of ρ with white noise:

$$\rho_p = \frac{1-p}{9} I_9 + p\rho, \;\; 0 \leq p \leq 1. \tag{21}$$

Numerical computations show that the existing criteria [6, 10, 11] based on MUMs cannot detect any entanglement of ρ_p. The CCNR criterion, the BR criterion, the RM criterion, the Schmidt CM criterion [23], and (10) can detect entanglement of ρ_p for $0.8897 \leq p \leq 1$, $0.9493 \leq p \leq 1$, $0.8822 \leq p \leq 1$, $0.8834 \leq p \leq 1$, and $0.8821 \leq p \leq 1$, respectively. Thus, (10) is the best one among these criteria.

For any ρ in $\mathbb{C}^d \otimes \mathbb{C}^d$ with full rank, by local filtering operations, there exist nonsingular matrices F_A, F_B on $\mathbb{C}^d$ such that the state ρ can be changed into its standard form [29]:

$$\tilde{\rho} = \frac{(F_A \otimes F_B)\rho(F_A \otimes F_B)^\dagger}{\mathrm{Tr}\left((F_A \otimes F_B)\rho(F_A \otimes F_B)^\dagger\right)} = \frac{1}{d^2}\left(I_{d^2} + \sum_{k=1}^{d^2-1} \xi_k G_k^A \otimes G_k^B\right),$$

where $\{G_k^A\}$ and $\{G_k^B\}$ are traceless orthogonal observables. Using the CM criterion [23, Proposition III.1] for $\tilde{\rho}$ we get the filter CM criterion [23, Proposition IV.13] that any separable state ρ satisfies

$$\sum_{i=1}^{d^2-1} \xi_i \leq d^2 - d.$$

For two-qubit case, this criterion becomes a sufficient and necessary criterion for separability. Moreover, it exhibits powerful ability in entanglement detection for other states; see some examples in [23].

From the filter CM criterion, the entanglement condition for ρ_p is $0.8723 \leq p \leq 1$. If we use (10) for the standard form $\tilde{\rho}_p$ of ρ_p, (10) can detect entanglement of ρ_p for $0.8722 \leq p \leq 1$. Thus, (10) is better than the filter CM criterion.

4 Separability criteria via GSIC-POVMs

Similar to Theorem 3.1, the separability criterion based on GSIC-POVMs can be derived for bipartite states.

Theorem 4.1. *Let $P = \{P_b\}_{b=1}^{d_1^2}$ and $Q = \{Q_b\}_{b=1}^{d_2^2}$ be two sets of GSIC-POVMs on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$ with parameters α_1 and α_2, respectively. If the quantum state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is separable, then*

$$||\mathcal{M}^{(P,Q)}(\rho)||_{\mathrm{tr}} \leq \sqrt{\frac{\alpha_1 d_1^2 + 1}{d_1(d_1+1)} - \sum_{i=1}^{d_1^2} (\mathrm{Tr}(P_i \rho^A))^2}\sqrt{\frac{\alpha_2 d_2^2 + 1}{d_2(d_2+1)} - \sum_{i=1}^{d_2^2} (\mathrm{Tr}(Q_i \rho^B))^2}, \tag{23}$$

where α_1, α_2 ....

By analogous analysis to Theorem 3.1, one can show that Theorems 4.1 is more efficient than the corresponding criteria in [18, Theorem 1], [11, Theorem 3] and [17, Theorem 2]. Since, by numerical computations, Theorems 4.1 has close performance with Theorem 3.1, numerical examples about Theorems 4.1 are omitted here.

5 Conclusions

We have proposed some separability criteria via MUMs and GSIC-POVMs. These criteria can be experimentally implemented, and stronger than the related criteria via MUMs and GSIC-POVMs. They can be seen as corollaries of the CM criteria based on MUMs and GSIC-POVMs. Moreover, these criteria can also outperform the well-known PPT, CCNR, BR, RM, and some evaluated CM criteria.

In the future, how to investigate genuine multipartite entanglement by using MUMs and GSIC-POVMs is an interesting problem. Suitable evaluations of the CM criterion via MUMs or GSIC-POVMs may contribute to generate more powerful separability criteria.

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