

**Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig**

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bases and mutually unbiased bases in
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by

*Ya-Jing Zhang, Hui Zhao, Naihuan Jing, and
Shao-Ming Fei*

Preprint no.: 6

2018



Unextendable maximally entangled bases and mutually unbiased bases in multipartite systems*

Ya-Jing Zhang¹, Hui Zhao^{1†}, Naihuan Jing^{2,3}, Shao-Ming Fei⁴

¹College of Applied Sciences, Beijing University of Technology, Beijing 100124, China

²Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

³Department of Mathematics, Shanghai University, Shanghai 200444, China

⁴School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

March 15, 2017

Abstract

We generalize the notion of unextendable maximally entangled basis from bipartite systems to multipartite quantum systems. It is proved that there do not exist unextendable maximally entangled bases in three-qubit systems. Moreover, two types of unextendable maximally entangled bases are constructed in tripartite quantum systems and proved to be not mutually unbiased.

Keywords: unextendable maximally entangled bases, mutually unbiased bases, multipartite quantum systems

PACS: 03.67.Mn, 03.67.Hk, 03.65.Ud

1. Introduction

Quantum entanglement has played an important role in various quantum information processings such as quantum teleportation,^[1] quantum cryptography,^[2] quantum dense coding,^[3] and parallel computing.^[4] As one of the intrinsic features of quantum computation and information, quantum entanglement is closely related to some of the fundamental problems in quantum mechanics such as reality and non-locality.^[5] An important issue concerns with the notion of unextendable product basis (UPB), which is a set of incomplete orthonormal product basis whose complementary space has no product states.^[6]

Corresponding to UPB is the unextendable maximally entangled basis (UMEB). It is known that there is no UMEB in the two-qubit system.^[7] The UMEB in bipartite systems was studied and some explicit constructions of UMEB were given in Refs. [8,9,10]. In addition, the UMEB problem was generalized to states with given Schmidt numbers.^[11,12] Though the bipartite case is well understood, the question of UMEB for multipartite systems is still an open problem.

*Project supported by the National Natural Science Foundation of China grant Nos. 11675113, 11281137, 11271138 and 11531004 and Simons Foundation grant No.198129.

[†]Corresponding author. E-mail:zhaohui@bjut.edu.cn

Another related interesting notion is that of mutually unbiased base (MUB), which also plays an important role in quantum information. The maximum number of MUB in \mathbb{C}^d is known to be no more than $d + 1$ for any given d . It has been confirmed that there are indeed $d + 1$ MUBs when d is a prime power. ^[13] However, for general d , the maximum number of MUB is still open.

In this paper, we study UMEB in three-qubit system and MUB in tripartite systems. The paper is organized as follows: In Sect.2, we first generalize UMEB in bipartite systems to multipartite systems, then prove that there does not exist UMEB in three-qubit systems. In Sect.3, we construct different UMEBs in tripartite systems and show they are not mutually unbiased. Conclusions and discussions are given in Sect.4.

2. UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

We first recall the definition of UMEB in the bipartite system. Let \mathbb{C}^{d_α} be the d_α -dimensional complex vector space. A set of states $\{|\phi_i\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}, i = 1, 2, \dots, n, n < d_1 d_2\}$ is called an n -number UMEB if and only if

- (i) $|\phi_i\rangle, i = 1, 2, \dots, n$, are maximally entangled;
- (ii) $\langle \phi_i | \phi_j \rangle = \delta_{ij}$;
- (iii) If $\langle \phi_i | \varphi \rangle = 0$ for all $i = 1, 2, \dots, n$, then $|\varphi\rangle$ cannot be maximally entangled.

Therefore the key component of the above UMEB is the concept of maximally entanglement of the bipartite system. To generalize the notion of the UMEB to multipartite systems, we first recall the definition of maximally entanglement in multipartite situation. As there is no unique way of characterizing the multipartite entanglement, different definition captures different features of this quantum phenomenon. In this paper, we employ the definition of maximally multipartite entangled states introduced by Facchi et al.^[14] For the existence of such maximally entangled states for qubit systems, see [15].

Consider a bipartition (A, \bar{A}) of system S, where $A \subset S, \bar{A} = S \setminus A, S = \{1, 2, \dots, n\}$ and $1 \leq n_A \leq n_{\bar{A}}$, with $n_A = |A|$, the cardinality of party A. At the level of Hilbert spaces, we get $H = H_A \otimes H_{\bar{A}}$.

Definition 1 State $\rho \in H_A \otimes H_{\bar{A}}, N_A = \dim(H_A) \leq \dim(H_{\bar{A}})$ is maximally entangled if and only if the reduced state is maximally mixed under all possible bipartite partitions (A, \bar{A}) :

$$\rho_A = Tr_{\bar{A}}(\rho) = \frac{I}{n_A}, \quad (1)$$

where I is corresponding identity matrix.

We now present the definition of UMEB in multipartite systems.

Definition 2 A set of states $\{|\phi_i\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_k}, i = 1, 2, \dots, n, n < d_1 d_2 \dots d_k\}$ is called an n -number UMEB if and only if

- (i) $|\phi_i\rangle, i = 1, 2, \dots, n$, are maximally entangled;
- (ii) $\langle \phi_i | \phi_j \rangle = \delta_{ij}$;
- (iii) If $\langle \phi_i | \varphi \rangle = 0$ for all $i = 1, 2, \dots, n$, then $|\varphi\rangle$ cannot be maximally entangled.

Next we focus on quantum states in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Theorem 1 UMEB does not exist in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Proof For three qubits, the maximally entangled states are local unitary equivalent to the GHZ state.^[14] Without loss of generality, a basis vector $|\phi\rangle$ of UMEB can be represented by a linear operator $U \otimes V$ acting

on $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that

$$|\phi\rangle = (I \otimes U \otimes V)|\phi_0\rangle, \quad (2)$$

where U, V are unitary matrices over \mathbb{C}^2 , and $|\phi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

We can construct eight vectors which are maximally entangled and orthogonal to each other.

$$\begin{aligned} |\phi_1\rangle &= (I \otimes I \otimes I)|\phi_0\rangle, \\ |\phi_{\alpha+1}\rangle &= (I \otimes I \otimes \sigma_\alpha)|\phi_0\rangle, \quad \alpha = 1, 2, 3 \\ |\phi_{\beta+4}\rangle &= (I \otimes \sigma_\beta \otimes I)|\phi_0\rangle, \quad \beta = 1, 2 \\ |\phi_{\gamma+6}\rangle &= (I \otimes \sigma_1 \otimes \sigma_\gamma)|\phi_0\rangle, \quad \gamma = 1, 2 \end{aligned} \quad (3)$$

by the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

Any three-qubit pure state can be generally written in the form:^[15]

$$|\varphi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \quad (5)$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i^2 = 1$ and $\theta \in [0, \pi]$. Next we will proof that if $\langle \phi_i | \varphi \rangle = 0$ for all $i = 1, 2, \dots, 8$, then $|\varphi\rangle$ cannot be maximally entangled.

Suppose $|\varphi\rangle$ is maximally entangled, consider $|\varphi\rangle$ as a three-qubit system associated to qubits A, B and C. Under the bipartition $A|BC$, the reduced state of $|\varphi\rangle$ is of the form:

$$\rho_A = \lambda_0^2|0\rangle\langle 0| + \lambda_0\lambda_1 e^{-i\theta}|0\rangle\langle 1| + \lambda_0\lambda_1 e^{i\theta}|1\rangle\langle 0| + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)|1\rangle\langle 1|, \quad (6)$$

where I is the identical operator in H_A . Setting $\rho_A = \frac{I}{2}$, we get

$$\lambda_0^2 = \frac{1}{2}, \quad \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \frac{1}{2}, \quad \lambda_1 = 0. \quad (7)$$

Let

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}. \quad (8)$$

Using Eq.(7) and $\langle \phi_i | \varphi \rangle = 0$ for all $i = 1, 2, \dots, n$, we have

$$\begin{aligned}
\langle \phi_1 | \varphi \rangle &= \frac{1}{\sqrt{2}}(\lambda_0 u_{11} v_{11} + \lambda_2 u_{21} v_{22} + \lambda_3 u_{22} v_{21} + \lambda_4 u_{22} v_{22}) = 0, \\
\langle \phi_2 | \varphi \rangle &= \frac{1}{\sqrt{2}}(\lambda_0 u_{11} v_{21} + \lambda_2 u_{21} v_{12} + \lambda_3 u_{22} v_{11} + \lambda_4 u_{22} v_{12}) = 0, \\
\langle \phi_3 | \varphi \rangle &= \frac{i}{\sqrt{2}}(\lambda_0 u_{11} v_{21} - \lambda_2 u_{21} v_{12} - \lambda_3 u_{22} v_{11} - \lambda_4 u_{22} v_{12}) = 0, \\
\langle \phi_4 | \varphi \rangle &= \frac{1}{\sqrt{2}}(\lambda_0 u_{11} v_{11} - \lambda_2 u_{21} v_{22} - \lambda_3 u_{22} v_{21} - \lambda_4 u_{22} v_{22}) = 0, \\
\langle \phi_5 | \varphi \rangle &= \frac{1}{\sqrt{2}}(\lambda_0 u_{21} v_{11} + \lambda_2 u_{11} v_{22} + \lambda_3 u_{12} v_{21} + \lambda_4 u_{12} v_{22}) = 0, \\
\langle \phi_6 | \varphi \rangle &= \frac{i}{\sqrt{2}}(\lambda_0 u_{21} v_{11} - \lambda_2 u_{11} v_{22} - \lambda_3 u_{12} v_{21} - \lambda_4 u_{12} v_{22}) = 0, \\
\langle \phi_7 | \varphi \rangle &= \frac{1}{\sqrt{2}}(\lambda_0 u_{21} v_{21} + \lambda_2 u_{11} v_{12} + \lambda_3 u_{12} v_{11} + \lambda_4 u_{12} v_{12}) = 0, \\
\langle \phi_8 | \varphi \rangle &= \frac{i}{\sqrt{2}}(\lambda_0 u_{21} v_{21} - \lambda_2 u_{11} v_{12} - \lambda_3 u_{12} v_{11} - \lambda_4 u_{12} v_{12}) = 0.
\end{aligned} \tag{9}$$

Hence

$$u_{11} v_{11} = u_{11} v_{21} = u_{21} v_{11} = u_{21} v_{21} = 0. \tag{10}$$

This result contradicts to the unitarity of U and V , then $|\varphi\rangle$ is not maximally entangled. Therefore we can complete the basis $|\phi_1\rangle, \dots, |\phi_8\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Hence UMEB does not exist in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

3. MUB in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

In this section we construct two UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ which are not mutually unbiased.

Definition 3^[8] Two orthonormal bases $B_1 = \{|b_i\rangle\}_{i=1}^d$ and $B_2 = \{|c_j\rangle\}_{j=1}^d$ of \mathbb{C}^d are said to be mutually unbiased if and only if

$$|\langle b_i | c_j \rangle| = \frac{1}{\sqrt{d}}, \quad \forall i, j = 1, \dots, d. \tag{11}$$

According to Ref.[8], we have two types of UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$. One is

$$\begin{aligned}
|\phi_0\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\phi_i\rangle &= (\sigma_i \otimes I_3)|\phi_0\rangle, \quad i = 1, 2, 3.
\end{aligned} \tag{12}$$

Another is

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}(\sigma_j \otimes I_3)(|0\rangle|x\rangle + |1\rangle|y\rangle), \quad j = 0, 1, 2, 3, \tag{13}$$

where $|x\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \frac{1+\sqrt{3}i}{2}|1\rangle + |2\rangle)$, $|y\rangle = \frac{1}{\sqrt{3}}(\frac{-\sqrt{3}+i}{2}|0\rangle + i|1\rangle - i|2\rangle)$ and $\sigma_0 = I$.

We now adopt the method in Ref.[12]. Suppose $\{|\psi_i\rangle\}$ is an unextendible entangled bases with Schmidt number k of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where $d_1 \leq d_2$, $1 \leq k \leq d_1$, $|\psi_i\rangle = \sum_{l=0}^k \lambda_l^{(i)} |\psi_l^{(i)}\rangle$, and $|\psi_l^{(i)}\rangle = |a_l^{(i)}\rangle |b_l^{(i)}\rangle$ with $\{|a_l^{(i)}\rangle : l = 1, \dots, d_1\}$ an orthonormal basis of subsystem \mathbb{C}^{d_1} , and $\{|b_l^{(i)}\rangle : l = 1, \dots, d_2\}$ an orthonormal basis of subsystem \mathbb{C}^{d_2} . If all Schmidt coefficients are equal to $\frac{1}{\sqrt{k}}$ and $k = d_1$, an unextendible entangled bases with Schmidt number k reduces to UMEB. Let

$$|\psi_{i,j}\rangle = \sum_{l=0}^{d_1-1} \lambda_l^{(i)} |\psi_l^{(i)}\rangle |j \oplus l\rangle, \tag{14}$$

where $\{|j\rangle\}$ is the standard computational basis of \mathbb{C}^{d_3} , $j = 0, 1, \dots, d_3 - 1$, $j \oplus l$ means $j + l \bmod d_3$. Then $\{|\psi_{i,j}\rangle\}$ is an UMEB of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$, $d_1 \leq d_2 \leq d_3$.^[12] We can obtain two UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

The first one is

$$\begin{aligned}
|\phi_{0,0}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \\
|\phi_{0,1}\rangle &= \frac{1}{\sqrt{2}}(|001\rangle + |112\rangle), \\
|\phi_{0,2}\rangle &= \frac{1}{\sqrt{2}}(|002\rangle + |110\rangle), \\
|\phi_{i,0}\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3 \otimes I_3)(|000\rangle + |111\rangle), \\
|\phi_{i,1}\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3 \otimes I_3)(|001\rangle + |112\rangle), \\
|\phi_{i,2}\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3 \otimes I_3)(|002\rangle + |110\rangle),
\end{aligned} \tag{15}$$

where $i = 1, 2, 3$.

The second one is

$$\begin{aligned}
|\psi_{j,0}\rangle &= \frac{1}{\sqrt{2}}(\sigma_j \otimes I_3 \otimes I_3)(|0\rangle|x\rangle|0\rangle + |1\rangle|y\rangle|1\rangle), \\
|\psi_{j,1}\rangle &= \frac{1}{\sqrt{2}}(\sigma_j \otimes I_3 \otimes I_3)(|0\rangle|x\rangle|1\rangle + |1\rangle|y\rangle|2\rangle), \\
|\psi_{j,2}\rangle &= \frac{1}{\sqrt{2}}(\sigma_j \otimes I_3 \otimes I_3)(|0\rangle|x\rangle|2\rangle + |1\rangle|y\rangle|0\rangle),
\end{aligned} \tag{16}$$

where $j = 0, 1, 2, 3$.

Due to $\langle\phi_{0,0}|\psi_{0,0}\rangle = \frac{1}{\sqrt{6}}$, the sets $\{|\phi_{i,l}\rangle\}$ and $\{|\psi_{j,l}\rangle\}$ are not the MUBs in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

4. Conclusion

We have first time generalized the notion of UMEB from bipartite systems to multipartite quantum systems. Based on this, we prove that there does not exist an UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Moreover, we have constructed two types of UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ which are not mutually unbiased. Our results may highlight the further investigation on UMEBs and MUBs for multipartite quantum states.

References

- [1] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
- [2] Fuchs C A, Gisin N, Griffiths R B, Niu C S and Peres A 1997 *Phys. Rev. A* **56** 1163
- [3] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
- [4] DiVincenzo D P 1995 *Quantum computation. Science* **270**(5234) 255-261
- [5] DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 2003 *Comm. Math. Phys.* **238** 379
- [6] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 *Phys. Rev. Lett.* **82** 5385
- [7] Bravyi S and Smolin J A 2011 *Phys. Rev. A* **84** 042306
- [8] Chen B and Fei S M 2013 *Phys. Rev. A* **88** 034301
- [9] Li M S, Wang Y L and Zheng Z J 2014 *Phys. Rev. A* **89** 062313
- [10] Guo Y 2016 *Phys. Rev. A* **94** 052302

- [11] Guo Y and Wu S 2014 *Phys. Rev. A* **90** 054303
- [12] Guo Y, Jia Y and Li X 2015 *Quantum Inf. Process.* **14** 3553
- [13] Wootters W K and Fields B D 1989 *Ann. Phys.* **191** 363
- [14] Facchi P, Florio G, Parisi G and Pascazio S 2008 *Phys. Rev. A* **77** 060304
- [15] Felix Huber, Otfried Gühne, Jens Siewert, arXiv:1608.06228.
- [16] Acín A, Andrianov A, Costa L, Jané E, Latorre J I and Tarrach R 2000 *Phys. Rev. Lett.* **85** 1560