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Divisor class groups of rational trinomial varieties

by

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DIVISOR CLASS GROUPS OF RATIONAL TRINOMIAL VARIETIES

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Abstract. We give an explicit description of the divisor class groups of rational trinomial varieties. As an application we show the connection between the iteration of Cox rings of varieties with torus action of complexity one of arbitrary dimension to the iteration of Cox rings of the Du Val surfaces.

1. Introduction

This article contributes to the explicit calculation of divisor class groups of affine varieties; see [Fle81, Lan83, SS84, SS07] for some previous work and Remark 2.19 for the relations to our results. We consider affine algebraic varieties X defined over the field $\mathbb{C}$ of complex numbers defined as the common vanishing set of trinomials

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \theta_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \ \ldots, \ \theta_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r},$$

with monomials $T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ and pairwise different $\theta_i \in \mathbb{C}^*$. We call such a variety a *trinomial variety*. Our first main result describes explicitly the divisor class groups of rational non-factorial trinomial varieties. For each exponent vector l_i set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$, denote $\mathfrak{l} := \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ and define

$$c(0) := \gcd(\mathfrak{l}_1, \mathfrak{l}_2), \quad c(1) := \gcd(\mathfrak{l}_0, \mathfrak{l}_2), \quad c(2) := \gcd(\mathfrak{l}_0, \mathfrak{l}_1),$$

$$c(i) := \frac{1}{\mathfrak{l}} \gcd(\mathfrak{l}_1, \mathfrak{l}_2) \gcd(\mathfrak{l}_0, \mathfrak{l}_2) \gcd(\mathfrak{l}_0, \mathfrak{l}_1) \quad \text{for} \quad i \geq 3.$$

Note that due to [ABHW18, Cor. 5.8] one can easily decide if a given trinomial variety is rational or factorial just in terms of the numbers $\mathfrak{l}_i$, see also Remark 2.2.

Theorem 1.1. *Let X be an affine, rational, non-factorial trinomial variety and set $\tilde{n} := \sum_{i=0}^r ((c(i) - 1)n_i - c(i) + 1)$.*

(i) *If $c := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1\}$, then the divisor class group $\mathrm{Cl}(X)$ is isomorphic to*

$$(\mathbb{Z}/\mathfrak{l}_2\mathbb{Z})^{c-1} \times \ldots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^{c-1} \times \mathbb{Z}^{\tilde{n}}.$$

(ii) *If $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1, 2\}$, then the divisor class group $\mathrm{Cl}(X)$ is isomorphic to*

$$\mathbb{Z}/(\mathfrak{l}_0\mathfrak{l}_1\mathfrak{l}_2/4)\mathbb{Z} \times (\mathbb{Z}/\mathfrak{l}_3\mathbb{Z})^3 \times \ldots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^3 \times \mathbb{Z}^{\tilde{n}}.$$

In order to prove this result we make use of the fact that rational trinomial varieties are $\mathbb{T}$-varieties of complexity one, i.e., they are endowed with an effective torus action $\mathbb{T} \times X \to X$ such that $\dim(\mathbb{T}) = \dim(X) - 1$ holds. We use the description of their total coordinate spaces, i.e. the spectrum of their Cox rings, given in [HW18, Prop. 2.6] to prove the above theorem and obtain as a by-product an explicit description of the divisor class group grading on the Cox ring of a rational trinomial variety; see Corollary 2.20.

Using Corollary 2.20 we can give a new perspective on the iteration of Cox rings for $\mathbb{T}$-varieties of complexity one. For this let X be a *hyperplatonic* trinomial

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variety, i.e., $\mathfrak{l}_0^{-1} + \ldots + \mathfrak{l}_r^{-1} > r - 1$ holds. This means that after reordering $\mathfrak{l}_0, \ldots, \mathfrak{l}_r$ decreasingly, $\mathfrak{l}_i = 1$ holds for all $i \geq 3$ and $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is a *platonic triple*, i.e., one of the triples $(5,3,2), (4,3,2), (3,3,2), (x,2,2), (x,y,1)$, where $x, y \in \mathbb{Z}_{\geq 1}$. We call this triple the *basic platonic triple* of X. Note that these varieties comprise all total coordinate spaces of affine log terminal varieties of complexity one; see [ABHW18] for the precise statement. Due to [HW18, Thm. 1.1] a hyperplatonic variety X admits *iteration of Cox rings*, i.e., there exists a chain

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 := X,$$

where X_p is a factorial affine variety, and in each step, X_{i+1} is the total coordinate space of X_i and $H_i := \operatorname{Spec} \mathbb{C}[\operatorname{Cl}(X_i)]$. Moreover any of the occurring total coordinate spaces is again hyperplatonic and there are exactly the following possible sequences of basic platonic triples arising from Cox ring iterations of hyperplatonic varieties, see [HW17, Cor. 1.4]:

(i) $(1,1,1) \to (2,2,2) \to (3,3,2) \to (4,3,2)$,
(ii) $(1,1,1) \to (x,x,1) \to (2x,2,2)$,
(iii) $(1,1,1) \to (x,x,1) \to (x,2,2)$,
(iv) $(\mathfrak{l}_{01}^{-1}\mathfrak{l}_0, \mathfrak{l}_{01}^{-1}\mathfrak{l}_1, 1) \to (\mathfrak{l}_0, \mathfrak{l}_1, 1)$, where $\mathfrak{l}_{01} := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$.

In the above iterations, the steps corresponding to $(1,1,1) \to (x,x,1)$ as well as the step of Case (iv) are exactly those steps, where H_i is a torus. The remaining parts of the iteration chains can be represented by Cox ring iterations of Du Val surfaces: Any platonic triple (a,b,c) defines a Du Val singularity by

$$Y(a,b,c) \ := \ V(T_1^a + T_2^b + T_3^c) \ \subseteq \ \mathbb{C}^3.$$

Case (i) corresponds to the chain $\mathbb{C}^2 \to A_1 \to D_4 \to E_6$ and $(x,x,1) \to (2x,2,2)$ resp. $(x,x,1) \to (2x,2,2)$ correspond to the chains $\mathbb{C}^2 \to A_n$ resp. $\mathbb{C}^2 \to A_{2n}$ with $n > 0$ odd. Overall we obtain the following structural result.

Corollary 1.2. *Let X be a hyperplatonic variety with basic platonic triple $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$. Denote by $(\mathfrak{l}_0', \mathfrak{l}_1', \mathfrak{l}_2')$ the basic platonic triple of the total coordinate space X' of X. Then there is a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\text{TCS}} & X \\ {\scriptstyle /\!\!/\mathbb{T}'}\downarrow & & \downarrow{\scriptstyle /\!\!/\mathbb{T}} \\ Y(\mathfrak{l}_0', \mathfrak{l}_1', \mathfrak{l}_2') & \xrightarrow[\text{TCS}]{} & Y(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2), \end{array}$$

where the horizontal arrows labelled "TCS" are total coordinate spaces and the downward arrows are good quotients by torus actions.

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2. Proof of the main results

We work in the notation of [HH13, HW17], where the Cox rings of rational T-varieties of complexity one are described. Note that the trinomial varieties defined in the introduction arise as the spectrum of these rings. We briefly recall the necessary results and constructions here. For a general introduction to the theory of Cox rings see e.g. [ADHL15].

Construction 2.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_0 + \ldots + n_r$ with positive integers n_i. For every $i = 0, \ldots, r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{C}[T_{ij}, S_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i,\ 1 \leq k \leq m].$$

We will also write $\mathbb{C}[T_{ij}, S_k]$ for the above polynomial ring. Let $A := (a_0, \ldots, a_r)$ be a $2 \times (r+1)$ matrix with pairwise linearly independent columns $a_i \in \mathbb{C}^2$. For every $i = 0, \ldots, r-2$ we define

$$g_i \ := \ \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \ \in \ \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_0, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Denote by P_0^* the transpose of P_0 and consider the projection

$$Q\colon \mathbb{Z}^{n+m} \ \to \ K_0 \ := \ \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*).$$

Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij}, S_k. Define a K_0-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K_0, \qquad \deg(S_k) \ := \ Q(e_k) \ \in \ K_0.$$

This is the finest possible grading of $\mathbb{C}[T_{ij}, S_k]$ leaving the variables and the g_i homogeneous and any other such grading coarsens this maximal one. In particular, we have a K_0-graded factor algebra

$$R(A, P_0) \ := \ \mathbb{C}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2}\rangle.$$

By the results of [HH13, HW17] the rings $R(A, P_0)$ are normal complete intersections and admit only constant homogeneous units. We use the following rationality criterion from [ABHW18, Cor. 5.8] for the spectrum of a ring $R(A, P_0)$ as above:

Remark 2.2. Let $R(A, P_0)$ be a ring as in Construction 2.1 and set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$. Then $\operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:

(i) We have $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $0 \leq i < j \leq r$, in other words, $R(A, P_0)$ is factorial.
(ii) There are $0 \leq i < j \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) > 1$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j\}$.
(iii) There are $0 \leq i < j < k \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = \gcd(\mathfrak{l}_i, \mathfrak{l}_k) = \gcd(\mathfrak{l}_j, \mathfrak{l}_k) = 2$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.

Definition 2.3. Let $R(A, P_0)$ be as above such that $\operatorname{Spec} R(A, P_0)$ is rational. We say that P_0 is gcd-*ordered* if it satisfies the following two properties

(i) $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $i = 0, \ldots, r$ and $j = 3, \ldots, r$,
(ii) $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$.

If $\operatorname{Spec} R(A, P_0)$ is rational, one can always achieve that P_0 is gcd-ordered by suitably reordering $l_0, \ldots, l_r$, which does not affect the K_0-graded algebra $R(A, P_0)$ up to isomorphy.

In order to prove our main results we make use of the explicit description of the total coordinate space of a rational trinomial variety given in [HW18]. We state the two necessary results here:

Lemma 2.4. [HW18, Lemma 2.5] *Let $R(A,P_0)$ be a ring as in Construction 2.1 and $X := \operatorname{Spec} R(A,P_0)$ be rational. Assume that P_0 is* gcd*-ordered. Then, with $\mathfrak{l} := \gcd(\mathfrak{l}_0,\mathfrak{l}_1,\mathfrak{l}_2)$, the number $c(i)$ of irreducible components of $V(X,T_{ij})$, where $j = 1,\ldots,n_i$, is given by*

i	0	1	2	≥ 3
$c(i)$	$\gcd(\mathfrak{l}_1,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_1)$	$\frac{1}{\mathfrak{l}}\gcd(\mathfrak{l}_1,\mathfrak{l}_2)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)\gcd(\mathfrak{l}_0,\mathfrak{l}_1)$

Proposition 2.5. [HW18, Prop. 2.6] *Let $R(A,P_0)$ be non-factorial with $\operatorname{Spec} R(A,P_0)$ rational. Assume that P_0 is* gcd*-ordered and set*

$$P_1 := \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}l_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}l_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}l_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}l_2 & 0 & 0 & & & \\ -l_0 & 0 & & l_3 & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & \\ -l_0 & 0 & \ldots & 0 & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Moreover, let $c(i)$ be as above and define numbers $n' := c(0)n_0 + \ldots + c(r)n_r$ and

$$n_{i,1},\ldots,n_{i,c(i)} := n_i, \qquad l_{ij,1},\ldots,l_{ij,c(i)} := \gcd((P_1)_{1,ij},\ldots,(P_1)_{r,ij}).$$

Then the vectors $l_{i,\alpha} := (l_{i1,\alpha},\ldots,l_{in_i,\alpha}) \in \mathbb{Z}^{n_{i,\alpha}}$ build up an $r' \times (n'+m)$ matrix P_0' with $r' = c(0)+\ldots+c(r)-1$. With a suitable matrix A', the affine variety $\operatorname{Spec} R(A',P_0')$ is the total coordinate space of the affine variety $\operatorname{Spec} R(A,P_0)$.

Construction 2.6. Let $R(A,P_0)$ be a ring as in Construction 2.1. Choose an integral $s \times (n+m)$ matrix d and build the $(r+s)\times(n+m)$ stack matrix

$$P := \begin{bmatrix} P_0 \\ d \end{bmatrix}.$$

We require the columns of P to be pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a vector space. Let P^* denote the transpose of P and consider the projection

$$Q\colon \mathbb{Z}^{n+m} \to K := \mathbb{Z}^{n+m}/\operatorname{im}(P^*).$$

Denoting as before by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij} and S_k, we obtain a K-grading on $\mathbb{K}[T_{ij},S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}) \in K, \qquad \deg(S_k) := Q(e_k) \in K.$$

This K-grading coarsens the K_0-grading of $\mathbb{K}[T_{ij},S_k]$ given in Construction 2.1 and thus defines a grading on $R(A,P_0)$.

Now, consider a rational trinomial variety $X := \operatorname{Spec} R(A,P_0)$. Let $\operatorname{Spec} R(A',P_0')$ be its total coordinate space and denote by $\mathcal{R}(X)$ its Cox ring. Then there exists a K'-grading on $R(A',P_0')$ such that $R(A',P_0') \cong \mathcal{R}(X)$ as graded rings. In particular $K' \cong \operatorname{Cl}(X)$ holds and there exists a good quotient

$$\operatorname{Spec} R(A',P_0') \xrightarrow{/\!\!/ H'} \operatorname{Spec} R(A,P_0)$$

with respect to the corresponding group action of $H' := \operatorname{Spec} \mathbb{C}[K']$. Moreover, due to [HW17, Thm. 1.7] we find a description of this grading via a stack matrix

$$P' := \begin{bmatrix} P_0' \\ d \end{bmatrix}$$

with $K' = \mathbb{Z}^{n'+m}/\mathrm{im}((P')^*)$ as in Construction 2.6. In particular the transpose $(P')^*$ defines an injective map. Now consider the group $K_0' := \mathbb{Z}^{n'+m}/\mathrm{im}((P_0')^*)$ and denote by $(K_0')^{\mathrm{tors}}$ the torsionsubgroup of K_0'. Then

$$(K_0')^{\mathrm{tors}} \subseteq \mathbb{Z}^{n'+m}/\mathrm{im}((P')^*) = K' \cong \mathrm{Cl}(X)$$

holds and we call $\mathrm{Cl}(X)^{\mathrm{ctors}} := (K_0')^{\mathrm{tors}}$ the *compulsory torsion* of the divisor class group of X.

Lemma 2.7. *Let $R(A, P_0)$ be a non factorial ring such that $X := \mathrm{Spec}\, R(A, P_0)$ is rational and assume that P_0 is* gcd*-ordered.*

(i) *If $c := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1\}$, then the compulsory torsion of the divisor class group of X is*

$$(\mathbb{Z}/\mathfrak{l}_2\mathbb{Z})^{c-1} \times \cdots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^{c-1}.$$

(ii) *If $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1, 2\}$, then the compulsory torsion of the divisor class group of X is*

$$\mathbb{Z}/(\mathfrak{l}_0/2)\mathbb{Z} \times \mathbb{Z}/(\mathfrak{l}_1/2)\mathbb{Z} \times \mathbb{Z}/(\mathfrak{l}_2/2)\mathbb{Z} \times (\mathbb{Z}/\mathfrak{l}_3\mathbb{Z})^3 \times \cdots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^3.$$

Proof. We prove (i). With our subsequent considerations we obtain that the divisor class group of X is given as $\mathbb{Z}^{n'+m}/\mathrm{im}((P')^*)$, where P' is some $(r'+s') \times (n'+m)$ stack matrix

$$\begin{bmatrix} P_0' \\ d' \end{bmatrix},$$

of full row rank, and with Proposition 2.5 we get that P_0' is the $r' \times (n'+m)$ matrix build up by the exponent vectors $c^{-1}l_0$, $c^{-1}l_1$ and c copies $l_{i,1}, \ldots, l_{i,c}$ of l_i for $i \geq 2$. Thus, to obtain the assertion, we compute the elementary divisors of P_0': Suitable elementary column operations transform P_0' into

$$\begin{bmatrix} c^{-1}\mathfrak{l}_0 & c^{-1}\mathfrak{l}_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ c^{-1}\mathfrak{l}_0 & 0 & \mathfrak{l}_{2,1} & & 0 & & & \\ \vdots & & & \ddots & \vdots & & & \\ c^{-1}\mathfrak{l}_0 & 0 & \ldots & & \mathfrak{l}_{r,c} & 0 & \ldots & 0 \end{bmatrix}.$$

As $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds for $i, j \notin \{0, 1\}$ we obtain for $1 \leq t \leq c$ that the $(r'-t+1)$-th determinantal divisor of P_0' equals $\mathfrak{l}_2^{c-t} \cdots \mathfrak{l}_r^{c-t}$. The assertion follows.

For the proof of (ii) we note that in this case P_0' is built up by 2 copies of $1/2l_0, 1/2l_1$ and $1/2l_2$ and 4 copies of each term l_i for $i \geq 3$. Then, applying the same arguments as above, we obtain the assertion. □

Construction 2.8. Let X be an irreducible, normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$ and finitely generated divisor class group. Denote by $\mathrm{WDiv}(X)$ the group of Weil-divisors of X and fix a finitely generated subgroup $\mathbb{Z}^n \cong \langle D_1, \ldots, D_n \rangle \leq \mathrm{WDiv}(X)$ such that the map $\pi\colon \mathbb{Z}^n \to \mathrm{Cl}(X)$ sending each Weil divisor D to its class $[D] \in \mathrm{Cl}(X)$ is surjective. Let $f_1, \ldots, f_r$ be any linear relations between the the classes of $D_1, \ldots, D_r$ with

$$f_j([D_1], \ldots, [D_n]) = \sum_{i=1}^{n} \alpha_{ij}[D_i] = [0] \in \mathrm{Cl}(X)$$

and set

$$P := \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{r1} & \ldots & \alpha_{rn} \end{bmatrix}.$$

Then there is a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\quad\pi\quad} & \mathrm{Cl}(X) \\ & \searrow \quad \nearrow & \\ & \mathbb{Z}^n/\mathrm{im}(P^*) & \end{array}$$

In particular $\mathrm{Cl}(X)$ is a factor group of $\mathbb{Z}^n/\mathrm{im}(P^*)$.

Lemma 2.9. *Let $l_i \in \mathbb{Z}^{n_i}_{>0}$ be any tuple, $k \in \mathbb{Z}_{\geq 1}$ and consider the matrix*

$$A(k, l_i) := \begin{bmatrix} l_i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_i \\ E_{n_i} & \dots & E_{n_i} \end{bmatrix} \in \mathrm{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}),$$

where E_{n_i} denotes the identity matrix of size n_i. Then $A(k, l_i)$ has rank $n_i - 1 + k$ and the $(n_i - 1 + k)$-th determinantal divisor divides $\mathfrak{l}_i^{k-1}$, where $\mathfrak{l}_i := \gcd(l_{i1}, \dots, l_{in_i})$.

Proof. Choose for any $2 \leq t \leq k$ an integer $1 \leq j_t \leq n_i$ and denote by e_{j_t} the column vector having 1 as j_t-th entry and all other entries equal zero. Consider the following $(n_i - 1 + k) \times (n_i - 1 + k)$ square matrix obtained by deleting the first row and several of the last $(k - 1) \cdot n_i$ columns of $A(k, l_i)$

$$\begin{bmatrix} 0 & \dots & 0 & l_{ij_2} & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & l_{ij_k} \\ & E_{n_i} & & e_{j_2} & \dots & e_{j_k} \end{bmatrix}.$$

The determinant of this matrix equals up to sign $l_{ij_2} \dots l_{ij_k}$.
With $\mathfrak{l}_i = \gcd(l_{i1}, \dots, l_{in_i})$ we obtain

$$\gcd(\prod_{t=2}^{k} l_{ij_t};\ j_t \in \{1, \dots, n_i\}) = \mathfrak{l}_i^{k-1}.$$

This shows that the $(n_i - 1 + k)$-th determinantal divisor divides $\mathfrak{l}_i^{k-1}$. Moreover, as $A(k, l_i)$ is obviously not of full rank this proves the assertions. □

The rings $R(A, P_0)$ as defined in Construction 2.1 are in general not unique factorization domains but have a similar property that will play an important role in our further considerations:

Definition 2.10. Let K be an abelian group and $R = \oplus_{w \in K} R_w$ a finitely generated integral K-graded $\mathbb{C}$-algebra. Set $H := \mathrm{Spec}\,\mathbb{C}[K]$ and $X := \mathrm{Spec}\,R$.

(i) A homogeneous element $0 \neq f \in R \setminus R^*$ is called *K-prime* if whenever $f|gh$ holds for homogeneous elements $g, h \in R$ we have $f|g$ or $f|h$.
(ii) We call R *factorially K-graded* if every homogeneous $0 \neq f \in R \setminus R^*$ is a product of K-prime elements.
(iii) An *H-prime divisor* on X is a Weil divisor $0 \neq \sum a_D D$, where $a_D \in \{0, 1\}$, the D are prime and those with $a_D = 1$ are transitively permuted by H.

Remark 2.11. Let $R(A, P_0)$ be as in Construction 2.1. Then due to [ADHL15, Thm. 3.4.2.3] $R(A, P_0)$ is factorially K_0-graded and the variables T_{ij} and S_k are K_0-prime. Due to [ADHL15, Prop. 1.5.3.3] this implies that the divisors $\mathrm{div}(T_{ij})$ and $\mathrm{div}(S_k)$ are H_0-prime, where $H_0 := \mathrm{Spec}\,\mathbb{C}[K_0]$ holds.

Remark 2.12. Let $R(A, P_0)$ be a K_0-graded ring as in Construction 2.1 defining a rational variety $X := \operatorname{Spec} R(A, P_0)$. Then X is endowed with an action of the torus $H_0^0 := \operatorname{Spec} \mathbb{C}[K_0/K_0^{\text{tors}}]$ of complexity one, where K_0^{tors} is the torsionsubgroup of K_0. Thus following the description of the Cox ring of a variety with torus action provided in [HS10] and used for the explicit calculation of the total coordinate space in Proposition 2.5, the variables $T_{ij,k}$ in the ring $R(A', P_0')$ correspond to the prime components in the exceptional fibers of the map $\pi\colon X_0 \to Y$, where $X_0 \subseteq X$ is the set of points with at most finite H_0^0-isotropy and the curve Y is the separation of X_0/H_0^0; see [ADHL15, Section 4.4.1]. In particular for fixed i, j the variables $T_{ij,1}, \ldots, T_{ij,c(i)}$ correspond to the prime divisors $D_{ij,1}, \ldots D_{ij,c(i)}$ inside $\mathrm{V}(X; T_{ij})$, where $1 \leq j \leq n_i$. Due to [HS10] the divisor class group grading on $R(A', P_0')$ is thus defined as

$$\deg(T_{ij,t}) = [D_{ij,t}] \in \mathrm{Cl}(X).$$

Moreover the free variables S_k' in $R(A', P_0')$ arise from the free variables S_k of the ring $R(A, P_0)$, which give rise to prime divisors $\mathrm{V}(X; S_k) = E_k$ with infinite H_0^0-isotropy. Due to Remark 2.11 the variable S_k is K_0-prime and thus K_0-factoriality of $R(A, P_0)$ implies

$$\deg(S_k') = [E_k] = [0] \in \mathrm{Cl}(X).$$

Note that all free variables of $R(A', P_0')$ arise this way.

Lemma 2.13. *Let $R(A, P_0)$ be a ring defining a rational variety $X := \operatorname{Spec} R(A, P_0)$. Assume that P_0 is* gcd*-ordered and* $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds, whenever $j \notin \{0, 1\}$. Then the defining relations of the Cox ring $R(A', P_0')$ of X have* $\mathrm{Cl}(X)$*-degree zero.*

Proof. Note that due to Lemma 2.4 there is at least one integer $i \in \{0, 1, 2\}$ such that $\mathrm{V}(X, T_{ij}) = D_{ij,1}$ is irreducible for $j = 1, \ldots, n_i$. As $R(A, P_0)$ is K_0-factorial, K_0-primeness of the variable T_{ij} implies that $D_{ij,1}$ is a principal divisor for $j = 1, \ldots, n_i$; see Remark 2.11. We conclude

$$\deg(T_{i,1}^{l_{i,1}}) = \sum_{j=1}^{n_i} l_{ij,1}[D_{ij,1}] = [0] \in \mathrm{Cl}(X).$$

As $T_{i,1}^{l_{i,1}}$ occurs as a term in at least one defining relation of $R(A', P_0')$ and all of the defining relations have the same degree, the assertion follows. □

Proof of Theorem 1.1, Case (i). Set $H_0^0 := H_0/H_0^{\text{tors}}$. We recall that the H_0^0-invariant prime divisors with finite isotropy generate the divisor class group of $X = \operatorname{Spec} R(A, P_0)$ and those are exactly the irreducible components of $\mathrm{V}(X, T_{ij})$, where $i = 0, \ldots, r$ and $1 \leq j \leq n_i$. Our aim is to determine some relations between the $\mathrm{Cl}(X)$-degrees of the divisors arising this way. Using Construction 2.8 this gives rise to an abelian group having $\mathrm{Cl}(X)$ as a factor group.

Let $D_{ij,1} \cup \cdots \cup D_{ij,c(i)}$ be the decomposition of $\mathrm{V}(X, T_{ij})$ into prime divisors. As $R(A, P_0)$ is K_0-factorial and T_{ij} is K_0-prime, [ADHL15, Prop. 1.5.3.3], see Remark 2.11, implies

$$\sum_{t=1}^{c(i)} [D_{ij,t}] = [0] \in \mathrm{Cl}(X). \tag{2.13.1}$$

Moreover, due to Lemma 2.13 the defining relations of $R(A', P_0')$ have degree zero. In particular, due to Proposition 2.5 for every $i = 0, \ldots, r$ and $1 \leq t \leq c(i)$ we obtain a term $T_{i,t}^{l_{i,t}} = T_{i1,t}^{l_{ij,t}} \cdots T_{ij,t}^{l_{in_i,t}}$ of degree zero occurring in the relations of

$R(A', P'_0)$. This gives rise to relations

$$\sum_{j=1}^{n_i} l_{ij,t}[D_{ij,t}] = [0] \in \mathrm{Cl}(X), \tag{2.13.2}$$

where $i = 0, \ldots, r$ and $t = 1, \ldots, c(i)$. As $l_{i,1} = \cdots = l_{i,c(i)}$ holds for any $i = 0, \ldots, r$, the relations (2.13.1) and (2.13.2) give rise to block matrices $A(c(i), l_{i,1})$ in a matrix P as in Construction 2.8. In particular we get an $m' \times n'$ matrix with $m' := \sum_{i=0}^{r}(n_i + c(i))$ and $n' := \sum_{i=0}^{r} c(i) \cdot n_i$ of the following form

$$P := \begin{bmatrix} A(c(0), l_{0,1}) & 0 & \cdots & 0 \\ 0 & A(c(1), l_{1,1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A(c(r), l_{r,1}) \end{bmatrix}. \tag{2.13.3}$$

Note that P is of rank $\sum_{i=0}^{r}(n_i - 1 + c(i))$ and the $\mathrm{rk}(P)$-th determinantal divisor of P equals the product of the $(n_i - 1 + c(i))$-th determinantal divisors of the block matrices $A(c(i), l_{i,1})$. With Lemma 2.9 we conclude that the divisor class group of X is isomorphic to a factor group of the group

$$\mathbb{Z}^{n'}/\mathrm{im}(P^*) \cong \mathbb{Z}^{n'-\mathrm{rk}(P)} \times G \tag{2.13.4}$$

with some finite abelian group G of order k with $k | (\mathfrak{l}_{0,1}^{c(0)-1} \cdots \mathfrak{l}_{r,1}^{c(r)-1})$.

We show that $\mathbb{Z}^{n'}/\mathrm{im}(P^*) \leq \mathrm{Cl}(X)$ and therefore equality holds. For this purpose we compare the dimensions of $X = \operatorname{Spec} R(A, P_0)$ and $\overline{X} = \operatorname{Spec} R(A', P'_0)$:

$$\begin{aligned} \dim(\overline{X}) - \dim(X) &= n' - (r' - 1) - (n - (r - 1)) \\ &= n' - \sum_{i=0}^{r} c(i) + 2 - \sum_{i=0}^{r} n_i + (r - 1) = n' - \mathrm{rk}(P). \end{aligned}$$

With $X = \overline{X} /\!\!/ \operatorname{Spec} \mathbb{C}[\mathrm{Cl}(X)]$ we conclude $\mathbb{Z}^{n'-\mathrm{rk}(P)} \leq \mathrm{Cl}(X)$. Using Lemma 2.7 we obtain

$$|\mathrm{Cl}(X)^{\mathrm{ctors}}| \leq |G| \leq |\mathrm{Cl}(X)^{\mathrm{ctors}}|$$

and the assertion follows. $\square$

We turn towards the proof of the second assertion of Theorem 1.1.

Definition 2.14. Let X be an irreducible normal variety and $Y \subseteq X$ a prime divisor. Let furthermore $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ be any ideal. Then we define the *order of* $\mathfrak{A}$ *along* Y to be $\min(\mathrm{ord}_Y(f_i);\ i = 1, \ldots, r) =: \mathrm{ord}_Y(\mathfrak{A})$.

Lemma 2.15. *Let X be an irreducible normal variety, $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ any ideal and $f \in \mathcal{O}(X)$. Then the following statements are equivalent:*

(i) $\mathrm{ord}_Y(\mathfrak{A}) = \mathrm{ord}_Y(f)$ *holds for all prime divisors $Y \subseteq X$.*
(ii) $\langle f \rangle = \mathfrak{A}$ *holds, i.e. $\mathfrak{A}$ is a principal ideal.*

In particular the Weil-divisor $D := \sum \mathrm{ord}_Y(\mathfrak{A})$, where the sum runs over all prime divisors $Y \subseteq X$, is principal if and only if $\mathfrak{A}$ is a principal ideal.

Proof. We prove (i) $\Rightarrow$ (ii). Observe that $f \mid f_i$ holds for $i = 1, \ldots, r$ as $\mathrm{div}(f) \leq \mathrm{div}(f_i)$ by construction. In particular $\langle f \rangle \supseteq \mathfrak{A}$. We prove the other inclusion. Consider the covering $\cup_{i=1}^{r} U_i$ of X where

$$U_i := X \setminus (Y_{i_1} \cup \cdots \cup Y_{i_{k_i}}),$$

where all prime divisors Y with $\mathrm{ord}_Y(f_i) \neq \mathrm{ord}_Y(\mathfrak{A})$ occur among the Y_{i_t}. Then inside U_i we have $f_i \mid f$. We obtain $c_i \cdot f_i = f$ with $c_i \in \mathcal{O}(U)^*$. Considering the associated sheaf $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ we obtain $f \in \tilde{\mathfrak{A}}(X) = \mathfrak{A}$. The other implication is clear. $\square$

Lemma 2.16. *Let $R(A, P_0)$ be a ring as in Construction 2.1 with g_0 of the form $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ and assume* $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds. Fix an integer $y \in \mathbb{Z}_{\geq 0}$ with $y \mid \mathfrak{l}_0$ and set*

$$\mathfrak{A}_y := \langle T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}, T_0^{1/y \cdot l_0} \rangle \leq R(A, P_0).$$

Then $\mathfrak{A}$ is a principal ideal if and only if $y = 1$ holds.

Proof. Note that $\mathfrak{A}_1 = \langle T_1^{1/2l_1} + iT_2^{1/2l_2} \rangle$ holds in $R(A, P_0)$. So let $y \neq 1$ and assume there is an $f \in \mathfrak{A}_y$ with $\langle f \rangle = \mathfrak{A}$. Then there exist $g_1, g_2, h_1, h_2 \in \mathbb{K}[T_{ij}, S_k]$ with $g_1 \cdot f + I = T_0^{1/y \cdot l_0} + I$ and $g_2 \cdot f + I = T_1^{1/2l_1} + iT_2^{1/2l_2} + I$ and

$$h_1 \cdot T_0^{1/y \cdot l_0} + h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) + I = f + I.$$

Inserting the third formula into the first one we obtain

$$T_0^{1/y \cdot l_0} + I = g_1 \cdot h_1 \cdot T_0^{1/y \cdot l_0} + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) + I$$

and so in particular

$$h := (g_1 \cdot h_1 - 1) \cdot T_0^{1/y \cdot l_0} + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) \in I. \tag{2.16.1}$$

As there can not occur any term $T_0^{1/y \cdot l_0}$ in I for $y \neq 1$, we conclude that g_1 and h_1 each have a constant term. Inserting the third formula above into the second, we obtain a constant term in g_2 and h_2 with similar arguments. But this leads to a term $\lambda \cdot (T_1^{1/2l_1} + i \cdot T_2^{1/2l_2})$ with $\lambda \neq 0$ in (2.16.1); a contradiction to $h \in I$. □

Proof of Theorem 1.1, Case (ii). With the same arguments as in the Case (i) we get relations of the form (2.13.1). Moreover since the degrees of the relations and thus all terms occurring in the Cox ring $R(A', P_0')$ of $X = \operatorname{Spec} R(A, P_0)$ coincide, we obtain

$$\sum_{j=1}^{n_0} l_{0j,1}[D_{0j,1}] = \sum_{j=1}^{n_i} l_{ij,t(i)}[D_{ij,t(i)}] \in \mathrm{Cl}(X), \tag{2.16.2}$$

where $i = 0, \dots, r$ and $1 \leq t(i) \leq c(i)$. Those replace the relations (2.13.2). Suitably ordered, this gives rise to a matrix

$$P := \left[\begin{array}{cc|ccc} -l_{0,1} & l_{0,2} & 0 & \cdots & 0 \\ E_{n_0} & E_{n_0} & 0 & \cdots & 0 \\ \hline * & 0 & A(c(1), l_{1,1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \dots & A(c(r), l_{r,1}) \end{array}\right], \tag{2.16.3}$$

where we use $c(0) = 2$ and the $*$ indicates that there might be some non-zero entries. By suitably swapping columns, applying elementary row operations and using $l_{0,1} = l_{0,2}$ one achieves a matrix

$$P' := \left[\begin{array}{c|ccc|c} -2l_{0,1} & 0 & \cdots & & 0 \\ \hline * & A(c(1), l_{1,1}) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \dots & A(c(r), l_{r,1}) & 0 \\ \hline E_{n_0} & 0 & \dots & 0 & E_{n_0} \end{array}\right].$$

The rank of P' equals $\sum_{i=0}^{r}(n_i - 1 + c(i))$. Using $l_{i,1} = l_{i,2} = l_i/2$ for $i = 0, 1, 2$, we obtain with Lemma 2.9 that the $(n_i - 1 + c(i))$-th determinantal divisors of $A(c(i), l_{i,1})$ divides $\mathfrak{l}_i/2$ for $i = 1, 2$. Using $l_{i,1} = \ldots = l_{i,4} = l_i$ for $i \geq 3$ we obtain that the $(n_i - 1 + c(i))$-th determinantal divisors of $A(c(i), l_{i,1})$ divides $\mathfrak{l}_i^3$ for $i \geq 3$. Thus considering the maximal square submatrices just including one of the first n_0 columns, Laplace expansion with respect to the first row shows that the $\mathrm{rk}(P')$-th determinantal divisor of P' divides $\mathfrak{l}_0$. If we delete all of the first n_0 columns we observe that the $(\mathrm{rk}(P')-1)$-th determinantal divisor of P' divides 1, i.e., it equals 1 up to sign. Thus $\mathrm{Cl}(X)$ is a factor group of

$$\mathbb{Z}^{n'}/\mathrm{im}(\mathrm{P}^*) \cong \mathbb{Z}^{n'-\mathrm{rk}(P')} \times G,$$

where G is a finite group of order k with $k|(\mathfrak{l}_0(\mathfrak{l}_1/2)(\mathfrak{l}_2/2)\mathfrak{l}_3^3 \ldots \mathfrak{l}_r^3)$.

We show equality of these groups. Observe that we may assume the relation g_0 of $R(A, P_0)$ to be of the form $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$. In particular the irreducible components $D_{0j,1}$ and $D_{0j,2}$ of $\mathrm{V}(X; T_{0j})$ are of the form

$$D_{0j,1} = \mathrm{V}(T_{0j}, T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}) \quad \text{and} \quad D_{0j,2} = \mathrm{V}(T_{0j}, T_1^{1/2l_1} - i \cdot T_2^{1/2l_2}).$$

We conclude that for $y \in \mathbb{Z}_{\geq 0}$ with $y \mid \mathfrak{l}_0$

$$D := \sum_{j=1}^{n_0} \frac{1}{y} l_{0j} D_{0j,1} = \sum_{Y} \mathrm{ord}_Y(\mathfrak{A}_y)$$

holds with $\mathfrak{A}_y$ as in Lemma 2.16. As $\mathfrak{A}_y$ is principal if and only if $y = 1$ holds, we obtain $\mathbb{Z}/\mathfrak{l}_0\mathbb{Z}$ as a factor of the divisor class group of X. Calculating the difference between the dimensions of $\mathrm{Spec}\, R(A, P_0)$ and $\mathrm{Spec}\, R(A', P_0')$ as in the proof of the case (i) we conclude $\mathbb{Z}^{n'-\mathrm{rk}(P')} \leq \mathrm{Cl}(X)$. As due to Lemma 2.7 and the assumption that $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ whenever $j \notin \{0, 1, 2\}$ holds, $\mathfrak{l}_0$ does not divide $|\mathrm{Cl}(X)^{\mathrm{tors}}|$ but $\mathfrak{l}_0/2$ does, we obtain

$$2 \cdot |\mathrm{Cl}(X)^{\mathrm{ctors}}| \leq |G| \leq 2 \cdot |\mathrm{Cl}(X)^{\mathrm{ctors}}|$$

and the assertion follows. □

Corollary 2.17. *Let X be an affine, rational, trinomial variety. Then the divisor class group of X is free abelian if and only if X is factorial or after reordering decreasingly we have $\mathfrak{l}_0 \geq \mathfrak{l}_1 \geq \mathfrak{l}_2 = \ldots = \mathfrak{l}_r = 1$.*

Proof. Assume the divisor class group of X is free abelian. Then either X is factorial and thus $\mathrm{Cl}(X) = \{0\}$ holds or we may apply Theorem 1.1 and conclude $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\mathfrak{l}_2 = \ldots = \mathfrak{l}_r = 1$ holds. The other direction is a direct consequence of Theorem 1.1. □

As an application, we consider trinomial varieties with an isolated singularity; recall that [LS13, Thm. 6.5] gives a complete description of all those with trivial divisor class group.

Corollary 2.18. *Let X be an affine, trinomial variety with an isolated singularity. Then $\dim(X) \leq 5$ holds and we are in one the following cases:*

(i) *If $\dim(X) = 2$ holds and X is rational then its divisor class group is a torsion group.*
(ii) *If $\dim(X) = 3$ holds then X is rational and its divisor class group is free abelian.*
(iii) *If $\dim(X) \geq 4$ holds then X is factorial.*

Proof. Assume X is two-dimensional. Then $n_i = 1$ holds for all $i = 0, \ldots, r$ and X has an isolated singularity at zero. Thus if X is rational, Theorem 1.1 implies that its divisor class group is a torsion group.

Assume $\dim(X) \geq 3$ holds. Then, considering the Jacobian of X, we conclude that X has an isolated singularity at zero if and only if X is a hypersurface with defining relation $g = T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$, where $1 \leq n_0 \leq n_1 \leq n_2 = 2$ and $l_{ij} = 1$ whenever $n_i = 2$, see also [LS13]. In particular $\dim(X) \leq 5$ holds and X is rational due to Remark 2.2. If $n_0 = n_1 = 1$ holds, i.e. X is of dimension three, we obtain $\mathfrak{l}_0, \mathfrak{l}_1 \geq \mathfrak{l}_2 = 1$. Applying Corollary 2.17 we conclude that X is free abelian. In the case that $n_0 \leq n_1 = n_2 = 2$ holds, i.e. $\dim(X) \geq 4$ holds, we obtain $\mathfrak{l}_1 = \mathfrak{l}_2 = 1$ and thus X is factorial due to Remark 2.2. □

Remark 2.19. We compare our results with the existing works already stated in the introduction.

In [Fle81] H. Flenner shows that rational three-dimensional quasihomogeneous complete intersections over algebraically closed fields of arbitrary characteristic with an isolated singularity have a free abelian divisor class group. Corollary 2.18 shows that this is as well true for all trinomial varieties with isolated singularity of dimension at least three.

Using Corollary 2.17 one can construct examples of affine, rational, trinomial varieties X with free abelian divisor class group having a higher dimensional singular locus: The three-dimensional variety

$$\mathrm{V}(T_{01}^4 + T_{11}^2 + T_{21}^3 T_{22}^2) \subseteq \mathbb{C}^4$$

has divisor class group $\mathbb{Z}$ and a one-dimensional singular locus. Note that not any three-dimensional trinomial variety has a free abelian divisor class group as for instance, we obtain divisor class group $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for the hypersurface

$$\mathrm{V}(T_{01}^4 + T_{11}^2 + T_{21}^3 T_{22}^3) \subseteq \mathbb{C}^4.$$

In [Lan83, SS84, SS07] J. Lang, A. Singh, S. Spiroff, G. Scheja and U. Storch present divisor class group computations for hypersurfaces of the form $\mathbb{K}[z, x_1, \ldots, x_d]/\langle z^n - g \rangle$, where g is a weighted homogeneous polynomial in $x_1, \ldots, x_d$ of degree relatively prime to n are treated. In particular using various methods they give explicit descriptions of the divisor class groups in any characteristic. In particular the divisor class groups of trinomial hypersurfaces of the form $\mathrm{V}(T_{01}^{l_{01}} + T_1^{l_1} + T_2^{l_2}) \subseteq \mathbb{C}^3$ with $\gcd(l_{01}, \mathfrak{l}_1) = 1 = \gcd(l_{01}, \mathfrak{l}_2)$ can be calculated with their results and are regained as part of our Theorem 1.1 (i). Note that any rational trinomial variety fulfilling Remark 2.2 (iii) leaves the framework of [Lan83, SS07, SS84] but can be treated via Theorem 1.1; explicit examples are the two hypersurfaces given above.

As a direct consequence of the proof of Theorem 1.1 we obtain the following description of the divisor class group grading on the Cox ring $R(A', P_0')$ of a rational trinomial variety $\operatorname{Spec} R(A, P_0)$:

Corollary 2.20. *Let $X := \operatorname{Spec} R(A, P_0)$ be a rational trinomial variety and assume that P_0 is* gcd*-ordered. Then the divisor class group grading on the Cox ring $R(A', P_0')$ is given as*

$$\deg(T_{ij,k}) = Q(e_{ij,k}), \quad \textit{with} \quad Q\colon \mathbb{Z}^{n'+m} \to \mathbb{Z}^{n'+m}/\mathrm{im}(P^*),$$

where P is one of the following:

(i) *If $c := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1\}$, then P is built up as in (2.13.3).*
(ii) *If $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1, 2\}$, then P is built up as in (2.16.3).*

Remark 2.21. As a direct consequence of Theorem 1.1, we can compute the divisor class groups of all affine varieties arising from a hyperplatonic Cox ring. We list the basic platonic tuple (bpt) of $R(A, P_0)$ and the divisor class group of $X := \operatorname{Spec} R(A, P_0)$ in a table:

Case	bpt of $R(A, P_0)$	divisor class group
(i)	$(4, 3, 2)$	$\mathbb{Z}^{n_1+n_3+\cdots+n_r-(r-1)} \times \mathbb{Z}/3\mathbb{Z}$
(ii)	$(3, 3, 2)$	$\mathbb{Z}^{2\cdot(n_2+\cdots+n_r-(r-1))} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
(iii)	$(x, y, 1)$	$\mathbb{Z}^{(\gcd(x,y)-1)\cdot(n_2+\cdots+n_r-(r-1))}$
(iv)	$(x, 2, 2)$ and $2 \nmid x$	$\mathbb{Z}^{n_0+n_3+\cdots+n_r-(r-1)} \times \mathbb{Z}/x\mathbb{Z}$
(v)	$(x, 2, 2)$ and $2 \mid x$	$\mathbb{Z}^{n_0+n_1+n_2+3\cdot(n_3+\cdots+n_r-(r-1))} \times \mathbb{Z}/x\mathbb{Z}$

.

With the explicit description of the grading of the Cox ring of a rational trinomial variety given via the matrices P as described in Corollary 2.20, we are able to prove our second main result.

Proof of Corollary 1.2. In a first step we show that for any hyperplatonic ring R with basic platonic triple $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$, there exists a good quotient $\mathbb{C}^{n+m} \supseteq \operatorname{Spec} R \to Y(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ with respect to a quasitorus $\mathbb{T}$. Setting

$$\tilde{P} := \begin{bmatrix} \frac{1}{\mathfrak{l}_0} l_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{\mathfrak{l}_r} l_r \end{bmatrix},$$

the map $Q\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\operatorname{im}(\tilde{P}^*)$ defines a grading on R, which coarsens the grading given by P_0 as in Construction 2.1. Moreover the Veronese subalgebra S with respect to the degree zero is generated by the monomials $T_0^{l_0/\mathfrak{l}_0}, \dots, T_r^{l_r/\mathfrak{l}_r}$ and we conclude $\operatorname{Spec} S \cong Y(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$.

Now denote by R' resp. S' the Cox rings of $\operatorname{Spec} R$ resp. $\operatorname{Spec} S$ as given in Proposition 2.5 with the grading given by matrices $P(R)$ resp. $P(S)$ as in Corollary 2.20. We claim that we obtain the following commutative diagram

$$\begin{array}{ccc} R' & \longleftarrow & R \\ \uparrow & & \uparrow \\ S' & \longleftarrow & S, \end{array}$$

where the upward arrow on the r.h.s. is the embedding of a Veronese subalgebra with respect to some grading group $\mathbb{Z}^k$ and the other arrows denote the embeddings of the Veronese subalgebras as defined above. This proves the assertion as considering the grading given by $P(S)$ on S' one directly checks that the isomorphism $S' \to \mathbb{C}[T_0, T_1, T_2]/\langle T_0^{\mathfrak{l}_0} + T_1^{\mathfrak{l}_1} + T_2^{\mathfrak{l}_2} \rangle$ deleting the redundant relations is a graded isomorphism with respect to the Cox ring grading on the latter ring.

To prove our result it is now only necessary to show that the composition of the embeddings $S \to S' \to R'$ given by the matrices $\tilde{P}$ and $P(S)$ factorizes over the embedding $R \to R'$ given by $P(R)$. Note that the grading giving rise to the composed Veronese embedding $S \to S' \to R'$ can be represented by a matrix of the same shape and with the same number of columns as $P(R)$ but replacing the matrices $A(c(i), l_{i,1})$ by matrices of the following form:

$$B(c(i), l_{i,1}) := \begin{bmatrix} l_{i,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_{i,1} \\ l_{i,1}/\mathfrak{l}_{i,1} & \dots & l_{i,1}/\mathfrak{l}_{i,1} \end{bmatrix} \in \operatorname{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}),$$

and in case of P as in (2.16.3) additionally replacing the rows 2 to $n_0 + 1$ with one row $(l_{0,1}/\mathfrak{l}_{0,1}, l_{0,1}/\mathfrak{l}_{0,1}, 0, \dots, 0)$. In particular the row lattice of this matrix is a sublattice of the row lattice of $P(R)$ and we only have to show that it is a saturated sublattice. By the structure of the occurring matrices this means that the row lattice generated by the matrix $B(c(i), l_{r,1})$ is a saturated sublattice of the row lattice of the matrix $A(c(i), l_{i,1})$. Note that the row lattice of $A(c(i), l_{i,1})$ is generated by the rows of

$$\begin{bmatrix} l_{i,1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & l_{i,1} & 0 \\ E_{n_i} & \dots & E_{n_i} & E_{n_i} \end{bmatrix} \in \mathrm{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}).$$

In particular the last n_i rows span a saturated sublattice of this row lattice. As the lattice generated by $(l_i/\mathfrak{l}_i, \dots, l_i/\mathfrak{l}_i)$ lies saturated in this sublattice, the assertion follows. □

References

[ABHW18] Ivan Arzhantsev, Lukas Braun, Jürgen Hausen, and Milena Wrobel, *Log terminal singularities, platonic tuples and iteration of Cox rings*, Eur. J. Math. **4** (2018), no. 1, 242–312. MR 3782223

[ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR 3307753

[Fle81] Hubert Flenner, *Divisorenklassengruppen quasihomogener Singularitäten*, J. Reine Angew. Math. **328** (1981), 128–160. MR 636200

[HH13] Jürgen Hausen and Elaine Herppich, *Factorially graded rings of complexity one*, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 414–428. MR 3077174

[HS10] Jürgen Hausen and Hendrik Süß, *The Cox ring of an algebraic variety with torus action*, Adv. Math. **225** (2010), no. 2, 977–1012. MR 2671185

[HW17] Jürgen Hausen and Milena Wrobel, *Non-complete rational T-varieties of complexity one*, Math. Nachr. **290** (2017), no. 5-6, 815–826. MR 3636380

[HW18] ———, *On iteration of Cox rings*, J. Pure Appl. Algebra **222** (2018), no. 9, 2737–2745. MR 3783016

[Lan83] Jeffrey Lang, *The divisor classes of the hypersurface $z^{p^m} = G(x_1, \cdots, x_n)$ in characteristic $p > 0$*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 613–634. MR 701514

[LS13] Alvaro Liendo and Hendrik Süss, *Normal singularities with torus actions*, Tohoku Math. J. (2) **65** (2013), no. 1, 105–130. MR 3049643

[SS84] Günter Scheja and Uwe Storch, *Zur Konstruktion faktorieller graduierter Integritätsbereiche*, Arch. Math. (Basel) **42** (1984), no. 1, 45–52. MR 751470

[SS07] Anurag K. Singh and Sandra Spiroff, *Divisor class groups of graded hypersurfaces*, Algebra, geometry and their interactions, Contemp. Math., vol. 448, Amer. Math. Soc., Providence, RI, 2007, pp. 237–243. MR 2389245

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