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by

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Monogamy relations characterize the distributions of entanglement in multipartite systems. We investigate monogamy relations related to the concurrence C, the entanglement of formation E, negativity N_c and Tsallis-q entanglement T_q. New α-th power of entanglement monogamy relations have been derived, which are tighter than the existing entanglement monogamy relations for some classes of quantum states. Detailed examples are presented.

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INTRODUCTION

Due to the essential roles played in quantum communication and quantum information processing, quantum entanglement [1–8] has been the subject of many recent studies in recent years. The study of quantum entanglement from various viewpoints has been a very active area and has led to many impressive results. As one of the fundamental differences between quantum and classical correlations, an essential property of entanglement is that a quantum system entangled with one of other subsystems limits its entanglement with the remaining ones. The monogamy relations give rise to the distribution of entanglement in the multipartite quantum systems. Moreover, the monogamy property has emerged as the ingredient in the security analysis of quantum key distribution [9].

For a tripartite system A, B and C, the usual monogamy of an entanglement measure $\mathcal{E}$ implies that [10] the entanglement between A and BC satisfies $\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}$. However, such monogamy relations are not always satisfied by all entanglement measures for all quantum states. In fact, it has been shown that the squared concurrence C^2 [11, 12] and entanglement of formation E^2 [13] satisfy the monogamy relations for multi-qubit states. The monogamy inequality was further generalized to various entanglement measures such as continuous-variable entanglement [14–16], squashed entanglement [10, 17, 18], entanglement negativity [19–23], Tsallis-q entanglement [24, 25], and Renyi-entanglement [26–28].

In this paper, we derive monogamy inequalities which are tighter than all the existing ones, in terms of the concurrence C, entanglement of formation E, negativity N_c and Tsallis-q entanglement T_q.

TIGHTER MONOGAMY RELATIONS FOR CONCURRENCE

We first consider the monogamy inequalities satisfied by the concurrence. Let $\mathbb{H}_X$ denote a discrete finite-dimensional complex vector space associated with a quantum subsystem X. For a bipartite pure state $|\psi\rangle_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$, the concurrence is given by [29–31], $C(|\psi\rangle_{AB}) = \sqrt{2\left[1-\mathrm{Tr}(\rho_A^2)\right]}$, where ρ_A is the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex roof extension, $C(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$, where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$ and $\sum_i p_i = 1$ and $|\psi_i\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance is defined by [32, 33]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where the maximum is taken over all possible decompositions of $\rho_{AB} = \mathrm{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. When $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ is a pure state, one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For an N-qubit state $\rho_{AB_1\cdots B_{N-1}} \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{A|B_1\cdots B_{N-1}}$, viewed as a bipartite state under the partition A and $B_1, B_2, \cdots, B_{N-1}$, satisfies [34]

$$C^\alpha(\rho_{A|B_1,B_2\cdots,B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}}), \quad (1)$$

for $\alpha \geq 2$, where $\rho_{AB_i} = \mathrm{Tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{AB_1\cdots B_{N-1}})$. It is further improved that for $\alpha \geq 2$, if $C(\rho_{AB_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \cdots, m$, and $C(\rho_{AB_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, then [35],

$$\begin{aligned} &C^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) \geq \\ &C^\alpha(\rho_{AB_1}) + \frac{\alpha}{2}C^\alpha(\rho_{AB_2}) + \cdots + \left(\frac{\alpha}{2}\right)^{m-1} C^\alpha(\rho_{AB_m}) \\ &+ \left(\frac{\alpha}{2}\right)^{m+1} \left(C^\alpha(\rho_{AB_{m+1}}) + \cdots + C^\alpha(\rho_{AB_{N-2}})\right) \\ &+ \left(\frac{\alpha}{2}\right)^{m} C^\alpha(\rho_{AB_{N-1}}) \end{aligned} \quad (2)$$

and for all $\alpha < 0$,

$$\begin{aligned} &C^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) < \\ &K(C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}})), \end{aligned} \quad (3)$$

where $K = \frac{1}{N-1}$.

In the following, we show that these monogamy inequalities satisfied by the concurrence can be further refined and become even tighter. For convenience, we denote $C_{AB_i} = C(\rho_{AB_i})$ the concurrence of ρ_{AB_i} and $C_{A|B_1,B_2\cdots,B_{N-1}} = C(\rho_{A|B_1\cdots B_{N-1}})$. We first introduce two Lemmas.

[**Lemma 1**]. For any real numbers x and t, $0 \leq t \leq 1$, $x \in [1, \infty)$, we have $(1+t)^x \geq 1 + (2^x - 1)t^x$.

[Proof]. Let $f(x, y) = (1+y)^x - y^x$ with $x \geq 1$, $y \geq 1$. Then $\frac{\partial f}{\partial y} = x[(1+y)^{x-1} - y^{x-1}] \geq 0$. Therefore, $f(x, y)$ is an increasing function of y, i.e., $f(x, y) \geq f(x, 1) = 2^x - 1$. Set $y = \frac{1}{t}$, $0 < t \leq 1$, we obtain $(1+t)^x \geq 1 + (2^x - 1)t^x$. When $t = 0$, the inequality is trivial. □

[**Lemma 2**]. For any $2 \otimes 2 \otimes 2^{n-2}$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$, if $C_{AB} \geq C_{AC}$, we have

$$C^\alpha_{A|BC} \geq C^\alpha_{AB} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AC}, \quad (4)$$

for all $\alpha \geq 2$.

[Proof]. It has been shown that $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$ for arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC} [11, 37]. Then, if $C_{AB} \geq C_{AC}$, we have

$$\begin{aligned} C^\alpha_{A|BC} &\geq (C^2_{AB} + C^2_{AC})^{\frac{\alpha}{2}} \\ &= C^\alpha_{AB}\left(1 + \frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}} \\ &\geq C^\alpha_{AB}\left[1 + (2^{\frac{\alpha}{2}} - 1)\left(\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}\right] \\ &= C^\alpha_{AB} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AC}, \end{aligned}$$

where the second inequality is due to Lemma 1. As the subsystems A and B are equivalent in this case, we have assumed that $C_{AB} \geq C_{AC}$ without loss of generality. Moreover, if $C_{AB} = 0$, we have $C_{AB} = C_{AC} = 0$. That is to say the lower bound becomes trivially zero. □

From Lemma 2 we have the following Theorem.

[**Theorem 1**]. For N-qubit mixed state, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, we have

$$\begin{aligned} &C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq \\ &C^\alpha_{AB_1} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AB_2} + \cdots + (2^{\frac{\alpha}{2}} - 1)^{m-1}C^\alpha_{AB_m} \\ &+(2^{\frac{\alpha}{2}} - 1)^{m+1}(C^\alpha_{AB_{m+1}} + \cdots + C^\alpha_{AB_{N-2}}) \\ &+(2^{\frac{\alpha}{2}} - 1)^{m}C^\alpha_{AB_{N-1}} \end{aligned} \quad (5)$$

for all $\alpha \geq 2$.

[Proof]. From the inequality (4), we have

$$\begin{aligned} &C^\alpha_{A|B_1B_2\cdots B_{N-1}} \\ &\geq C^\alpha_{AB_1} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{A|B_2\cdots B_{N-1}} \\ &\geq C^\alpha_{AB_1} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AB_2} + (2^{\frac{\alpha}{2}} - 1)^2C^\alpha_{A|B_3\cdots B_{N-1}} \\ &\geq \cdots \\ &\geq C^\alpha_{AB_1} + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AB_2} + \cdots + (2^{\frac{\alpha}{2}} - 1)^{m-1}C^\alpha_{AB_m} \\ &\quad +(2^{\frac{\alpha}{2}} - 1)^{m}C^\alpha_{A|B_{m+1}\cdots B_{N-1}}. \end{aligned} \quad (6)$$

Similarly, as $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, we get

$$\begin{aligned} &C^\alpha_{A|B_{m+1}\cdots B_{N-1}} \\ &\geq (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AB_{m+1}} + C^\alpha_{A|B_{m+2}\cdots B_{N-1}} \\ &\geq (2^{\frac{\alpha}{2}} - 1)(C^\alpha_{AB_{m+1}} + \cdots + C^\alpha_{AB_{N-2}}) \\ &\quad +C^\alpha_{AB_{N-1}}. \end{aligned} \quad (7)$$

Combining (6) and (7), we have Theorem 1. □

As for $\alpha \geq 2$, $(2^{\frac{\alpha}{2}}-1)^m \geq (\alpha/2)^m$ for all $1 \leq m \leq N-3$, our formula (5) in Theorem 1 gives a tighter monogamy relation with larger lower bounds than (1) and (2). In Theorem 1 we have assumed that some $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ and some $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for the $2\otimes 2\otimes\cdots\otimes 2$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$. If all $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, N-2$, then we have the following conclusion:

[Theorem 2]. If $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$, then we have

$$\begin{aligned} &C^{\alpha}_{A|B_1\cdots B_{N-1}} \geq \\ &C^{\alpha}_{AB_1} + (2^{\frac{\alpha}{2}}-1)C^{\alpha}_{AB_2} + \cdots + (2^{\frac{\alpha}{2}}-1)^{N-3}C^{\alpha}_{AB_{N-2}} \\ &+(2^{\frac{\alpha}{2}}-1)^{N-2}C^{\alpha}_{AB_{N-1}}. \end{aligned} \tag{8}$$

Example 1. Let us consider the three-qubit state $|\psi\rangle$ in the generalized Schmidt decomposition form [38, 39],

$$\begin{aligned} |\psi\rangle &= \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle \\ &\quad +\lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \tag{9}$$

where $\lambda_i \geq 0$, $i = 0, 1, 2, 3, 4$ and $\sum_{i=0}^{4}\lambda_i^2 = 1$. From the definition of concurrence, we have $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$, and $C_{AC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_1 = \frac{1}{2}$, $\lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$, one has $C_{A|BC} = \frac{\sqrt{2}}{2}$, $C_{AB} = C_{AC} = \frac{\sqrt{6}}{6}$, then $C^{\alpha}_{A|BC} = (\frac{\sqrt{2}}{2})^{\alpha}$, $C^{\alpha}_{AB} + C^{\alpha}_{AC} = 2(\frac{\sqrt{6}}{6})^{\alpha}$, $C^{\alpha}_{AB} + \frac{\alpha}{2}C^{\alpha}_{AC} = (1+\frac{\alpha}{2})(\frac{\sqrt{6}}{6})^{\alpha}$, $C^{\alpha}_{AB} + (2^{\frac{\alpha}{2}}-1)C^{\alpha}_{AC} = 2^{\frac{\alpha}{2}}(\frac{\sqrt{6}}{6})^{\alpha}$. One can see that our result is better than the results in [34] and [35] for $\alpha \geq 2$, see Fig 1.

TIGHTER MONOGAMY REALATIONS FOR EOF

The entanglement of formation (EoF) [40, 41] is a well defined important measure of entanglement for bipartite systems. Let $\mathbb{H}_A$ and $\mathbb{H}_B$ be m and n dimensional ($m \leq n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A), \tag{10}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = -\mathrm{Tr}(\rho\log_2\rho)$. For a bipartite mixed state $\rho_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$, the entanglement of formation is given by,

$$E(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{11}$$

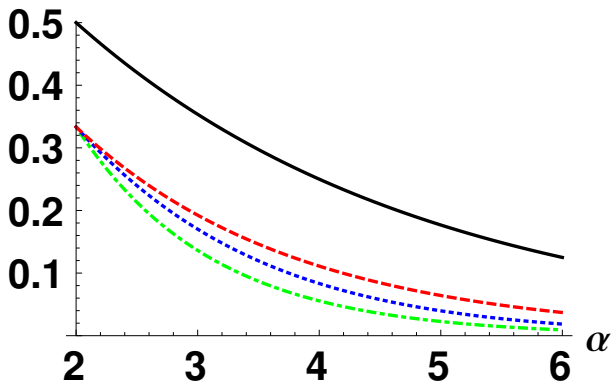

FIG. 1: The axis y is the concurrence of $|\psi\rangle$ and its lower bound, which are functions of α. The black solid line represents the concurrence of $|\psi\rangle$ in Example 1, red dashed line represents the lower bound from our result, blue dotted (green dotdashed) line represents lower bound from the result in [35] ([34]).

with the minimum taking over all possible pure stae decompositions of ρ_{AB}.

Denote $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$, where $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. From (10) and (11), one has $E(|\psi\rangle) = f\left(C^2(|\psi\rangle)\right)$ for $2\otimes m$ ($m \geq 2$) pure state $|\psi\rangle$, and $E(\rho) = f\left(C^2(\rho)\right)$ for two-qubit mixed state ρ [42]. It is obvious that $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$. $f(x)$ satisfies the following relations:

$$f^{\sqrt{2}}(x^2+y^2) \geq f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{12}$$

where $f^{\sqrt{2}}(x^2+y^2) = [f(x^2+y^2)]^{\sqrt{2}}$.

It has been shown that the EoF does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{A|BC}$ [43]. In [44] the authors showed that EoF is a monotonic function satisfying $E^2(C^2_{A|B_1B_2\cdots B_{N-1}}) \geq E^2(\sum_{i=1}^{N-1} C^2_{AB_i})$. For $N-$qubit systems, one has [34],

$$E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\alpha}_{AB_1} + E^{\alpha}_{AB_2} + \cdots + E^{\alpha}_{AB_{N-1}} \tag{13}$$

for $\alpha \geq \sqrt{2}$, where $E_{A|B_1B_2\cdots B_{N-1}}$ is the entanglement of formation of ρ in bipartite partition $A|B_1B_2\cdots B_{N-1}$, and E_{AB_i}, $i = 1, 2, \cdots, N-1$, is the EoF of the mixed states $\rho_{AB_i} = \mathrm{Tr}_{B_1B_2\cdots B_{i-1},B_{i+1}\cdots B_{N-1}}(\rho)$. It is further improved that for $\alpha \geq \sqrt{2}$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j =$

$m+1,\cdots,N-2$, $\forall\ 1\leq m\leq N-3$, $N\geq 4$, then [35]

$$\begin{aligned}&E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\geq\\&E^{\alpha}_{AB_1}+(\alpha/\sqrt{2})E^{\alpha}_{AB_2}\cdots+(\alpha/\sqrt{2})^{m-1}E^{\alpha}_{AB_m}\\&+(\alpha/\sqrt{2})^{m+1}(E^{\alpha}_{AB_{m+1}}+\cdots+E^{\alpha}_{AB_{N-2}})\\&+(\alpha/\sqrt{2})^{m}E^{\alpha}_{AB_{N-1}}.\end{aligned}\tag{14}$$

In fact, generally we can prove the following results.

[Theorem 3]. For any N-qubit mixed state $\rho\in\mathbb{H}_A\otimes\mathbb{H}_{B_1}\otimes\cdots\otimes\mathbb{H}_{B_{N-1}}$, if $C_{AB_i}\geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\cdots,m$, and $C_{AB_j}\leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\cdots,N-2$, $\forall\ 1\leq m\leq N-3$, $N\geq 4$, the entanglement of formation $E(\rho)$ satisfies

$$\begin{aligned}&E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\geq\\&E^{\alpha}_{AB_1}+(2^t-1)E^{\alpha}_{AB_2}\cdots+(2^t-1)^{m-1}E^{\alpha}_{AB_m}\\&+(2^t-1)^{m+1}(E^{\alpha}_{AB_{m+1}}+\cdots+E^{\alpha}_{AB_{N-2}})\\&+(2^t-1)^{m}E^{\alpha}_{AB_{N-1}},\end{aligned}\tag{15}$$

for $\alpha\geq\sqrt{2}$, where $t=\alpha/\sqrt{2}$.

[Proof]. For $\alpha\geq\sqrt{2}$, we have

$$\begin{aligned}&f^{\alpha}(x^2+y^2)\\&=\left(f^{\sqrt{2}}(x^2+y^2)\right)^t\\&\geq\left(f^{\sqrt{2}}(x^2)+f^{\sqrt{2}}(y^2)\right)^t\\&\geq\left(f^{\sqrt{2}}(x^2)\right)^t+(2^t-1)\left(f^{\sqrt{2}}(y^2)\right)^t\\&=f^{\alpha}(x^2)+(2^t-1)f^{\alpha}(y^2),\end{aligned}\tag{16}$$

where the first inequality is due to the inequality (12), and the second inequality is obtained from a similar consideration in the proof of the second inequality in (4).

Let $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|\in\mathbb{H}_A\otimes\mathbb{H}_{B_1}\otimes\cdots\otimes\mathbb{H}_{B_{N-1}}$ be the optimal decomposition of $E_{A|B_1B_2\cdots B_{N-1}}(\rho)$ for the N-qubit mixed state ρ, we have

$$\begin{aligned}&E_{A|B_1B_2\cdots B_{N-1}}(\rho)\\&=\sum_i p_iE_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\\&=\sum_i p_if\left(C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right)\\&\geq f\left(\sum_i p_iC^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right)\\&\geq f\left(\left[\sum_i p_iC_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right]^2\right)\\&\geq f\left(C^2_{A|B_1B_2\cdots B_{N-1}}(\rho)\right),\end{aligned}$$

where the first inequality is due to that $f(x)$ is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}}\geq\sum_i x_iy_i$, with $x_i=\sqrt{p_i}$ and $y_i=\sqrt{p_i}C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of concurrence and that $f(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned}&E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}(\rho)\\&\geq f^{\alpha}(C^2_{AB_1}+C^2_{AB_2}+\cdots+C^2_{AB_{m-1}})\\&\geq f^{\alpha}(C^2_{AB_1})+(2^t-1)f^{\alpha}(C^2_{AB_2})+\cdots+(2^t-1)^{m-1}f^{\alpha}(C^2_{AB_m})\\&\quad+(2^t-1)^{m+1}(f^{\alpha}(C^2_{AB_{m+1}})+\cdots+f^{\alpha}(C^2_{AB_{N-2}}))\\&\quad+(2^t-1)^{m}f^{\alpha}(C^2_{AB_{N-1}})\\&=E^{\alpha}_{A|B_1}+(2^t-1)E^{\alpha}_{AB_2}+\cdots+(2^t-1)^{m-1}E^{\alpha}_{AB_m}\\&\quad+(2^t-1)^{m+1}(E^{\alpha}_{AB_{m+1}}+\cdots+E^{\alpha}_{AB_{N-2}})\\&\quad+(2^t-1)^{m}E^{\alpha}_{AB_{N-1}},\end{aligned}$$

where we have used the monogamy inequality in (1) for $N-$qubit states ρ to obtain the first inequality. By using (16) and the similar consideration in the proof of Theorem 1, we get the second inequality. Since for any $2\otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i})=f\left[C^2(\rho_{AB_i})\right]$, one gets the last equality. □

As for $(2^{\alpha/\sqrt{2}}-1)\geq\alpha/\sqrt{2}$ for $\alpha\geq\sqrt{2}$, (15) is obviously tighter than (13) and(14). Moreover, similar to the concurrence, for the case that $C_{AB_i}\geq C_{A|B_{i+1}\cdots B_{N-1}}$ for all $i=1,2,\cdots,N-2$, we have a simple tighter monogamy relation for entanglement of formation:

[Theorem 4]. If $C_{AB_i}\geq C_{A|B_{i+1}\cdots B_{N-1}}$ for all $i=1,2,\cdots,N-2$, we have

$$\begin{aligned}E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\geq\quad&E^{\alpha}_{AB_1}+(2^{\alpha/\sqrt{2}}-1)E^{\alpha}_{AB_2}+\cdots\\&+(2^{\alpha/\sqrt{2}}-1)^{N-2}E^{\alpha}_{AB_{N-1}},\end{aligned}\tag{17}$$

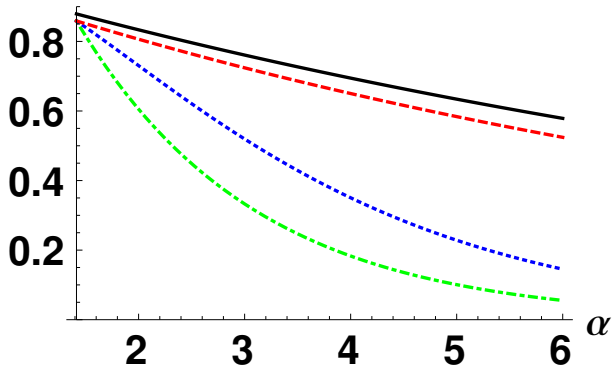

FIG. 2: The axis y is the EOF of the W state $|W\rangle$ and its lower bounds, which are functions of α. The black solid line represents the EOF of the state $|W\rangle$ in Example 2, red dashed line represents the lower bound from our result, blue dotted (green dotdashed) line represents the lower bound from the result in [35] ([34]).

for $\alpha \geq \sqrt{2}$.

Example 2. Let us consider the W state, $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. We have $E_{AB} = E_{AC} = 0.550048$, $E_{A|BC} = 0.918296$, then $E^{\alpha}_{A|BC} = (0.918296)^{\alpha}$, $E^{\alpha}_{AB} + E^{\alpha}_{AC} = 2(0.550048)^{\alpha}$, $E^{\alpha}_{AB} + \frac{\alpha}{\sqrt{2}}E^{\alpha}_{AC} = (1+\frac{\alpha}{\sqrt{2}})(0.550048)^{\alpha}$, $E^{\alpha}_{AB} + (2^{\frac{\alpha}{\sqrt{2}}}-1)E^{\alpha}_{AC} = 2^{\frac{\alpha}{\sqrt{2}}}(0.550048)^{\alpha}$. It is easily verified that our results is better than the results in [34] and [35] for $\alpha \geq \sqrt{2}$, see Fig 2.

TIGHTER MONOGAMY RELATIONS FOR NEGATIVITY

Another well-known quantifier of bipartite entanglement is the negativity. Given a bipartite state ρ_{AB} in $\mathbb{H}_A \otimes \mathbb{H}_B$, the negativity is defined by [45], $N(\rho_{AB}) = (||\rho^{T_A}_{AB}|| - 1)/2$, where $\rho^{T_A}_{AB}$ is the partial transpose with respect to the subsystem A, $||X||$ denotes the trace norm of X, $||X|| = \mathrm{Tr}\sqrt{XX^{\dagger}}$. Negativity is a computable measure of entanglement, and is a convex function of ρ_{AB}. It vanishes if and only if ρ_{AB} is separable for the $2\otimes 2$ and $2\otimes 3$ systems [46]. For the purpose of discussion, we use the following definition of negativity, $N(\rho_{AB}) = ||\rho^{T_A}_{AB}|| - 1$. For any bipartite pure state $|\psi\rangle_{AB}$, the negativity $N(\rho_{AB})$ is given by $N(|\psi\rangle_{AB}) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} = (\mathrm{Tr}\sqrt{\rho_A})^2 - 1$, where λ_i are the eigenvalues for the reduced density matrix of $|\psi\rangle_{AB}$. For a mixed state ρ_{AB}, the convex-roof extended negativity (CREN) is defined as

$$N_c(\rho_{AB}) = \min \sum_i p_i N(|\psi_i\rangle_{AB}), \tag{18}$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle_{AB}\}$ of ρ_{AB}. CREN gives a perfect discrimination of positive partial transposed bound entangled states and separable states in any bipartite quantum systems [47, 48].

Let us consider the relation between CREN and concurrence. For any bipartite pure state $|\psi\rangle_{AB}$ in a $d\otimes d$ quantum system with Schmidt rank 2, $|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$, one has $N(|\psi\rangle_{AB}) = \| \ |\psi\rangle\langle\psi|^{T_B} \ \| -1 = 2\sqrt{\lambda_0\lambda_1} = \sqrt{2(1-\mathrm{Tr}\rho_A^2)} = C(|\psi\rangle_{AB})$. In other words, negativity is equivalent to concurrence for any pure state with Schmidt rank 2, and consequently it follows that for any two-qubit mixed state $\rho_{AB} = \sum p_i|\psi_i\rangle_{AB}\langle\psi_i|$,

$$\begin{aligned} N_c(\rho_{AB}) &= \min \sum_i p_i N(|\psi_i\rangle_{AB}) \\ &= \min \sum_i p_i C(|\psi_i\rangle_{AB}) \\ &= C(\rho_{AB}). \end{aligned} \tag{19}$$

With a similar consideration of concurrence, we obtain the following result.

[Theorem 5]. For any N-qubit state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, if $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $N_{cAB_j} \leq N_{cA|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, we have

$$\begin{aligned} & N^{\alpha}_{c\ A|B_1B_2\cdots B_{N-1}} \geq N^{\alpha}_{c\ AB_1} \\ & +(2^{\frac{\alpha}{2}}-1)N^{\alpha}_{c\ AB_2} + \cdots + (2^{\frac{\alpha}{2}}-1)^{m-1}N^{\alpha}_{c\ AB_m} \\ & +(2^{\frac{\alpha}{2}}-1)^{m+1}(N^{\alpha}_{c\ AB_{m+1}} + \cdots + N^{\alpha}_{c\ AB_{N-2}}) \\ & +(2^{\frac{\alpha}{2}}-1)^{m}N^{\alpha}_{c\ AB_{N-1}} \end{aligned} \tag{20}$$

for all $\alpha \geq 2$.

In Theorem 5 we have assumed that some $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ and some $N_{cAB_j} \leq N_{cA|B_{j+1}\cdots B_{N-1}}$. If all $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, N-2$, then we have the following conclusion:

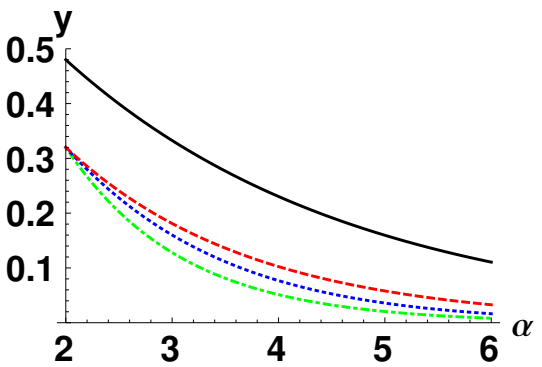

FIG. 3: The axis y is the concurrence of $|\psi\rangle$ and its lower bound, which are functions of α. The black solid line represents the concurrence of $|\psi\rangle$ in Example 3, red dashed line represents the lower bound from our result, blue dotted (green dotdashed) line represents lower bound from the result in [36] ([34]).

[Theorem 6]. If $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$, we have

$$\begin{aligned} & N^{\alpha}_{cA|B_1\cdots B_{N-1}} \geq \\ & N^{\alpha}_{cAB_1} + (2^{\frac{\alpha}{2}} - 1)N^{\alpha}_{cAB_2} + \cdots \\ & +(2^{\frac{\alpha}{2}} - 1)^{N-2} N^{\alpha}_{cAB_{N-1}}. \end{aligned} \tag{21}$$

Example 3. Let us consider again the three-qubit state $|\psi\rangle$ (9). From the definition of CREN, we have $N_{cA|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $N_{cAB} = 2\lambda_0\lambda_2$, and $N_{cAC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{5}$. One gets $N^{\alpha}_{cA|BC} = (\frac{2\sqrt{3}}{5})^{\alpha}$, $N^{\alpha}_{cAB} + N^{\alpha}_{cAC} = 2(\frac{2}{5})^{\alpha}$, $N^{\alpha}_{cAB} + \frac{\alpha}{2}N^{\alpha}_{cAC} = \left(1 + \frac{\alpha}{2}\right)(\frac{2}{5})^{\alpha}$, $N^{\alpha}_{cAB} + (2^{\frac{\alpha}{2}} - 1)N^{\alpha}_{cAC} = 2^{\frac{\alpha}{2}}(\frac{2}{5})^{\alpha}$. One can see that our result is better than the results in [34] and [36] for $\alpha \geq 2$, see Fig 3.

TIGHTER MONOGAMY RELATIONS FOR TSALLIS-Q ENTANGLEMENT

For a bipartite pure state $|\psi\rangle_{AB}$, the Tsallis-q entanglement is defined by [24],

$$T_q(|\psi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1 - tr\rho_A^q), \tag{22}$$

for any $q > 0$ and $q \neq 1$. If q tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, $\lim_{q\to 1} T_q(\rho) = -tr\rho\log\rho = S_q(\rho)$. For a bipartite mixed state ρ_{AB}, Tsallis-q entanglement is defined via the convex-roof extension, $T_q(\rho_{AB}) = \min\sum_i p_i T_q(|\psi_i\rangle_{AB})$, with the minimum taken over all possible pure state decompositions of ρ_{AB}.

In [49], the author has proved an analytic relationship between Tsallis-q entanglement and concurrence for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$,

$$T_q(|\psi\rangle_{AB}) = g_q(C^2(|\psi\rangle_{AB})), \tag{23}$$

where the function $g_q(x)$ is defined by

$$g_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(\frac{1-\sqrt{1-x}}{2}\right)^q\right] \tag{24}$$

It has been shown that $T_q(|\psi\rangle) = g_q\left(C^2(|\psi\rangle)\right)$ for $2 \otimes m$ $(m \geq 2)$ pure state $|\psi\rangle$, and $T_q(\rho) = g_q\left(C^2(\rho)\right)$ for two-qubit mixed state ρ [24]. Hence (23) holds for any q such that $g_q(x)$ in (24) is monotonically increasing and convex. In particular, $g_q(x)$ satisfies the following relations for $2 \leq q \leq 3$,

$$g_q(x^2 + y^2) \geq g_q(x^2) + g_q^2(y^2). \tag{25}$$

The Tsallis-q entanglement satisfies [24]

$$T_{q_{A|B_1B_2\cdots B_{N-1}}} \geq \sum_{i=1}^{N-1} T_{q_{AB_i}}, \tag{26}$$

where $i = 1, 2, \cdots N-1$, $2 \leq q \leq 3$. It is futher proved in [49],

$$T^2_{q\ A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T^2_{q\ AB_i}, \tag{27}$$

with $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$. In fact, generally we can prove the following results.

[Theorem 7]. For an arbitrary N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall$ $1 \leq m \leq N-3$, $N \geq 4$, the α-th power of Tsallis-q entanglement satisfies the monogamy relation

$$\begin{aligned} & T^{\alpha}_{q_{A|B_1B_2\cdots B_{N-1}}} \geq \\ & T^{\alpha}_{q_{AB_1}} + (2^{\alpha} - 1)T^{\alpha}_{q_{AB_2}} + \cdots + (2^{\alpha} - 1)^{m-1}T^{\alpha}_{q_{AB_m}} \\ & +(2^{\alpha} - 1)^{m+1}(T^{\alpha}_{q_{AB_{m+1}}} + \cdots + T^{\alpha}_{q_{AB_{N-2}}}) \\ & +(2^{\alpha} - 1)^{m}T^{\alpha}_{q_{AB_{N-1}}}, \end{aligned} \tag{28}$$

where $\alpha \geq 1$, $T_{q_{A|B_1B_2\cdots B_{N-1}}}$ quantifies the Tsallis-q entanglement in the partition $A|B_1B_2\cdots B_{N-1}$ and

$T_{q_{AB_i}}$ quantifies that in two-qubit subsystem AB_i with $2 \leq q \leq 3$.

[Proof]. For $\alpha \geq 1$, we have

$$\begin{aligned} g_q^\alpha(x^2+y^2) &\geq \left(g_q(x^2)+g_q(y^2)\right)^\alpha \\ &\geq g_q^\alpha(x^2)+(2^\alpha-1)g_q^\alpha(y^2), \end{aligned} \tag{29}$$

where the first inequality is due to the inequality (25), and the second inequality is obtained from a similar consideration in the proof of the second inequality in (4).

Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_N-1}$ be the optimal decomposition for the N-qubit mixed state ρ, we have

$$\begin{aligned} &T_{q_{A|B_1B_2\cdots B_{N-1}}}(\rho) \\ &= \sum_i p_i T_q(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}) \\ &= \sum_i p_i g_q\left[C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \\ &\geq g_q\left[\sum_i p_i C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \\ &\geq g_q\left(\left[\sum_i p_i C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\ &= g_q\left[C^2_{A|B_1B_2\cdots B_{N-1}}(\rho)\right], \end{aligned} \tag{30}$$

where the first inequality is due to that $g_q(x)$ is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}} \geq \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$ and $y_i = \sqrt{p_i}C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of Tsallis-q entanglement and that $g_q(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned} &T^\alpha_{q\ A|B_1B_2\cdots B_{N-1}}(\rho) \\ &\geq g_q^\alpha\left[\sum_i C^2(\rho_{AB_i})\right] \\ &\geq {g_q}^\alpha(C_{AB_1}) + (2^\alpha-1){g_q}^\alpha(C_{AB_2}) + \cdots \\ &\quad +(2^\alpha-1)^{m-1}{g_q}^\alpha(C_{AB_m}) \\ &\quad +(2^\alpha-1)^{m+1}\left({g_q}^\alpha(C_{AB_{m+1}}) + \cdots + {g_q}^\alpha(C_{AB_{N-2}})\right) \\ &\quad +(2^\alpha-1)^m{g_q}^\alpha(C_{AB_{N-1}}) \\ &= T^\alpha_{q_{AB_1}} + (2^\alpha-1)T^\alpha_{q_{AB_2}} + \cdots + (2^\alpha-1)^{m-1}T^\alpha_{q_{AB_m}} \\ &\quad +(2^\alpha-1)^{m+1}(T^\alpha_{q_{AB_{m+1}}} + \cdots + T^\alpha_{q_{AB_{N-2}}}) \\ &\quad +(2^\alpha-1)^m T^\alpha_{q_{AB_{N-1}}}, \end{aligned} \tag{31}$$

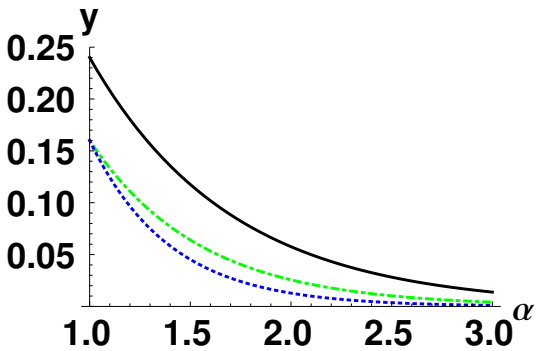

FIG. 4: The axis y is the concurrence of $|\psi\rangle$ and its lower bound, which are functions of α. The black solid line represents the concurrence of $|\psi\rangle$ in Example 4, green dotdashed line represents the lower bound from our result, blue dotted line represents lower bound from the result in [24].

where we have used the monogamy inequality in (1) for $N-$qubit states ρ to obtain the first inequality. By using (29) and the similar consideration in the proof of Theorem 1, we get the second inequality. Since for any $2 \otimes 2$ quantum state ρ_{AB_i}, $T_q(\rho_{AB_i}) = g_q\left[C^2(\rho_{AB_i})\right]$, one gets the last equality. □

Example 4. Let us consider again the three-qubit state $|\psi\rangle$ (9). From the definition of Tsallis-q entanglement, when $q=2$, we have $T_{2\,A|BC} = 2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2)$, $T_{2\,AB} = 2\lambda_0^2\lambda_2^2$, and $T_{2\,AC} = 2\lambda_0^2\lambda_3^2$. Set $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{5}$. One gets $T^\alpha_{2\,A|BC} = (\frac{6}{25})^\alpha$, $T^\alpha_{2\,AB} + T^\alpha_{2\,AC} = 2(\frac{2}{25})^\alpha$, $T^\alpha_{2\,AB} + (2^{\frac{\alpha}{2}}-1)T^\alpha_{2\,AC} = 2^\alpha(\frac{2}{25})^\alpha$. One can see that our result is better than that in [34] for $\alpha \geq 2$, see Figure 4.

CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We have presented monogamy relations related to the α-power of concurrence C, entanglement of formation E, negativity N_c and Tsallis-q entanglement T_q, which are tighter, at least for some classes of quantum states, than the existing entanglement monogamy relations for $\alpha > 2$, $\alpha > \sqrt{2}$, $\alpha > 2$ and $\alpha > 1$, respectively. The necessary conditions that our new inequalities are strictly tighter can been seen from our monogamy relations. For instance, (8) is tighter than the existing ones for $\alpha > 2$, for all quantum states

that at least one of the C_{AB_i}s $(i = 2, ..., N-1)$ is not zero, which excludes the fully separable states that have no entanglement distribution at all among the subsystems. Another case that $C_{AB_i} = 0$ for all $i = 2, ..., N-1$ is the N-qubit GHZ state [50], which is genuine multipartite entangled. However, for the genuine entangled N-qubit W-state [51], one has $C_{AB_i} = \frac{2}{N}$, $i = 2, ..., N-1$. In general, most of states have at least one non-zero C_{AB_i} $(i = 2, ..., N-1)$.

Monogamy relations characterize the distributions of entanglement in multipartite systems. Tighter monogamy relations imply finer characterizations of the entanglement distribution. Our approach may also be used to study further the monogamy properties related to other quantum correlations.

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