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Separability criteria based on Heisenberg-Weyl representation of density matrices

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Separability is an important problem in theory of quantum entanglement. By using the Bloch representation of quantum states in terms of the Heisenberg-Weyl observable basis, we present a new separability criterion for bipartite quantum systems. It is shown that this criterion can be better than the previous ones in detecting entanglement. The results are generalized to multipartite quantum states.

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INTRODUCTION

Quantum entanglement is a fascinating phenomenon in quantum physics. In recent decades, much works have been devoted to understand entanglement as it plays important roles in many quantum information processing. Nevertheless, there are still many problems remain unsolved in the theory of quantum entanglement. One basic problem is to determine whether a given bipartite state is entangled or separable. Although the problem is believed to be a nondeterministic polynomial-time hard problem, there are a number of operational criteria to deal with the problem, for example, the positive partial transpose (PPT) criterion [2, 3], realignment criteria [4–8], covariance matrix criteria [9–11], correlation matrix criteria [12–14] and so on. More recently, some more separability criteria have been proposed [15–20]. Among them, Li et al. [15] presented separability criteria based on correlation matrices and the Bloch vectors of reduced density matrices. And by adding some extra parameters, Ref.[20] presents a more general separability criterion for bipartite states in terms of the Bloch representation of density matrices.

The state of two quantum systems A and B, acting on the finite-dimensional Hilbert space $H = H_A \otimes H_B$, is described by the density operator ρ. A state ρ is said to be separable if ρ can be written as a convex combination of product vectors [1], i.e.

$$\rho = \sum_i p_i |\psi_i, \varphi_i\rangle\langle\psi_i, \varphi_i|, \qquad (1)$$

where $0 \leqslant p_i \leqslant 1$, $\sum_i p_i = 1$, and $|\psi_i, \varphi_i\rangle = |\psi_i\rangle_A \otimes |\varphi_i\rangle_B$ ($|\psi\rangle_A \in H_A$ and $|\varphi\rangle_B \in H_B$). The state ρ is said to be entangled, when ρ cannot be written as in form of Eq.(1).

In this article, we put forward a new Bloch representation in terms of the Heisenberg-Weyl (HW) observable basis [21]. It is one of the standard Hermitian generalization of Pauli operators, constructed from HW operators [22–25]. They have distinct properties from those of Gell-Mann matrices [21], Based on the Heisenberg-Weyl representation of density matrices, we give a new separability criterion for bipartite quantum states and multipartite states. By example, we show that this criterion has advantages in determining whether a quantum state is separable or entangled.

HW observable basis. First, we briefly introduce the HW-operator basis [21]. The generalized Pauli "phase" and "shift" operators are given by $Z = e^{\frac{i2\pi Q}{N}}$ and $X = e^{\frac{-i2\pi P}{N}}$, respectively, $X|j\rangle = |j + 1 \text{ mod } N\rangle$ and $Z|j\rangle = e^{\frac{i2\pi j}{N}}|j\rangle$. Q and P are the discrete position and momentum operators describing a $N \times N$ grid.

The phase-space displacement operators for N-level systems are defined by

$$\mathcal{D}(l, m) = Z^l X^m e^{\frac{-i\pi lm}{N}} \ , \qquad (2)$$

i.e. [26]

$$\mathcal{D}(l, m) = \sum_{k=0}^{N-1} e^{\frac{2i\pi kl}{N}} |k\rangle\langle(k + m) mod\ N|, \qquad (3)$$

$l, m = 0, 1, \ldots, N - 1$. These non-Hermitian orthogonal basis operators satisfy the following orthogonality condition, $Tr\{\mathcal{D}(l, m)\mathcal{D}^\dagger(l', m')\} = N\delta_{l,l'}\delta_{m,m'}$.

The complete set of Hermitian operators can be constructed from the HW operators $\mathcal{D}(l, m)$ by defining

$$\mathcal{Q}(l, m) = \mathcal{X}\mathcal{D}(l, m) + \mathcal{X}^*\mathcal{D}^\dagger(l, m) \ , \qquad (4)$$

where $\mathcal{X} = (1 \pm i)/2$. $\mathcal{Q}(l, m)$ are the so called HW observable basis and satisfy the orthogonality condition,

$$Tr\{\mathcal{Q}(l, m)\mathcal{Q}^\dagger(l', m')\} = N\delta_{l,l'}\delta_{m,m'} \ . \qquad (5)$$

This basis simply reduces to the pauli matrices for $N = 2$. When $N = 3$, we have

$$\mathcal{Q}(0,1) = \frac{1}{2}\begin{pmatrix} 0 & 1+i & 1-i \\ 1-i & 0 & 1+i \\ 1+i & 1-i & 0 \end{pmatrix},$$

$$\mathcal{Q}(0,2) = \frac{1}{2}\begin{pmatrix} 0 & 1-i & 1+i \\ 1+i & 0 & 1-i \\ 1-i & 1+i & 0 \end{pmatrix},$$

$$\mathcal{Q}(1,0) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1-\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3}-1 \end{pmatrix},$$

$$\mathcal{Q}(1,1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{\frac{\pi}{4}i} & e^{\frac{5}{12}\pi i} \\ e^{-\frac{\pi}{4}i} & 0 & e^{\frac{11}{12}\pi i} \\ e^{-\frac{5}{12}\pi i} & e^{-\frac{11}{12}\pi i} & 0 \end{pmatrix},$$

$$\mathcal{Q}(1,2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{-\frac{11}{12}\pi i} & e^{\frac{\pi}{4}i} \\ e^{\frac{11}{12}\pi i} & 0 & e^{\frac{5}{12}\pi i} \\ e^{-\frac{\pi}{4}i} & e^{-\frac{5}{12}\pi i} & 0 \end{pmatrix},$$

$$\mathcal{Q}(2,0) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3}-1 & 0 \\ 0 & 0 & -1-\sqrt{3} \end{pmatrix},$$

$$\mathcal{Q}(2,1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{\frac{\pi}{4}i} & e^{-\frac{11}{12}\pi i} \\ e^{-\frac{\pi}{4}i} & 0 & e^{-\frac{5}{12}\pi i} \\ e^{\frac{11}{12}\pi i} & e^{\frac{5}{12}\pi i} & 0 \end{pmatrix},$$

$$\mathcal{Q}(2,2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{\frac{5}{12}\pi i} & e^{\frac{\pi}{4}i} \\ e^{-\frac{5}{12}\pi i} & 0 & e^{-\frac{11}{12}\pi i} \\ e^{-\frac{\pi}{4}i} & e^{\frac{11}{12}\pi i} & 0 \end{pmatrix}.$$

BLOCH REPRESENTATION UNDER HEISENBERG-WELY OBSERVABLES

A state $\rho \in \mathbb{C}^N$ of single quantum system can be expressed in terms of the $N \times N$ identity operator I_N and the $N^2 - 1$ traceless Hermitian HW observable operators $\mathcal{Q}(l,m)$,

$$\begin{aligned} \rho &= \frac{1}{N}\sum_{l,m=0}^{N-1} r_{lm}\mathcal{Q}(l,m) \\ &= \frac{1}{N}(I_N + \sum_{\substack{l,m=0 \\ (l,m)\neq(0,0)}}^{N-1} r_{lm}\mathcal{Q}(l,m)) , \end{aligned} \tag{6}$$

where $\mathcal{Q}(0,0) = I_N$. The coefficients r_{lm} in Eq.(6) are given by

$$r_{lm} = Tr(\rho\mathcal{Q}(l,m)) ,$$

where $l = 0, 1, \cdots, N-1$, $m = 0, 1, \cdots, N-1$ and $(l,m) \neq (0,0)$. We denote

$$\begin{aligned} \mathbf{r} \;=\; & (r_{0,1}, r_{0,2}, \cdots, r_{0,N-1}, r_{1,0}, \cdots, r_{1,N-1}, \\ & \cdots, r_{N-1,0}, \cdots, r_{N-1,N-1}). \end{aligned} \tag{7}$$

Lemma 1 *For pure states,*

$$\| \boldsymbol{r} \|_2 = \sqrt{N-1} , \tag{8}$$

where $\| \cdot \|_2$ *is the Euclidean norm on* $\mathbb{R}^{N^2-1}$.

Proof: According to the Eq.(6), we have

$$\begin{aligned} Tr\rho^2 &= \frac{1}{N^2}\sum_{l,m,l',m'=0}^{N-1} r_{lm}r_{l'm'}Tr\{\mathcal{Q}(l,m)\mathcal{Q}(l',m')\} \\ &= \frac{1}{N^2}\cdot N \cdot \sum_{l,m=0}^{N-1} r_{lm}^2 \\ &= \frac{1}{N}(1 + \sum_{\substack{l,m=0 \\ (l,m)\neq(0,0)}}^{N-1} r_{lm}^2) = \frac{1}{N}(1 + \| \mathbf{r} \|_2^2). \end{aligned}$$

Since ρ is a pure state, one has $Tr\rho = Tr\rho^2 = 1$. Therefore $\| \mathbf{r} \|_2^2 = N - 1$. □

Now consider bipartite states $\rho \in \mathbb{C}^M \otimes \mathbb{C}^N$. Any state ρ can be similarly represented as [27]

$$\begin{aligned} \rho =& \frac{1}{MN}(I_M \otimes I_N + \sum_{(l,m)\neq(0,0)} r_{lm}\mathcal{Q}(l,m) \otimes I_N \\ &+ \sum_{(k,n)\neq(0,0)} s_{kn} I_M \otimes \mathcal{Q}(k,n) \\ &+ \sum_{(l,m),(k,n)\neq(0,0)} t_{lmkn}\mathcal{Q}(l,m) \otimes \mathcal{Q}(k,n)), \end{aligned} \tag{9}$$

In particular,

$$t_{lmkn} = Tr\{\rho\mathcal{Q}(l,m)) \otimes \widetilde{\mathcal{Q}}(k,n)\},$$

where $l, m = 0, \cdots, M-1$, $k, n = 0, \cdots, N-1$ and $(l,m), (k,n) \neq (0,0)$.

SEPARABILITY CRITERIA FOR BIPARTITE STATES

Similarly to (7), we denote $\mathbf{r} = (r_{0,1}, \cdots, r_{0,M-1}, \cdots, r_{M-1,0}, \cdots, r_{M-1,M-1})^t$ and $\mathbf{s} = (s_{0,1}, \cdots, s_{0,N-1}, \cdots, s_{N-1,0}, \cdots, s_{N-1,N-1})^t$, where t stands for transpose. Set $T = (t_{ij})$, where the entries t_{ij} are given by the coefficients

t_{lmkn}, $l,m = 0,\cdots,M-1$, $k,n = 0,\cdots,N-1$, $(l,m),(k,n) \neq (0,0)$), with the first two indices lm associated with the array index i, and the last two indices kn with the column index j of T.

Let us consider the following matrix,

$$\mathcal{S}^m_{\alpha,\beta}(\rho) = \begin{pmatrix} \alpha\beta E_{m\times m} & \beta\omega_m(\mathbf{s})^t \\ \alpha\omega_m(\mathbf{r}) & T \end{pmatrix}, \tag{10}$$

where

$$\omega_m(\mathbf{x}) = \underbrace{\begin{pmatrix} \mathbf{x} & \cdots & \mathbf{x} \end{pmatrix}}_{m\ columns},$$

α and β are nonnegative real numbers, m is a given natural number, $E_{m\times m}$ is an $m\times m$ matrix with all entries being 1. We have the following theorem,

Theorem 1 *If the state ρ in $\mathbb{C}^M \otimes \mathbb{C}^N$ is separable, then*

$$\| \mathcal{S}^m_{\alpha,\beta}(\rho) \|_{tr} \leq \sqrt{(m\beta^2 + M - 1)(m\alpha^2 + N - 1)}\ , \tag{11}$$

where $||\cdot||_{tr}$ stands for the trace norm (the sum of singular values).

Proof: Since ρ is separable, from [14] there exist vectors $\mathbf{u}_i \in \mathbb{R}^{M^2-1}$, $\mathbf{v}_i \in \mathbb{R}^{N^2-1}$ satisfying Eq.(8), and weights p_i satisfying $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ such that

$$T = \sum_i p_i \mathbf{u}_i \mathbf{v}_i^t\ ,\ \mathbf{r} = \sum_i p_i \mathbf{u}_i\ ,\ \mathbf{s} = \sum_i p_i \mathbf{v}_i.$$

From Lemma 1, we have

$$\| \mathbf{u}_i \|_2 = \sqrt{M-1}\ ,\ \| \mathbf{v}_i \|_2 = \sqrt{N-1}\ .$$

The matrix (10) has the form,

$$\begin{aligned} \mathcal{S}^m_{\alpha,\beta}(\rho) &= \sum_i p_i \begin{pmatrix} \alpha\beta E_{m\times m} & \beta\omega_m(\mathbf{v}_i)^t \\ \alpha\omega_m(\mathbf{u}_i) & \mathbf{u}_i\mathbf{v}_i^t \end{pmatrix}, \\ &= \sum_i p_i \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{u}_i \end{pmatrix} \begin{pmatrix} \alpha E_{1\times m} & \mathbf{v}_i^t \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \| \mathcal{S}^m_{\alpha,\beta}(\rho) \|_{tr} &= \| \sum_i p_i \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{u}_i \end{pmatrix} \begin{pmatrix} \alpha E_{1\times m} & \mathbf{v}_i^t \end{pmatrix} \|_{tr} \\ &\leq \sum_i p_i \| \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{u}_i \end{pmatrix} \begin{pmatrix} \alpha E_{1\times m} & \mathbf{v}_i^t \end{pmatrix} \|_{tr}\ . \end{aligned}$$

Accounting to that for any vectors $|i\rangle = (i_1, i_2, \cdots, i_m)^t$ and $|j\rangle = (j_1, j_2, \cdots, j_n)^t$, one has

$$\begin{aligned} \| |i\rangle\langle j| \|_{tr} &= Tr\sqrt{(|i\rangle\langle j|)^\dagger |i\rangle\langle j|} \\ &= Tr\sqrt{|j\rangle(\langle i||i\rangle)\langle j|} \\ &= \sqrt{(i_1^2 + i_2^2 + \cdots + i_m^2)(j_1^2 + j_2^2 + \cdots + j_n^2)} \\ &= \| |i\rangle \|_2 \| |j\rangle \|_2\ , \end{aligned} \tag{12}$$

we have

$$\begin{aligned} &\sum_i p_i \| \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{u}_i \end{pmatrix} \begin{pmatrix} \alpha E_{1\times m} & \mathbf{v}_i^t \end{pmatrix} \|_{tr} \\ &= \| \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{u}_i \end{pmatrix} \|_2 \| \begin{pmatrix} \alpha E_{m\times 1} \\ \mathbf{v}_i \end{pmatrix} \|_2 \\ &= \sqrt{(m\beta^2 + M - 1)(m\alpha^2 + N - 1)}\ , \end{aligned}$$

which gives rise to (11). □

Remark Theorem 1 implies that a pure bipartite quantum state in Bloch representation Eq. (9) is separable if and only if

$$\begin{aligned} \mathcal{S}^m_{\alpha,\beta}(\rho) &= \begin{pmatrix} \beta E_{m\times 1} \\ \mathbf{r} \end{pmatrix} \begin{pmatrix} \alpha E_{1\times m} & \mathbf{s}^t \end{pmatrix} \\ &= \begin{pmatrix} \alpha\beta E_{m\times m} & \beta\omega_m(\mathbf{s})^t \\ \alpha\omega_m(\mathbf{r}) & \mathbf{r}\mathbf{s}^t \end{pmatrix}. \end{aligned}$$

Note that Eq.(9) can be rewritten as

$$\rho = \rho_A \otimes \rho_B + \frac{1}{MN}[(t_{lmkn} - r_{lm}s_{kn})\mathcal{Q}(l,m) \otimes \mathcal{Q}(k,n)],$$

where ρ_A and ρ_B are the reduced density matrices. Since $\mathcal{Q}(l,m) \otimes \widetilde{\mathcal{Q}}(k,n)$ are linearly independent, $(t_{lmkn} - r_{lm}s_{kn})\mathcal{Q}(l,m) \otimes \widetilde{\mathcal{Q}}(k,n) = 0$ if and only if $t_{lmkn} - r_{lm}s_{kn} = 0$, i.e. $T = \mathbf{r}\mathbf{s}^t$, for any l,m,k,n. Moreover, for $N = M = 2$, the HW observable basis is equivalent to Pauli matrices. In this case Theorem 1 is equivalent to the Theorem 1 in [20].

For high dimensional quantum states, let us consider the following 2×4 bound entangled state [28] as an example,

$$\rho = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+b) & 0 & 0 & \frac{1}{2}\sqrt{1-b^2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{1}{2}\sqrt{1-b^2} & 0 & 0 & \frac{1}{2}(1+b) \end{pmatrix}.$$

where $0 < b < 1$. We mix the above state with state $|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\rho_x = x|\xi\rangle\langle\xi| + (1-x)\rho.$$

By choosing

$$\alpha = \sqrt{\frac{1}{d_1 - 1}},\quad \beta = \sqrt{\frac{1}{d_2 - 1}},\quad m = 1,\quad b = 0.9,$$

our Theorem 1 can detect the entanglement in ρ_x for $0.22349 \leq x \leq 1$, while the Theorem 1 in [20], the V-B criterion [14] and the L-B criterion [15] can only detect the entanglement in ρ_x for $0.2283 \leq x \leq 1$, $0.2293 \leq$

$x \le 1$ and $0.2841 \le x \le 1$, respectively. In this case, our criterion is better in detecting entanglement.

Here, instead of (4), if we define $\mathcal{Q}(l,m) = k(\mathcal{X}\mathcal{D}(l,m) + \mathcal{X}^*\mathcal{D}^\dagger(l,m))$, $k = \sqrt{2/N}$, then $\| \mathbf{r} \|_2 = \sqrt{\frac{N(N-1)}{2}}$, and the conclusion becomes

$$\| \mathcal{S}^m_{\alpha,\beta}(\rho) \|_{tr} \le \frac{1}{2}\sqrt{(2m\beta^2 + M^2 - M)(2m\alpha^2 + N^2 - N)}\,.$$

In this case, the least upper bound in Theorem 1 is equal to the Theorem 1 [20].

SEPARABILITY CRITERIA FOR MULTIPARTITE STATES

We now generalize our result in Theorem 1 to multipartite case. Let $\mathcal{S}$ be an $f_1 \times \cdots \times f_N$ tensor, A and $\bar{A}$ be two nonempty subsets of $\{1,\ldots,N\}$ satisfying $A \cup \bar{A} = 1,\ldots,N$. Let $\mathcal{S}^{A|\bar{A}}$ denote the A, $\bar{A}$ matricization of $\mathcal{S}$, see [20, 29] for detail.

For any state ρ in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N}$, we import a natural number m and nonnegative real parameters $\alpha_1,\ldots\alpha_N$, and define

$$\delta^{(d_i)}_{(k_i,n_i)} = \begin{cases} \alpha_i I_{d_i}, & 1 \le k_i = n_i \le m \\ Q^{(d_i)}(k_i - m, n_i - m), & \begin{array}{l} m \le k_i, \\ n_i \le d_i + m - 1, \\ (k_i,n_i) \ne (m,m), \end{array} \end{cases}$$

where $i = 1,\ldots,N$. Denote $\mathcal{W}^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\rho)$ the tensor given by elements of the following form,

$$w_{(k_1,n_1)\ldots(k_N,n_N)} = Tr(\rho\delta^{(d_1)}_{(k_1,n_1)} \otimes \cdots \otimes \delta^{(d_N)}_{(k_N,n_N)}),$$

where $1 \le k_i = n_i \le m$, $m \le k_i, n_i \le d_i + m - 1$. Below we give the full separability criterion based on $\mathcal{W}^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\rho)$.

Theorem 2 *If a state ρ in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N}$ is fully separable, then for any subset A of $\{1,\ldots,N\}$, we have*

$$\| (\mathcal{W}^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\rho))^{A|\bar{A}} \|_{tr} \le \prod_{k=1}^{N} \sqrt{(m\alpha_k^2 + d_k - 1)}. \tag{13}$$

Proof: Without loss of generality, we assume

$$\begin{aligned} A &= \{q_1,\ldots,q_M\}, \qquad q_1 < \cdots < q_M, \\ \bar{A} &= \{q_{M+1},\ldots,q_N\}, \qquad q_M + 1 < \cdots < q_N. \end{aligned}$$

Since ρ is fully separable, from [30] there exist vectors $\mathbf{u}^{(k)}_i \in \mathbb{R}^{d_k^2-1}$ such that

$$\mathcal{W}^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\rho) = \sum_i p_i \begin{pmatrix} \alpha_1 E_{m\times 1} \\ \mathbf{u}^{(1)}_i \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \alpha_N E_{m\times 1} \\ \mathbf{u}^{(N)}_i \end{pmatrix},$$

where $\| \mathbf{u}^{(k)}_i \|_2 = \sqrt{d_k - 1}$. Thus

$$\begin{aligned}
&\| (\mathcal{W}^{(m)}_{\alpha_1,\alpha_2,\ldots,\alpha_N}(\rho))^{A|\bar{A}} \|_{tr} \\
&= \| \sum_i p_i \overset{M}{\underset{l=1}{\otimes}} \begin{pmatrix} \alpha_{ql} E_{m\times 1} \\ \mathbf{u}^{(q_l)}_i \end{pmatrix} \overset{N}{\underset{p=M+1}{\otimes}} \begin{pmatrix} \alpha_{q_p} E_{m\times 1} \\ \mathbf{u}^{(q_p)}_i \end{pmatrix}^t \|_{tr} \\
&\le \sum_i p_i \| \overset{M}{\underset{l=1}{\otimes}} \begin{pmatrix} \alpha_{q_l} E_{m\times 1} \\ \mathbf{u}^{(q_l)}_i \end{pmatrix} \overset{N}{\underset{p=M+1}{\otimes}} \begin{pmatrix} \alpha_{q_p} E_{m\times 1} \\ \mathbf{u}^{(q_p)}_i \end{pmatrix}^t \|_{tr} \\
&= \sum_i p_i \| \overset{M}{\underset{l=1}{\otimes}} \begin{pmatrix} \alpha_{q_l} E_{m\times 1} \\ \mathbf{u}^{(q_l)}_i \end{pmatrix} \|_2 \| \overset{N}{\underset{p=M+1}{\otimes}} \begin{pmatrix} \alpha_{q_p} E_{m\times 1} \\ \mathbf{u}^{(q_p)}_i \end{pmatrix} \|_2 \\
&= \sum_i p_i \sqrt{tr(\overset{M}{\underset{l=1}{\otimes}}(\begin{pmatrix} \alpha_{q_l} E_{m\times 1} \\ \mathbf{u}^{(q_l)}_i \end{pmatrix}^\dagger \begin{pmatrix} \alpha_{q_l} E_{m\times 1} \\ \mathbf{u}^{(q_l)}_i \end{pmatrix}))} \\
&\quad \cdot \sqrt{tr(\overset{N}{\underset{p=M+1}{\otimes}}(\begin{pmatrix} \alpha_{q_p} E_{m\times 1} \\ \mathbf{u}^{(q_p)}_i \end{pmatrix}^\dagger \begin{pmatrix} \alpha_{q_p} E_{m\times 1} \\ \mathbf{u}^{(q_p)}_i \end{pmatrix}))} \\
&= \sum_i p_i \sqrt{\sum_{k=1}^{N} tr(\begin{pmatrix} \alpha_k E_{m\times 1} \\ \mathbf{u}^{(k)}_i \end{pmatrix}^\dagger \begin{pmatrix} \alpha_k E_{m\times 1} \\ \mathbf{u}^{(k)}_i \end{pmatrix})} \\
&= \prod_{k=1}^{N} \sqrt{(m\alpha_k^2 + d_k - 1)},
\end{aligned}$$

where we have used the equality Eq.(12) and $tr(A \otimes B) = trA \cdot trB$.

CONCLUSION

We have studied the separability problem based on the Bloch representation of density matrices in terms of the Heisenberg-Weyl observable basis. New separability criteria have been derived for both bipartite and multipartite quantum systems, which provide more efficient ways in detecting quantum entanglement for certain kinds of quantum states. These criteria can experimentally implemented.

In [20] the traceless Hermitian generators of $SU(d)$ satisfying the orthogonality relations have been used in the Bloch representation of density matrices. While in this paper we have adopted the same approach as the one used in [20] but used the Heisenberg-Weyl observable basis in the Bloch representation of density matrices. An interesting fact here is that the ability of detecting quantum entanglement can be improved by using different observable basis in the Bloch representation. Hence our results are complementary to the ones obtained in [20] in detecting entanglement for some quantum states. Just like the case that one needs different witnesses to detect the entanglement of different quantum states, we need to measure the quantum systems with suitable local observable sets in entanglement detection. The choices of suitable observable basis depend on the detailed entangled states to be detected. It would be more interesting if such state-dependent choices of observable basis can be analytically derived optimally. It is also possible to improve such separability criteria by taking into account measurement outcomes of more observable bases simultaneously, similar to the cases that involve all the mutually unbiased bases [31, 32], or mutually unbiased measurements [33], or general symmetric informationally complete positive operator-valued measurements [34].

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