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Approximations of Qubit States under
 B_2 Distance

by

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Complete Optimal Convex Approximations of Qubit States under B_2 Distance

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We consider the optimal approximation of arbitrary qubit states with respect to an available states consisting the eigenstates of two of three Pauli matrices, the B_2 -distance of an arbitrary target state. Both the analytical formulae of the B_2 -distance and the corresponding complete optimal decompositions are obtained. The tradeoff relations for both the sum and the squared sum of the B_2 -distances have been analytically and numerically investigated.

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I. INTRODUCTION

Quantifying correlations among multipartite systems is one of the most important problems in quantum theory. However, most correlation measures become notorious difficult to calculate with the increasing partite and dimension. An alternative way to deal with the problem is to consider the distance of a given state to the so called free states in resource theory. For example, entanglement is considered as the minimal distance of a given state to the set of separable states in quantum systems [? ? ? ?]. The quantum discord is regarded as the minimal distance of a given state to classically correlated states [?]. And quantum coherence can be quantified by the optimal convex approximation of the given state to the reference orthogonal base [?].

While convexity is a very important property in mathematics and has been studied for long time, several related recent developments in quantum information have stimulated new interest in this topic [? ?]. The problem of optimal approximation to an unavailable quantum channel or state by the available channels or states was considered in [? ?] recently. It was shown that the optimally approximated distance has a natural operational interpretation. It can quantify the least distinguishable channel (state) from the given convex set to the target channel (state). The trace distance measure of coherence can be regarded as convex approximation to the target state with respect to a fixed base of the system, where the fixed base can be either orthogonal or nonorthogonal [? ? ? ? ?]. In Ref. [?], the author considered the the B_3 -distance, the distance from a target qubit state to the convex approximation of bases containing the eigenstates of all Pauli matrices. The optimal convex approximation on the B_3 -distance has been obtained.

In this work, we focus on B_2 -distance, the distance corresponding to the convex approximation of bases containing the eigenstates from one of the pairs of Pauli matrices. We investigate all the optimal convex decompositions for the desired quantum state. The paper is organized as follows. In ??, we calculate the B_2 -distance in eight different cases, with the parameter regions achieving each optimal approximation explicitly given. In ??, we study tradeoff relations for both the sum and the square sum of the B_2 -distance.

II. THE PAULI B_2 -DISTANCE OF QUBIT STATE

For an equal *priori* probability of two given quantum states ρ and ρ_0 , the optimal discrimination between them can be quantified by the following probability $p_{discr}(\rho, \rho_0)$,

$$p_{discr}(\rho, \rho_0) = 1/2 + 1/4 \|\rho - \rho_0\|_1,$$

where $\|\rho\|_1$ denotes the trace norm of ρ , $\|\rho\|_1 = Tr \sqrt{\rho^\dagger \rho} = \sum_i \sqrt{r_i}$, r_i are the eigenvalues of $\rho^\dagger \rho$.

The optimal convex approximation of the quantum state ρ with respect to a given set ρ_i is quantified by $D_{\{\rho_i\}}(\rho) = \min_{\{p_i\}} \{\|\rho - \sum_i p_i \rho_i\|_1\}$, and the best approximated points are the set $S(\rho^{opt}) = \{\rho^{opt} | D_{\{\rho_i\}}(\rho) = \|\rho - \rho^{opt}\|_1\}$.

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This optimal convex approximation provides the worst probability of discriminating the desired state ρ from any of the available states $\sum_i p_i \rho_i$. For any other figure of merit that quantifies the distance between quantum states, the optimal convex approximation can be similarly defined (e.g., by a decreasing function of the fidelity). We remind that the best approximation can be arrived at many points and $S(\rho^{opt})$ represents the set of all the optimal points achieving the minimum distance.

Let $|0\rangle$ and $|1\rangle$, $|2\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|3\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and $|4\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + \sqrt{-1}|1\rangle)$ and $|5\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - \sqrt{-1}|1\rangle)$ be the eigenstates of the Pauli matrices σ_z , σ_x , and σ_y , respectively. We consider the following available set of states,

$$\begin{aligned} \mathbf{B}'_2 &= \left\{ |0\rangle, |1\rangle, |2\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |3\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}, \\ \mathbf{B}''_2 &= \left\{ |0\rangle, |1\rangle, |4\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + \sqrt{-1}|1\rangle), |5\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - \sqrt{-1}|1\rangle) \right\}, \\ \mathbf{B}'''_2 &= \left\{ |2\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |3\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), |4\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + \sqrt{-1}|1\rangle), |5\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - \sqrt{-1}|1\rangle) \right\}, \end{aligned} \quad (1)$$

where \mathbf{B}'_2 contains the eigenstates of σ_x, σ_z , \mathbf{B}''_2 the eigenstates of σ_y, σ_z and \mathbf{B}'''_2 the eigenstates of σ_x, σ_y . The target qubit state ρ can be parameterized by

$$\rho = \begin{pmatrix} 1-a & k\sqrt{a(1-a)}e^{-\sqrt{-1}\phi} \\ k\sqrt{a(1-a)}e^{\sqrt{-1}\phi} & a \end{pmatrix}, \quad (2)$$

with $a \in [0, 1]$, $\phi \in [0, 2\pi]$, and $k \in [0, 1]$ [?]. Since the \mathbf{B}_2 -distance is invariant under the state transformations $\rho(a, k, \phi) \rightarrow \rho(1-a, k, \phi)$ and $\rho(a, k, n\pi/2 \pm \phi) \rightarrow \rho(a, k, \phi)$ (with integer n), we can restrict our study on the case $a \in [0, 1/2]$ and $\phi \in [0, \pi/2]$.

For any given target quantum state ρ and available basis set in Eq.(??), we reduce the optimal approximation problem to find the minimum $D_{B_2}(\rho) = \min_{\{p_i\}} \{\|\rho - \sum p_i |e_i\rangle\langle e_i|\|_1\}$ with respect to the probabilities $\{p_i\}$, where $|e_i\rangle$ represent the states of \mathbf{B}'_2 , \mathbf{B}''_2 or \mathbf{B}'''_2 in Eq.(??). Denote $\rho_i = |i\rangle\langle i|$. The original problem is reduced to the optimal approximation problem of finding the minimum distance

$$D_{B_2}(\rho) = \min_{\{p_i\}} \{2\sqrt{|Det(\rho - \sum_i p_i \rho_i)|}\}$$

such that $p_i \geq 0$, $\sum_j p_j = 1$.

We first consider the \mathbf{B}'_2 distance, i.e., $D_{B'_2}(\rho) = \min_{\{p_i\}} \{2\sqrt{|Det(\rho - \sum_{i=0}^3 p_i \rho_i)|}\}$. Set

$$f(p_0, p_1, p_2, p_3) = |Det(\rho - \sum_{i=0}^3 p_i \rho_i)| - \sum_{i=0}^3 \lambda_i p_i - \lambda \sum_{i=0}^3 p_i.$$

Since the constraint inequality condition sets $p_i \geq 0$ is convex and the equality constraint $\sum_j p_j = 1$ is linear, by the Karush-Kuhn-Tucker Theorem[?], the following KKT condition must be satisfied while solving the above optimization problem.

$$\frac{\partial f}{\partial p_i} = 0, \quad \lambda_i p_i = 0, \quad \lambda_i \geq 0, \quad p_i \geq 0, \quad \sum_{j=0}^3 p_j = 1, \quad i = 0, 1, 2, 3, \quad (3)$$

Eq.(??) reduces to the following equations,

$$\begin{aligned} p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_3 + \lambda_0 + \lambda - a &= 0, \\ p_0 + \frac{1}{2}p_1 + \frac{1}{2}p_3 + \lambda_1 + \lambda - 1 + a &= 0, \\ \frac{1}{2}p_0 + \frac{1}{2}p_1 + p_3 + \lambda_2 + \lambda + k\sqrt{a(1-a)}\cos\phi - \frac{1}{2} &= 0, \\ \frac{1}{2}p_0 + \frac{1}{2}p_1 + p_2 + \lambda_3 + \lambda - k\sqrt{a(1-a)}\cos\phi - \frac{1}{2} &= 0, \\ \lambda_i p_i = 0, \quad \lambda_i \geq 0, \quad p_i \geq 0, \quad i = 0, 1, 2, 3, \\ \sum_i p_i &= 1. \end{aligned}$$

Solving the above equations, we can obtain the complete analytical solutions to the optimal convex approximation ρ^{opt} of ρ . The $S(\rho^{opt})$ of ρ with respect to \mathcal{B}'_2 is given as

i) For $a \geq k\sqrt{a(1-a)}\cos\phi$, we have

$$D_{\mathcal{B}'_2}(\rho) = 2k\sqrt{a(1-a)}\sin\phi = \langle\sigma_y\rangle, \quad (4)$$

which is attained at

$$\begin{aligned} p_0 &= 1 - a - k\sqrt{a(1-a)}\cos\phi - t, \\ p_1 &= a - k\sqrt{a(1-a)}\cos\phi - t, \\ p_2 &= 2k\sqrt{a(1-a)}\cos\phi + t, \\ p_3 &= t, \end{aligned} \quad (5)$$

where t satisfies $a - k\sqrt{a(1-a)}\cos\phi \geq t \geq 0$. Let $A'_1 = \{\sum_{i=0}^3 p_i \rho_i\}$ denote the set of states with p_i given by Eq.(??). Then A'_1 contains all the optimal points achieving the distance $D_{\mathcal{B}'_2}(\rho)$ in Eq.(??).

ii) For $a < k\sqrt{a(1-a)}\cos\phi$, we have the the optimal convex approximation distance

$$\begin{aligned} D_{\mathcal{B}'_2}(\rho) &= \sqrt{2(1 + \sin^2\phi)k^2(a(1-a)) - 4a\cos\phi k\sqrt{a(1-a)} + 2a^2} \\ &= \sqrt{\langle\sigma_y\rangle^2 + \frac{1}{2}(\langle\sigma_x\rangle + \langle\sigma_z\rangle - 1)^2}, \end{aligned} \quad (6)$$

which is attained with

$$\begin{aligned} p_0 &= 1 - a - k\sqrt{a(1-a)}\cos\phi, \\ p_2 &= a + k\sqrt{a(1-a)}\cos\phi, \\ p_1 &= p_3 = 0. \end{aligned} \quad (7)$$

Denote $A'_2 = \{p_0\rho_0 + p_2\rho_2\}$, with p_0, p_2 given by Eq.(??). Then A'_2 contains all the optimal states achieving the distance $D_{\mathcal{B}'_2}(\rho)$ in Eq.(??). Therefore $S(\rho^{opt})$ is given by $S(\rho^{opt}) = A'_1 \cup A'_2$, which is the set of optimal states that gives rise to the optimal convex approximations.

Next we consider the optimal convex approximation of ρ with respect to \mathcal{B}''_2 . Namely, $D_{\mathcal{B}''_2}(\rho) = \min_{\{p_i\}}\{2\sqrt{|\text{Det}(\rho - \sum_{i=0,1,4,5} p_i \rho_i)|}\}$. Similar to the case of \mathcal{B}'_2 , we have

i) For $a \geq k\sqrt{a(1-a)}\sin\phi$, the optimal convex approximated distance is given by

$$D_{\mathcal{B}''_2}(\rho) = 2k\sqrt{a(1-a)}\cos\phi = \langle\sigma_x\rangle. \quad (8)$$

The with the optimal probability weights are given by

$$\begin{aligned} p_0 &= 1 - a - k\sqrt{a(1-a)}\sin\phi - t, \\ p_1 &= a - k\sqrt{a(1-a)}\sin\phi - t, \\ p_4 &= 2k\sqrt{a(1-a)}\sin\phi + t, \\ p_5 &= t, \end{aligned} \quad (9)$$

where t satisfies $a - k\sqrt{a(1-a)}\sin\phi \geq t \geq 0$. Denote $A''_1 = \{\sum_{i=0}^3 p_i \rho_i\}$ with p_i given by Eq.(??). Then A''_1 contains all the optimal states achieving the distance $D_{\mathcal{B}''_2}(\rho)$ in Eq.(??).

ii) For $a < k\sqrt{a(1-a)}\sin\phi$, we have the optimal convex approximated distance

$$\begin{aligned} D_{\mathcal{B}''_2}(\rho) &= \sqrt{2(1 + \cos^2\phi)k^2(a(1-a)) - 4a\sin\phi k\sqrt{a(1-a)} + 2a^2} \\ &= \sqrt{\langle\sigma_x\rangle^2 + \frac{1}{2}(\langle\sigma_y\rangle + \langle\sigma_z\rangle - 1)^2}, \end{aligned} \quad (10)$$

with the optimal probability weights given by

$$\begin{aligned} p_0 &= 1 - a - k\sqrt{a(1-a)}\sin\phi, \\ p_4 &= a + k\sqrt{a(1-a)}\sin\phi, \\ p_1 &= p_5 = 0. \end{aligned} \quad (11)$$

Let $A_2'' = \{p_0\rho_0 + p_4\rho_4\}$ be the set of states with p_0 and p_4 given by Eq.(??). Then $S(\rho^{opt})$ is given by $S(\rho^{opt}) = A_1'' \cup A_2''$.

For the optimal approximation of ρ with respect to the basis in B_2''' , we have

i) For $1/2 \geq k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$, the optimal convex approximated distance has the form

$$D_{B_2'''}(\rho) = (1 - 2a) = \langle\sigma_z\rangle, \quad (12)$$

with the optimal probability weights

$$\begin{aligned} p_2 &= 1/2 + k\sqrt{a(1-a)}(\cos\phi - \sin\phi) - t, \\ p_3 &= 1/2 - k\sqrt{a(1-a)}(\cos\phi + \sin\phi) - t, \\ p_4 &= 2k\sqrt{a(1-a)}\sin\phi + t, \\ p_5 &= t, \end{aligned} \quad (13)$$

where t is given by $1/2 > k\sqrt{a(1-a)}(\sin\phi + \cos\phi) \geq t \geq 0$. Hence $A_1''' = \{\sum_{i=0}^3 p_i\rho_i\}$, with p_i given by Eq.(??), contains all the optimal states achieving the distance $D_{B_2'''}(\rho)$ in Eq.(??).

ii) For $\frac{1}{2} < k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$, we have

$$\begin{aligned} D_{B_2'''}(\rho) &= \sqrt{(1 - 2a)^2 + 2(k\sqrt{a(1-a)}(\cos\phi + \sin\phi) - 1/2)^2} \\ &= \sqrt{\langle\sigma_z\rangle^2 + \frac{1}{2}(\langle\sigma_y\rangle + \langle\sigma_x\rangle - 1)^2}, \end{aligned} \quad (14)$$

with the optimal probability weights given by

$$\begin{aligned} p_2 &= 1/2 + k\sqrt{a(1-a)}(\cos\phi - \sin\phi), \\ p_4 &= 1/2 - k\sqrt{a(1-a)}(\cos\phi - \sin\phi), \\ p_3 &= p_5 = 0. \end{aligned} \quad (15)$$

Denoting $A_2''' = \{p_2\rho_2 + p_4\rho_4\}$, with p_2 and p_4 given by Eq.(??), we have $S(\rho^{opt}) = A_1''' \cup A_2'''$, which is the set of states achieving all the optimal convex approximations.

In Fig. ??, we plot the distance $D_{B_2'}(\rho)$ for fixed parameters of k and ϕ . One can see that for the fixed value $\phi = \frac{\pi}{4}$, Fig.??(a) shows that the optimal distance $D_{B_2'}(\rho)$ increases with k and decreases with the parameter a . Fig.??(c) shows the interface such that the region above the surface corresponds to the case *i)*, namely, $a \geq k\sqrt{a(1-a)}\cos\phi$; and the region below the surface is the case *ii)*, $a < k\sqrt{a(1-a)}\cos\phi$. In Fig. ?? and ??, the distances $D_{B_2'}(\rho)$ and $D_{B_2'''}(\rho)$ with the fixed values are also plotted, respectively. The corresponding interface is plotted in Fig.??(c) (Fig.??(c)): the region above the surface corresponds to the case $a \geq k\sqrt{a(1-a)}\sin\phi$ ($1/2 \geq k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$), the region below the surface is the case $a < k\sqrt{a(1-a)}\sin\phi$ ($1/2 < k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$), respectively.

III. TRADEOFF RELATIONS AMONG THE B_2 -DISTANCES

We have calculated the optimal distances $D_{B_2'}(\rho)$, $D_{B_2''}(\rho)$ and $D_{B_2'''}(\rho)$, with explicit formulae for arbitrary qubit-state ρ classified in two parameter regions each. Interestingly, we find that the sum and the squared sum of $D_{B_2'}(\rho)$, $D_{B_2''}(\rho)$ and $D_{B_2'''}(\rho)$ display some tradeoff relations in each parameter region.

Let ① represents the parameter region of the state ρ with $a \geq k\sqrt{a(1-a)}\cos\phi$, and ② the parameter region $a < k\sqrt{a(1-a)}\cos\phi$. Similarly, ③ (④) represents the parameter region with $a \geq k\sqrt{a(1-a)}\sin\phi$ ($a <$

(a) $\phi = \pi/4$ (b) $k = 4/5$ (c) Area of the interface

FIG. 1: Optimal convex approximation of a qubit mixed state ρ w.r.t. the set B_2' spanned by the eigenstates of the Pauli matrices σ_z and σ_x . The distance $D_{B_2'}(\rho)$ is plotted vs the target state parameters a , k and ϕ , for fixed value of the parameter $\phi = \frac{\pi}{4}$ [FIG. 1.(a)], for fixed value of the parameter $k = 4/5$ [FIG. 1.(b)]. The interface of the regions of the two cases i) and ii) is plotted in FIG. 1.(c), the region above the surface corresponds to the case $a \geq k\sqrt{a(1-a)}\cos\phi$, the region below the surface is the case $a < k\sqrt{a(1-a)}\cos\phi$.

(a) $\phi = \pi/4$ (b) $k = 4/5$ (c) Area of the interface

FIG. 2: Optimal convex approximation of a qubit mixed state ρ w.r.t. the set B_2'' spanned by the eigenstates of the Pauli matrices σ_z and σ_y . The distance $D_{B_2''}(\rho)$ is plotted for fixed value of the parameter $\phi = \frac{\pi}{4}$ [FIG.2.(a)], and for fixed value of the parameter $k = 4/5$ [FIG. 2.(b)]. The interface of the regions of the two cases i) and ii) is plotted in FIG. 2.(c), the region above the surface corresponds to the case $a \geq k\sqrt{a(1-a)}\sin\phi$, the region below the surface is the case $a < k\sqrt{a(1-a)}\sin\phi$.

$k\sqrt{a(1-a)}\sin\phi$), and ⑤ (⑥) represents the parameter region with $\frac{1}{2} \geq k\sqrt{a(1-a)}\cos\phi$ ($\frac{1}{2} < k\sqrt{a(1-a)}\cos\phi$). For every $D_{B_2'}(\rho)$ ($D_{B_2''}(\rho)$, $D_{B_2'''}(\rho)$) there are two parameter regions: i) and ii). A state ρ may belong to the region ① in calculating the distance $D_{B_2'}(\rho)$, but to region ④ in calculating $D_{B_2''}(\rho)$, and to region ⑤ in calculating $D_{B_2'''}(\rho)$. Therefore we have all together eight cases of parameter regions

region 1	①③④⑤	region 3	①③④⑥	region 5	②③④⑤	region 7	①④⑤⑥
region 2	②③④⑤	region 4	①④⑤⑥	region 6	②③④⑥	region 8	②③④⑥

For each case, the three B_2 -distances display different values. In Fig.??, we plot the minimum $\min D_{B_2}(\rho) = \min\{D_{B_2'}(\rho), D_{B_2''}(\rho), D_{B_2'''}(\rho)\}$. One can see from Fig.?? that, for fixed $\phi = \pi/4$ and $k = 4/5$, $\min D_{B_2}(\rho)$ is always nonzero for nonzero parameters a and k or ϕ , namely, all the three Pauli B_2 -distances are nonzero.

Concerning the tradeoff relations of the three Pauli B_2 -distances, for convenience, we denote

$$D_{B_2}(\rho) = D_{B_2'}(\rho) + D_{B_2''}(\rho) + D_{B_2'''}(\rho), \quad (16)$$

$$D_{B_2}(\rho)^2 = D_{B_2'}(\rho)^2 + D_{B_2''}(\rho)^2 + D_{B_2'''}(\rho)^2. \quad (17)$$

By the numerical calculation, we obtain the tradeoff relation among $D_{B_2'}(\rho)$, $D_{B_2''}(\rho)$ and $D_{B_2'''}(\rho)$, see the following

(a) $\phi = \pi/4$ (b) $k = 4/5$ (c) Area of the interface

FIG. 3: Optimal convex approximation of a qubit mixed state ρ w.r.t. the set B_2''' spanned by the eigenstates of the Pauli matrices σ_x and σ_y . The distance $D_{B_2'''}(\rho)$ is plotted for fixed value of the parameter $\phi = \frac{\pi}{4}$ [FIG.3.(a)] and for fixed value of the parameter $k = 4/5$ [FIG. 3.(b)]. The interface of the regions of the two cases i) and ii) is plotted in FIG. 3.(c), the region above the surface corresponds to the case $1/2 \geq k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$, the region below the surface is the case $1/2 < k\sqrt{a(1-a)}(\sin\phi + \cos\phi)$.

(a) $\phi = \pi/4, \min D(B_2(\rho))$ (b) $k = 4/5, \min D(B_2(\rho))$

FIG. 4: Illustrations of the minimum Pauli B_2 -distance, $\min D(B_2(\rho)) = \min\{D(B_2'(\rho)), D(B_2''(\rho)), D(B_2'''(\rho))\}$

table:

region	$D_{B_2}(\rho)$	$D_{B_2}(\rho)^2$	region	$D_{B_2}(\rho)$	$D_{B_2}(\rho)^2$
1	[0,1.742)	[0.1]	5	[1,1.742)	(0.501,1.086)
2	(1.006,1.750)	(0.666,1.068)	6	(1,1.742)	(0.666,1.068)
3	[1,1.742)	(0.501,1.086)	7	(1,1.742)	(0.666,1.068)
4	[1,1.742)	(0.501,1.086)	8	(1.5,1.765)	(0.750,1.060)

Fig.?? shows all the parameter regions of a, k, ϕ such that the three B_2 -distances are achieved. These regions completely characterize all the optimal convex approximations of a state ρ w.r.t. B_2 -distance.

It has been shown that, for a given state, the three optimal distances to the bases in B_2', B_2'' and B_2''' satisfy a kind of tradeoff relations. In fact, the bounds on $D_{B_2}(\rho) = D_{B_2'}(\rho) + D_{B_2''}(\rho) + D_{B_2'''}(\rho)$ or $D_{B_2}(\rho)^2 = D_{B_2'}(\rho)^2 + D_{B_2''}(\rho)^2 + D_{B_2'''}(\rho)^2$ are tightly related to the quantum uncertainty relations satisfied by the three Pauli operators, since both the distances and the standard deviations of the Pauli operators are given by the mean values of the the Pauli operators. From $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 4k^2a(1-a)\sin^2\varphi + 4k^2a(1-a)\cos^2\varphi + (1-2a)^2 \leq 1$, one gets $(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq 1/2$, where ΔS_x ($\Delta S_y, \Delta S_z$) is the standard deviation and $\Delta S_x = \frac{\sqrt{1-\langle\sigma_x\rangle^2}}{2}$. On the other hand $|\langle\frac{\sigma_x}{2}\rangle| + |\langle\frac{\sigma_y}{2}\rangle| + |\langle\frac{\sigma_z}{2}\rangle|$

(a)Over view (b)upward view (c)rolling-over view

FIG. 5: Complete classification on the state parameter regions.

$= \frac{2\sqrt{2}a(1-a)+1-2a}{2} \leq \frac{3\sqrt{2}}{8}$. Therefore, we have

$$(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq \frac{\tau}{2} (|\langle \frac{\sigma_x}{2} \rangle| + |\langle \frac{\sigma_y}{2} \rangle| + |\langle \frac{\sigma_z}{2} \rangle|),$$

where $\tau = \frac{2}{\sqrt{3}}$ is the triple constant given in the uncertain relations in [? ?]. From formulae (4), (8) and (12), we immediately get that in region 1, our Pauli B_2 -distances is in accordance with the uncertainty relation.

IV. CONCLUSION

In summary, we have shown that a qubit mixed state ρ can be approximated by a number of effectively available pure states spanned by the eigenstates of the Pauli matrices. It is well known that correlation limits the extractable information [? ? ?], where one does want to minimize the probability of discrimination. The advantage of our results is that we presented the complete set of optimal decompositions of a given state. In [?] for a given state, only one particular optimal decomposition has been elegantly derived, in which p_3 and p_5 are chosen to be zero. Hence, basically it is the minimal distance with respect to four of six eigenvectors of the Pauli matrices. As a simple example, consider the following mixed qubit state, $\rho = \begin{pmatrix} 1/2 & 1/5 \\ 1/5 & 1/2 \end{pmatrix}$. One can verify that $D_{B_2'}(\rho) = 0$. All the optimal convex approximation points with respect to the basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ are given by Eq. (??). If we choose $t = 0$, then we obtain $\rho = 0.3\rho_0 + 0.3\rho_1 + 0.4\rho_3$. Moreover, we also have $D_{B_2''}(\rho) = 0$. The optimal convex approximation points with respect to the basis $\{|2\rangle, |3\rangle, |4\rangle, |5\rangle\}$ are given by Eq. (??). For $t = 0.3$, one obtains the optimal decomposition, $\rho = 0.7\rho_2 + 0.3\rho_3$. In [?], only one optimal decomposition $\rho = 0.3\rho_0 + 0.3\rho_1 + 0.4\rho_3$ is obtained. Other optimal decompositions like $\rho = 0.7\rho_2 + 0.3\rho_3$ can not be obtained even considering the optimal convex approximation with respect to the full bases $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$. Therefore, the research o the best convex approximation about B_2 -distance may provide an alternative way to analyze the optimal convex approximation about B_3 -distance. Our approach may be also used to study other kinds of optimal decompositions associated with other bases.

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