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Preprint no.: 95 2018

FLABBY AND INJECTIVE OBJECTS IN TOPOSES

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Abstract. We introduce a general notion of *flabby objects* in elementary toposes and study their basic properties. In the special case of localic toposes, this notion reduces to the common notion of flabby sheaves, yielding a site-independent characterization of flabby sheaves. Continuing a line of research started by Roswitha Harting, we use flabby objects to show that an internal notion of injective objects coincides with the corresponding external notion, in stark contrast with the situation for projective objects. We show as an application that higher direct images can be understood as internal cohomology, and we study flabby objects in the effective topos.

As is nowadays well-established, any topos supports an *internal language* which can be used to reason about the objects and morphisms of the topos in a naive element-based language, allowing us to pretend that the objects are plain sets (or types) and that the morphisms are plain maps between those sets ([13, Chapter 6], [14, Section 1.3], [15, Chapter 14], [25, Chapter VI]). The internal language is sound with respect to intuitionistic reasoning, whereby any intuitionistic theorem holds in any topos.

The internal language of a sheaf topos enables *relativization by internalization.* For instance, by interpreting the proposition

> "in any short exact sequence of modules, if the two outer ones are finitely generated then so is the middle one"

of intuitionistic commutative algebra internally to the topos of sheaves over a space X, we obtain the geometric analogue

> "in any short exact sequence of sheaves of modules over X, if the two outer ones are of finite type then so is the middle one".

This way of deducing geometric theorems provides conceptual clarity, reduces technical overhead and justifies certain kinds of "fast and loose reasoning" typical of informal algebraic geometry. As soon as we go beyond the fragment of geometric sequents and consider more involved first-order or even higher-order statements, also significant improvements in proof length and proof complexity can be gained. For instance, Grothendieck's generic freeness lemma admits a short and simple proof in this framework, while previously-published proofs proceed in a somewhat involved series of reduction steps and require a fair amount of prerequisites in commutative algebra [12, 10].

The practicality of this approach hinges on the extent to which the dictionary between internal and external notions has been worked out. For instance, the simple example displayed above hinges on the dictionary entry stating that a sheaf of modules is of finite type if and only if it looks like a finitely generated module from the internal point of view. The motivation for this note was to find internal characterizations of flabby sheaves and of higher direct images, and the resulting entries are laid out in Section 3 and in Section 4: A sheaf is flabby if and only if, from the internal point of view, it is a flabby set, a notion introduced in Section 2 below; and higher direct images look like sheaf cohomology from the internal point of view.

As a byproduct, we demonstrate how the notion of flabby sets is a useful organizing principle in the study of injective objects. We employ flabby sets to give a new proof of Roswitha Harting's results that injectivity of sheaves is a local notion [17] and that a sheaf is injective if and only if it is injective from the internal point of view [20], which she stated (in slightly different language) for sheaves of abelian groups. We use the opportunity to correct a small mistake of hers, namely claiming that the analogous results for sheaves of modules would be false.

When employing the internal language of a topos, we are always referring to Mike Shulman's extension of the usual internal language, his *stack semantics* [28]. This extension allows to internalize unbounded quantification, which among other things is required to express the internal injectivity condition and the internal construction of sheaf cohomology.

A further motivation for this note was our desire to seek a constructive account of sheaf cohomology. Sheaf cohomology is commonly defined using injective resolutions, which can fail to exist in the absence of the axiom of choice [8], but flabby resolutions can also be used in their stead, making them the obvious candidate for a constructively sensible replacement of the usual definition. However, we show in Section 5 and in Section 6 that flabby resolutions present their own challenges, and in summary we failed to reach our goal. To the best of our knowledge, the problem of giving a constructive account of sheaf cohomology is still open.

In view of almost 80 years of sheaf cohomology, this state of affairs is slightly embarrassing, challenging the call that "once [a] subject is better understood, we can hope to refine its definitions and proofs so as to avoid [the law of excluded middle]" [30, Section 3.4].

A constructive account of sheaf cohomology would be highly desirable, not only out of a philosophical desire to obtain a deeper understanding of the foundations of sheaf cohomology, but also to: use the tools of sheaf cohomology in the internal setting of toposes, thereby extending their applicability by relativization by internalization; and to carry out *integrated developments* of algorithms for computing sheaf cohomology, where we would extract algorithms together with termination and correctness proofs from a hypothetical constructive account.

Acknowledgments. We are grateful to Thorsten Altenkirch, Simon Henry and Maria Emilia Maietti for insightful discussions and pointers to prior work, to Jürgen Jost, Marc Nieper-Wißkirchen and Peter Schuster for valuable guidance, and to Daniel Albert and Caterina Cozzi for their careful readings of earlier drafts. Most of the work for this note was carried out at the University of Augsburg and the Max Planck Institute for Mathematics in the Sciences in Leipzig.

1. Flabby sheaves

A sheaf F on a topological space or a locale X is *flabby* (flasque) if and only if all restriction maps $F(X) \to F(U)$ are surjective. The following properties of flabby sheaves render them fundamental to the theory of sheaf cohomology:

(1) Let $(U_i)_i$ be an open covering of X. A sheaf F on X is flabby if and only if all of its restrictions $F|_{U_i}$ are flabby as sheaves on U_i.

(2) Let $f : X \to Y$ be a continuous map. If F is a flabby sheaf on X, then $f_*(F)$ is a flabby sheaf on Y.
(3) Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves of modules.
  (a) If F is flabby, then this sequence is also exact as a sequence of presheaves.
  (b) If F and H are flabby, then so is G.
  (c) If F and G are flabby, then so is H.
(4a) Any sheaf embeds into a flabby sheaf.
(4b) Any sheaf of modules embeds into a flabby sheaf of modules.

Since we want to develop an analogous theory for flabby objects in elementary toposes, it is worthwhile to analyze the logical and set-theoretic commitments which are required to establish these properties. The standard proofs of properties (1), (3a), (3b) and (3c) require Zorn's lemma to construct maximal extensions of given sections. The standard proof of property (4b) requires the law of excluded middle, to ensure that the Godement construction actually yields a flabby sheaf. Properties (2) and (4a) can be verified purely intuitionistically.

There is an alternative definition of flabbiness, to be introduced below, which is equivalent to the usual one in presence of Zorn's lemma and which requires different commitments: For the alternative definition, properties (1), (3b) and (4a) can be verified purely intuitionistically. There is a substitute for property (3a) which can be verified purely intuitionistically. We do not know whether property (4b) can be established purely intuitionistically, but we give a rudimentary analysis in Section 6.

Both definitions can be generalized to yield notions of flabby objects in elementary toposes; but for toposes which are not localic, the two resulting notions will differ, and only the one obtained from the alternative definition is stable under pullback and can be characterized in the internal language. We therefore adopt in this paper the alternative one as the official definition.

Definition 1.1. A sheaf F on a topological space (or locale) X is *flabby* if and only if for all opens U and all sections $s \in F(U)$, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s can be extended to a section on $U \cup U_i$.

If F is a flabby sheaf in the traditional sense, then F is obviously also flabby in the sense of Definition 1.1 – singleton coverings will do. Conversely, let F be a flabby sheaf in the sense of Definition 1.1. Let $s \in F(U)$ be a local section. Zorn's lemma implies that there is a maximal extension $s' \in F(U')$. By assumption, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s' can be extended to $U' \cup U_i$. Since s' is maximal, $U' \cup U_i = U'$ for all i. Therefore $X = \bigcup_i U_i \subseteq U'$; hence s' is a global section, as desired.

We remark that unlike the traditional definition of flabbiness, Definition 1.1 exhibits flabbiness as a manifestly local notion.

2. Flabby sets

We intend this section to be applied in the internal language of an elementary topos; we will speak about sets and maps between sets, but intend our arguments to be applied to objects and morphisms in toposes. We will therefore be careful to reason purely intuitionistically. We adopt the terminology of [24] regarding subterminals and subsingletons: A subset $K \subseteq X$ is *subterminal* if and only if any given elements are equal ($\forall x, y \in K.\ x = y$), and it is a *subsingleton* if and only if there is an element $x \in X$ such that $K \subseteq \{x\}$. Any subsingleton is trivially subterminal, but the converse might fail.

Definition 2.1. A set X is *flabby* if and only if any subterminal subset of X is a subsingleton, that is, if and only if for any subset $K \subseteq X$ such that $\forall x, y \in K.\ x = y$, there exists an element $x \in X$ such that $K \subseteq \{x\}$.

In the presence of the law of excluded middle, a set is flabby if and only if it is inhabited. This characterization is a constructive taboo:

Proposition 2.2. *If any inhabited set is flabby, then the law of excluded middle holds.*

Proof. Let φ be a truth value. The set $X := \{0\} \cup \{1 \mid \varphi\} \subseteq \{0, 1\}$ is inhabited by 0 and contains 1 if and only if φ holds. Let K be the subterminal $\{1 \mid \varphi\} \subseteq X$. Flabbiness of X implies that there exists an element $x \in X$ such that $K \subseteq \{x\}$. We have $x \neq 1$ or $x = 1$. The first case entails $\neg\varphi$. The second case entails $1 \in X$, so φ. $\square$

Let $\mathcal{P}_{\leq 1}(X)$ be the set of subterminals of X.

Proposition 2.3. *A set X is flabby if and only if the canonical map $X \to \mathcal{P}_{\leq 1}(X)$ which sends an element x to the singleton set $\{x\}$ is final.*

Proof. By definition. $\square$

The set $\mathcal{P}_{\leq 1}(X)$ of subterminals of X can be interpreted as the set of *partially-defined elements* of X. In this view, the empty subset is the maximally undefined element and a singleton is a maximally defined element. A set is flabby if and only if any of its partially-defined elements can be refined to an honest element.

Remark 2.4. Although we will see in Section 5 that there is some relation between flabby sets and $\neg\neg$-separated sets, neither notion encompasses the other. The set Ω is flabby, but might fail to be $\neg\neg$-separated; the set $\mathbb{Z}$ is $\neg\neg$-separated, even discrete, but might fail to be flabby. This can abstractly be seen by adapting the proof of Proposition 2.2. An explicit model in which $\mathbb{Z}$ is not flabby can be obtained by picking any topological space T such that $H^1(T, \underline{\mathbb{Z}}) \neq 0$. Then the constant sheaf $\underline{\mathbb{Z}}$ is not flabby and hence, by Proposition 3.3 below, not a flabby set from the internal point of view of $\mathrm{Sh}(T)$.

Definition 2.5.
1. A set I is *injective* if and only if, for any injection $i : A \to B$, any map $f : A \to I$ can be extended to a map $B \to I$.
2. An R-module I is *injective* if and only if, for any linear injection $i : A \to B$ between R-modules, any linear map $f : A \to I$ can be extended to a linear map $B \to I$, as in the diagram below.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

In the presence of the law of excluded middle, a set is injective if and only if it is inhabited. In the presence of the axiom of choice, an abelian group is injective (as a $\mathbb{Z}$-module) if and only if it is divisible. Injective sets and modules have been intensively studied in the context of foundations before [8, 20, 23, 1]; the following properties are well-known:

Proposition 2.6. (1) *Any set embeds into an injective set.*
(2) *Any injective module is also injective as a set.*
(3) *Assuming the axiom of choice, any module embeds into an injective module.*

Proof. (1) One can check that, for instance, the full powerset $\mathcal{P}(X)$ and the set of subterminals $\mathcal{P}_{\leq 1}(X)$ are each injective.
(2) The forgetful functor from modules to sets possesses a left exact left adjoint. More explicitly, if $i : A \to B$ is an injective map between sets and if $f : A \to I$ is an arbitrary map, then the induced map $R\langle A\rangle \to R\langle B\rangle$ between free modules is also injective, the given map f lifts to a linear map $R\langle A\rangle \to I$, and an R-linear extension $R\langle B\rangle \to I$ induces an extension $B \to I$ of f.
(3) One verifies that any abelian group embeds into a divisible abelian group. By Baer's criterion (which requires the axiom of choice), divisible abelian groups are injective. The result for modules over arbitrary rings then follows purely formally, since the functor $A \mapsto \mathrm{Hom}(R, A)$ from abelian groups to R-modules has a left exact left adjoint with monic unit. □

Proposition 2.7. *Any injective set is flabby.*

Proof. Let I be an injective set. Let $K \subseteq I$ be a subterminal. The inclusion $f : K \to I$ extends along the injection $K \to 1 = \{\star\}$ to a map $1 \to I$. The unique image x of that map has the property that $K \subseteq \{x\}$. □

Corollary 2.8. *Any set embeds into a flabby set.*

Proof. Immediate by Proposition 2.6(1) and Proposition 2.7. □

A further corollary of Proposition 2.7 is that the statement "any inhabited set is injective" is a constructive taboo: If any inhabited set is injective, then any inhabited set is flabby, thus the law of excluded middle follows by Proposition 2.2.

Proposition 2.9. *Any singleton set is flabby. The cartesian product of flabby sets is flabby.*

Proof. Immediate. □

Subsets of flabby sets are in general not flabby, as else any set would be flabby in view of Corollary 2.8.

Proposition 2.10. (1) *Let I be an injective set. Let T be an arbitrary set. Then the set I^T of maps from T to I is flabby.*
(2) *Let I be an injective R-module. Let T be an arbitrary R-module. Then the set $\mathrm{Hom}_R(T, I)$ of linear maps from T to I is flabby.*

Proof. We first cover the case of sets. Let $K \subseteq I^T$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} T' \arrow[r, hook] \arrow[d] & T \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where T' is the subset $\{s \in T \mid K \text{ is inhabited}\} \subseteq T$ and the solid vertical map sends $s \in T'$ to $g(s)$, where g is an arbitrary element of K. This association is well-defined. Since I is injective, a dotted lift as indicated exists. If K is inhabited, this lift is an element of K.

The same kind of argument applies to the case of modules. If K is a subterminal of $\mathrm{Hom}_R(T, I)$, we define T' to be the submodule $\{s \in T \mid s = 0 \text{ or } K \text{ is inhabited}\}$ and consider the analogous injectivity diagram, where the solid vertical map $f : T' \to I$ is now defined by cases: Let $s \in T'$. If $s = 0$, then we set $f(s) = 0$; if K is inhabited, then we set $f(s) := g(s)$, where g is an arbitrary element of K. This association is again well-defined, and a dotted lift yields the desired element of $\mathrm{Hom}_R(T, I)$. $\square$

Proposition 2.10 can be used to give an alternative proof of Proposition 2.7 and to generalize Proposition 2.7 to modules: If I is an injective set, then the set $I^1 \cong I$ is flabby. If I is an injective module, then the set $\mathrm{Hom}_R(R, I) \cong I$ is flabby.

Lemma 2.11. (1) *Let I be an injective set. Let $i : A \to B$ be an injection. Let $f : A \to I$ be an arbitrary map. Then the set of extensions of f to B is flabby.*

(2) *Let I be an injective R-module. Let $i : A \to B$ be a linear injection. Let $f : A \to I$ be an arbitrary linear map. Then the set of linear extensions of f to B is flabby.*

Proof. For the first claim, we set $X := \{\bar{f} \in I^B \mid \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} i[A] \cup B' \arrow[r, hook] \arrow[d, "g"'] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where B' is the set $\{s \in B \mid K \text{ is inhabited}\}$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] \cup B'$. If $s \in i[A]$, we set $g(s) := f(a)$, where $a \in A$ is an element such that $s = i(a)$. If $s \in B'$, we set $g(s) := \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. If K is inhabited, this map is an element of K.

The proof of the second claim is similar. We set $X := \{\bar{f} \in \mathrm{Hom}_R(B, I) \mid \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} i[A] + B' \arrow[r, hook] \arrow[d, "g"'] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where B' is the submodule $\{t \in B \mid t = 0 \text{ or } K \text{ is inhabited}\} \subseteq B$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] + B'$. Then $s = i(a) + t$ for an element $a \in A$ and an element $t \in B'$. Since $t \in B'$, $t = 0$ or K is inhabited. If $t = 0$, we set $g(s) := f(a)$. If K is inhabited, we set $g(s) := f(a) + \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. Furthermore, if K is inhabited, then this map is an element of K. $\square$

Proposition 2.12. *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ *be a short exact sequence of modules. Let* $s \in M''$*. If* M' *is flabby, then the set of preimages of* s *under* p *is flabby.*

Proof. Let $X := \{u \in M \mid p(u) = s\}$. Let $K \subseteq X$ be a subterminal. Since p is surjective, there is an element $u_0 \in X$. The translated set $K - u_0 \subseteq M$ is still a subterminal, and its preimage under i is as well. Since M' is flabby, there is an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$. We verify that $K \subseteq \{u_0 + i(v)\}$.

Thus let $u \in K$ be given. Then $p(u - u_0) = 0$, so by exactness the set $i^{-1}[K - u_0]$ is inhabited. It therefore contains v. Thus $i(v) \in K - u_0$. Since $K = \{u\}$, it follows that $i(v) = u - u_0$, so $u \in \{u_0 + i(v)\}$ as claimed. ☐

Toby Kenney stressed that the notion of an injective set should be regarded as an interesting strengthening of the constructively rather ill-behaved notion of a nonempty set [23]. For instance, while the statements "there is a choice function for every set of nonempty sets" and even "there is a choice function for every set of inhabited sets" are constructive taboos, the statement "there is a choice function for every set of injective sets" is constructively neutral. Proposition 2.12 demonstrates that the notion of a flabby set can be regarded as an interesting intermediate notion: In the situation of Proposition 2.12, the set of preimages is not only not empty or inhabited, but even flabby.

Proposition 2.13. *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ *be a short exact sequence of modules. If* M' *and* M'' *are flabby, so is* M*.*

Proof. Let $K \subseteq M$ be a subterminal. Then its image $p[K] \subseteq M''$ is a subterminal as well. Since M'' is flabby, there is an element $s \in M''$ such that $p[K] \subseteq \{s\}$.

Since p is surjective, there is an element $u_0 \in M$ such that $p(u_0) = s$.

The preimage $i^{-1}[K - u_0] \subseteq M'$ is a subterminal. Since M' is flabby, there exists an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$.

Thus $K \subseteq \{u_0 + i(v)\}$. ☐

Noticeably missing here is a statement as follows: "Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M are flabby, so is M''." Assuming Zorn's lemma in the metatheory, this statement is true in every topos of sheaves over a locale, but we do not know whether it has an intuitionistic proof and in fact we surmise that it has not.

3. Flabby objects

Definition 3.1. An object X of an elementary topos $\mathcal{E}$ is *flabby* if and only if the statement "X is a flabby set" holds in the stack semantics of $\mathcal{E}$.

This definition amounts to the following: An object X of an elementary topos $\mathcal{E}$ is flabby if and only if, for any monomorphism $K \to A$ and any morphism $K \to X$, there exists an epimorphism $B \to A$ and a morphism $B \to X$ such that the following diagram commutes.

$$\begin{tikzcd} K \times_A B \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed, bend left] \\ X & \end{tikzcd}$$

Instead of referencing arbitrary stages $A \in \mathcal{E}$, one can also just reference the generic stage: Let $\mathcal{P}_{\leq 1}(X)$ denote the *object of subterminals* of X; this object is a certain subobject of $\mathcal{P}(X) = [X, \Omega_{\mathcal{E}}]_{\mathcal{E}}$, the powerobject of X. The subobject K_0 of $X \times \mathcal{P}_{\leq 1}(X)$ classified by the evaluation morphism $X \times \mathcal{P}_{\leq 1}(X) \to X \times \mathcal{P}(X) \to \Omega_{\mathcal{E}}$ is the *generic subterminal* of X. The object X is flabby if and only if there exists an epimorphism $B \to \mathcal{P}_{\leq 1}(X)$ and a morphism $B \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
K_0 \times_{\mathcal{P}_{\leq 1}(X)} B \arrow[r, hook] \arrow[d] & \mathcal{P}_{\leq 1}(X) \arrow[dl, dashed, bend left] \\
X &
\end{tikzcd}$$

Proposition 3.2. *Let X and T be objects of an elementary topos $\mathcal{E}$.*

(1) *If X is flabby, so is $X \times T$ as an object of $\mathcal{E}/T$.*
(2) *The converse holds if the unique morphism $T \to 1$ is an epimorphism.*

Proof. This holds for any property which can be defined in the stack semantics [28, Lemma 7.3]. □

Proposition 3.3. *Let F be a sheaf on a topological space X (or a locale). Then F is flabby as a sheaf if and only if F is flabby as an object of the sheaf topos* $\mathrm{Sh}(X)$.

Proof. The proof is routine; we only verify the "only if" direction. Let F be flabby as a sheaf. It suffices to verify the defining condition for stages of the form $A = \mathrm{Hom}(\cdot, U)$, where U is an open of X. A monomorphism $K \to A$ then amounts to an open $V \subseteq U$ (the union of all opens on which K is inhabited). A morphism $K \to F$ amounts to a section $s \in F(V)$. Since F is flabby as a sheaf, there is an open covering $X = \bigcup_{i \in I} V_i$ such that, for all i, the section s can be extended to a section s_i of $V \cup V_i$. The desired epimorphism is $B := \coprod_i \mathrm{Hom}(\cdot, (V \cup V_i) \cap U) \to A$, and the desired morphism $B \to X$ is given by the sections $s_i|_{(V \cup V_i) \cap U}$.

As stated, the argument in the previous paragraph requires the axiom of choice to pick the extensions s_i; this can be avoided by a standard trick of expanding the index set of the coproduct to include the choices: We redefine $B := \coprod_{(i,t) \in I'} \mathrm{Hom}(\cdot, (V \cup V_i) \cap U)$, where $I' = \{(i \in I, t \in F(V \cup V_i)) \mid t|_V = s\}$ and define the morphism $B \to X$ on the (i,t)-summand by $t|_{(V \cup V_i) \cap U}$. □

Proposition 3.4. *Let X be a flabby object of a localic topos $\mathcal{E}$. If Zorn's lemma is available in the metatheory, then X possesses a global element (a morphism $1 \to X$).*

Proof. This is a restatement of the discussion following Definition 1.1. □

Proposition 3.5. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. If f_* preserves epimorphisms, then f_* preserves flabby objects.*

Proof. Let $X \in \mathcal{F}$ be a flabby object. Let $k : K \to A$ be a monomorphism in $\mathcal{E}$ and let $x : K \to f_*(X)$ be an arbitrary morphism. Without loss of generality, we may assume that A is the terminal object 1 of $\mathcal{E}$. Then $f^*(k) : f^*(K) \to 1$ is a monomorphism in $\mathcal{F}$ and the adjoint transpose $x^t : f^*(K) \to X$ is a morphism in $\mathcal{F}$. Since X is flabby, there is an epimorphism $B \to 1$ in $\mathcal{F}$ and a morphism $y : B \to X$ such that the morphism $f^*(K) \times B \to X$ factors over y. Hence x factors over $f_*(y) : f_*(B) \to f_*(X)$. We conclude because the morphism $f_*(B) \to f_*(1)$ is an epimorphism by assumption. □

The assumption on f_* of Proposition 3.5 is for instance satisfied if f is a local geometric morphism.

Definition 3.6. An object I of an elementary topos $\mathcal{E}$ is *externally injective* if and only if for any monomorphism $A \to B$ in $\mathcal{E}$, the canonical map $\mathrm{Hom}_{\mathcal{E}}(B, I) \to \mathrm{Hom}_{\mathcal{E}}(A, I)$ is surjective. It is *internally injective* if and only if for any monomorphism $A \to B$ in $\mathcal{E}$, the canonical morphism $[B, I] \to [A, I]$ between Hom objects is an epimorphism in $\mathcal{E}$.

If R is a ring in an elementary topos $\mathcal{E}$, a similar definition can be given for R-modules in $\mathcal{E}$, referring to the set respectively the object of linear maps. The condition for an object to be internally injective can be rephrased in various ways. The following proposition lists five of these conditions. The equivalence of the first four is due to Roswitha Harting [20].

Proposition 3.7. *Let $\mathcal{E}$ be an elementary topos. Then the following statements about an object $I \in \mathcal{E}$ are equivalent.*

(1) *I is internally injective.*
(1') *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (1) as an object of $\mathcal{E}/A$.*
(2) *The functor $[\cdot, I] : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ maps monomorphisms in $\mathcal{E}$ to morphisms for which any global element of the target locally (after change of base along an epimorphism) possesses a preimage.*
(2') *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (2) as an object of $\mathcal{E}/A$.*
(3) *The statement "I is an injective set" holds in the stack semantics of $\mathcal{E}$.*

Proof. The implications (1) $\Rightarrow$ (2), (1') $\Rightarrow$ (2'), (1') $\Rightarrow$ (1) and (2') $\Rightarrow$ (2) are trivial.

The equivalence (1') $\Leftrightarrow$ (3) follows directly from the interpretation rules of the stack semantics.

The implication (2) $\Rightarrow$ (2') employs the extra left adjoint $p_! : \mathcal{E}/A \to \mathcal{E}$ of $p^* : \mathcal{E} \to \mathcal{E}/A$ (which maps an object $(X \to A)$ to X), as in the usual proof that injective sheaves remain injective when restricted to smaller open subsets: We have that $p_* \circ [\cdot, p^*(I)]_{\mathcal{E}/A} \cong [\cdot, I]_{\mathcal{E}} \circ p_!$, the functor $p_!$ preserves monomorphisms, and one can check that p_* reflects the property that global elements locally possess preimages. Details are in [20, Thm. 1.1].[1]

The implication (2') $\Rightarrow$ (1') follows by performing an extra change of base, exploiting that any non-global element becomes a global element after a suitable change of base. ☐

Let R be a ring in $\mathcal{E}$. Then the analogue of Proposition 3.7 holds for R-modules in $\mathcal{E}$, if $\mathcal{E}$ is assumed to have a natural numbers object. The extra assumption is needed in order to construct the left adjoint $p_! : \mathrm{Mod}_{\mathcal{E}/A}(R \times A) \to \mathrm{Mod}_{\mathcal{E}}(R)$. Phrased in the internal language, this adjoint maps a family $(M_a)_{a \in A}$ of R-modules to the direct sum $\bigoplus_{a \in A} M_a$. Details on this construction, phrased in the language of sets but interpretable in the internal language, can for instance be found in [26, page 54].

Somewhat surprisingly, and in stark contrast with the situation for internally projective objects (which are defined dually), internal injectivity coincides with

[1] Harting formulates her theorem for abelian group objects, and has to assume that $\mathcal{E}$ contains a natural numbers object to ensure the existence of an abelian version of $p_!$.

external injectivity for localic toposes. In the special case of sheaves of abelian groups, this result is due to Roswitha Harting [20, Proposition 2.1].

Theorem 3.8. *Let I be an object of an elementary topos $\mathcal{E}$. If I is externally injective, then I is also internally injective. The converse holds if $\mathcal{E}$ is localic and Zorn's lemma is available in the metatheory.*

Proof. For the "only if" direction, let I be an object which is externally injective. Then I satisfies Condition (2) in Proposition 3.7, even without having to pass to covers.

For the "if" direction, let I be an internally injective object. Let $i : A \to B$ be a monomorphism in $\mathcal{E}$ and let $f : A \to I$ be an arbitrary morphism. We want to show that there exists an extension $B \to I$ of f along i. To this end, we consider the object of such extensions, defined by the internal expression

$$F := \{\bar{f} \in [B, I] \mid \bar{f} \circ i = f\}.$$

Global elements of F are extensions of the kind we are looking for. By Lemma 2.11(1), interpreted in $\mathcal{E}$, this object is flabby. By Proposition 3.4, it has a global element. □

The analogue of Theorem 3.8 for modules holds as well, if $\mathcal{E}$ is assumed to have a natural numbers object. The proof carries over word for word, only referencing Lemma 2.11(2) instead of Lemma 2.11(1). It seems that Roswitha Harting was not aware of this generalization, even though she did show that injectivity of sheaves of modules over topological spaces is a local notion [17, Remark 5], as she (mistakenly) states in [17, page 233] that "the notions of injectivity and internal injectivity do not coincide" for modules.

It is worth noting that, because the internal language machinery was at that point not as well-developed as it is today, Harting had to go to considerable length to construct internal direct sums of abelian group objects [19], and in order to verify that taking internal direct sums is faithful she felt the need to employ Barr's metatheorem [18, Theorem 1.7]. Nowadays we can verify both statements by simply carrying out an intuitionistic proof in the case of the topos of sets and then trusting the internal language to obtain the generalization to arbitrary elementary toposes with a natural numbers object.

Since we were careful in Section 2 to use the law of excluded middle and the axiom of choice only where needed, most results of that section carry over to flabby and internally injective objects. Specifically, we have:

Scholium 3.9. *For any elementary topos $\mathcal{E}$:*

1. *Any object embeds into an internally injective object.*
2. *(If $\mathcal{E}$ has a natural numbers object.) The underlying unstructured object of an internally injective module is internally injective.*
3. *Any internally injective object is flabby.*
4. *Any object embeds into a flabby object.*
5. *The terminal object is flabby. The product of flabby objects is flabby.*
6. *Let I be an internally injective object. Let T be an arbitrary object. Then $[T, I]$ is a flabby object.*
7. *(If $\mathcal{E}$ has a natural numbers object.) Let I be an internally injective R-module. Let T be an arbitrary R-module. Then $[T, I]_R$, the subobject of the internal Hom consisting only of the linear maps, is a flabby object.*

(8) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in $\mathcal{E}$. If M' and M'' are flabby objects, so is M.*

Proof. We established the analogous statements for sets and modules purely intuitionistically in Section 2, and the stack semantics is sound with respect to intuitionistic logic. □

Scholium 3.10. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in a localic topos $\mathcal{E}$. Let M' be a flabby object. Assuming Zorn's lemma in the metatheory, the induced sequence $0 \to \Gamma(M') \to \Gamma(M) \to \Gamma(M'') \to 0$ of $\Gamma(R)$-modules is exact, where $\Gamma(X) = \mathrm{Hom}_{\mathcal{E}}(1, X)$.*

Proof. We only have to verify exactness at $\Gamma(M'')$, so let $s \in \Gamma(M'')$. Interpreting Proposition 2.12 in $\mathcal{E}$, we see that the object of preimages of s is flabby. Since $\mathcal{E}$ is localic, this object is a flabby sheaf; since Zorn's lemma is available, it possesses a global element. Such an element is the desired preimage of s in $\Gamma(M)$. □

If $\mathcal{E}$ is not necessarily localic or Zorn's lemma is not available, only a weaker substitute for Scholium 3.10 is available: Given $s \in \Gamma(M'')$, the object of preimages of s is flabby. In particular, given any point of $\mathcal{E}$, we can extend any local preimage of s to a preimage which is defined on an open neighborhood of that point. We believe that there are situations in which this weaker substitute is good enough, similar to how in constructive algebra often the existence of a sufficiently large field extension is good enough where one would classically blithely pass to an algebraic closure.

Remark 3.11. A direct generalization of the traditional notion of a flabby sheaf, as opposed to our reimagining in Definition 1.1, to elementary toposes is the following. An object X of an elementary topos $\mathcal{E}$ is *strongly flabby* if and only if, for any monomorphism $K \to 1$ in $\mathcal{E}$, any morphism $K \to X$ lifts to a morphism $1 \to X$.

One can verify, purely intuitionistically, that a sheaf F on a space T is flabby in the traditional sense if and only if F is a strongly flabby object of $\mathrm{Sh}(T)$.

The notion of strongly flabby objects is, however, not local (in the same sense that the notion of flabby objects is, as stated in Proposition 3.2) and therefore cannot be characterized in the internal language. A specific example is the G-set G (with the translation action), considered as an object of the topos BG of G-sets, where G is a nontrivial group. This object is not strongly flabby, since the morphism $\emptyset \to 1$ does not lift, but its pullback to the slice $BG/G \simeq \mathrm{Set}$ is (assuming the law of excluded middle in the metatheory), and the unique morphism $G \to 1$ is indeed an epimorphism.

4. Higher direct images as internal sheaf cohomology

Let X be a locale and let $f : Y \to X$ be an over-locale. By the fundamental relation between locales and topological spaces, this situation arises for instance, when given a sober topological space and a topological space over it, as is often the case in algebraic topology or algebraic geometry. Let a sheaf $\mathcal{O}_Y$ of rings on Y be given. Then the traditional way to define the *higher direct images* of a sheaf E of $\mathcal{O}_Y$-modules is to pick an injective resolution $0 \to E \to I^\bullet$ and set $R^n f_*(E) := H^n(f_*(I^\bullet))$.

Assuming the axiom of choice, there are enough injective sheaves of modules so that this recipe can be carried out. The resulting sheaf of $f_*\mathcal{O}_Y$-modules is well-defined in the following sense: Given a further injective resolution $0 \to E \to J^\bullet$, there is up to homotopy precisely one morphism $I^\bullet \to J^\bullet$ compatible with the identity on E, and this morphism induces an isomorphism on cohomology.

Higher direct images are pictured as a "relative" version of sheaf cohomology. Due to the result that injectivity of sheaves of modules can be characterized in the internal language, we can give a precise rendering of this slogan: We can understand higher direct images as internal sheaf cohomology.

The details are as follows. The over-locale Y corresponds to a locale $I(Y)$ internal to $\mathrm{Sh}(X)$, in such a way that the category of internal sheaves over this internal locale coincides with $\mathrm{Sh}(Y)$; in particular, a given sheaf E of $\mathcal{O}_Y$-modules can be regarded as a sheaf over $I(Y)$. Under this equivalence, the morphism $f : Y \to X$ corresponds to the unique morphism $I(Y) \to \mathrm{pt}$ to the internal one-point locale. Hence it makes sense to construct, from the internal point of view of $\mathrm{Sh}(X)$, the sheaf cohomology $H^n(I(Y), E)$ of E.

Usually one would not expect an internal construction which depends on arbitrary choices to yield a globally-defined sheaf over X – following the definition of the stack semantics we only obtain a family of sheaves defined on members of some open covering of X; but we verify in Theorem 4.2 below that in our case, it does, and that the resulting sheaf coincides with $R^n f_*(E)$.

Lemma 4.1. *Let Y be a ringed locale over a locale X. Let I be a sheaf of modules over Y. Assuming Zorn's lemma in the metatheory, the following statements are equivalent:*

(1) *I is an injective sheaf of modules.*
(2) *From the point of view of $\mathrm{Sh}(Y)$, I is an injective module.*
(3) *From the point of view of $\mathrm{Sh}(X)$, I is an injective module from the point of view of $\mathrm{Sh}(I(Y))$.*
(4) *From the point of view of $\mathrm{Sh}(X)$, I is an injective sheaf of modules on $I(Y)$.*

Proof. The equivalence of the first two statements is by Theorem 3.8. The equivalence (2) $\Leftrightarrow$ (3) is by the idempotency of the stack semantics: $\mathrm{Sh}(Y) \models \varphi$ if and only if $\mathrm{Sh}(X) \models (\mathrm{Sh}(I(Y)) \models \varphi)$. (Shulman stated and proved a restricted version of this idempotency property in his original paper on the stack semantics [28, Lemma 7.20]. A proof of the general case is slightly less accessible [11, Lemma 1.20].) The equivalence (3) $\Leftrightarrow$ (4) is by interpreting Theorem 3.8 internally to $\mathrm{Sh}(X)$. This requires Zorn's lemma to hold internally to $\mathrm{Sh}(X)$; this is indeed the case since we assume Zorn's lemma in the metatheory and since the validity of Zorn's lemma passes from the metatheory to localic toposes [22, Proposition D4.5.14]. □

Theorem 4.2. *Let $f : Y \to X$ be a ringed locale over a locale X. Let E be a sheaf of modules over Y. Assuming the axiom of choice in the metatheory, the expression "$H^n(I(Y), E)$" of the internal language of $\mathrm{Sh}(X)$ denotes a globally-defined sheaf over X, and this sheaf coincides with $R^n f_*(E)$.*

Proof. By Lemma 4.1 and by the fact that every sheaf of modules over Y admits an injective resolution, every sheaf of modules over $I(Y)$ admits an injective resolution from the point of view of $\mathrm{Sh}(X)$. Hence we can, internally to $\mathrm{Sh}(X)$, carry out the construction of $H^n(I(Y), E)$. Externally, this yields an open covering of X such that we have, for each member U of that covering

- a sheaf M over U,

- a module structure on M,
- a resolution $0 \to E|_{f^{-1}U} \to I^\bullet$ by sheaves of modules which are internally and hence externally injective and
- data exhibiting M as the n-th cohomology of $(f|_{f^{-1}U})_*(I^\bullet)$.

On intersections of such opens U and U', there is exactly one isomorphism $M|_{U\cap U'} \to M'|_{U\cap U'}$ of sheaves of modules induced by a morphism of resolutions which is compatible with the identity on E. Hence the cocycle condition for these isomorphisms is satisfied, ensuring that the individual sheaves M glue to a globally-defined sheaf of modules on X. (The individual injective resolutions need not glue to a global injective resolution.)

The claim that this sheaf coincides with $R^n f_*(E)$ follows from the fact that we can pick as internal resolution of E (considered as a sheaf over $I(Y)$) the particular injective resolution of E (considered as a sheaf over Y) used to define $R^n f_*(E)$. $\square$

The internal characterization provided by Theorem 4.2 gives, as a simple application, a logical explanation that higher direct images along the identity $\mathrm{id} : X \to X$ vanish: From the internal point of view of $\mathrm{Sh}(X)$, the over-locale X corresponds to the one-point locale, and the higher cohomology of the one-point locale vanishes.

In algebraic geometry, the internal characterization can be used to immediately deduce the explicit description of the higher direct images of Serre's twisting sheaves along the projection $\mathbb{P}^n_S \to S$, where S is an arbitrary base scheme (or even base locally ringed locale), from a computation of the cohomology of projective n-space. Background on carrying out scheme theory internally to a topos is given in [12, Section 12].

5. Flabby objects in the effective topos

The notion of flabby objects originates from the notion of flabby sheaves and is therefore closely connected to Grothendieck toposes. Hence it is instructive to study flabby objects in elementary toposes which are not Grothendieck toposes, away from their original conceptual home. We begin this study with establishing the following observations on flabby objects in the effective topos. We follow the terminology of Martin Hyland's survey on the effective topos [21].

Proposition 5.1. *Let X be a flabby object in the effective topos. Let $f : X \to X$ be a morphism. If X is effective, the statement "f has a fixed point" holds in the effective topos.*

Proposition 5.2. *Assuming the law of excluded middle in the metatheory, any $\neg\neg$-separated module in the effective topos embeds into a flabby module.*

The intuitive reason for why Proposition 5.1 holds is the following. Let X be a flabby object in the effective topos. Then there is a procedure which computes for any subterminal $K \subseteq X$ an element x_K such that $K \subseteq \{x_K\}$. This element might not depend extensionally on K, but this fine point is not important for this discussion. Let $f : X \to X$ be a morphism. We construct the self-referential subset $K := \{f(x_K)\}$; the formal proof below will indicate how this can be done. Then $K \subseteq \{x_K\}$, so $f(x_K) = x_K$.

A corollary of Proposition 5.1 is that the trivial module is the only flabby module in the effective topos whose underlying unstructured object is an effective set: Given such a flabby module M, let $v \in M$ be an arbitrary element. Then the morphism $x \mapsto v + x$ has a fixed point; thus $v + x = x$ for some element x, and hence $v = 0$.

It is the self-referentiality which makes the proof of Proposition 5.1 work, but the blame for paucity of flabby objects in the effective topos is to put on the realizers for statements of the form "$K = K$", where $(=)$ is the nonstandard equality predicate of the powerobject $\mathcal{P}(X)$. A procedure witnessing flabbiness has to compute a reflexivity realizer for a suitable element x_K from a reflexivity realizer for a given element K. However, such realizers are not very informative. Metaphorically speaking, a procedure witnessing flabbiness has to conjure elements out of thin air.

This problem does not manifest with objects X which are not effective sets. Reflexivity realizers for these objects are themselves not very informative; a procedure witnessing flabbiness therefore only has to turn one kind of non-informative realizers into another kind. The flabby modules featuring in the proof of Proposition 5.2 will accordingly not be effective sets.

Proof of Proposition 5.1. For any Turing machine e, let $v_e : |X| \to \Sigma$ be the nonstandard predicate given by

$$\begin{aligned} v_e(x) = \{m \in \mathbb{N} \,|\, & \text{there is an element } x_0 \in |X| \text{ such that} \\ & e \text{ terminates with an element of } [\![x_0 = x_0]\!] \text{ and } m \in [\![x = x_0]\!]\} \end{aligned}$$

and let $K_e \in \Sigma^{|X| \times \Sigma}$ be the nonstandard predicate given by

$$K_e(x, u) = [\![(x = x) \wedge (u \leftrightarrow v_e(x))]\!].$$

One can explicitly construct a realizer a_e of the statement "$K_e = K_e$", where $(=)$ is the nonstandard equality predicate of the object $\mathcal{P}_{\leq 1}(X)$ of subterminals of X. This is where the assumption that X is effective is important; without it, we could only verify "$K_e = K_e$" where $(=)$ is the nonstandard equality predicate of the full powerobject $\mathcal{P}(X)$.

Since X is flabby, there is a realizer r for the statement "$\forall K \in \mathcal{P}_{\leq 1}(X).\ \exists x \in X.\ \forall y \in X.\ (y \in K \Rightarrow y = x)$". Let s be a realizer for the statement "$\forall x \in X.\ \exists y \in Y.\ y = f(x)$". Let e be the particular Turing machine which proceeds as follows:

1. Simulate r on input a_e in order to obtain a realizer $b \in [\![x = x]\!]$ for some $x \in |X|$.
2. Simulate s on input b in order to obtain a realizer $c \in [\![f(x) = f(x)]\!]$.
3. Output c.

This description of the machine e makes use of the number e coding it; the recursion theorem yields a general reason why this self-referentiality is possible. Here we can even do without this theorem, since a close inspection of the construction of a_e shows that a_e is actually independent of e. This should not come as a surprise, as reflexivity realizers of $\mathcal{P}(X)$ and $\mathcal{P}_{\leq 1}(X)$ are known to be not very informative.

Passing a_e to r yields a reflexivity realizer of some element $x_{K_e} \in |X|$. Therefore the Turing machine e does terminate, with a reflexivity realizer for $f(x_{K_e})$. Thus the statement "$f(x_{K_e}) \in K_e$" is realized; hence "$f(x_{K_e}) = x_{K_e}$" is as well. $\square$

Proof of Proposition 5.2. Let $(\Gamma \dashv \Delta) : \mathrm{Set} \to \mathrm{Eff}$ be the inclusion of the double-negation sheaves. For a $\neg\neg$-separated module M in the effective topos, the canonical morphism $M \to \Delta(\Gamma(M))$ is a monomorphism; the set $\Gamma(M)$ is flabby by virtue of being inhabited; and Δ preserves flabby objects by Proposition 3.5. $\square$

Remark 5.3. The analogues of Proposition 5.1 and Proposition 5.2 for the realizability topos constructed using infinite time Turing machines [5, 16] are true as well, with the same proofs.

6. Conclusion

We originally set out to develop an intuitionistic account of Grothendieck's sheaf cohomology. Čech methods can be carried out constructively, and there are constructive accounts of special cases, resulting even in efficient-in-practice algorithms [3, 4], but it appears that there is not a general framework for sheaf cohomology which would work in an intuitionistic metatheory.

The main obstacle preventing Grothendieck's theory of derived functors to be interpreted constructively is its reliance on injective resolutions. It is known that in the absence of the axiom of choice, much less in a purely intuitionistic context, there might not be any nontrivial injective abelian group [8].

In principle, this problem could be remedied by employing flabby resolutions instead of injective ones. There are, however, two problems with this suggestion. Firstly, all proofs known to us that flabby sheaves are acyclic for the global sections functor require Zorn's lemma. This problem might be mitigated by relying on the substitute property discussed following Scholium 3.10. But secondly, it is an open question whether one can show, purely intuitionistically, that any sheaf of modules embeds into a flabby sheaf of modules. The following is known about this problem:

(1) There is a purely intuitionistic proof that any sheaf of sets embeds into a flabby sheaf of sets (Scholium 3.9(4)).
(2) The existence of enough flabby modules, and even the existence of enough injective modules, is *not* a constructive taboo, that is, these statements do not entail a classical principle like the law of excluded middle or the principle of omniscience. This is because assuming the axiom of choice, any Grothendieck topos has enough injective (and therefore flabby) modules.
(3) There is a way of embedding any module into a flabby module if *quotient inductive types*, as suggested by Altenkirch and Kaposi [2], are available. These generalize ordinary inductive W-types, which exist in any topos [27, 7, 6] and whose existence can indeed be verified in an intuitionistic set theory like IZF, by allowing to give constructors and state identifications at the same time. More specifically, given an R-module M, we can construct a flabby envelope T of M as the quotient inductive type generated by the following clauses: $0 \in T$ (where 0 is a formal symbol); if $t, s \in T$, then $t + s \in T$; if $t \in T$ and $r \in R$, then $rt \in T$; if $x \in M$, then $\underline{x} \in T$; if $K \subseteq T$ is a subterminal, then $\varepsilon_K \in T$; if $t, s, u \in T$ and $r, r' \in R$, then $t + (s + u) = (t + s) + u$, $t + s = s + t$, $t + 0 = t = t + 0$, $0t = 0$, $1t = t$, $r(t + u) = rt + ru$, $(r + r')t = rt + r't$; if $x, y \in M$ and $r \in R$, then $\underline{0} = 0$, $\underline{x + y} = \underline{x} + \underline{y}$, $\underline{rx} = r\underline{x}$; and if $t \in T$, then $\varepsilon_{\{t\}} = t$.
However, it is an open question under which circumstances quotient inductive types can be shown to exist. Zermelo–Fraenkel with choice certainly suffices, while Zermelo–Fraenkel without choice does not [29, Section 9],

hence IZF also does not.[2] The existence of quotient inductive types seem to be, as the existence of enough injective modules, *constructively neutral.*

(4) There are a number of simple constructions which come close to providing flabby envelopes for arbitrary modules M. For instance, we could equip the set $T := \mathcal{P}_{\leq 1}(X)/{\sim}$, where $K \sim L$ if and only if $K = L$ or $K \cup L \subseteq \{0\}$, with a module structure given by $0 := [\{0\}]$, $[K]+[L] := [K+L]$ and $r[K] := rK$. The resulting module admits a linear injection from M, sending an element x to $[\{x\}]$. However, it fails to be flabby. Given a subterminal $E \subseteq \mathcal{P}_{\leq 1}(X)/{\sim}$, there is the well-defined element $v := [\{x \in M \mid x \in K \text{ for some } [K] \in E\}]$, but we cannot verify $E \subseteq \{v\}$.

(5) There appears to be some tension regarding the effective topos: Proposition 5.2 shows that at least $\neg\neg$-separated modules in the effective topos always embed into flabby modules, assuming the law of excluded middle in the metatheory, while Proposition 5.1 shows that no nontrivial effective module is flabby.

We currently believe that it is not possible to give a constructive account of a global cohomology functor which would associate to any sheaf of modules its cohomology. However, it should be possible to do so for a restricted class of sheaves, while still preserving the good formal properties expected from derived functors.

References

[1] P. Aczel, B. van den Berg, J. Granstroem, and P. Schuster. "Are there enough injective sets?" In: *Studia Logica* 101.3 (2013), pp. 467–482.

[2] T. Altenkirch and A. Kaposi. "Type theory in type theory using quotient inductive types". In: *Proceedings of POPL '16.* ACM, 2016, pp. 18–29.

[3] M. Barakat and M. Lange-Hegermann. "An axiomatic setup for algorithmic homological algebra and an alternative approach to localization". In: *J. Algebra Appl.* 10.2 (2011), pp. 269–293.

[4] M. Barakat and M. Lange-Hegermann. "On the Ext-computability of Serre quotient categories". In: *J. Algebra* 240 (2014), pp. 333–349.

[5] A. Bauer. "An injection from the Baire space to natural numbers". In: *Math. Structures Comput. Sci.* 25.7 (2015), pp. 1484–1489.

[6] B. van den Berg and C. Kouwenhoven-Gentil. *W-types in the effective topos.* 2004. URL: `https://www.staff.science.uu.nl/~ooste110/realizability/WinEff.ps.gz`.

[7] B. van den Berg and I. Moerdijk. *W-types in sheaves.* 2008. URL: `https://arxiv.org/abs/0810.2398`.

[8] A. Blass. "Injectivity, projectivity, and the axiom of choice". In: *Trans. Amer. Math. Soc.* 255 (1979), pp. 31–59.

[9] A. Blass. "Words, free algebras, and coequalizers". In: *Fund. Math.* 117.2 (1983), pp. 117–160.

[10] I. Blechschmidt. *An elementary and constructive proof of Grothendieck's generic freeness lemma.* 2018. URL: `https://arxiv.org/abs/1807.01231`.

[2]With quotient inductive types, any infinitary algebraic theory admits free algebras. However, it is consistent with Zermelo–Fraenkel set theory that some such theories do not admit free algebras [9].

[11] I. Blechschmidt. “Topostheorie und algebraische Geometrie”. MA thesis. University of Augsburg, 2012. URL: `https://www.ingo-blechschmidt.eu/masters-thesis.pdf`.

[12] I. Blechschmidt. “Using the internal language of toposes in algebraic geometry”. PhD thesis. University of Augsburg, 2017. URL: `https://rawgit.com/iblech/internal-methods/master/notes.pdf`.

[13] F. Borceux. *Handbook of Categorical Algebra: Volume 3, Sheaf Theory.* Encyclopedia Math. Appl. Cambridge University Press, 1994.

[14] O. Caramello. *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic ‘bridges’.* Oxford University Press, 2018.

[15] R. Goldblatt. *Topoi: The Categorical Analysis of Logic.* Vol. 98. Stud. Logic Found. Math. Elsevier, 1984.

[16] J. Hamkins and A. Lewis. “Infinite time Turing machines”. In: *J. Symbolic Logic* 65.2 (2000), pp. 567–604.

[17] R. Harting. “A remark on injectivity of sheaves of abelian groups”. In: *Archiv der Mathematik* 36.1 (1981), pp. 230–233.

[18] R. Harting. “Abelian groups in a topos: injectives and injective effacements”. In: *J. Pure Appl. Algebra* 30 (1983), pp. 247–260.

[19] R. Harting. “Internal coproduct of abelian groups in an elementary topos”. In: *Comm. Algebra* 10.11 (1982), pp. 1173–1237.

[20] R. Harting. “Locally injective abelian groups in a topos”. In: *Comm. Algebra* 11.4 (1983), pp. 349–376.

[21] M. Hyland. “The effective topos”. In: *The L. E. J. Brouwer Centenary Symposium.* Ed. by A. S. Troelstra and D. van Dalen. North-Holland, 1982, pp. 165–216.

[22] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Oxford University Press, 2002.

[23] T. Kenney. “Injective power objects and the axiom of choice”. In: *J. Pure Appl. Algebra* 215 (2 2011), pp. 131–144.

[24] A. Kock. “Algebras for the partial map classifier monad”. In: *Category Theory. Proceedings of the International Conference held in Como, Italy, July 22–28, 1990.* Ed. by A. Carboni, M. Pedicchio, and G. Rosolini. Vol. 1488. Lecture Notes in Math. Springer, 1991, pp. 262–278.

[25] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory.* Universitext. Springer, 1992.

[26] R. Mines, F. Richman, and W. Ruitenburg. *A Course in Constructive Algebra.* Universitext. Springer, 1988.

[27] I. Moerdijk and E. Palmgren. “Wellfounded trees in categories”. In: *Ann. Pure Appl. Logic* 104 (2000), pp. 189–218.

[28] M. Shulman. “Stack semantics and the comparison of material and structural set theories”. 2010. URL: `https://arxiv.org/abs/1004.3802`.

[29] M. Shulman and P. Lumsdaine. *Semantics of higher inductive types.* 2017. URL: `https://arxiv.org/abs/1705.07088`.

[30] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* Institute for Advanced Study, 2013. URL: `https://homotopytypetheory.org/book`.