Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

How rough path lifts affect expected return and volatility: a rough model under transaction cost

(revised version: May 2023)

by

Hoang Duc Luu and Jürgen Jost

Preprint no.: 101 2019

How rough path lifts affect expected return and volatility: a rough model under transaction cost

Luu Hoang Duc [*], Jürgen Jost [†]

Abstract

We develop a general mathematical framework, based on rough path theory, that can incorporate the empirically observed nonlinear mean-variance relation of the logarithmic return in a systematic manner. This model offers the possibility of an additional noise hidden in the rough path lift, hence supporting the idea of mixture of a Gaussian noise that is close to a standard Brownian motion and another source of long memory noise (a fractional Brownian motion for instance), that can account for the multi-scaling phenomenon in financial data. The no-arbitrage principle is then satisfied under the assumption of transaction costs as long as the driving noise is a sticky process. We also discuss the potential risk of model uncertainty where the ambiguity comes from the rough path lifts, as well as the problem of cooperation. Our models are supported by empirical evidence from financial data and in particular, can explain some stylized fact (a parabolic lower bound of a mean-variance relation) that has not been explained before.

Keywords: stock price, expected return, volatility, noise, rough path theory, rough path lifts, rough differential equations, no-arbitrage, risk.

1 Introduction

It is well-known that the original Samuelson stock model [44],

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.1}$$

for a stock price S_t at time t with growth factor μ, volatility σ and a stochastic integral in the sense of Itô with respect to a standard Brownian motion B_t, does not reproduce certain rather universal features of empirical stock price data (the so-called stylized facts); hence many modifications have been suggested ever since. First, the Hull & White model [28] suggests that the growth factor and the volatility should be time-dependent and stochastic, leading to a model of the form

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t, \tag{1.2}$$

where $\mu : [0,T] \to \mathbb{R}$ is assumed to be Lebesgue integrable and deterministic for simplicity, and σ_t satisfies a stochastic differential equation

$$d \log \sigma_t = k(\theta - \log \sigma_t)dt + \gamma dW_t \tag{1.3}$$

[*]Luu H. Duc is with the Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany, & Institute of Mathematics, Viet Nam Academy of Science and Technology, Viet Nam, duc.luu@mis.mpg.de, lhduc@math.ac.vn

[†]Jürgen Jost is with the Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany & the Santa Fe Institute, Santa Fe, USA, jjost@mis.mpg.de.

with parameters k, θ, γ and another Brownian motion W_t, with an instantaneous correlation $dB_t dW_t = \rho dt$ with a parameter $\rho \in [0,1]$ between the two different Brownian motions. The Heston model [26] proposes that $\sigma_t = \sqrt{\nu_t}$ where ν_t follows a Cox-Ingersoll-Ross equation

$$d\nu_t = k\Big[\theta - \nu_t\Big]dt + \lambda\sqrt{\nu_t}dW_t. \tag{1.4}$$

Here $k, \theta, \lambda > 0$ are parameters, and the two Brownian motions again possess an instantaneous correlation $dB_t dW_t = \rho dt$ with $\rho \in [0,1]$.

Still, the stock model (1.2) does not account for certain memory effects in $\log S_t$. This seems to require a more radical solution than simply making the coefficients time dependent, but keeping standard Brownian motion as the underlying stochastic process. This issue was raised already very early in [32], which suggested that the standard Brownian motion in (1.2) should be generalized to self similar processes, including fractional Brownian motions B^H, i.e., a family of centered Gaussian processes $B^H = \{B^H(t)\}$, $t \in \mathbb{R}$ or $\mathbb{R}_+$ with continuous sample paths and covariance function

$$R_H(s,t) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad \forall t, s \in \mathbb{R}.$$

In [11], ordinary Brownian motion is replaced by fractional Brownian motion in the model (1.3) for the variance σ_t, resulting in

$$d\log\sigma_t = k(\theta - \log\sigma_t)dt + \gamma dB_t^H. \tag{1.5}$$

In order to obtain a process with long memory, a Hurst exponent $H > \frac{1}{2}$ is needed. Recent empirical studies [4], [21] however showed that if we assume the model (1.5), the log-volatility behaves essentially as a fractional Brownian motion with Hurst exponent H of order 0.1 at any reasonable time scale, and thus, we do not have a long memory process. (As we shall see below, however, when the empirical data are fit to other models, larger Hurst exponents emerge. Thus, this is a feature of the model and not of the data). This observation motivates a study in [5], in which the authors suggest a more general dynamic model of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= f(Z_t)(\rho dW_t + \sqrt{1-\rho^2}dW_t') \\ Z_t &= z + \int_0^t K(s,t)v(Z_s)ds + \int_0^t K(s,t)u(Z_s)dW_s, \end{aligned} \tag{1.6}$$

where K is a kernel, W, W' are independent Brownian motions and f, u, v are sufficiently smooth functions.

Another important stylized fact is the multi-scaling phenomenon in financial data, and in particular that the generalized Hurst exponent varies depending on the time scale (see e.g. [3], [15], [2], [7], [8]). The multi-scaling issue can be explained either by considering a random time change of the Brownian noise B_t through the time change process I_t, i.e. $X_t = B_{I_t}$ for all $t \geq 0$ (see [2]); or by assuming that the noise X_t has the form

$$X_h(t) = \sum_{k=1}^{\lfloor \frac{t}{h} \rfloor} e^{\omega_h(k)}\Big(B^H_{(k+1)h} - B^H_{kh}\Big) \tag{1.7}$$

for some Hurst exponent $H \geq \frac{1}{2}$, $\omega_h(\cdot) \sim \mathcal{N}(0, \lambda^2\log(\frac{L}{h}))$ with the intermittency parameter λ and the autocorrelation length L, and the time scale h such that the $\omega_h(k)$ are correlated up to the distance L, i.e.

$$\mathrm{Cov}(\omega_h(k_1), \omega_h(k_2)) = \lambda^2\rho_h(|k_1-k_2|), \quad \rho_h(|k_1-k_2|) = \begin{cases} \frac{L}{(|k_1-k_2|+1)h} & \text{for} \quad |k_1-k_2| \leq \frac{L}{h} - 1 \\ 1 & \text{otherwise} \end{cases}$$

(see [3] and [8]).

Studies in [15] and [7] (see also our computations in Subsection 4.1) reveal that the log return of stock indices (for instance Nasdaq, Dow Jones, SP500) or various exchange rates has the Hurst exponents ranging from 0.45 to 0.7 in some period. This leaves it open whether or not the noise should have long memory, and why the Hurst exponents often look smaller than $\frac{1}{2}$ for data quoted on timescales of minutes to hours but increase to values significantly larger than $\frac{1}{2}$ for daily to monthly quoted data. It is generally agreed, however, that the Hurst exponent for the log return of the stock price is always bigger than $\frac{1}{3}$.

Of course, one can introduce models with many parameters, in order to match the empirical data. Our approach is different, because we want to develop a general mathematical framework within which the observed phenomena find a conceptual explanation. In fact, the basic and robust relation that emerges from our numerical investigation of a wide range of stocks and other indices is that when we plot the daily mean-variance relation of the logarithmic return, we find a parabola shaped curve as a lower envelope. In particular, the relation is not linear, as basic models suggest, and the nonlinearity exhibits a clear structure. This asks for a systematic explanation. The mathematical theory we shall draw upon for that purpose is rough path theory [31], [20], [19], a well-established and very powerful mathematical approach to stochastic processes. The theory can handle rather general driving noises in a systematic manner. The key idea consists in directly incorporating higher order information about the noise to define certain integrals in an algebraic manner. While we are not the first to apply rough path theory to model financial data, we should nevertheless take this opportunity to point out some conceptual aspects. First, the theory provides a unified and elegant treatment for all kinds of driving noises, whose effects can then be compared in an efficient manner. A rough path can be defined solely on the basis of simple algebraic relations that are naturally satisfied by all kinds of stochastic integrals. While the traditional stochastic view is based on unpredictability or uncertainty, the rough path approach emphasizes the control of a path by some underlying noise and its properties. Secondly, the noise naturally needs to operate only along the considered or observed path, and no assumption about noise along unobserved paths is required. The pathwise approach is therefore very well compatible with data oriented methods.

Thus, we propose a new model using rough path theory, which covers all well-known cases and phenomena. That is, we do not interpret the stochastic system in (1.2) in the sense of Itô, but attempt to solve in the pathwise sense the general form

$$dS_t = \mu_t S_t dt + \sigma_t S_t dX_t$$

for a certain Gaussian noise X, where the second integral $\int \sigma S dX$ is understood as a rough integral as defined by Gubinelli [24]. In that framework, the information from a realization x of X is not enough and additional information of a *rough path lift* $\mathbf{x} = (x, [\omega, dx], [x, dx])$ is required, where ω is a realization of another independent source of Gaussian noise W on which the volatility σ depends, and $[\omega, dx], [x, dx]$ are the corresponding *Levy areas* satisfying Chen's relation. Our rough path approach therefore needs more information than just a driving noise X and its truncated signature $(X, [X, dX])$ as often seen in the literature on rough path theory. As proved in Theorem 2.5, the equation can be solved to have an explicit pathwise solution

$$\log S_t(\cdot) = \log S_\tau(\cdot) + \int_\tau^t \mu_u du - \frac{1}{2}\int_\tau^t \sigma_u^2(\cdot) d[X]_u(\cdot) + \int_\tau^t \sigma_u(\cdot) dX_u(\cdot),$$

where $\int \sigma^2 d[X]$ is a pathwise Young integral with respect to the bracket process $[X]$ defined in [19] and $\int \sigma dX$ is a pathwise rough integral defined in the Gubinelli sense with respect to the rough path lift $\mathbf{X} = (X, [W, dX], [X, dX])$. The advantage of the model is that, while the price process S itself is controlled in the pathwise sense by X and its realizations, σ is naturally driven in the pathwise sense by both X and W, thus its driving noises are correlated with the noise X of the price process S as suggested by the Hull & White model, and moreover its regularity can be as low as that of W, which is captured by empirical evidence in [4, 21] (recall that we mentioned above that a Hurst exponent can be as low as .1). For a class of such functions σ we refer to Remark 2.6. And in particular, without additional effort of detecting signatures of higher orders, this will allow us to go below the Hölder exponent 1/3 that naturally appears in rough path theory.

The no-arbitrage principle has been viewed as the fundamental requirement for a model to satisfy the *efficient market hypothesis* (EMH). For models based on standard Brownian motion like (1.1), (1.2) or (1.6), this is usually not a problem. When the stochastic noise in the asset price model comes from fractional Brownian motions or in general Gaussian processes which display the long rang dependence observed in empirical data, it has however been shown, e.g. in [43] or in [9] that the model allows for arbitrage. But the model in [34] which uses the Skorohod-Wick-Itô integral shows that the existence of arbitrage can be avoided. Another solution to this arbitrage problem comes from [10], [30], [33, Chapter 5, pp. 305-306] which assumes that the noise is the mixture of a standard Brownian noise B and a fractional Brownian motion B^H for $H \in (\frac{1}{2}, 1)$. Subsequently, the no simple arbitrage statement was proved in [6] for the wider class $\exp\{Z_t + \sigma B_t\}$ of geometric mixed noise, where $(B, \mathcal{F})$ is a standard Brownian motion and Z is an $\mathcal{F}$-adapted process independent of B, although there might still exist arbitrage in the weak form if one does not specify assumptions on available trading strategies.

In contrast to that approach, in this paper, we follow [22], [23] and assume transaction costs (see also the related works [13] and the references therein in the context of fractional price processes). The reason is that a sufficient condition for no-arbitrage only requires the log-price $\log S_t$ to be a *sticky process*. Also, it turns out that the class of sticky processes is very large, since it contains strong Markov processes or any stochastic process with *conditional full support* (CFS). The stickiness has also been studied later in [42] for a larger class of stochastic processes. The CFS criterion was extended in [37] for a class of mixed noise $Z_t + X_t$, where Z is an arbitrary continuous process, and X is a process independent of Z that has CFS. This class includes also mixed forms of mutually independent standard Brownian motions and fractional Brownian motions. In particular, in Section 3 we show in Theorem 3.1 that, in the presence of transaction costs, the condition on CFS of two independent processes $X, [X]$ leads to the CFS of the logarithm price process $\log S_t$ when the (stochastic) volatility has a positive lower bound. Consequently, the process S_t is arbitrage-free.

The above model also shows that there is a freedom to first allow $[X]$ to vary in a stochastic way and then use $X, [X]$ to define the Levy area $[X, dX]$ through Chen's relation, thus leading to a mixture of noises. We thus proceed in Section 4 to explain empirical data within our framework. The multi-scaling phenomenon can be explained by a rough model in Subsection 4.1, where the Hurst exponent is computed from the linear regression method between the logarithms of the variance and the duration. Moreover, our empirical analysis in Subsection 4.2 finds that there is a nonlinear mean-variance relation, which could not be explained by using the classical model (1.1) or (1.2) but naturally follows in the framework of the rough model with such an additional noise $[X]$. It is important to note that the model works under the assumption on the stochasticity of the bracket path $[X]$, which turns out to match very well with Peng's theory of G-Brownian motion $\hat{B}$ [38], which has uncertainty in its quadratic variation process $\langle \hat{B} \rangle$ (see Example 2.2). Also, it suggests to make a connection to related work [14] on the pricing of contingent claims under volatility uncertainty.

Finally, we show in Section 5 by a simple example that the well-known negative effect of the variance on the expected log-return can indeed come from the stochastic process $[X]$. Namely, the appearance of the additional noise $[X]$ does increase the variance of the log-return. We therefore raise the problem of the model risk, in which the volatility increases from stock model ambiguity, and the uncertainty comes from the rough path lifts. At the end, we also consider the cooperation problem in ergodicity economics [40] for the rough model, and discuss the fact that different stochastic formulations might result in different conclusions on cooperation effects. In particular, in one setting, we can show that cooperation can mitigate the effects of volatility and is therefore beneficial.

2 Rough model of stock prices

2.1 Rough path theory

This section presents a brief introduction to rough path theory for Hölder continuous paths. For a compact time interval $[a,b] \subset \mathbb{R}$, define $\Delta^2([a,b]) := \{(s,t) : a \leq s \leq t \leq b\}$. Let $C([a,b],\mathbb{R})$ denote the space of all continuous paths $y : [a,b] \to \mathbb{R}$ equipped with the sup-norm $\|\cdot\|_{\infty,[a,b]}$ given by $\|y\|_{\infty,[a,b]} = \sup_{t\in[a,b]} \|y_t\|$, where $\|\cdot\|$ is simply the absolute value.[1] We write $y_{s,t} := y_t - y_s$. For $0 < \alpha < 1$, we denote by $C^\alpha([a,b],\mathbb{R})$ the space of Hölder continuous functions with exponent α on $[a,b]$ equipped with the norm

$$\|y\|_{\alpha,[a,b]} := \|y_a\| + \vert\!\vert\!\vert y\vert\!\vert\!\vert_{\alpha,[a,b]} = \|y_a\| + \sup_{s<t\in[a,b]} \frac{\|y_{s,t}\|}{(t-s)^\alpha},$$

Similar to [31], [24] and [19], a path $x \in C^\alpha([a,b],\mathbb{R})$ can be lifted to a tuple $\mathbf{x} = (x, [\omega, dx], [x, dx]) \in C^\alpha([a,b],\mathbb{R}) \oplus C^{\alpha+\beta}([a,b]^2, \mathbb{R}\otimes\mathbb{R}) \oplus C^{2\alpha}([a,b]^2, \mathbb{R}\otimes\mathbb{R})$ with respect to an additional path $\omega \in C^\beta([a,b],\mathbb{R})$ with $0 < \alpha, \beta \leq \frac{1}{2}$, where

$$\begin{aligned}
[\omega, dx] \in C^{\alpha+\beta}([a,b]^2, \mathbb{R}\otimes\mathbb{R}) &:= \Big\{\Xi_{\cdot,\cdot} \in C([a,b]^2, \mathbb{R}\otimes\mathbb{R}) : \sup_{s,t\in[a,b],\ s\neq t} \frac{\|\Xi_{s,t}\|}{|t-s|^{\alpha+\beta}} < \infty\Big\}, \\
[x, dx] \in C^{2\alpha}([a,b]^2, \mathbb{R}\otimes\mathbb{R}) &:= \Big\{\Xi'_{\cdot,\cdot} \in C([a,b]^2, \mathbb{R}\otimes\mathbb{R}) : \sup_{s,t\in[a,b],\ s\neq t} \frac{\|\Xi'_{s,t}\|}{|t-s|^{2\alpha}} < \infty\Big\},
\end{aligned} \tag{2.1}$$

are called *Levy areas* if they satisfy Chen's relation

$$\begin{aligned}
\delta[\omega, dx]_{s,u,t} &= [\omega, dx]_{s,t} - [\omega, dx]_{s,u} - [\omega, dx]_{u,t} = \omega_{s,u} x_{u,t}, \\
\delta[x, dx]_{s,u,t} &= [x, dx]_{s,t} - [x, dx]_{s,u} - [x, dx]_{u,t} = x_{s,u} x_{u,t}, \quad \forall a \leq s \leq u \leq t \leq b.
\end{aligned} \tag{2.2}$$

We shall call $\mathbf{x}$ a *rough path lift*. The set $\mathcal{C}^{\alpha,\beta}([a,b], \mathbb{R}\oplus(\mathbb{R}\otimes\mathbb{R})\oplus(\mathbb{R}\otimes\mathbb{R})) \subset C^\alpha([a,b],\mathbb{R}) \oplus C^{\alpha+\beta}([a,b]^2, \mathbb{R}\otimes\mathbb{R}) \oplus C^{2\alpha}([a,b]^2, \mathbb{R}\otimes\mathbb{R})$ of all rough path lifts $\mathbf{x}$ on $[a,b]$ is then a closed set (but not a linear space), equipped with the rough path semi-norm

$$\begin{aligned}
\vert\!\vert\!\vert \mathbf{x}\vert\!\vert\!\vert_{\alpha,\beta,[a,b]} &:= \vert\!\vert\!\vert x\vert\!\vert\!\vert_{\alpha,[a,b]} + \vert\!\vert\!\vert [\omega, dx]\vert\!\vert\!\vert^{\frac{1}{2}}_{\alpha+\beta,[a,b]^2} + \vert\!\vert\!\vert [x, dx]\vert\!\vert\!\vert^{\frac{1}{2}}_{2\alpha,[a,b]^2} < \infty, \quad \text{where} \\
\vert\!\vert\!\vert [\omega, dx]\vert\!\vert\!\vert_{\alpha+\beta,[a,b]^2} &:= \sup_{s,t\in[a,b];s\neq t} \frac{\|[\omega, dx]_{s,t}\|}{|t-s|^{\alpha+\beta}}, \quad \vert\!\vert\!\vert [x, dx]\vert\!\vert\!\vert_{2\alpha,[a,b]^2} := \sup_{s,t\in[a,b];s\neq t} \frac{\|[x, dx]_{s,t}\|}{|t-s|^{2\alpha}}.
\end{aligned} \tag{2.3}$$

In the absence of such an ω, this simply reduces to the traditional rough path $\mathbf{x} = (x, [x, dx])$ as in [31].

Remark 2.1 One can view $[\omega, dx]$ as *postulating* the value of the quantity

$$\int_s^t \omega_{s,r} dx_r := [\omega, dx]_{s,t}, \tag{2.4}$$

where the right hand side is taken as a definition for the left hand side, so that $\int \omega dx$ is written only symbolically. For a non-trivial example, consider two independent centered Gaussian processes W and X satisfying

$$E|W_{s,t}|^{2p} \leq C_W |t-s|^{2pH}, \quad E|X_{s,t}|^{2q} \leq C_W |t-s|^{2qH'}, \quad \forall 0 \leq s \leq t \leq T \tag{2.5}$$

[1] The theory can be extended to vector-valued paths in $C([a,b],\mathbb{R}^d)$ without undue difficulties, but for our purposes the scalar-valued case suffices.

where $0 < H, H' < 1 < p, q$ such that $pH, qH' > 1$ and $H + H' > \frac{1}{2}$. Then one can follows [19, Chapter 10] to prove that, given the condition on the regularity of the incremental covariances

$$R_X \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} := EX_{s,t}X_{s',t'}, \quad R_W \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} := EW_{s,t}W_{s',t'}, \quad \forall s < t, s' < t'$$

such that for all $a \le s \le t \le b$

$$\begin{aligned} \|R_X\|_{p-\mathrm{var},[s,t]^2} &= \left(\sup_{\mathcal{P}(s,t),\mathcal{P}'(s,t)} \left| R_X \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} \right|^p \right)^{\frac{1}{p}} \le M_X |t-s|^{2H}, \\ \|R_W\|_{q-\mathrm{var},[s,t]^2} &= \left(\sup_{\mathcal{P}(s,t),\mathcal{P}'(s,t)} \left| R_W \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} \right|^q \right)^{\frac{1}{q}} \le M_W |t-s|^{2H'}, \end{aligned}$$

where the suprema are taken over finite partitions $\mathcal{P}, \mathcal{P}'$ of $[s,t]$, it is possible to define the stochastic integral

$$[W, dX]_{s,t} = \int_s^t W_{s,u} dX_u := \lim_{|\mathcal{P}(s,t)| \to 0} \sum_{t_i \in \mathcal{P}(s,t)} W_{s,t_i} X_{t_i,t_{i+1}} \tag{2.6}$$

as the L^2-limit of the Stieltjes integrals on finite partitions $\mathcal{P}(s,t) = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of $[s,t]$ with $|\mathcal{P}(s,t)| := \max\limits_{[u,v] \in \mathcal{P}(s,t)} |v-u|$. Moreover,

$$E\left(\int_s^t W_{s,u} dX_u \right)^2 \le C_{H,H'} \|R_X\|_{p-\mathrm{var},[s,t]^2} \|R_W\|_{q-\mathrm{var},[s,t]^2} \le C_{H,H',X,W} |t-s|^{2(H+H')}$$

holds. As a result, for any $\alpha \in (0,H), \beta \in (0,H')$ such that $\alpha + \beta > \frac{1}{2}$ and $p\alpha, q\beta > 1$, there exists a version of W, X and $[W, dX]$ such that a.s. all realizations $\omega \in C^\beta, x \in C^\alpha, [\omega, dx] \in C^{\alpha+\beta}$ respectively. Moreover, $[\omega, dx]$ satisfies the Chen relation (2.2).

Below, let us review some specific examples for $[x, dx]$.

Example 2.2 1. When x is a realization of a local martingale X, for instance $X_t = \int_0^t a_s dB_s$, we define the stochastic integral $\int y dX$ as the integral w.r.t. the local martingale X [18, Section 2.5]. We can then apply the Itô formula [18, Section 2.8, p.64]

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) dX_u + \frac{1}{2} \int_s^t f''(X_u) d\langle X \rangle_u$$

for any function $f \in C^2$, where $\langle X \rangle_t$ is the quadratic variance process, to compute explicitly

$$[X, dX]_{s,t} = \int_s^t X_{s,u} dX_u = \frac{1}{2} X_{s,t}^2 - \frac{1}{2} \Big(\langle X \rangle_t - \langle X \rangle_s \Big).$$

In particular if $X_t = \int_0^t a_s dB_s$, then $\langle X \rangle_t = \int_0^t a_u^2 du$. Hence, in the pathwise sense, it is easy to check that $[X, dX]$ satisfies Chen's relation (2.2) a.s.

2. When $X = B^H$ is a fractional Brownian motion which is not a semi-martingale [43], we cannot apply the classical Itô calculus, but define the stochastic integral $\int y \delta B^H$ in the sense of Skorohod-Wick-Itô by using the Wick product as in [34, Chapter 5]. Then by using the Wick-Itô formula [34] for the Skorohod-Wick-Itô integral

$$f(B_t^H) - f(B_s^H) = \int_s^t H u^{2H-1} f''(B_u^H) du + \int_s^t f'(B_u^H) \delta B_u^H \tag{2.7}$$

for any function $f \in C^2$, we can compute explicitly

$$[B^H, dB^H]_{s,t} \;\; := \;\; \int_s^t B^H_{s,u} \delta B^H_u = \frac{1}{2}(B^H_{s,t})^2 - \frac{1}{2}\Big(t^{2H} - s^{2H}\Big).$$

In general, the signatures $[X, dX]$ can also be defined for a scalar centered Gaussian process of the form $X_t = \int_0^t K(t,s)dB_s$ where B is a standard Brownian motion, and $K(t,s)$ is a square integrable kernel. In particular, using the Itô-type formula

$$f(X_t) - f(X_s) = \int_s^t f'(X_u)\delta X_u + \frac{1}{2}\int_s^t f''(X_u)dR_u \tag{2.8}$$

for a function $f \in C^2$, where $R_u = E(X_u)^2 = \int_0^u K^2(u,s)ds$ and the stochastic integral $\int \delta X$ can be computed as the limit of Riemann sums defined w.r.t. the Wick product [1]. Thus, $[X, dX]_{s,t} := \int_s^t X_{s,u}\delta X_u$ can be computed explicitly, which satisfies Chen's relation (2.2) a.s.

3. When $X = \hat{B}$ is a G-Brownian motion together with its quadratic variation process $\langle \hat{B} \rangle$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{B}_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ [38], then one can prove (see e.g. [39, Proposition 3.2]) that

$$\|\hat{B}_{s,t}\|_{L^q_G} \le C_{q,\overline{\sigma}}|t-s|^{\frac{1}{2}}, \quad \|[\hat{B}, d\hat{B}]_{s,t}\|_{L^{\frac{q}{2}}_G} \le C_{q,\overline{\sigma}}|t-s|, \quad \forall q > 2,$$

where $[\hat{B}, d\hat{B}]_{s,t} = \int_s^t \hat{B}_{s,u}d\hat{B}_u$. As a result, one has the G-sublinear expectation version of the Kolmogorov criterion [39, Theorem 3.1] that for all $\nu \in [0, \frac{1}{2} - \frac{1}{q})$, $(\hat{B}, [\hat{B}, d\hat{B}])$ has a continuous modification and there exists $K_\nu \in L^q_G, \mathbb{K}_\nu \in L^{\frac{q}{2}}_G$ such that for any $s,t \in [0,T]$ one has inequalities

$$|\hat{B}_{s,t}| \le K_\alpha |t-s|^\nu, \quad |[\hat{B}, d\hat{B}]_{s,t}| \le \mathbb{K}_\nu |t-s|^{2\nu}.$$

In particular choosing $q > 6$ and $\nu \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{q})$, one can then compute explicitly

$$[\hat{B}, d\hat{B}]_{s,t} = \frac{1}{2}(\hat{B}^2_{s,t} - \langle \hat{B} \rangle_{s,t}).$$

The definition is such that in general $[\omega, dx] \neq [x, d\omega]$. We also define $\{\omega, x\}$ by

$$\{\omega, x\}_{s,t} = \omega_{s,t}x_{s,t} - [\omega, dx]_{s,t} - [x, d\omega]_{s,t}, \quad \forall a \le s \le t \le b. \tag{2.9}$$

Then it is easy to check that $\{\omega, x\} \in C^{\alpha+\beta}$ with

$$\{\omega, x\}_{s,t} = \{\omega, x\}_{s,u} + \{\omega, x\}_{u,t}, \quad \forall a \le s \le u \le t \le b,$$

so one can redefine $\{\omega, x\}_{s,t} = \{\omega, x\}_{a,t} - \{\omega, x\}_{a,s}$, where $\{\omega, x\}_{a,\cdot} \in C^{\alpha+\beta}([a,b], \mathbb{R})$ is a Hölder continuous path. In Remark 2.1, the integrals $\int W dX$ and $\int X dW$ are well-defined as in (2.6) for two independent centered Gaussian processes and moreover satisfy

$$\int_s^t W_{s,u}dX_u + \int_s^t X_{s,u}dW_u = W_{s,t}X_{s,t}$$

hence $\{W, X\} \equiv 0$ a.s.
In particular, for $\omega \equiv x$, $\{x, x\}$ coincides with the bracket path in [19]

$$[x]_{s,t} := x^2_{s,t} - 2[x, dx]_{s,t}, \quad \forall s \le t. \tag{2.10}$$

which satisfies Chen's relation (2.2), hence $[x]_{s,t} = [x]_{a,t} - [x]_{a,s}$ where $[x]_{a,\cdot} \in C^{2\alpha}([a,b], \mathbb{R})$ is a Hölder continuous path. It is important to note that, using (2.10) one can use x and $[x]$ as inputs to define (uniquely) $[x, dx]_{s,t} = \frac{1}{2}(x^2_{s,t} - [x]_{s,t})$.

Remark 2.3 In Example 2.2, we can compute $[X]$ explicitly as follows

$$[X]_{s,t} := \begin{cases} t-s & \text{if} \quad X_t = B_t \\ \langle X\rangle_t - \langle X\rangle_s & \text{if} \quad X_t \quad \text{is a local martiangle} \\ \int_0^t K(t,u)^2 du - \int_0^s K(s,u)^2 du & \text{if} \quad X_t = \int_0^t K(t,s) dB_s \\ 0 & \text{if} \quad X_\cdot \in C^\nu, \nu > \frac{1}{2} \\ \langle \hat{B}\rangle_t - \langle \hat{B}\rangle_s & \text{if} \quad X_t = \hat{B}_t \end{cases} . \tag{2.11}$$

We observe that $[X]$ is non-random for the four first cases in (2.11). But the fifth case is special since $[X] = \langle \hat{B}\rangle$ is truly stochastic, and even of mean-uncertainty. In fact, $\langle \hat{B}\rangle$ has stationary increments and is independent of $\hat{B}$ with

$$\hat{\mathbb{E}}[\langle \hat{B}\rangle_t - \langle \hat{B}\rangle_s] = \overline{\sigma}^2(t-s) \geq -\hat{\mathbb{E}}[-\Big(\langle \hat{B}\rangle_t - \langle \hat{B}\rangle_s\Big)] = \underline{\sigma}^2(t-s), \quad \forall t \geq s, \tag{2.12}$$

where $\underline{\sigma}^2 := -\hat{\mathbb{E}}[-\hat{B}_1^2] \leq \hat{\mathbb{E}}[\hat{B}_1^2] =: \overline{\sigma}^2$, and $\hat{\mathbb{E}}$ is the sub-linear expectation, see [38].

2.2 Rough integrals

For $y \in \mathcal{C}^\beta([a,b],\mathbb{R})$ and $x \in \mathcal{C}^\nu([a,b],\mathbb{R})$ with $\beta + \nu > 1$, the Young integral $\int_a^b y_t dx_t$ can be defined as

$$\int_a^b y_s dx_s := \lim_{|\Pi[a,b]|\to 0} \sum_{[u,v]\in\Pi[a,b]} y_u x_{u,v}, \tag{2.13}$$

where the limit is taken over all finite partitions $\Pi[a,b]$ of $[a,b]$ (see [47, p. 264–265]). This integral satisfies the additivity property by construction, as well as the so-called Young-Loeve estimate [20, Theorem 6.8, p. 116]

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} \Big\| \leq K_{\beta,\nu} |t-s|^{\beta+\nu} \, \|y\|_{\beta,[s,t]} \, \|x\|_{\nu,[s,t]} , \tag{2.14}$$

for all $[s,t] \subset [a,b]$, where $K_{\beta,\nu} := (1 - 2^{1-\beta-\nu})^{-1}$.
Next, we introduce the construction of rough integrals in case $0 < \beta, \alpha < \frac{1}{2}$ such that

$$3\alpha > \beta + 2\alpha > 1. \tag{2.15}$$

Following [24], a path $y \in C^\beta([a,b],\mathbb{R})$ is said to be *controlled by* $(\omega,x)^{\mathrm{T}} \in C^\beta([a,b],\mathbb{R}) \otimes C^\alpha([a,b],\mathbb{R})$ if there exists a tube $(\partial_\omega y, \partial_x y, R^y)$ with a path $\partial_\omega y \in C^\alpha([a,b],\mathcal{L}(\mathbb{R},\mathbb{R})), \partial_x y \in C^\beta([a,b],\mathcal{L}(\mathbb{R},\mathbb{R}))$ and a remainder $R^y \in C^{\alpha+\beta}([a,b]^2,\mathbb{R})$ such that

$$y_{s,t} = (\partial_\omega y)_s \omega_{s,t} + (\partial_x y)_s x_{s,t} + R^y_{s,t}, \qquad \forall a \leq s \leq t \leq b. \tag{2.16}$$

$\partial_\omega y, \partial_x y$ are called the Gubinelli (partial) derivatives of y with respect to ω, x respectively, which are uniquely defined as long as ω, x are *truly rough* [19, Proposition 6.4], i.e. $\omega \in C^\beta([a,b],\mathbb{R}) \setminus C^{2\beta}([a,b],\mathbb{R})$ and $x \in C^\alpha([a,b],\mathbb{R}) \setminus C^{2\alpha}([a,b],\mathbb{R})$.

Example 2.4 Consider $y_t = g(x_t)\omega_t$ on $[a,b]$ with $g \in C^2(\mathbb{R})$, then it is easy to check that

$$\begin{aligned} y_{s,t} &= \omega_s(g(x_t) - g(x_s)) + g(x_s)\omega_{s,t} + (g(x_t) - g(x_s))\omega_{s,t} \\ &= \omega_s g'(x_s) x_{s,t} + g(x_s)\omega_{s,t} + \mathcal{O}(|x_{s,t}||\omega_{s,t}|) + \mathcal{O}(|x_{s,t}|^2) \\ &= \omega_s g'(x_s) x_{s,t} + g(x_s)\omega_{s,t} + \mathcal{O}(|t-s|^{\alpha+\beta}), \end{aligned}$$

where $g(x_\cdot), g'(x_\cdot) \in C^\alpha([a,b],\mathbb{R})$ and $\omega.g'(x_\cdot) \in C^\beta([a,b],\mathbb{R})$. Hence $y = g(x)\omega$ is controlled by $(\omega,x)^{\mathrm{T}}$ in the sense of (2.16). This example shows that there is flexibility in the regularity of Gubinelli partial derivatives of y, as long as (2.15) is satisfied.

Denote by $\mathcal{D}^{\alpha,\beta}([a,b])$ the space of all the tuple $(y, \partial_\omega y, \partial_x y)$ controlled by $(\omega, x)^{\mathrm{T}}$, then $\mathcal{D}^{\alpha,\beta}([a,b])$ is a Banach space equipped with the norm

$$\begin{aligned}
\|(y,\partial_\omega y,\partial_x y)\|_{\alpha,\beta,[a,b]} &:= \|y_a\| + \|(\partial_\omega y)_a\| + \interleave(y,\partial_\omega y,\partial_x y)\interleave_{\alpha,\beta,[a,b]}, \\
\interleave(y,\partial_\omega y,\partial_x y)\interleave_{\alpha,\beta,[a,b]} &:= \interleave\partial_\omega y\interleave_{\alpha,[a,b]} + \interleave\partial_x y\interleave_{\beta,[a,b]} + \interleave R^y\interleave_{\alpha+\beta,[a,b]^2}.
\end{aligned}$$

For a rough path lift $\mathbf{x} = (x, [\omega, dx], [x, dx])$ and any controlled rough path $(y, \partial_\omega y, \partial_x y) \in \mathcal{D}^{\alpha,\beta}([a,b])$, one can easily check from Chen's relation (2.2) that $F_{s,t} := y_s x_{s,t} + (\partial_\omega y)_s[\omega, dx]_{s,t} + (\partial_x y)_s[x, dx]_{s,t}$ satisfies

$$\begin{aligned}
\delta F_{s,u,t} &= -y_{s,u}x_{u,t} - (\partial_\omega y)_{s,u}[\omega,dx]_{u,t} + (\partial_\omega y)_s\omega_{s,u}x_{u,t} - (\partial_x y)_{s,u}[x,dx]_{u,t} + (\partial_x y)_s x_{s,u}x_{u,t} \\
&= -R^y_{s,u}x_{u,t} - (\partial_\omega y)_{s,u}[\omega,dx]_{u,t} - (\partial_x y)_{s,u}[x,dx]_{u,t} = \mathcal{O}(|t-s|^{2\alpha+\beta}).
\end{aligned}$$

Because of condition (2.15), the integral $\int_s^t y_u dx_u$ can be defined, thanks to the sewing lemma [24], [12], for any $a \leq s \leq t \leq b$ as

$$\int_s^t y_u dx_u = \int_s^t y_u d\mathbf{x}_u := \lim_{|\mathcal{P}[s,t]|\to 0} \sum_{[u,v]\in\mathcal{P}[s,t]} \Big(y_u x_{u,v} + (\partial_\omega y)_u[\omega,dx]_{u,v} + (\partial_x y)_u[x,dx]_{u,v} \Big) \quad (2.17)$$

where the limit is taken over all finite partitions $\mathcal{P}[s,t]$ of $[s,t]$ and is thus independent of $\mathcal{P}$. This rough integral is additive, i.e. $\int_s^t ydx = \int_s^\tau ydx + \int_\tau^t ydx$ for any $s \leq \tau \leq t$. Moreover, there exists a constant $C_\alpha > 1$, such that

$$\begin{aligned}
&\Big\| \int_s^t y_u dx_u - y_s x_{s,t} - (\partial_\omega y)_s[\omega,dx]_{s,t} - (\partial_x y)_s[x,dx]_{s,t} \Big\| \\
&\leq C_\alpha |t-s|^{2\alpha+\beta}\Big(\interleave x\interleave_{\alpha,[s,t]} \interleave R^y\interleave_{\alpha+\beta,[s,t]^2} + \interleave\partial_\omega y\interleave_{\alpha,[s,t]} \interleave[\omega,dx]\interleave_{\alpha+\beta,[s,t]^2} \\
&\qquad + \interleave\partial_x y\interleave_{\beta,[s,t]} \interleave[x,dx]\interleave_{2\alpha,[s,t]^2} \Big).
\end{aligned} \quad (2.18)$$

Note that if $\partial_\omega y \equiv 0$, then (2.16) implies that $y \in C^\alpha([a,b],\mathbb{R})$. In that case if $\beta = \alpha$ one goes back to the Gubinelli rough integral $\int ydx$ for controlled paths y by the driving rough path x (see [24]). Obviously, when y is constant, (2.17) reduces to $\int_s^t y_u dx_u = yx_{s,t}$. In addition, if $x \in C^1$, then the rough integral (2.17) becomes the ordinary Riemann-Stieltjes integral.

2.3 Time dependent rough models

The second equation in model (1.6) suggests that the volatility is time dependent and controlled by another source of noise W which might be of low regularity. This leads to the question how to understand the stochastic integral in the first equation of (1.6) when the Itô calculus may not be applicable. This motivates us to extend the construction of rough integrals in the Gubinelli sense in Subsection 2.2. Accordingly, given parameters $\frac{1}{2} > \alpha > \beta > 0$ which also satisfy (2.15), let us consider the time dependent model

$$dS_t = \mu_t S_t dt + \sigma_t S_t dx_t \quad (2.19)$$

where $\mu : [0,T] \to \mathbb{R}$ is Lebesgue integrable and $\sigma \in C^\beta([0,T],\mathbb{R})$ is controlled by a realization $(\omega, x)^{\mathrm{T}} \in C^\beta([0,T],\mathbb{R}) \otimes C^\alpha([0,T],\mathbb{R})$ in the sense (2.16). Equation (2.19) is understood in the integral form

$$S_t = S_0 + \int_0^t \mu_u S_u du + \int_0^t \sigma_u S_u dx_u, \quad (2.20)$$

where the second integral in (2.20) is well-defined as a rough integral (2.17). Indeed, one should look for a solution $S \in C^{\alpha}([0,T],\mathbb{R})$ that is controlled by $(\omega,x)^{\mathrm{T}}$ in the sense of (2.16) such that $S_{\tau,t} = (\partial_x S)_\tau x_{\tau,t} + R^S_{\tau,t}$ for some $\partial_x S \in C^{\beta}([0,T],\mathbb{R}), R^S \in C^{\alpha+\beta}([0,T],\mathbb{R}), \partial_\omega S \equiv 0$. Since $\alpha > \beta$,

$$
\begin{aligned}
\sigma_t S_t - \sigma_\tau S_\tau &= \Big(\sigma_\tau + (\partial_\omega \sigma)_\tau \omega_{\tau,t} + (\partial_x \sigma)_\tau x_{\tau,t} + R^\sigma_{\tau,t}\Big)\Big(S_\tau + (\partial_x S)_\tau x_{\tau,t} + R^S_{\tau,t}\Big) - \sigma_\tau S_\tau \\
&= (\partial_\omega \sigma)_\tau S_\tau \omega_{\tau,t} + \Big(\sigma_\tau (\partial_x S)_\tau + (\partial_x \sigma)_\tau S_\tau\Big) x_{\tau,t} + R^\sigma_{\tau,t}\Big(S_\tau + (\partial_x S)_\tau x_{\tau,t} + R^S_{\tau,t}\Big) + \\
&\quad + \Big(\sigma_\tau + (\partial_\omega \sigma)_\tau \omega_{\tau,t} + (\partial_x \sigma)_\tau x_{\tau,t}\Big) R^S_{\tau,t} + \Big((\partial_\omega \sigma)_\tau \omega_{\tau,t} + (\partial_x \sigma)_\tau x_{\tau,t}\Big)(\partial_x S)_\tau x_{\tau,t} \\
&= (\partial_\omega \sigma)_\tau S_\tau \omega_{\tau,t} + \Big(\sigma_\tau (\partial_x S)_\tau + (\partial_x \sigma)_\tau S_\tau\Big) x_{\tau,t} + R^{\sigma S}_{\tau,t},
\end{aligned} \tag{2.21}
$$

for some $R^{\sigma S} \in C^{\alpha+\beta}([0,T]^2,\mathbb{R})$. Hence, $\sigma S \in C^{\beta}([0,T],\mathbb{R})$ is controlled by $(\omega,x)^{\mathrm{T}}$ in the sense (2.16). Therefore the rough integral $\int_a^b \sigma S dx$ is well-defined as

$$
\begin{aligned}
&\int_a^b \sigma_u S_u dx_u \\
&= \lim_{|\mathcal{P}(a,b)|\to 0} \sum_{u,v \in \mathcal{P}(a,b)} \Big(\sigma_u S_u x_{u,v} + (\partial_\omega \sigma)_u S_u [\omega, dx]_{u,v} + \Big(\sigma_u (\partial_x S)_u + (\partial_x \sigma)_u S_u\Big)[x,dx]_{u,v}\Big),
\end{aligned} \tag{2.22}
$$

and in particular $\partial_x S = \sigma S \in C^{\beta}([0,T],\mathbb{R})$. In other words, the rough differential equation (2.19) or the rough integral equation (2.20) is well-posed. In the following result, we obtain an explicit solution for equation (1.1) using rough path calculus.

Theorem 2.5 *Given $\frac{1}{2} > \alpha > \frac{1}{3} \geq \beta > 0$ satisfying* (2.15), *a coefficient path $\sigma \in C^{\beta}([0,T],\mathbb{R})$ controlled by $(\omega,x)^{\mathrm{T}}$ and a driving path $x \in C^{\alpha}([0,T],\mathbb{R})$, there exists a unique rough solution of equation* (2.19) *on $[0,T]$. Moreover, the explicit pathwise solution of* (2.19) *is given by $S_t = e^{Y_t}$ where*

$$
Y_b = Y_a + \int_a^b \mu_u du - \frac{1}{2}\int_a^b \sigma_u^2 d[x]_u + \int_a^b \sigma_u dx_u \tag{2.23}
$$

and the path $[x] \in C^{2\alpha}([0,T],\mathbb{R})$ is defined in (2.10).

Proof: The existence and uniqueness part is proved by following similar arguments in [24] or [17]. For the benefit of the reader, we provide a sketch of the proof in the Appendix. To derive formula (2.23), observe that zero is the trivial solution of (2.19), hence it follows from the existence and uniqueness parts that $S_t \neq 0$ for all $t \geq \tau$ whenever $S_\tau \neq 0$. That implies $S_t > 0$ for all $t \geq \tau$ whenever $S_\tau > 0$. Note that $\partial_x S = \sigma S$ while $\partial_\omega S = 0$. Now applying the Taylor expansion for the function $Y_t = \log S_t$, we obtain from the fact $S, Y \in C^{\alpha}([0,T])$ that

$$
\log S_t = \log S_\tau + \frac{1}{S_\tau} S_{\tau,t} - \frac{1}{2S_\tau^2} S_{\tau,t}^2 + O(|t-\tau|^{3\alpha}), \tag{2.24}
$$

for $0 < t - \tau < h$ on $[0,T]$, where $0 < h \ll 1$ is small enough. The discretized scheme for equation (2.19) using (2.18) and (2.22) yields

$$
\begin{aligned}
S_{\tau,t} &= \mu_\tau S_\tau (t-\tau) + \sigma_\tau S_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau S_\tau [\omega, dx]_{\tau,t} \\
&\quad + \Big(\sigma_\tau (\partial_x S)_\tau + (\partial_x \sigma)_\tau S_\tau\Big)[x,dx]_{\tau,t} + O(|t-\tau|^{2\alpha+\beta}) \\
&= \mu_\tau S_\tau (t-\tau) + \sigma_\tau S_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau S_\tau [\omega, dx]_{\tau,t} \\
&\quad + \Big(\sigma_\tau^2 S_\tau + (\partial_x \sigma)_\tau S_\tau\Big)[x,dx]_{\tau,t} + O(|t-\tau|^{2\alpha+\beta}),
\end{aligned} \tag{2.25}
$$

for all $|t-\tau| < h$ on $[0,T]$. Combining (2.24) and (2.25), and using the fact that $x \in C^\alpha$, $[\omega, dx] \in C^{\alpha+\beta}$, $[x,dx] \in C^{2\alpha}$, we obtain

$$\begin{aligned} Y_{\tau,t} &= \mu_\tau(t-\tau) + \sigma_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau [\omega, dx]_{\tau,t} + \big(\sigma_\tau^2 + (\partial_x \sigma)_\tau\big)[x,dx]_{\tau,t} + O(|t-\tau|^{2\alpha+\beta}) \\ &\quad -\frac{1}{2}\Big(\mu_\tau(t-\tau) + \sigma_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau [\omega, dx]_{\tau,t} + \big(\sigma_\tau^2 + (\partial_x \sigma)_\tau\big)[x,dx]_{\tau,t} + O(|t-\tau|^{2\alpha+\beta})\Big)^2 \\ &\quad + O(|t-\tau|^{3\alpha}) \\ &= \mu_\tau(t-\tau) + \sigma_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau [\omega, dx]_{\tau,t} + (\partial_x \sigma)_\tau [x,dx]_{\tau,t} \\ &\quad -\sigma_\tau^2 \Big(\frac{1}{2} x_{\tau,t}^2 - [x,dx]_{\tau,t}\Big) + O(|t-\tau|^{2\alpha+\beta}) \\ &= \mu_\tau(t-\tau) + \sigma_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau [\omega, dx]_{\tau,t} + (\partial_x \sigma)_\tau [x,dx]_{\tau,t} - \frac{1}{2}\sigma_\tau^2 [x]_{\tau,t} + O(|t-\tau|^{2\alpha+\beta}) \end{aligned}$$

for all $|t-\tau| < h$ on $[0,T]$. Note that $\mu_\tau(t-\tau)$, $\sigma_\tau x_{\tau,t} + (\partial_\omega \sigma)_\tau [\omega, dx]_{\tau,t} + (\partial_x \sigma)_\tau [x,dx]_{\tau,t}$ and $\sigma_\tau^2 [x]_{\tau,t}$ are respectively the discrete approximation of the integrals $\int_\tau^t \mu_u du$, $\int_\tau^t \sigma_u dx_u$ and $\int_\tau^t \sigma_u^2 d[x]_u$. Hence using (2.18) and (2.14), one finally obtains

$$Y_{\tau,t} = \int_\tau^t \mu_u du + \int_\tau^t \sigma_u dx_u - \frac{1}{2}\int_\tau^t \sigma_u^2 d[x]_u + O(|t-\tau|^{2\alpha+\beta}) \tag{2.26}$$

for all $|t-\tau| < h$ on $[0,T]$. Next for any $0 \le a < b \le T$, we discretize the interval $[a,b]$ into sub-intervals of length $h = \frac{b-a}{N}$ with end points $a = t_0 < t_1 < \ldots < t_N = b$ for N large enough. Then by using (2.26) and the additivity of the integrals, we obtain

$$\begin{aligned} Y_{a,b} &= \sum_{i=0}^{N-1} y_{t_i,t_{i+1}} \\ &= \sum_{i=0}^{N-1} \Big(\int_{t_i}^{t_{i+1}} \mu_u du + \int_{t_i}^{t_{i+1}} \sigma_u dx_u - \frac{1}{2}\int_{t_i}^{t_{i+1}} \sigma_u^2 d[x]_u + O(h^{2\alpha+\beta})\Big) \\ &= \int_a^b \mu_u du + \int_a^b \sigma_u dx_u - \frac{1}{2}\int_a^b \sigma_u^2 d[x]_u + (b-a)O(h^{2\alpha+\beta-1}). \end{aligned} \tag{2.27}$$

Letting $h \to 0$ in (2.27) and using the fact that $2\alpha+\beta > 1$, one derives (2.23).

□

Remark 2.6 (i) Let us describe the solution formula (2.23). The first integral is of classical Riemann type, the second is of Young type for $\sigma^2 \in C^\beta$ with respect to the path $[x] \in C^{2\alpha}$ because of (2.15). The third one is a rough integral (2.17) for σ controlled by $(\omega, x)^{\mathrm{T}}$.

(ii) It is important to note that the stochastic process σ can be chosen as $\sigma = g(X)W$ for X, W in Remark 2.1, so that it satisfies Example 2.4. In particular, its β-Hölder continuity can be chosen as small as possible provided that (2.15) is satisfied. This should capture the empirical evidence in [4, 21] in which $\beta < H' < 0.1$. The same conclusion holds if W has the form $W = h(W')$ where X, W' are independent Gaussian processes satisfying Remark 2.1 and $h \in C^2(\mathbb{R})$ such that h and its derivative h' are bounded. In that case, one can formally write the equation for σ as

$$d\sigma_t = g'(X_t)h(W_t')dX_t + g(X_t)h'(W_t')dW_t'. \tag{2.28}$$

Equation (2.28) shows that there is a possible explanation for the fact that there is a correlation between the noise $(X, [W,dX], [X,dX])$ that drives the log-price process Y and the noise (X, W') that drives σ, as suggested in the Hull & White model [28]. For example, a choice of $g(X) = e^{\rho X}$ and $h(W') = \frac{e^{\sqrt{1-\rho^2}W'}}{e^{\sqrt{1-\rho^2}W'}+1}$ for $\rho \in (0,1)$ will rewrite (2.28) in the form

$$d\sigma = \sigma(\rho dX + \frac{\sqrt{1-\rho^2}}{e^{\sqrt{1-\rho^2}W'}+1} dW'),$$

which has a correlated noise to X and implies further that the log-volatility $\log \sigma$ has its very low regularity from W'.

Due to (2.10), y is determined once we know the information on $x, [x]$ and $[\omega, dx]$. In particular, when σ constant, (2.23) reduces to

$$y_b = y_a + \int_a^b \mu_u du - \frac{1}{2}\sigma^2[x]_{a,b} + \sigma x_{a,b}.$$

Example 2.7 Below we review several special cases.

- When $X = B$ a standard Brownian motion B, then Example 2.2(1) shows that $[B]_{a,b}(\cdot) = b-a$ a.s. and we go back to solving the classical model (1.2) using Itô calculus, so that the log-price has the form
$$Y_{s,t} = \int_s^t \Big(\mu_u - \frac{\sigma_u^2}{2}\Big)du + \int_s^t \sigma_u dB_u. \tag{2.29}$$

- Also, if $X = B^H$ is a fractional Brownian motion for $H \in (0,1)$, we go back to the model $dS_t = \mu S_t dt + \sigma S_t \delta B_t^H$ with the Skorohod-Wick-Itô integral $\int y \delta B^H$ proposed in [34] as discussed in Example 2.2(2), hence $[B^H]_{a,b}(\cdot) := (b^{2H} - a^{2H})$ a.s. and the solution is obtained explicitly from [35] as
$$Y_{s,t} = \mu(t-s) - \frac{\sigma^2}{2}(t^{2H} - s^{2H}) + \sigma B^H_{s,t}; \tag{2.30}$$

- When $X = \hat{B}$ is the G-Brownian motion with respect to the sublinear expectation $\hat{\mathbb{E}}$, then it follows from [38] that $[\hat{B}] = \langle \hat{B} \rangle$ and one can solve the stochastic differential equation
$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u d\hat{B}_u \tag{2.31}$$
explicitly as
$$Y_{s,t} = \mu(t-s) + \sigma \hat{B}_{s,t} - \frac{1}{2}\sigma^2\Big(\langle \hat{B}\rangle_t - \langle \hat{B}\rangle_s\Big). \tag{2.32}$$

3 No arbitrage under transaction costs

3.1 Rough models in stochastic settings

Motivated by the G-Brownian motions in Remark 2.3 and Example 2.7, we propose additional hypotheses in this section. The first one deals with stochastic processes $X, [X], W$ of which $x, [x], \omega$ are realizations respectively.

Hypothesis A $X, [X], W$ are mutually independent stochastic processes with stationary increments on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that are $\mathcal{F}_t$-adapted, such that $[X] \in C^{2\nu}([0,T], \mathbb{R})$ a.s. and that X, W are centered Gaussian satisfying condition (2.5) in Remark 2.1 for $\frac{1}{2} > \nu > \alpha > \frac{1}{3} \geq \beta$ in (2.15), thus $[W, dX]$ is well-defined. In addition, $[X, dX]_{s,t} = \frac{1}{2}\Big(X^2_{s,t} - [X]_{s,t}\Big)$ for all $s, t \in [0,T]$ a.s.

It is easy to check that X in Example 2.2 satisfies Hypothesis A. In this situation, for a stochastic process σ satisfying the conditions in Remark 2.6, the logarithm price process $Y_t := \log S_t$ can be written explicitly in the pathwise sense as

$$Y_t(\cdot) = Y_s(\cdot) + \int_s^t \mu_u du - \frac{1}{2}\int_s^t \sigma_u(\cdot)^2 d[X]_u(\cdot) + \int_s^t \sigma_u(\cdot) dX_u(\cdot), \quad \forall s,t \in [0,T]. \tag{3.1}$$

To handle the no arbitrage problem, as in [22] we consider a realistic assumption on transaction costs. Namely, consider the model with a riskless asset price process A_t and a risky asset price process $(S_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions of right continuity and saturatedness and S_t is a continuous path, strictly positive almost surely, adapted and quasi-left continuous w.r.t. $\mathcal{F}_t$. An investor trades in the risky asset according to the strategy $(\theta_t)_{t\in[0,T]}$, which represents the number of shares held at time t, and each unit traded in the risky asset generates a transaction cost of $\kappa > 0$ units, which is charged to the riskless asset account. Consider a simple strategy θ which requires a finite number of transactions at stopping times $(\tau_i)_{i=1}^n$, then $\theta = \sum_{i=1}^n \theta_i 1_{]\tau_{i-1},\tau_i]}$ for some random variables $(\theta_i)_{i=1}^n$, where θ_i is $\mathcal{F}_{\tau_i}$ - measurable, and $\theta_0 = 0$ conventionally. The liquidation value of a portfolio with zero initial capital is

$$V_t(\theta) = \sum_{i=1}^n \theta_i (S_{\tau_i\wedge t} - S_{\tau_{i-1}\wedge t}) - \kappa \sum_{\tau_i \le t} S_{\tau_i} |\theta_i - \theta_{i-1}| - \kappa S_t |\theta_t|. \tag{3.2}$$

This discrete model is then proved to converge to the continuous model

$$V_t(\theta) = \langle \theta, S\rangle_t - \kappa \int_{[0,t]} S_u d\|\theta\|_u - \kappa S_t |\theta_t|, \tag{3.3}$$

where $\|\theta\|_t$ is the total variation of θ on $[0,t]$ and $\langle \theta, S\rangle_t$ is a certain type of pathwise integral. According to [22], a strategy θ is *admissible* if $V_t(\theta) \ge -M$ a.s. for some $M > 0$ and for all $t > 0$. It is called an *arbitrage opportunity* on $[0,T]$ if it is admissible with $V_T(\theta) \ge 0$ a.s. and $\mathbb{P}(V_T(\theta) > 0) > 0$. A market is *arbitrage free on* $[0,T]$ if, for all admissible strategies θ, $V_T(\theta) \ge 0$ a.s. only if $V_T(\theta) = 0$ a.s. The market is arbitrage free with transaction costs κ if S_t satisfies the condition that for all stopping times τ such that $\mathbb{P}(\tau < T) > 0$, we have

$$\mathbb{P}\Big(\sup_{t\in[\tau,T]} \Big| \frac{S_\tau}{S_t} - 1 \Big| < \kappa, \tau < T \Big) > 0. \tag{3.4}$$

Condition (3.4) is satisfied when the asset logarithm price process Y_t is *sticky* w.r.t. the filtration $\mathcal{F}_t$, i.e., for all $\epsilon, T > 0$ and all stopping times τ such that $\mathbb{P}(\tau < T) > 0$, one has

$$\mathbb{P}(\sup_{t\in[\tau,T]} |Y_\tau - Y_t| < \epsilon, \tau < T) > 0. \tag{3.5}$$

According to [22], any *strong Markov process* (i.e., for every finite $\mathcal{F}_\tau$-stopping time τ under the conditional law $\mathbb{P}(\cdot|X_\tau = y)$, the process $(X_{\tau+t})_{t\ge 0}$ is independent of $\mathcal{F}_\tau$ and has the law $\mathbb{P}_y$) is sticky. Another sticky class consists of adapted stochastic processes w.r.t. a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that have *conditional full support* (CFS), i.e.

$$\forall t \in [0,T),\ (\mathbb{P}\text{-a.s.}): \quad \mathrm{supp}\Big(\mathrm{law}\big[(X_u)_{u\in[t,T]} | \mathcal{F}_t \big](\cdot) \Big) = C_{X_t(\cdot)}([t,T], \mathbb{R}) \tag{3.6}$$

where $C_\eta([t,T],\mathbb{R})$ is the space of continuous functions $f \in C([t,T],\mathbb{R})$ taking values in $\mathbb{R}$ with initial value $f(t) = \eta$, and we regard $\mathrm{law}\big[(X_u)_{u\in[t,T]}|\mathcal{F}_t\big]$ as a regular conditional law (a random Borel probability measure) on $C([t,T], I)$ [29, pp. 106-107]. Furthermore, any stochastic process with CFS is proved in [23, Theorem 1.2] to admit an ϵ-consistent pricing system for all $\epsilon > 0$, i.e. there exists a pair $(\tilde S, \tilde P)$ where $\tilde P$ is an equivalent probability w.r.t. $\mathbb{P}$ and $\tilde S$ is a $\tilde P$-martingale (adapted to $\mathcal{F}_t$) such that

$$\frac{1}{1+\epsilon} \le \frac{\tilde S_t}{S_t} \le 1 + \epsilon, \quad \forall t \in [0,T].$$

We therefore need another assumption.

Hypothesis B Either $X_t, [X]_t$ are strong Markov processes, or they have CFS in the sense of (3.6).

One can easily check that all processes in Example 2.7 satisfy Hypothesis B. Our next main result of this paper is formulated as follows.

Theorem 3.1 *Under Hypotheses A, B and the situation of transaction costs $\kappa > 0$, assume further that $\sigma \in C^\beta([0,T],\mathbb{R})$ is controlled by $(W,X)^{\mathrm{T}}$ a.s. such that*

$$\inf_{t\in[0,T]} |\sigma_t| > 0 \quad a.s. \tag{3.7}$$

Then the logarithm price process Y_t in (3.1) *is sticky. Consequently, S_t is arbitrage free under transaction costs κ on any interval $[0,T]$.*

Proof: In case $X,[X]$ are strong Markov processes, the stickiness follows from [22, Proposition 3.1]. Assume now that X and $[X]$ have CFS, we are going to prove that Y also has CFS. Since σ is controlled by W a.s. and $X,[X],W$ are mutually independent, it then suffices to prove that: *If X has CFS on $[0,T]$, then so does the stochastic process $-\frac{1}{2}\int_0^\cdot \sigma_u^2 d[X]_u + \int_0^\cdot \sigma_u dX_u$.* Since the proof is lengthy, we only sketch the ideas here. In any case, the proof will proceed by a repeated application of Egorov's theorem [25, Theorem A, p.88]. Due to [37, Lemma 2.1, Theorem 3.3] and [23, Lemma 4.5], it suffices to prove an equivalent property to the CFS that, for every $\epsilon > 0, B \in \mathcal{F}, t \in [0,T), g \in C_0([t,T],\mathbb{R})$, the following inequality holds

$$\mathbb{P}\Big(\Big\| -\frac{1}{2}\int_0^\cdot \sigma_u^2 d[X]_u + \int_t^\cdot \sigma_u dX_u - g \Big\|_{\infty,[t,T]} < \epsilon \;|\mathcal{F}_t\Big)(\cdot) > 0 \quad \text{a.s. on } B. \tag{3.8}$$

Since $C_0^1([t,T],\mathbb{R})$ is dense in $C_0([t,T],\mathbb{R})$, it is enough to prove (3.8) for $g \in C_0^1([t,T],\mathbb{R})$. To do that, one first needs to approximate the Young integral $(\int_t^\tau \sigma_u^2 d[X]_u)_{\tau\in[t,T]}$ and the rough integral $(\int_t^\tau \sigma_u dX_u)_{\tau\in[t,T]}$ respectively by Riemann-Stieltjes integrals $(\int_t^\tau \sigma_u^2 d[X]_u^\delta)_{\tau\in[t,T]}$ and $(\int_t^\tau \sigma_u dX_u^\delta)_{\tau\in[t,T]}$, where

$$[X]_u^\delta := [X]_{\delta m} + \frac{u-\delta m}{\delta}[X]_{\delta m,\delta(m+1)}, \quad X_u^\delta := X_{\delta m} + \frac{u-\delta m}{\delta} X_{\delta m,\delta(m+1)}, \quad \forall u \in [\delta m, \delta(m+1)] \tag{3.9}$$

are piecewise linear approximations of $[X], X$ for sufficiently small $\delta \in (0,1)$ (see e.g. [20, Prop. 5.5, Lemma 5.19, Prop. 5.20, Theorem 5.23] for an approximation in the Hölder norms). In this case, $[W, dX^\delta]$ and $[X, dX^\delta]$ are well-defined as Riemann-Stieltjes integrals a.s., thus satisfying Chen's relation (2.2). In fact, since W and X are independent Gaussian processes satisfying Remark 2.1, one can prove that, for any $0 \le s \le \delta m \le \delta l \le t \le T$,

$$\begin{aligned}
E(X_{s,t} - X_{s,t}^\delta)^{2p} &\le C_p\Big(E(X_{s,\delta m} - X_{s,\delta m}^\delta)^{2p} + E(X_{\delta l,t} - X_{\delta l,t}^\delta)^{2p}\Big) \\
&\le C_p\Big(E(X_{s,\delta m} - \frac{\delta m - s}{\delta}X_{\delta m,\delta(m+1)})^{2p} + E(X_{\delta l,t} - \frac{t-\delta l}{\delta}X_{\delta l,\delta(l+1)})^{2p}\Big) \\
&\le C_{p,X}\Big((s-\delta m)^2 H + (t-\delta l)^{2H}\Big) \\
&\le C_{p,X}\delta^{2H-2\nu}(t-s)^{2\nu}
\end{aligned}$$

for a generic constant $C_{p,X}$ and $\alpha < \nu < H$ such that $p\nu > 1$. In addition, $X - X^\delta$ is also a Gaussian process which is independent of W. Hence by following similar constructions in [19, Chapter 10] (see also Remark 2.1), one can proves that the stochastic integral $\int_s^t W_{s,u} d(X-X^\delta)_u$ is well defined and satisfies

$$\begin{aligned}
E\left(\int_s^t W_{s,u} d(X-X^\delta)_u\right)^2 &\le C_{H,H'}\delta^{2H-2\nu} \left\| R_{X-X^\delta} \right\|_{p-\text{var},[s,t]^2} \| R_W \|_{q-\text{var},[s,t]^2} \\
&\le C_{H,H',X,W}\delta^{2H-2\nu}|t-s|^{2(\nu+H')}
\end{aligned}$$

By choosing $\delta = \frac{1}{2^r}, r \in \mathbb{N}$ (so that the approximation is dyadic) and applying the Borel-Catelli lemma, one follows that for $\alpha < \nu < H$ and $\beta < H'$ satisfying (2.15) the following limit holds a.s.

$$\lim_{\delta=\frac{1}{2^r}, r\to\infty} \left(\left\| X - X^\delta \right\|_{\alpha,[0,T]} + \left\| [W,dX] - [W,dX^\delta] \right\|_{\alpha+\beta,[0,T]^2} + \left\| X^2_{\cdot,\cdot} - (X^\delta_{\cdot,\cdot})^2 \right\|_{2\alpha,[0,T]^2} \right) = 0.$$

Similarly, one can use [20, Lemma 5.27] to prove that

$$\lim_{\delta=\frac{1}{2^r}, r\to\infty} \left\| [X] - [X]^\delta \right\|_{2\alpha,[0,T]} = 0 \quad \text{a.s.}$$

Now using inequalities (2.14) and (2.18), one obtains

$$\begin{aligned} & \sup_{\tau\in[t,T]} \left\| \int_t^\tau \Big(\sigma_u^2 + (\partial_x\sigma)_u\Big) d[X]_u - \int_t^\tau \Big(\sigma_u^2 + (\partial_x\sigma)_u\Big) d[X]^\delta_u \right\| \\ \leq & \ \Big(T^{2\alpha} \|\sigma^2 + \partial_x\sigma\|_{\infty,[0,T]} + K_\alpha T^{2\alpha+\beta} \left\|\sigma^2 + \partial_x\sigma\right\|_{\beta,[0,T]} \Big) \left\| [X] - [X]^\delta \right\|_{2\alpha,[0,T]} \\ \to & \ 0 \quad \text{a.s. as } \delta = \frac{1}{2^r} \to 0. \end{aligned} \tag{3.10}$$

On the other hand, by assigning

$$\begin{aligned} F_{u,v} \ := & \ \sigma_u(X_{u,v} - X^\delta_{u,v}) + (\partial_\omega\sigma)_u \Big([W,dX]_{u,v} - [W,dX^\delta]_{u,v} \Big) \\ & + (\partial_x\sigma)_u \left([X,dX]_{u,v} - [X,dX^\delta]_{u,t} + \frac{1}{2}[X]_{u,v} \right), \end{aligned} \tag{3.11}$$

it is easy to check, due to Hypothesis A, that

$$\begin{aligned} \delta F_{s,u,v} \ = & \ -R^\sigma_{s,u}(X_{u,v} - X^\delta_{u,v}) - (\partial_\omega\sigma)_{s,u} \Big([W,dX]_{u,v} - [W,dX^\delta]_{u,v} \Big) \\ & - (\partial_x\sigma)_{s,u} \left([X,dX]_{u,v} - [X,dX^\delta]_{u,v} + \frac{1}{2}[X]_{u,v} \right) \\ = & \ -R^y_{s,u}(X_{u,v} - X^\delta_{u,v}) - (\partial_\omega\sigma)_{s,u} \Big([W,dX]_{u,v} - [W,dX^\delta]_{u,v} \Big) \\ & - (\partial_x\sigma)_{s,u} \left(\frac{1}{2}(X^2_{u,v} - (X^\delta_{u,v})^2) - [X - X^\delta, dX^\delta]_{u,v} \right). \end{aligned}$$

Hence

$$\begin{aligned} \|\delta F_{s,u,v}\| \ \leq & \ |v-s|^{\beta+2\alpha} \Big(\|R^\sigma\|_{\alpha+\beta} \left\| X - X^\delta \right\|_\alpha + \left\| [W,dX] - [W,dX^\delta] \right\|_{\alpha+\beta} \\ & + \|\partial_x\sigma\|_\beta \left\| X^2_{\cdot,\cdot} - (X^\delta_{\cdot,\cdot})^2 \right\|_{2\alpha} + \|\partial_x\sigma\|_\beta \left\| X - X^\delta \right\|_\alpha \left\| X^\delta \right\|_\alpha \Big). \end{aligned}$$

We are now in the position to apply the sewing lemma [24], so that by taking the Darboux sum $\sum_{[u,v]\in\mathcal{P}(s,t)} F_{u,v}$ for any finite partition $\mathcal{P}(s,t)$ of $[s,t]$ and let $|\mathcal{P}(s,t)|$ tends to zero, the right hand side of (3.11) converges in the pathwise sense, thanks to (2.13), (2.17) and the fact that $X^\delta \in C^1$, to

$$\int_s^t \sigma_u dX_u - \int_s^t \sigma_u dX^\delta_u + \frac{1}{2}\int_s^t (\partial_x\sigma)_u d[X]_u.$$

Moreover, one obtains from the sewing lemma that

$$\begin{aligned}
&\sup_{\tau\in[t,T]}\left\|\int_t^\tau \sigma_u dX_u - \int_t^\tau \sigma_u dX^\delta_u + \frac{1}{2}\int_t^\tau (\partial_x\sigma)_u d[X]_u\right\| \\
\leq\ & \Big(T^\alpha\|\sigma\|_{\infty,[0,T]} + T^{\alpha+\beta}\|\partial_\omega\sigma\|_{\infty,[0,T]} + T^{2\alpha}\|\partial_x\sigma\|_{\infty,[0,T]} + C_\alpha T^{2\alpha+\beta}\, |\!|\!|(\sigma,\partial_\omega\sigma,\partial_x\sigma)|\!|\!|_{\alpha,\beta,[0,T]}\Big)\times \\
&\times\Bigg(\Big|\!\Big|\!\Big|X-X^\delta\Big|\!\Big|\!\Big|_{\alpha,[0,T]} + \Big|\!\Big|\!\Big|[W,dX]-[W,dX^\delta]\Big|\!\Big|\!\Big|_{\alpha+\beta,[0,T]^2} + \Big|\!\Big|\!\Big|X^2_{\cdot,\cdot} - (X^\delta_{\cdot,\cdot})^2\Big|\!\Big|\!\Big|_{2\alpha,[0,T]^2} \\
&\qquad + \Big|\!\Big|\!\Big|X-X^\delta\Big|\!\Big|\!\Big|_{\alpha,[0,T]}\Big|\!\Big|\!\Big|X^\delta\Big|\!\Big|\!\Big|_{\alpha,[0,T]}\Bigg) \to 0 \quad \text{a.s. as } \delta\to 0 \quad \text{exponentially fast.}
\end{aligned}\tag{3.12}$$

Hence by Egorov's theorem, there exists a measurable set $B_1\in\mathcal{F}$ with $\mathbb{P}(B_1)$ smaller but very close to 1 such that the limits in (3.10) and (3.12) are uniform on B_1, thus the sum of the left hand sides of both (3.10) and (3.12) are bounded by $\frac{\epsilon}{4}$ uniformly on B_1 by choosing a sufficiently small $\delta=\frac{1}{2^r}$.

Next, assign $Q(\sigma)=\sigma^2+\partial_x\sigma$. By fixing this small $\delta=\frac{1}{2^r}$, one would like to approximate σ by a sequence of piecewise constant functions $\sigma^{(n)}$ such that $\int_0^T|Q(\sigma_u)-Q(\sigma^{(n)}_u)|du\to 0$ and $\int_0^T|\sigma_u-\sigma^{(n)}_u|du\to 0$ as $n\to\infty$ almost surely. As a result, as n tends to infinity

$$\begin{aligned}
\sup_{\tau\in[t,T]}\left\|\int_t^\tau (Q(\sigma_u)-Q(\sigma^{(n)}_u))d[X]^\delta_u\right\| &\leq \frac{1}{\delta^{1-2\alpha}}\,|\!|\!|[X]|\!|\!|_{2\alpha,[0,T]}\int_0^T|Q(\sigma_u)-Q(\sigma^{(n)}_u)|du\to 0; \\
\sup_{\tau\in[t,T]}\left\|\int_t^\tau (\sigma_u-\sigma^{(n)}_u)dX^\delta_u\right\| &\leq \frac{1}{\delta^{1-\alpha}}\,|\!|\!|X|\!|\!|_{\alpha,[0,T]}\int_0^T|\sigma_u-\sigma^{(n)}_u|du\to 0.
\end{aligned}\tag{3.13}$$

Again by Egorov's theorem, one can choose a measurable set $B_2\in\mathcal{F}$ with $\mathbb{P}(B_2)$ smaller than but very close to 1, such that the left hand sides of the two inequalities in (3.13) tend to zero uniformly on B_2, thus their sum can be bounded by $\frac{\epsilon}{4}$ by choosing $n\geq N(\epsilon,B_2)$ big enough. With that fixed n, assume that the piecewise constant function $\sigma^{(n)}$ takes values $\sigma^{(n)}_{\tau_i}, i=1,\ldots,k_n$ on $[t,T]$. Then

$$\begin{aligned}
&\left\|-\frac{1}{2}\int_t^\tau Q(\sigma^{(n)}_u)d[X]^\delta_u + \int_t^\tau \sigma^{(n)}_u dX^\delta_u - g_{t,\tau}\right\| \\
\leq\ & \sum_{\tau_i\leq\tau, m\in\mathbb{N}}\left\|\int_{[\tau_i,\tau_{i+1}]\cap[\delta m,\delta(m+1)]}\left(-\frac{1}{2}Q(\sigma^{(n)}_{\tau_i})\frac{[X]_{\delta m,\delta(m+1)}}{\delta} + \sigma^{(n)}_{\tau_i}\frac{X_{\delta m,\delta(m+1)}}{\delta} - g'_u\right)du\right\| \\
\leq\ & \sum_{\tau_i\leq\tau, m\in\mathbb{N}}\left\|\int_{[\tau_i,\tau_{i+1}]\cap[\delta m,\delta(m+1)]}\left(-\frac{1}{2}Q(\sigma^{(n)}_{\tau_i})\frac{[X]_{\delta m,\delta(m+1)}}{\delta} + \sigma^{(n)}_{\tau_i}\frac{X_{\delta m,\delta(m+1)}}{\delta} - \frac{g_{\delta m,\delta(m+1)}}{\delta}\right)du\right\| \\
&+(T-t)\sup_{u,v\in[t,T],|u-v|\leq\delta}\|g'_u-g'_v\| \\
\leq\ & \frac{2(T-t)}{\delta}\sup_{i=1,\ldots,k_n}\sup_{u\in[t,T+\delta]}\left\|-\frac{1}{2}Q(\sigma^{(n)}_{\tau_i})[X]_{t,u} + \sigma^{(n)}_{\tau_i}X_{t,u} - g_u\right\| \\
&+(T-t)\sup_{u,v\in[t,T],|u-v|\leq\delta}\|g'_u-g'_v\|.
\end{aligned}\tag{3.14}$$

Observe that the second term on the right hand side of (3.14) tends to zero a.s. as $\delta=\frac{1}{2^r}\to 0$, hence it can be bounded by $\frac{\epsilon}{4}$ by changing $\delta=\frac{1}{2^r}$ to small enough.

Meanwhile due to condition (3.7), for a fixed finite sequence of random variables $\sigma^{(n)}_{\tau_i}, i=1,\ldots,k_n$, we construct for each random variable $\sigma^{(n)}_{\tau_i}$ and a fixed $m\in\mathbb{N}$ a countable partition $(\Omega^i_{m,j,l})_{j,l\in\mathbb{N}} = \Big(\{\omega:\sigma^{(n)}_{\tau_i}\in[\frac{j}{2^m},\frac{j+1}{2^m}), Q(\sigma^{(n)}_{\tau_i})\in[\frac{l}{2^m},\frac{l+1}{2^m})\}\Big)_{j,l\in\mathbb{N}}$. On $\Omega^i_{m,j,l}$

$$\left\|\Big(\sigma^{(n)}_{\tau_i}-\frac{j}{2^m}\Big)X_{t,u}\right\| \ \leq\ \frac{1}{2^m}\sup_{0<u<v<T+\delta}\|X_{u,v}\| \to 0 \quad \text{uniformly in } j \text{ as } m\to\infty$$

$$\left\|\frac{1}{2}\Big(Q(\sigma^{(n)}_{\tau_i})-\frac{l}{2^m}\Big)[X]_{t,u}\right\| \;\leq\; \frac{1}{2}\frac{1}{2^m}\sup_{0<u<v<T+\delta}\|[X]_{u,v}\|\to 0 \quad \text{uniformly in } l \text{ as } m\to\infty.$$

Due to Egorov's theorem, one can choose a set $B_3\in\mathcal{F}$ with $\mathbb{P}(B_3)$ smaller than but very close to 1 and $m>M(\epsilon)$ large enough, such that on $B_3\cap\Omega^i_{m,j,l}$,

$$\sup_{u\in[t,T+\delta]}\left\|-\frac{1}{2}\Big(Q(\sigma^{(n)}_{\tau_i})-\frac{l}{2^m}\Big)[X]_{t,u}+\Big(\sigma^{(n)}_{\tau_i}-\frac{j}{2^m}\Big)X_{t,u}\right\|\leq\frac{\epsilon\delta}{8T}. \tag{3.15}$$

Since $[X]$ has CFS, so does $-\frac{1}{2}[X]$ (see [37, Remark 2.1]), hence it follows from the CFS of X and $-\frac{1}{2}[X]$ on the interval $[t,T+\delta]$ and the fact that X and $-\frac{1}{2}[X]$ are independent that, for any set $\Omega^i_{m,j,l}$ with positive probability and the function $\frac{2^m}{j}\frac{g}{2},\frac{2^m}{l}\frac{g}{2}\in C_0([t,T+\delta],\mathbb{R})$, one obtains the following inequality a.s. on $\Omega^i_{m,j,l}$

$$\begin{aligned}
&\mathbb{P}\left(\left\|\frac{l}{2^m}(-\frac{1}{2}[X])_{t,\cdot}+\frac{j}{2^m}X_{t,\cdot}-g\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{8T}\;|\mathcal{F}_t\right)(\cdot)\\
\geq\;&\mathbb{P}\left(\left\{\left\|\frac{l}{2^m}(-\frac{1}{2}[X])_{t,\cdot}-\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\right\}\cap\left\{\left\|\frac{j}{2^m}X_{t,\cdot}-\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\right\}\;|\mathcal{F}_t\right)(\cdot)\\
=\;&\mathbb{P}\left(\left\{\left\|-\frac{1}{2}[X]_{t,\cdot}-\frac{2^m}{l}\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\frac{2^m}{l}\right\}\cap\left\{\left\|X_{t,\cdot}-\frac{2^m}{j}\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\frac{2^m}{j}\right\}\;|\mathcal{F}_t\right)(\cdot)\\
=\;&\mathbb{P}\left(\left\{\left\|-\frac{1}{2}[X]_{t,\cdot}-\frac{2^m}{l}\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\frac{2^m}{l}\right\}\;|\mathcal{F}_t\right)(\cdot)\times\\
&\times\mathbb{P}\left(\left\{\left\|X_{t,\cdot}-\frac{2^m}{j}\frac{g}{2}\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{16T}\frac{2^m}{j}\right\}\;|\mathcal{F}_t\right)(\cdot)>0.
\end{aligned}$$

By taking the refined partition $\mathcal{P}=\{\cap_{i=1,\dots,k_n}\Omega^i_{m,j,l}\}_{j,l\in\mathbb{N}}$, then for every set $\tilde{\Omega}_{m,j,l}\in\mathcal{P}$ with $\mathbb{P}(\tilde{\Omega}_{m,j,l})>0$ one can apply the CFS of $X,[X]$ on $\tilde{\Omega}_{m,j,l}$ and $\frac{2^m}{j}\frac{g}{2},\frac{2^m}{j}\frac{g}{2}\in C_0([t,T+\delta],\mathbb{R})$ to obtain

$$\mathbb{P}\left(\left\|-\frac{1}{2}\frac{l}{2^m}[X]_{t,\cdot}+\frac{j}{2^m}X_{t,\cdot}-g\right\|_{\infty,[t,T+\delta]}\leq\frac{\epsilon\delta}{8T}\;|\mathcal{F}_t\right)(\cdot)>0,\quad\text{a.s. on }\tilde{\Omega}_{m,j,l}. \tag{3.16}$$

Combining (3.15) and (3.16) and the triangle inequality, one can prove that for any set $\tilde{\Omega}_{m,j,l}\in\mathcal{P}$ with $\mathbb{P}(\tilde{\Omega}_{m,j,l})>0$

$$\mathbb{P}\left(\frac{2(T-t)}{\delta}\sup_{i=1,\dots,k_n}\sup_{u\in[t,T+\delta]}\left\|-\frac{1}{2}Q(\sigma^{(n)}_{\tau_i})[X]_{t,u}+\sigma^{(n)}_{\tau_i}X_{t,u}-g_u\right\|\leq\frac{\epsilon}{4}\;|\mathcal{F}_t\right)(\cdot)>0\quad\text{a.s. on }\tilde{\Omega}_{m,j,l}. \tag{3.17}$$

Finally, (3.10), (3.12), (3.13), (3.14), (3.17) and the triangle inequality proves (3.8) for a.s. all $\omega\in B\cap B_1\cap B_2\cap B_3$. Since B_1,B_2,B_3 can be chosen arbitrarily close to Ω of full probability, this proves (3.8) for a.s. all $\omega\in B$.

□

4 Some empirical evidence

Model (3.1) implies that one can not rule out the appearance of an additional source of noise $[X]$ in the log-price process. In this section, we would like to investigate the empirical facts that can be explained easily if we assume the stochasticity of $[X]$.

4.1 Hurst exponents and the multi-scaling phenomenon

Table 1 shows how the Hurst exponents vary w.r.t. different time scales h, for $h = 1$ corresponding to daily quotes, by using the rescale range test [R/S] with minimal size of 20 to avoid numerical errors in estimating the linear regression [15]. As seen in Tables 1,2,3, the Hurst exponents for daily,

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.5158	0.5103	0.5234	0.5218	0.5201	0.5278	0.5613	0.5816	0.6105
Dow Jones	0.5199	0.5177	0.5301	0.5410	0.5437	0.5534	0.5764	0.5952	0.5990
Nasdaq100	0.5177	0.5193	0.5245	0.5256	0.5324	0.5490	0.5694	0.6069	0.6305

Table 1: Hurst exponents for different time scales using [R/S] analysis. Data source: Dukascopy Bank SA

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.4898	0.4911	0.4969	0.4879	0.4838	0.4818	0.5649	0.6006	0.5284
Dow Jones	0.4937	0.4952	0.5021	0.4909	0.4927	0.4951	0.5049	0.5386	0.5015
Nasdaq100	0.4907	0.4964	0.5048	0.4974	0.5031	0.4755	0.5188	0.5651	0.4649

Table 2: Hurst exponents for different time scales using Spectral analysis. Data source: Dukascopy Bank SA

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.5202	0.5202	0.5157	0.5197	0.5181	0.5149	0.5549	0.5410	0.5483
Dow Jones	0.5198	0.5208	0.5174	0.5127	0.5161	0.5081	0.5651	0.5734	0.5655
Nasdaq100	0.5271	0.5294	0.5224	0.5185	0.5307	0.5288	0.6112	0.6490	0.6491

Table 3: Hurst exponents for different time scales using Higuchi method. Data source: Dukascopy Bank SA

weekly and monthly data are significantly bigger than $\frac{1}{2}$. On the other hand, the Hurst exponents are very close to 0.5 for a time scale $h \ll 1$ (minute quotes), implying that the effect of standard Brownian motion dominates, and thus X_t is essentially B_t which corresponds to $H = \frac{1}{2}$.
Tables 2 and 3 show the results for the same data but for different methods of computing the fractal dimension, namely using spectral analysis and the Higuchi method [27]. We see that quite often the spectral method gives Hurst exponents smaller than $\frac{1}{2}$ for smaller time-scales. In contrast, the Higuchi method gives quite stable results which are comparable with the rescale range method in Table 1.

In addition, our numerical computations, in Table 4 and Figure 1 with data from the stock index SP500 for different timescales, show a non-linear dependence of the variance on the time duration,

$$\text{Var } Y_{t,t+\tau} = \sigma^2 \tau^{2H} \Leftrightarrow \log\Big(\text{Var } Y_{t,t+\tau}\Big) = 2H \log \tau + 2 \log \sigma, \tag{4.1}$$

where $1 > H > 0$ for all data of timescales from minute to monthly quotes. Moreover, H seems to depend increasingly on the time scale h. Relation (4.1) is tested by choosing $\tau = 2^k, k = 0, \ldots, m$ where $m = \log_2 \frac{N}{100}$ and N is the length of the data. The variance Var $Y_{t,t+\tau}$ can be computed based on the sequence $\{Y_{i\tau,(i+1)\tau}\}$ with length no less than 100 to be able to neglect potential noises from small data.

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
σ	0.0003	0.0006	0.0010	0.0014	0.0020	0.0036	0.0121	0.0246	0.0417
H	0.4872	0.4841	0.4810	0.4832	0.4798	0.4673	0.5237	0.5663	0.6191

Table 4: Linear regression coefficients of relation (4.1) for SP500, from minute to monthly quotes. Data source: Dukascopy Bank SA

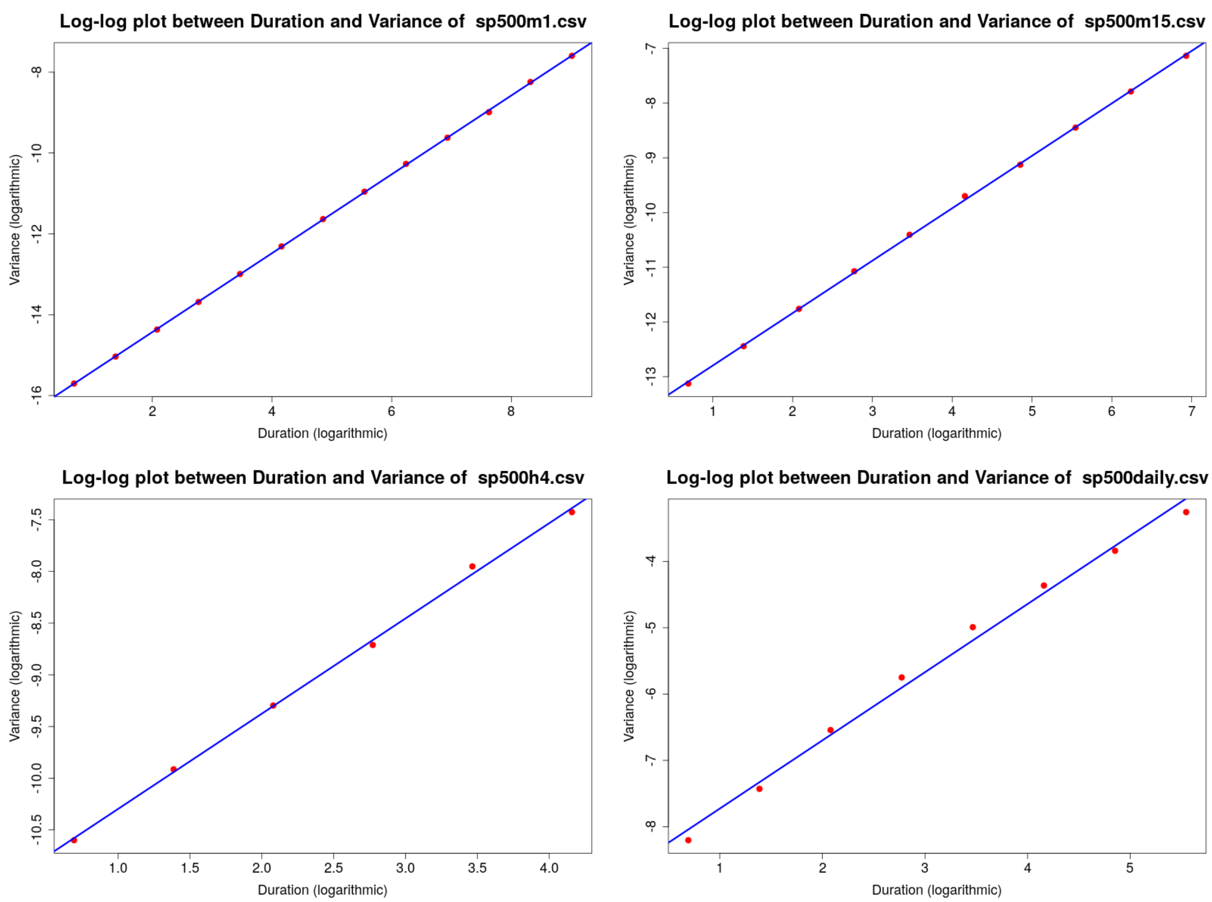

Figure 1: Linear regression of relation (4.1) for SP500, with time scales: 1 minute, 15 minutes, 4 hours and daily. Data source: Dukascopy Bank SA

It is not clear how this nonlinearity can arise from the time dependent model (1.2) with additional assumptions on the processes μ_t and σ_t. However, we can give a simple explanation for the numerical results in Table 4 and Figure 1 by using model (2.23) for a constant process σ. Indeed, we can simply assume that $[x] \in C^{2\alpha}$ is a realization of the stochastic process $[X]$. Then it follows from (3.1) that

$$Y_{t,t+h} = \mu h + \sigma X_{t,t+h} - \frac{\sigma^2}{2}[X]_{t,t+h}, \tag{4.2}$$

where X and $[X]$ are independent. As a result,

$$EY_{t,t+h} = \mu h - \frac{\sigma^2}{2} E[X]_{t,t+h}, \quad \text{Var } Y_{t,t+h} = \sigma^2 \text{Var } X_{t,t+h} + \frac{\sigma^4}{4} \text{Var } [X]_{t,t+h}. \tag{4.3}$$

Since $X, [X]$ have stationary increments, assume further that

$$\text{Var } X_{t,t+h} = EX^2_{t,t+h} = Ch, \quad \text{Var } [X]_{t,t+h} = C_H h^{2H} \tag{4.4}$$

for some $H \in (\frac{1}{2}, 1)$ and constants C, C_H. The definition of $[X]$ in (2.10) shows the advantage of rough path lifts that we have some freedom to use the driving processes X and $[X]$ as an input to define $[X, dX]$. Therefore assumption (4.4) will be satisfied if we choose $X = B$ and $[X] = B^H$ (so that one can define the Levy area $[X, dX]_{s,t} = \frac{1}{2}(B^2_{s,t} - B^H_{s,t})$ as presented in Section 2). Then it follows from (4.3) that

$$\log\Big(\text{Var } Y_{t,t+h}\Big) = \log\Big(C\sigma^2 h + \frac{C_H}{4}\sigma^4 h^{2H}\Big) \approx \begin{cases} \log(h) + \log(C\sigma^2) & \text{if} \quad h \ll 1 \\ 2H\log(h) + \log(\frac{C_H}{4}\sigma^4) & \text{if} \quad h \gg 1 \end{cases}. \tag{4.5}$$

Therefore for different time scales h ranging from $\frac{1}{1440}$ for minute quotes to 30 for monthly quotes, the regression coefficient for the relation between $\log\Big(\text{Var } Y_{t,t+h}\Big)$ and $\log h$ in (4.5) can increase from smaller than (but also close to) $\frac{1}{2}$ to H for $H > \frac{1}{2}$, as observed in Table 4. We emphasize here that the Hurst index in Table 4 is in the range $(0.46, 0.62)$ does not rule out the rough model for $H \in (\frac{1}{3}, \frac{1}{2})$ with the rough path lift $\mathbf{x} = (x, [x, dx])$ (which often requires $[x]$ to have regularity of $C^{2\nu}$ for $2\nu > \frac{2}{3} > 0.66$), because it might happen that the maximal time-scale of monthly data is not enough to avoid numerical problems of mixture between small and large time scales in (4.5) (we expect that the yearly data would solve the problem). Another reason is that X and $[X]$ might not be independent in reality, which makes the equality in (4.3) for the variance an inequality, thereby a creating numerical error in the approximation (4.5).

4.2 Upper-parabolic mean-variance relation of the logarithmic return

A drawback of model (1.1) is the fact that

$$EY_{t,t+h} = h\Big(\mu - \frac{\sigma^2}{2}\Big), \quad \text{Var } Y_{t,t+h} = \sigma^2 h,$$

which implies that the variance depends linearly and negatively on the expected return

$$\text{Var } Y_{t,t+h} = 2\mu h - 2EY_{t,t+h}. \tag{4.6}$$

This linear relation also occurs in the time dependent model (1.2). Our numerical computations with empirical data show a different picture. We collect time series $\{Y_j\}_{j=1}^N$ of 1 minute logarithmic quotes, so that the time step h is small. Then for any set $Y^{(h)}_k := \{Y_{km+i}\}_{i=1}^m$ of daily period where $k = 0 \ldots [\frac{N}{m}] - 1$, we calculate the mean $EY^{(h)}_k = \frac{1}{m}\sum_{i=1}^m Y_{km+i,km+i+1}$ and its variance $\text{Var } Y^{(h)}_k = \frac{1}{m-1}\sum_{i=1}^{m-1}\Big(Y_{km+i,km+i+1} - EY^{(h)}_k\Big)^2$ of the 1-minute logarithmic return during that day. Figures 2 and 3 show that, for all types of financial asset prices from stocks to stock indices, the set of daily mean-variances $(EY^{(h)}_k, \text{Var } Y^{(h)}_k)$ has a parabola-shaped left envelope, which cannot be explained by model (4.6).

While it seems complicated to theoretically explain this parabolic relation using a time dependent Itô model, our rough model under Hypothesis A easily accounts for that. Indeed, consider again

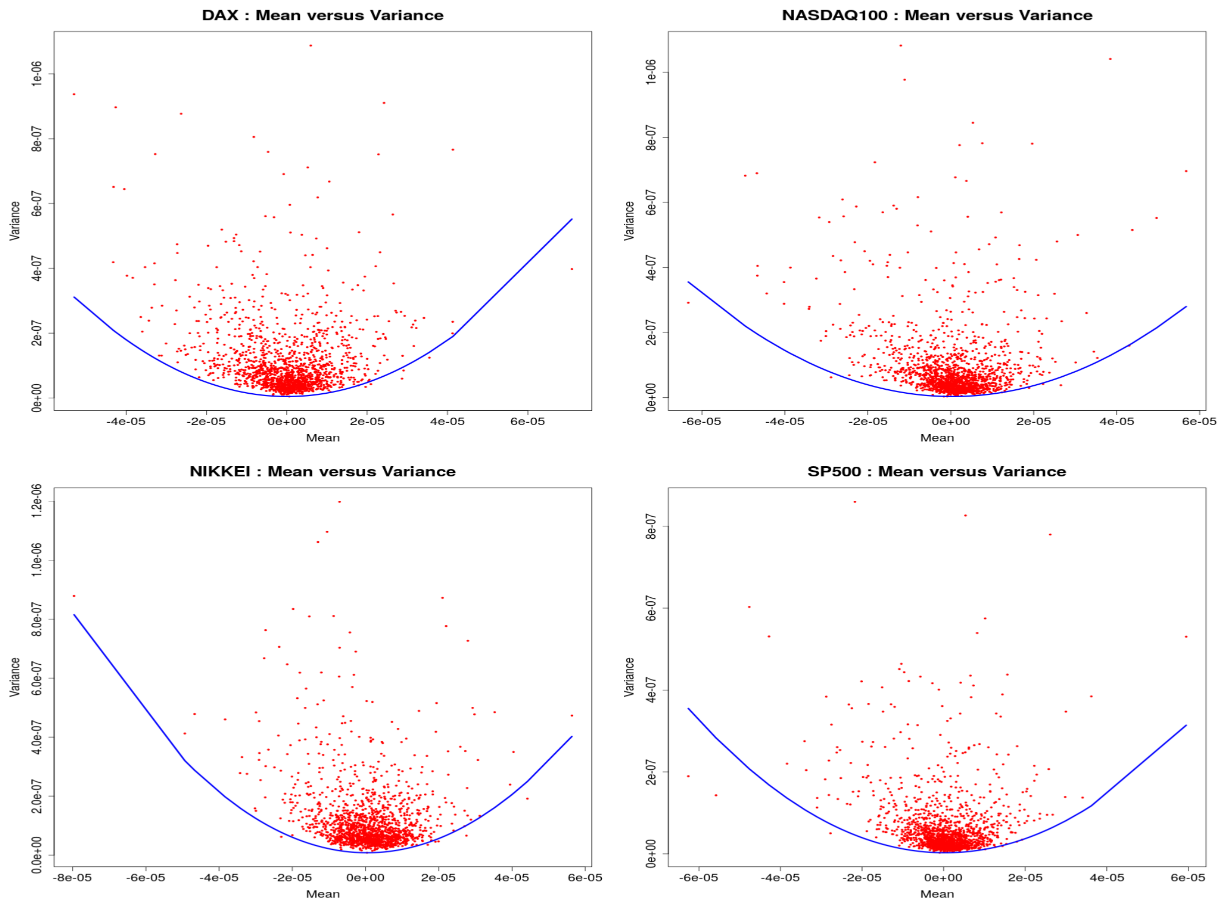

Figure 2: Upper-parabolic mean-variance relation (4.7) for 1-minute quotes of stock indices. Data: Dax, Nasdaq100, Nikkei, SP500. Data source: Dukascopy Bank SA

model (4.2) with expectation and variance computed in (4.3). By solving for σ^2 in terms of $EY_{t,t+h}$ and μ in the first equality and inserting it into the second equality we obtain a parabolic relation

$$\text{Var } Y_{t,t+h} = 2\frac{\text{Var } X_{t,t+h}}{E[X]_{t,t+h}}\Big(\mu h - EY_{t,t+h}\Big) + \frac{\text{Var } [X]_{t,t+h}}{(E[X]_{t,t+h})^2}\Big(\mu h - EY_{t,t+h}\Big)^2. \tag{4.7}$$

In particular, since $\sigma^2 = \frac{2}{E[X]_{t,t+h}}\Big(\mu h - EY_{t,t+h}\Big) \geq 0$, it follows that

$$\text{Var } Y_{t,t+h} \geq \frac{\text{Var } [X,2]_{t,t+h}}{(E[X]_{t,t+h})^2}\Big(\mu h - EY_{t,t+h}\Big)^2 \tag{4.8}$$

$$\Leftrightarrow \quad (EY_{t,t+h}, \text{Var } Y_{t,t+h}) \text{ lies inside parabola } \mathcal{P} := \Big\{(x,y) \in \mathbb{R}^2 : y = \frac{\text{Var } [X]_{t,t+h}}{(E[X]_{t,t+h})^2}(\mu h - x)^2\Big\},$$

which explains Figures 2 and 3, because in reality μ might be time dependent. Since $[X]$ has stationary increments, $E[X]_{t,t+h}$ and Var $[X]_{t,t+h}$ are independent of t, thus the parabola $\mathcal{P}$ depends only on the time step h. The symmetry axis of the parabola $\mathcal{P}$ is $x_0 = \mu h \approx 0$ since $\mu h \approx 0$ for small time steps h (due to high frequency data of minute quotes). It is important to note that the parameters of the parabola $\mathcal{P}$ depend only on the noise $[X, 2]$ and are independent of X. Moreover, in case $[X]$ is non-random, Var $[X]_{t,t+h} \equiv 0$ so the parabola reduces to the flat line

$$\text{Var } Y_{t,t+h} = 2\frac{\text{Var } X_{t,t+h}}{E[X]_{t,t+h}}\Big(\mu h - EY_{t,t+h}\Big)$$

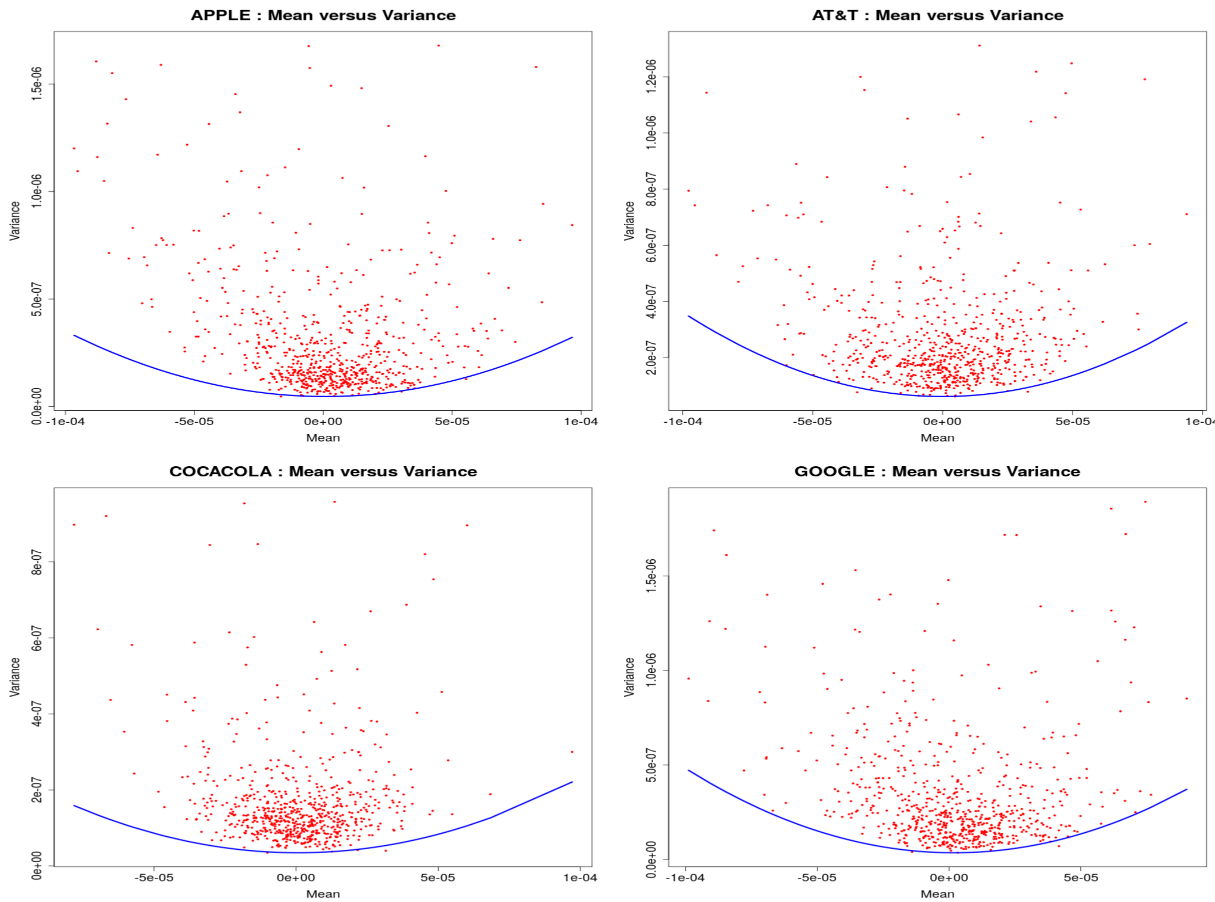

Figure 3: Upper-parabolic mean-variance relation (4.7) for 1-minute quotes of several stocks. Data: Apple, AT&T, Coca-cola, Google. Data source: Dukascopy Bank SA

which includes the special case (4.6) by assigning $X := B$ so that $[B]_{s,t} := t - s$.

5 Uncertainty from rough path lifts

In this section, we are going to discuss open problems that arise as model risk when dealing with the possible uncertainty coming from the rough path lifts. As there are many ways of modeling and interpreting stochastic noises in a model, it might affect the final result and many aspects in the evaluation process.

5.1 Stochastic stock model ambiguity

The rough model in Section 4 implies that there would be an additional source of noise coming from $[X]$ that affects the asset price. To show how negative the effect of the rough path signatures on the expected log-return could be, let us consider again model (4.2) and assign $h := 1$ for simplicity and denote by $R_t = \log \frac{S_t}{S_{t-h}} = Y_{t-1,t}$ the log-return. It then follows from (4.2) that

$$R_t = Y_{t-1,t} = \mu - \frac{\sigma^2}{2}[X]_{t-1,t} + \sigma X_{t-1,t}. \tag{5.1}$$

Table 5.1 shows the expectation and variance of the log-return in the considered models. Observe

	ER_t	Var R_t
Itô Model (1.1)	$\mu - \frac{1}{2}\sigma^2$	σ^2
Wick Model (2.30)	$\mu - \frac{\sigma^2}{2}\left[t^{2H} - (t-1)^{2H}\right]$	σ^2
G-Brownian Model (2.32)	$\mu - \frac{\sigma^2}{2}E[X]_{t-1,t}$	$\sigma^2\text{Var } X_{t-1,t} + \frac{1}{4}\sigma^4\text{Var } [X]_{t-1,t}$

Table 5: Comparison of expected value and variance among models.

that for model (5.1) we have additional nonlinear terms that are quadratic in σ and which will be important for our subsequent discussion. Observe from the classical model (1.1) that

$$ER_t = \mu - \frac{1}{2}\text{Var}R_t. \tag{5.2}$$

For the Wick model we get

$$\begin{aligned} ER_t &= \mu - \frac{1}{2}\left[t^{2H} - (t-1)^{2H}\right]\text{Var}R_t \\ \text{which implies that} \quad & \begin{cases} ER_t > \mu - \frac{1}{2}\text{Var}R_t & \text{if} \quad H < \frac{1}{2} \\ ER_t \leq \mu - \frac{1}{2}\text{Var}R_t & \text{if} \quad H \geq \frac{1}{2} \end{cases}. \end{aligned} \tag{5.3}$$

In comparison, it follows that for the G-Brownian model

$$\begin{aligned} ER_t \leq \mu - \frac{1}{2}\text{Var}R_t \quad &\Leftrightarrow \quad \frac{\sigma^2}{2}\text{Var } X_{t-1,t} + \frac{1}{8}\sigma^4\text{Var } [X]_{t-1,t} \leq \frac{\sigma^2}{2}E[X]_{t-1,t} \\ &\Leftrightarrow \quad \text{Var } X_{t-1,t} + \frac{\sigma^2}{4}\text{Var } [X]_{t-1,t} \leq E[X]_{t-1,t}. \end{aligned} \tag{5.4}$$

In all three models, the expected return is decreasing when the variance is increasing, which matches with classical results. The negative effect of the variance on the expected return is however very different for all three models.
Indeed, the variance of R_t in the classical model (1.1) is different from that of the rough model (5.1); in fact,

$$\text{Var}\left[R_t^{\text{Itô}}\right] = \text{Var}\left[R_t^{\text{Wick}}\right] = \sigma^2 \leq \text{Var}\left[R_t^{\text{GBM}}\right]. \tag{5.5}$$

Due to that reason, the information is somehow hidden in the rough path lift, which can increase the uncertainty of the model and result in a bigger risk.

To see how this leads to a model risk, let us now review the strategy for selling an asset in the portfolio, which is discussed in [45] and [46]. Assume that the growth rate μ, which depends mainly on the intrinsic (fundamental) value, is a piecewise constant function, and that the volatility parameter σ is unknown. According to [46], the criterion of the trading strategy is to sell the asset when its expected value is negative. When we follow (5.2), this would mean in practice that we sell when the variance crosses the threshold 2μ

$$ER_t < 0 \Leftrightarrow 2\mu < \text{Var}R_t. \tag{5.6}$$

When we applied the criterion (5.6) also for the Wick model for $H < \frac{1}{2}$ or for the G-Brownian model with Var $X_{t-1,t} + \frac{\sigma^2}{4}$Var $[X]_{t-1,t} > E[X]_{t-1,t}$, then because the effect of the variance on the expected return in (5.3) and (5.4) is less negative, we would sell too early. On the other hand, in the case of $H \geq \frac{1}{2}$ in (5.3) for the Wick model or in the case of (5.4) for the G-Brownian model, the criterion is not optimal, because it underestimates the larger effect of the variance on the expected return in (5.3) as well as in (5.4), thus we would sell too late. Failure to use the right model can therefore create a model risk of mis-calculating the expected value, which then affects the trading strategies.

5.2 Cooperation game via rough path lifts

Following [40, Section 4.1], we consider the cooperation problem in the simplest form. Namely, consider two individuals who meet at every time step to sum up their total wealth and then to split the whole in ratio $(\rho, 1-\rho)$ for some $\rho \in (0,1)$, before going back to their business. Each individual asset grows according to a discrete geometric rough model

$$\Delta S^i_t = S^i_t\Big(\mu\Delta t + \sigma\Delta X^i_t\Big), \qquad S^i_{t+\Delta t} = S^i_t + \Delta S^i_t, \quad i = 1,2, \tag{5.7}$$

where X^i are two independent scalar Gaussian processes that belongs to C^ν for some $\nu \in (\frac{1}{3}, \frac{1}{2})$. Meanwhile the cooperation model, assigned by $S^{1\oplus 2}$, grows according to the average rough model

$$\Delta S^{1\oplus 2,\rho}_t = S^{1\oplus 2,\rho}_t\Big(\mu\Delta t + \sigma\Delta X^{1\oplus 2,\rho}_t\Big), \qquad S^{1\oplus 2,\rho}_{t+\Delta t} = S^{1\oplus 2,\rho}_t + \Delta S^{1\oplus 2,\rho}_t, \quad i = 1,2, \tag{5.8}$$

where $X^{1\oplus 2,\rho} = \rho X^1 + (1-\rho)X^2$. The limiting equations for (5.7) and (5.8) are, respectively

$$\begin{aligned} dS^i_t &= S^i_t\Big(\mu dt + \sigma dX^i_t\Big); \\ dS^{1\oplus 2,\rho}_t &= S^{1\oplus 2,\rho}_t\Big(\mu dt + \sigma dX^{1\oplus 2,\rho}_t\Big). \end{aligned} \tag{5.9}$$

As such, one can solve (5.9) explicitly using the rough model (2.23)

$$S^i_b \;=\; S^i_a \exp\Big\{\mu(b-a) - \frac{\sigma^2}{2}[X^i]_{a,b} + \sigma X^i_{a,b}\Big\}; \tag{5.10}$$

$$S^{1\oplus 2,\rho}_b \;=\; S^{1\oplus 2,\rho}_a \exp\Big\{\mu(b-a) - \frac{\sigma^2}{2}[X^{1\oplus 2,\rho}]_{a,b} + \sigma X^{1\oplus 2,\rho}_{a,b}\Big\}. \tag{5.11}$$

To evaluate the effectiveness of the cooperation strategy, we consider the continuous versions of (5.7) and (5.8) in the rough form to rough paths $(X^i, [X^i, dX^i])$ and $(X^{1\oplus 2,\rho}, [X^{1\oplus 2,\rho}, dX^{1\oplus 2,\rho}])$, where

$$\begin{aligned} &\int_s^t X^{1\oplus 2,\rho} dX^{1\oplus 2,\rho} = [X^{1\oplus 2,\rho}, dX^{1\oplus 2,\rho}]_{s,t} \\ = \;\; &\rho^2[X^1, dX^1]_{s,t} + (1-\rho)^2[X^2, dX^2]_{s,t} + \rho(1-\rho)\left(\int_s^t X^1_{s,u}dX^2_u + \int_s^t X^2_{s,u}dX^1_u\right) \end{aligned} \tag{5.12}$$

and the two integrals on the right hand side of (5.12) are understood as realizations of two stochastic integrals $\int_s^t X^1_{s,u}dX^2_u$ and $\int_s^t X^2_{s,u}dX^1_u$ of two independent Gaussian processes X^1, X^2, as presented in Remark 2.1. Since $\int_s^t X^1_{s,u}dX^2_u + \int_s^t X^2_{s,u}dX^1_u = X^1_{s,t}X^2_{s,t}$, it is easy to show that

$$[X^{1\oplus 2,\rho}] = \rho^2[X^1] + (1-\rho)^2[X^2]. \tag{5.13}$$

As a result, even in the case of independent identical Gaussian processes with the same form of Levy areas $[X^i, dX^i]$ so that the $[X^i]$ are the same, the solution (5.11) of the cooperation scheme would grow at a rate that is different from that of the solution (5.10) of the individual scheme, because in general

$$[X^1] = [X^2] \neq \big(\rho^2[X^1] + (1-\rho)^2[X^2]\big)\,.$$

In particular, in the specific scenarios of Remark 2.3, it holds that

$$[X^1]_{a,b} = [X^2]_{a,b} \geq \big(\rho^2[X^1]_{a,b} + (1-\rho)^2[X^2]_{a,b}\big) = \big(\rho^2 + (1-\rho)^2\big)\,[X^1]_{a,b} \quad \forall 0 \leq a \leq b, \tag{5.14}$$

thus the solution (5.11) of the cooperation scheme would grow at a higher exponential rate than the solution (5.10) of the individual scheme.

It is important to note that this cooperation game discussed in [40, Section 4.1] is a direct result of Shannon's Demon phenomenon [41, p. 201] or Kelly's fractional betting strategy (see e.g. [36, Example 11.2.5, p. 254] for a continuous version of the optimal portfolio selection problem). It works under the assumption that $(X^i)_{i=1,2}$ are independent Brownian motions and system (5.9) is solved via Itô calculus (which is the first case of (2.11)). In this situation, the inequality (5.14) is strict, which indicates that the individuals really benefit from cooperation. In the rough path setting, cooperation can mitigate the effect of volatility.
However, in general the inequality (5.14) can become an equality, for instance if $[X^i, dX^i]_{s,t} = \int_s^t X^i_{s,u} \circ dX^i_u$ where the stochastic integral is understood in the Stratonovich form for two standard Brownian motions X^i_t, because $[X^i] \equiv 0$ in this case. It can also happen if $[X^i, dX^i]_{s,t} = \int_s^t X^i_{s,u} dX^i_u$ where the stochastic integral is understood in the Young sense for two fractional Brownian motion X^i_t with the same Hurst index $H > \frac{1}{2}$ (which is the fourth case of (2.11)). That being said, it might be premature to claim that cooperation leads to a better result, since the answer depends on how we model the different stochastic fluctuations and interpret the stochastic system (5.9).

Acknowledgments

The authors would like to thank Dukascopy bank SA for permission to use their data for academic research. This work was carried out at the Max Planck Institute for Mathematics in the Sciences (MPI MIS Leipzig). A part of this work was done while L.H.D. visited Vietnam Institute for Advanced Studies in Mathematics (VIASM) for a three month research stay in summer 2022.
We should also thank the reviewers for their detailed and helpful comments.

Data Availability Statement

The data that support the findings of this study are available from Dukascopy Bank SA. Restrictions apply to the availability of these data, which were used under permission for this study. Data are available at https://www.dukascopy.com/swiss/english/home/ with the permission of Dukascopy Bank SA.

Disclaimer from Dukascopy bank SA

The data from Dukascopy Bank are not meant to indicate the actual value at any given point in time but represent a discretionary assessment by Dukascopy Bank SA only. The market data assessment system is in constant development and is provided "AS IS", "AS AVAILABLE", "WITH ALL ITS FAULTS" and is offered without any covenants or any express, implied or statutory warranties including (without limitation and qualification) any warranties as to accuracy, functionality, performance, merchantability, quiet enjoyment, system integration, data accuracy or fitness for any particular purpose and any warranties arising from trade usage, course of dealing or course of performance.

6 Appendix: proof of Theorem 2.5

Existence. To prove the existence part, we first look for a solution $(S, 0, \partial_x S) \in \mathcal{D}^{\alpha,\beta}([0,\tau])$ that is controlled by $(\omega, x)^{\mathrm{T}}$ such that $\partial_\omega S \equiv 0$ (and will be neglected) for a small enough $\tau \in (0,1)$ that will be specified later. Consider the solution mapping

$$\mathcal{M}(S)_t := S_0 + \int_0^t f(u,S_u)du + \int_0^t \sigma_u S_u dx_u, \quad \forall t \in [0,\tau].$$

Since the rough integral and the Riemann integral are both additive, it follows from (2.18) and (2.22) that $\mathcal{M}(S)$ is controlled by $(\omega,x)^{\mathrm{T}}$ with $(\partial_x \mathcal{M}(S))_r = \sigma_r S_r$ and $(\partial_\omega \mathcal{M}(S))_r = 0$. In addition a direct computation shows that

$$\begin{aligned}
&\|\!|(\mathcal{M}(S), \partial_x \mathcal{M}(S))\|\!|_{\alpha,\beta,[0,\tau]} \\
\leq\;& C_\alpha\Big(\tau^{1-2\alpha} + \|\!|\omega\|\!|_{\beta,[0,\tau]} + \|\!|x\|\!|_{\alpha,[0,\tau]} + \|\!|[\omega,dx]\|\!|_{\alpha+\beta,[0,\tau]} + \|\!|[x,dx]\|\!|_{2\alpha,[0,\tau]}\Big) \times \\
&\times\Big(\|(\sigma,\partial_\omega\sigma,\partial_x\sigma)\|_{\alpha,\beta,[0,\tau]} + \|\mu\|_{\infty,[0,T]}\Big)\Big(\|S_0\| + \|(\partial_x S)_0\| + \|\!|(S,\partial_x S)\|\!|_{\alpha,\beta,[0,\tau]}\Big),
\end{aligned} \tag{6.1}$$

for some generic constant $C_\alpha > 1$. Note that $\mathcal{M}(S)_0 = S_0, \partial_x \mathcal{M}(S)_0 = \sigma_0 S_0$. Choose $\tau \in (0,1)$ small enough such that

$$\begin{aligned}
&\tau^{1-2\alpha} + \|\!|\omega\|\!|_{\beta,[0,\tau]} + \|\!|x\|\!|_{\alpha,[0,\tau]} + \|\!|[\omega,dx]\|\!|_{\alpha+\beta,[0,\tau]} + \|\!|[x,dx]\|\!|_{2\alpha,[0,\tau]} \\
=\;& \frac{1}{2}\Big[1 + C_\alpha\Big(\|(\sigma,\partial_\omega\sigma,\partial_x\sigma)\|_{\alpha,\beta,[0,T]} + \|\mu\|_{\infty,[0,T]}\Big)\Big]^{-1},
\end{aligned}$$

and consider the subset of the Banach space $\mathcal{D}^{\alpha,\beta}([0,\tau])$

$$\begin{aligned}
\mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau) := \Big\{(\bar{S},\partial_\omega\bar{S},\partial_x\bar{S}) \in \mathcal{D}^{\alpha,\beta}([0,\tau]) \quad &: \quad \partial_\omega \bar{S} \equiv 0, \bar{S}_0 = S_0, (\partial_x\bar{S})_0 = \sigma_0 S_0 \\
&\text{and} \quad \Big\|\!\Big|(\bar{S},\partial_x\bar{S})\Big\|\!\Big|_{\alpha,\beta,[0,\tau]} \leq \|S_0\| + \|\sigma_0 S_0\|\Big\},
\end{aligned}$$

then $\mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau)$ is a compact set in a Banach space with respect to the metric

$$d(S^1,S^2) = \Big\|\!\Big|(S^1 - S^2, \partial_x S^1 - \partial_x S^2)\Big\|\!\Big|_{\alpha,\beta,[0,\tau]}.$$

It also follows from (6.1) that $\mathcal{M} : \mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau) \to \mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau)$. Moreover, from the linearity of $\mathcal{M}$ w.r.t. S and similar estimates to (6.1), it is easy to check that on $\mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau)$

$$\begin{aligned}
&\Big\|\!\Big|(\mathcal{M}(S-\bar{S}), \partial_x \mathcal{M}(S-\bar{S}))\Big\|\!\Big|_{\alpha,\beta,[0,\tau]} \\
\leq\;& C_\alpha\Big(\tau^{1-2\alpha} + \|\!|\omega\|\!|_{\beta,[0,\tau]} + \|\!|x\|\!|_{\alpha,[0,\tau]} + \|\!|[\omega,dx]\|\!|_{\alpha+\beta,[0,\tau]} + \|\!|[x,dx]\|\!|_{2\alpha,[0,\tau]}\Big) \times \\
&\times\Big(\|(\sigma,\partial_\omega\sigma,\partial_x\sigma)\|_{\alpha,\beta,[0,\tau]} + \|\mu\|_{\infty,[0,T]}\Big) \Big\|\!\Big|(S-\bar{S},\partial_x(S-\bar{S}))\Big\|\!\Big|_{\alpha,\beta,[0,\tau]} \\
\leq\;& \frac{1}{2}\Big\|\!\Big|(S-\bar{S},\partial_x(S-\bar{S}))\Big\|\!\Big|_{\alpha,\beta,[0,\tau]}.
\end{aligned} \tag{6.2}$$

Hence $\mathcal{M}$ is a contraction mapping on $\mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau)$ and by the Banach fixed point theorem, there exists a unique solution on $\mathcal{D}^{\alpha,\beta}(S_0,\sigma_0 S_0,\tau)$.

Uniqueness. Assume that there exist two solutions S and $\bar{S}$ starting from the same initial values S_0 and $\sigma_0 S_0$, then $\partial_\omega S = \partial_\omega \bar{S} = 0$. By taking the difference $(S - \bar{S})$ and repeat the same arguments as above with noting that $\mathcal{M}(S-\bar{S}) = S - \bar{S}, \partial_x \mathcal{M}(S-\bar{S}) = \partial_x(S-\bar{S})$, we obtain from (6.2) that

$$\Big\|\!\Big|(S-\bar{S},\partial_x(S-\bar{S}))\Big\|\!\Big|_{\alpha,\beta,[0,\tau]} \leq \frac{1}{2}\Big\|\!\Big|(S-\bar{S},\partial_x(S-\bar{S}))\Big\|\!\Big|_{\alpha,\beta,[0,\tau]}$$

or $S - \bar{S} \equiv 0$ on $[0,\tau]$. This proves the uniqueness of the solution with given initial values.

Concatenation. Next, for a given interval $[0,T]$ we construct a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ such that $\tau_0 = 0$ and

$$\begin{aligned}
&(\tau_{k+1}-\tau_k)^{1-2\alpha} + \|\!|\omega\|\!|_{\beta,[\tau_k,\tau_{k+1}]} + \|\!|x\|\!|_{\alpha,[\tau_k,\tau_{k+1}]} + \|\!|[\omega,dx]\|\!|_{\alpha+\beta,[\tau_k,\tau_{k+1}]} + \|\!|[x,dx]\|\!|_{2\alpha,[\tau_k,\tau_{k+1}]} \\
=\;& \frac{1}{2}\Big[1 + C_\alpha\Big(\|(\sigma,\partial_\omega\sigma,\partial_x\sigma)\|_{\alpha,\beta,[0,T]} + \|\mu\|_{\infty,[0,T]}\Big)\Big]^{-1}.
\end{aligned} \tag{6.3}$$

It is easy to check (see e.g. [17]) that $\tau_k \to \infty$ as $k \to \infty$, and there exists a unique solution on each interval $[\tau_k, \tau_{k+1}]$. The unique solution on $[0, T]$ is then constructed by concatenation. The conclusion also holds for the backward equation by the same arguments.

References

[1] E. Alòs, O. Mazet, D. Nualart. Stochastic calculus with respect to Gaussion processes. *Annals of Probability*, **29**, 2, (2001), 766–801.

[2] A. Andreoli, F. Caravenna, P. Dai Pra, G. Posta. Scaling and multiscaling in financial series: a simple model. *Adv. Appl. Prob.*, **44**, (2012), 1018–1051.

[3] E. Bacry, J. Delour, J.F.Muzy. Multifractal random walk. *Physical Review E*, **64**, (2001), 026103.

[4] C. Bayer, P. Friz, J. Gatheral. Pricing under rough volatility. *Quantitative Finance.*, Vol. **16 (6)**, (2016), 887–904.

[5] C. Bayer, P. Friz, P. Gassiat, J. Martin, B. Stemper. A regularity structure for rough volatility. *Mathematical Finance*, (2018). DOI: 10.1111/mafi.12233.

[6] C. Bender. Simple arbitrage. *Annals of Applied Prob.*, **22** (5), (2012), 2067–2085.

[7] S. Bianchi, A. Pianese. Multi-fractional processes in finance. *Risk and Decision Analysis*, **5**, (2014), 1–22.

[8] R.J. Buonocore, T. Aste, T. Di Matteo. Measuring multiscaling in financial time series. *Chaos, Solitons and Fractals*, **88**, (2016), 38–47.

[9] P. Cheridito. Mixed fractional Brownian motion. *Bernoulli* **7**(6), (2001), 913–934.

[10] P. Cheridito. Arbitrage in fractional Brownian motion models. *Finance Stoch.* **7**(4), (2003), 533–553.

[11] F. Comte, E. Renault. Long memory in continuous-time stochastic volatility models. *Mathematical Finance*, Vol. **8**, No. 4, (1998), 291–323.

[12] L.Coutin. Rough paths via sewing lemma. *ESAIM: Probability and Statistics.*, **16**, (2012), 479–526.

[13] C. Czichowsky, W. Schachermayer, J. Yang. Shadow prices for continuous processes. *Mathematical Finance*, Vol. **27**, No. 3, (2017), 623–658.

[14] L. Denis, C. Martini. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Annals of Applied Prob.*, **16** (2), (2006), 827–852.

[15] T. Di Matteo. Multi-scaling in finance. *Quantitative Finance*, **7** (1), (2007), 21–36.

[16] L. H. Duc. Random attractors for dissipative systems with rough noises. *Disc. Cont. Dyn. Syst.*, Vol. **42**(4), (2022), 1873–1902.

[17] L. H. Duc. Controlled differential equations as rough integrals. *Pure and Applied Functional Analysis*, Vol. **7**(4), (2022), 1245–1271.

[18] R. Durrett. *Brownian motion and martingales in analysis.* Wadsworth Pub. Co., 1st edition, 1984.

[19] P. Friz, M. Hairer. *A course on rough path with an introduction to regularity structure.* Universitext, Vol. **XIV**, Springer, Berlin, 2014.

[20] P. Friz, N. Victoir. *Multidimensional stochastic processes as rough paths: theory and applications.* Cambridge Studies in Advanced Mathematics, 120. Cambridge Unversity Press, Cambridge, 2010.

[21] J. Gatheral, T. Jaisson, M. Rosenbaum. Volatility is rough. *Quantitative Finance*, **18** (6), (2018), 933–949.

[22] P. Guasoni. No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16** (3), (2007), 569–582.

[23] P. Guasoni, M. Rásonyi, W. Schachermayer. Consistent price systems and face-lifting pricing under transaction cost. *Annals of Applied Probability*, **18**, (2008), 491–520.

[24] M. Gubinelli. Controlling rough paths. *J. Funtional Analysis*, **216** (1), (2004), 86–140.

[25] P. Halmos. *Measure theory.* Graduate Texts in Mathematics, Springer-New York, 1950.

[26] S. L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies.* **6 (2)**, (1993), 327–343

[27] T. Higuchi. Approach to an irregular time series on the basis of the fractal theory. *Physica D*, **31**, (1988), 277–283.

[28] J.Hull and A.White. The Pricing of Options on Assets with Stochastic Volatilities. *J. Finance* **3**, (1987) 281–300.

[29] O. Kallenberg. *Foundations of modern probability.* 2nd edn., Springer, New York, (2002).

[30] Y. A. Kuznetsov. The absence of arbitrage in a model with fractal Brownian motion. *Russ. Math. Surv.*, **54**, (1999), 847–848.

[31] T. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoam.*, Vol. **14** (2), (1998), 215–310.

[32] B. Mandelbrot, J. van Ness. Fractional Brownian motion, fractional noises and applications. *SIAM Review*, **4**, No. 10, (1968), 422–437.

[33] Y. Mishura. *Stochastic calculus for fractional Brownian motion and related processes.* Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2008.

[34] G. Di Nunno, B. Øksendal, F. Proske. *Malliavin calculus for Lévy processes with applications to finance.* Universitext, Springer, (2009).

[35] B. Oksendal. Fractional Brownian motion in finance. In: Jensen, B.S., Palokangas, T. (eds) *Stochastic economic dynamics, CBS Press Copenhagen*, (2007).

[36] B. Oksendal. *Stochastic differential equations. An introduction with applications. 6th edition* Springer, 2013.

[37] M. Pakkanen. Stochastic integrals and conditional full support. *J. Appl. Prob.*, **47**, (2010), 650–667.

[38] Sh. Peng. *Nonlinear expectations and stochastic calculus under uncertainty.* Probability Theory and Stochastic Modelling, Springer, 2010.

[39] Sh. Peng, H. Zhang. Stochastic calculus with respect to G-Brownian motion viewed through rough paths. *Science China, Mathematics*, Vol. **59**(1), (2016), 1–xx.

[40] O. Peters, A. Adamou. *Ergodicity economics.* London Mathematical Laboratory, 2018.

[41] W. Poundstone. *Fortune's Formula: The untold story of the scientific betting system that beat the casinos and Wall Street.* Hill and Wang - a division of Farrar, Straus and Giroux, New York, 2006.

[42] M. Rásonyi, H. Sayit. Sticky processes, local and true martingales. *Bernoulli*, **24** (4A), (2018), 2752–2775.

[43] L.C.G. Rogers. Arbitrage with fractional Brownian motion. *Math. Finance* **7**(1), (1997), 95–105.

[44] P. A. Samuelson. Rational theory of warrant pricing. Indust. Management Rev., **6**(2), (1965), 13–31.

[45] A. Shiryaev, Z. Xu, X. Zhou. Thou shalt buy and hold. *Quantiative Finance*, **8**(8), (2008), 765–776.

[46] A. Shiryaev, A. Novikov. On a stochastic version of the trading rule "Buy and Hold". *Statistics and Decisions*, **26**, (2008), 289–302.

[47] L.C. Young. An integration of Hölder type, connected with Stieltjes integration. *Acta Math.* **67**, (1936), 251–282.