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by

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# Monogamy Properties of Qubit Systems

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We investigate monogamy relations related to quantum entanglement for  $n$ -qubit quantum systems. General monogamy inequalities are presented to the  $\beta$ th ( $\beta \in (0, 2)$ ) power of concurrence, negativity and the convex-roof extended negativity, as well as the  $\beta$ th ( $\beta \in (0, \sqrt{2})$ ) power of entanglement of formation. These monogamy relations are complementary to the existing ones with different regions of parameter  $\beta$ . In additions, new monogamy relations are also derived which include the existing ones as special cases.

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## I. INTRODUCTION

Quantum entanglement [1–5] lies at the heart of quantum information processing and quantum computation[6]. Accordingly its quantification has drawn much attention in the last decade. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of other systems limits its entanglement with the remaining systems. The monogamy relations give rise to the structures of entanglement distribution in multipartite systems. Monogamy is also an essential feature allowing for security in quantum key distribution [8].

For a tripartite system  $A, B$  and  $C$ , the monogamy of an entanglement measure  $\varepsilon$  implies that [9], the entanglement between  $A$  and  $BC$  satisfies  $\varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}$ . Such monogamy relations are not always satisfied by any entanglement measures. It has been shown that the squared concurrence  $C^2$  [10, 11] and the squared entanglement of formation  $E^2$  [12] do satisfy such monogamy relations. In Ref.[13] it has been shown that the general monogamy inequalities are satisfied by the  $\alpha$  ( $\alpha \geq 2$ )th power of concurrence  $C^\alpha$  and the  $\alpha$  ( $\alpha \geq \sqrt{2}$ )th power of entanglement of formation  $E^\alpha$  for  $n$ -qubit mixed states. Another useful entanglement measure is the negativity[7], a quantitative version of Peres's criterion for separability. The authors in Ref.[14] studied the monogamy property of the  $\alpha$ th power of negativity  $N^\alpha$  ( $\alpha \geq 2$ ) and discussed tighter  $\alpha$ th ( $\alpha \geq 2$ ) power of the convex-roof extended negativity (CREN)  $\tilde{N}^\alpha$ . In Ref.[15] tighter monogamy inequalities for concurrence, entanglement of formation and CREN has been investigated for  $\alpha \geq 2$ .

However, it is not clear for the monogamy properties of the  $\alpha$ th ( $0 < \alpha < 2$ ) power of concurrence, negativity and CREN, and the  $\alpha$ th ( $0 < \alpha < \sqrt{2}$ ) power of entanglement of formation. In this paper, we study the general monogamy inequalities of  $C^\beta$ ,  $N^\beta$ ,  $\tilde{N}^\beta$  and  $E^\beta$  for  $\beta \in [0, M]$ , where  $M$  is any real number greater than zero.

## II. MONOGAMY PROPERTY OF CONCURRENCE

For a bipartite pure state  $|\psi\rangle_{AB}$ , the concurrence is given by [16–18],

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \quad (1)$$

where  $\rho_A$  is reduced density matrix by tracing over the subsystem  $B$ ,  $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ . The concurrence is extended to mixed states  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ ,  $p_i \geq 0$ ,  $\sum_i p_i = 1$ , by the convex roof construction,

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle). \quad (2)$$

For  $n$ -qubit quantum states, the concurrence satisfies [13]

$$C_{A|B_1 B_2 \dots B_{n-1}}^\alpha \geq C_{AB_1}^\alpha + \dots + C_{AB_{n-1}}^\alpha, \quad (3)$$

for  $\alpha \geq 2$ , where  $C_{A|B_1 B_2 \dots B_{n-1}}$  is the concurrence of  $\rho$  under bipartite partition  $A|B_1 B_2 \dots B_{n-1}$ , and  $C_{AB_i}$ ,  $i = 1, 2, \dots, n-1$ , is the concurrence of the mixed states  $\rho_{AB_i} = \text{Tr}_{B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_{n-1}}(\rho)$ . For  $C_{AB_i} \neq 0$ ,  $i = 1, \dots, n-1$ , the concurrence satisfies

$$C_{A|B_1 \dots B_{n-1}}^\alpha < C_{AB_1}^\alpha + \dots + C_{AB_{n-1}}^\alpha, \quad (4)$$

for  $\alpha \leq 0$ . Further, in Ref. [15] tighter monogamy inequalities than (3) are derived for the  $\alpha$ th ( $\alpha \geq 2$ ) power of concurrence.

**Lemma 1** For real numbers  $x \in [0, 1]$  and  $t \geq 1$ , we have  $(1+t)^x \geq 1 + (2^x - 1)t^x$ .

[Proof] Let  $g_x(t) = \frac{(1+t)^x - 1}{t^x}$  with  $x \in [0, 1]$  and  $t \in [1, +\infty)$ . Since  $\frac{dg_x(t)}{dt} = xt^{-(x+1)}[1 - (1+t)^{x-1}] \geq 0$ , we obtain that  $g_x(t)$  is an increasing function of  $t$ . Hence,  $g_x(t) \geq g_x(1)$ , i.e,  $(1+t)^x \geq 1 + (2^x - 1)t^x$ .

**Theorem 1** For any  $2 \otimes 2 \otimes 2^{n-2}$  tripartite mixed state: (1) if  $C_{AB} \leq C_{AC}$ , the concurrence satisfies

$$C_{A|BC}^\beta \geq C_{AB}^\beta + (2^{\frac{\beta}{\alpha}} - 1)C_{AC}^\beta, \quad (5)$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

(2) if  $C_{AB} \geq C_{AC}$ , the concurrence satisfies

$$C_{A|BC}^\beta \geq (2^{\frac{\beta}{\alpha}} - 1)C_{AB}^\beta + C_{AC}^\beta, \quad (6)$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

[Proof] For arbitrary  $2 \otimes 2 \otimes 2^{n-2}$  tripartite state  $\rho_{ABC}$ , one has [13],

$$C_{A|BC}^\alpha \geq C_{AB}^\alpha + C_{AC}^\alpha.$$

If  $\max\{C_{AB}, C_{AC}\} = 0$ , i.e.,  $C_{AB} = C_{AC} = 0$ , obviously we have the inequalities (5) or (6); If  $\min\{C_{AB}, C_{AC}\} = 0$ , obviously  $C_{A|BC} \geq \max\{C_{AB}, C_{AC}\} \geq (2^{\frac{\beta}{\alpha}} - 1)\max\{C_{AB}, C_{AC}\}$  with  $0 \leq \beta \leq \alpha$ , we also have the inequalities (5) or (6).

If  $\max\{C_{AB}, C_{AC}\} > 0$  and  $\min\{C_{AB}, C_{AC}\} \neq 0$ , assuming  $0 < C_{AB} \leq C_{AC}$ , we have

$$\begin{aligned} C_{A|BC}^{\alpha x} &\geq (C_{AB}^\alpha + C_{AC}^\alpha)^x \\ &= C_{AB}^{\alpha x} \left(1 + \frac{C_{AC}^\alpha}{C_{AB}^\alpha}\right)^x \\ &\geq C_{AB}^{\alpha x} \left(1 + (2^x - 1) \left(\frac{C_{AC}^\alpha}{C_{AB}^\alpha}\right)^x\right) \\ &= C_{AB}^{\alpha x} + (2^x - 1)C_{AC}^{\alpha x}, \end{aligned}$$

where the second inequality is due to the inequality  $(1+t)^x \geq 1 + (2^x - 1)t^x$  for  $0 \leq x \leq 1$  and  $t = \frac{C_{AC}^\alpha}{C_{AB}^\alpha} \geq 1$ . Denote  $\alpha x = \beta$ . Then  $\beta \in [0, \alpha]$  since  $x \in [0, 1]$  and one gets the inequality (5). If  $C_{AB} \geq C_{AC}$ , similar proof gives the inequality (6).

One can see that Theorem 1 reduces to the monogamy inequality (3) if  $\beta = \alpha \geq 2$ . In particular, if we take  $\beta = 1$ , we have  $C_{A|BC} \geq \min\{C_{AB}, C_{AC}\} + (2^{\frac{1}{\alpha}} - 1)\max\{C_{AB}, C_{AC}\}$  for  $\alpha \geq 2$ . And the tighter relation is  $C_{A|BC} \geq \min\{C_{AB}, C_{AC}\} + (\sqrt{2} - 1)\max\{C_{AB}, C_{AC}\}$ .

*Example 1.* Let us consider the three-qubit case. Any three-qubit state  $|\psi\rangle$  can be written in the generalized Schmidt decomposition [13, 19, 20],

$$\begin{aligned} |\psi\rangle &= \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle \\ &\quad + \lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \quad (7)$$

where  $\lambda_i \geq 0$ ,  $i = 0, \dots, 4$ , and  $\sum_{i=0}^4 \lambda_i^2 = 1$ . From Eq.(1) and Eq.(2), we have  $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$ ,  $C_{AB} = 2\lambda_0\lambda_2$ , and  $C_{AC} = 2\lambda_0\lambda_3$ . Without loss of generality, we set  $\lambda_0 = \cos\theta_0$ ,  $\lambda_1 = \sin\theta_0 \cos\theta_1$ ,  $\lambda_2 = \sin\theta_0 \sin\theta_1 \cos\theta_2$ ,  $\lambda_3 = \sin\theta_0 \sin\theta_1 \sin\theta_2 \cos\theta_3$ , and  $\lambda_4 = \sin\theta_0 \sin\theta_1 \sin\theta_2 \sin\theta_3$ ,  $\theta_i \in [0, \frac{\pi}{2}]$ . Assume  $\lambda_3 \geq \lambda_2$ , i.e.  $C_{AC} \geq C_{AB}$ :

(a) if  $\theta_2 = \frac{\pi}{2}$ , we have

$$\begin{aligned} C_{A|BC}^\beta - C_{AB}^\beta - (2^{\frac{\beta}{\alpha}} - 1)C_{AC}^\beta &= (2\lambda_0)^\beta \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^{\frac{\beta}{2}} - \lambda_2^\beta - (2^{\frac{\beta}{\alpha}} - 1)\lambda_3^\beta \right] \\ &= (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 \left[ 1 - (2^{\frac{\beta}{\alpha}} - 1)\cos^\beta \theta_3 \right] \\ &\geq (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 (2 - 2^{\frac{\beta}{\alpha}}) \\ &\geq 0, \end{aligned}$$

where  $0 \leq \beta \leq \alpha$ ,  $\alpha \geq 2$  and the first inequality is due to  $\cos\theta_3 \leq 1$ ;

(b) if  $\theta_2 \neq \frac{\pi}{2}$ , we denote  $t_1 = \frac{\sin^\alpha \theta_2 \cos^\alpha \theta_3}{\cos^\alpha \theta_2}$ . We have

$$\begin{aligned} C_{A|BC}^\beta - C_{AB}^\beta - (2^{\frac{\beta}{\alpha}} - 1)C_{AC}^\beta &= (2\lambda_0)^\beta \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^{\frac{\beta}{2}} - \lambda_2^\beta - (2^{\frac{\beta}{\alpha}} - 1)\lambda_3^\beta \right] \\ &= (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 \left[ 1 - \cos^\beta \theta_2 - (2^{\frac{\beta}{\alpha}} - 1)\sin^\beta \theta_2 \cos^\beta \theta_3 \right] \\ &= (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 \left[ 1 - \cos^\beta \theta_2 \left( 1 + (2^{\frac{\beta}{\alpha}} - 1)t_1^{\frac{\beta}{\alpha}} \right) \right] \\ &\geq (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 \left[ 1 - \cos^\beta \theta_2 (1 + t_1)^{\frac{\beta}{\alpha}} \right] \\ &= (2\lambda_0)^\beta \sin^\beta \theta_0 \sin^\beta \theta_1 \left[ 1 - (\cos^\alpha \theta_2 + \sin^\alpha \theta_2 \cos^\alpha \theta_3)^{\frac{\beta}{\alpha}} \right] \\ &\geq 0, \end{aligned}$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ . The first inequality is due to Lemma 1 with  $0 \leq x = \frac{\beta}{\alpha} \leq 1$  and the second inequality is due to  $\cos^\alpha \theta_2 + \sin^\alpha \theta_2 \cos^\alpha \theta_3 \leq 1$  for  $\alpha \geq 2$ .

Therefore, for this case we have  $C_{A|BC}^\beta \geq C_{AB}^\beta + (2^{\frac{\beta}{\alpha}} - 1)C_{AC}^\beta$  for  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ . For the case  $\lambda_3 \leq \lambda_2$ , i.e.,  $C_{AB} \geq C_{AC}$ , similarly one obtains that  $C_{A|BC}^\beta \geq (2^{\frac{\beta}{\alpha}} - 1)C_{AB}^\beta + C_{AC}^\beta$  with  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

By using the Theorem 1 repeatedly, we have the following theorem for multipartite qubit systems.

**Theorem 2** For any  $n$ -qubit quantum state  $\rho$  such that  $C_{AB_i} \leq C_{A|B_{i+1} \dots B_{n-1}}$  for  $i = 1, \dots, m$ , and  $C_{AB_j} \geq C_{A|B_{j+1} \dots B_{n-1}}$  for  $j = m+1, \dots, n-2$ ,  $\forall 1 \leq m \leq n-3$ ,  $n \geq 4$ , we have

$$\begin{aligned} C^\beta(\rho_{A|B_1 B_2 \dots B_{n-1}}) &\geq \sum_{i=1}^m (2^{\frac{\beta}{\alpha}} - 1)^{i-1} C^\beta(\rho_{AB_i}) \quad (8) \\ &\quad + (2^{\frac{\beta}{\alpha}} - 1)^{m+1} \sum_{i=m+1}^{n-2} C^\beta(\rho_{AB_i}) \\ &\quad + (2^{\frac{\beta}{\alpha}} - 1)^m C^\beta(\rho_{AB_{n-1}}), \end{aligned}$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

[Proof] For convenience, we denote  $r = 2^{\frac{\beta}{\alpha}} - 1$ . For any  $2 \otimes 2 \otimes 2 \otimes \dots \otimes 2$  quantum states  $\rho_{AB_1 \dots B_{n-1}}$ , we

have

$$\begin{aligned}
& C_{A|B_1 B_2 \dots B_{n-1}}^\beta(\rho) \\
& \geq C_{AB_1}^\beta + r C_{A|B_2 \dots B_{n-1}}^\beta \\
& \geq C_{A|B_1}^\beta + r C_{A|B_2}^\beta + r^2 C_{A|B_3 \dots B_{n-2}}^\beta \\
& \geq \dots \\
& \geq \sum_{i=1}^m r^{i-1} C_{AB_i}^\beta + r^m C_{A|B_{m+1} \dots B_{n-1}}^\beta \\
& \geq \sum_{i=1}^m r^{i-1} C_{AB_i}^\beta + r^m [r C_{AB_{m+1}}^\beta + C_{A|B_{m+2} \dots B_{n-1}}^\beta] \\
& \geq \dots \\
& \geq \sum_{i=1}^m r^{i-1} C_{AB_i}^\beta + r^{m+1} \sum_{i=m+1}^{n-2} C_{AB_i}^\beta + r^m C_{AB_{n-1}}^\beta,
\end{aligned}$$

where the first four inequalities are due to  $C_{AB_i} \leq C_{A|B_{i+1} \dots B_{n-1}}$  ( $i = 1, \dots, m$ ) and the inequality (5), the last three inequalities are due to  $C_{AB_j} \leq C_{A|B_{j+1} \dots B_{n-1}}$  ( $j = m+1, \dots, n-2$ ) and the inequality (6).  $\square$

For an  $n$ -qubit quantum state  $\rho_{AB_1 \dots B_{n-1}}$ , in Ref.[13] it has been shown that the  $\beta$ th concurrence  $C^\beta$  ( $0 < \beta < 2$ ) does not satisfy monogamy inequalities like  $C^\beta(|\psi\rangle_{A|B_1 B_2 \dots B_{n-1}}) \geq \sum_{i=1}^{n-1} C^\beta(\rho_{AB_i})$ . Theorem (2) first time gives the monogamy inequality satisfied by the  $\beta$ -th concurrence  $C^\beta$  for the case of ( $0 < \beta < 2$ ), a problem that was not solved in Refs.[13, 15]. Specifically, if  $\beta = 1$  and  $\alpha = 2$ , we get the monogamy relation satisfied by the concurrence  $C$ :

$$\begin{aligned}
C(\rho_{A|B_1 B_2 \dots B_{n-1}}) & \geq \sum_{i=1}^m (\sqrt{2} - 1)^{i-1} C(\rho_{AB_i}) \\
& + (\sqrt{2} - 1)^{m+1} \sum_{i=m+1}^{n-2} C(\rho_{AB_i}) \\
& + (\sqrt{2} - 1)^m C(\rho_{AB_{n-1}}).
\end{aligned}$$

*Example 2.* Let us consider the pure state  $|\psi\rangle$  (7) in the Example 1. Set  $\lambda_0 = \frac{\sqrt{2}}{2}$ ,  $\lambda_1 = \frac{1}{2}$ ,  $\lambda_2 = \frac{1}{4}$ ,  $\lambda_3 = \frac{3\sqrt{3}}{20}$  and  $\lambda_4 = \frac{\sqrt{3}}{5}$ . We have  $C_{A|BC} = \frac{\sqrt{2}}{2} \approx 0.707$  and  $C_{AB} + C_{AC} = \frac{5\sqrt{2} + 3\sqrt{6}}{20} \approx 0.721$ . One can see that  $C_{A|BC} < C_{AB} + C_{AC}$ . Denoting  $u(\beta, \alpha) = C_{A|BC}^\beta - C_{AB}^\beta - (2^{\frac{\beta}{\alpha}} - 1)C_{AC}^\beta = (\frac{\sqrt{2}}{2})^\beta - (\frac{\sqrt{2}}{4})^\beta - (2^{\frac{1}{\alpha}} - 1)(\frac{3\sqrt{6}}{20})^\beta$  with  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ , we have  $u(1, \alpha) \geq 0.201$  for all  $\alpha \geq 2$ . Furthermore, our result shows that  $u(\beta, \alpha) \geq 0$  for all  $0 \leq \beta \leq 2$  and  $\alpha \geq 2$ , see Fig. 1.

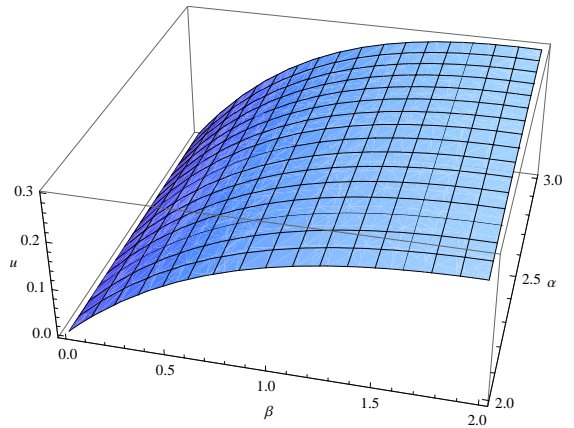


FIG. 1:  $u(\beta, \alpha)$  for  $0 \leq \beta \leq 2$  and  $\alpha \geq 2$ .

### III. MONOGAMY INEQUALITY FOR NEGATIVITY

Given a bipartite state  $\rho_{AB}$ , the negativity is defined by [23]

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2},$$

where  $\rho_{AB}^{T_A}$  is the partially transposed matrix of  $\rho_{AB}$  with respect to the subsystem  $A$ ,  $\|X\| = \text{Tr}\sqrt{XX^\dagger}$  denotes the trace norm of  $X$ . For the convenience of discussion, we use the following definition of negativity:

$$N(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1.$$

It has been shown that for any  $n$ -qubit pure state  $|\psi\rangle_{A|B_1 \dots B_{n-1}}$ , the negativity satisfies the monogamy inequality holds[14]:

$$N_{A|B_1 \dots B_{n-1}}^\alpha(|\psi\rangle) \geq N_{AB_1}^\alpha + \dots + N_{AB_{n-1}}^\alpha,$$

for  $\alpha \geq 2$ , and the polygamy inequality

$$N_{A|B_1 \dots B_{n-1}}^\alpha(|\psi\rangle) < N_{AB_1}^\alpha + \dots + N_{AB_{n-1}}^\alpha,$$

for  $\alpha \leq 0$ . Here  $N_{A|B_1 \dots B_{n-1}}(|\psi\rangle)$  is the negativity of  $|\psi\rangle$  under bipartite partition  $A|B_1 \dots B_{n-1}$ , and  $N_{AB_i}$  is the negativity of the quantum state  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{n-1}}(|\psi\rangle)$ . In the following we study the monogamy property of the  $\beta$ th power of negativity  $N^\beta$  for  $\beta \in (0, 2)$ .

**Theorem 3** For any  $n$ -qubit quantum pure state  $|\psi\rangle$  such that  $C_{AB_i} \leq C_{A|B_{i+1} \dots B_{n-1}}$  for  $i = 1, \dots, m$ , and  $C_{AB_j} \geq C_{A|B_{j+1} \dots B_{n-1}}$  for  $j = m+1, \dots, n-2$ ,  $\forall 1 \leq m \leq n-3$  and  $n \geq 4$ , we have

$$\begin{aligned}
N^\beta(|\psi\rangle_{A|B_1 B_2 \dots B_{n-1}}) & \geq \sum_{i=1}^m (2^{\frac{\beta}{\alpha}} - 1)^{i-1} N^\beta(\rho_{AB_i}) \quad (9) \\
& + (2^{\frac{\beta}{\alpha}} - 1)^{m+1} \sum_{i=m+1}^{n-2} N^\beta(\rho_{AB_i}) \\
& + (2^{\frac{\beta}{\alpha}} - 1)^m N^\beta(\rho_{AB_{n-1}}),
\end{aligned}$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

Theorem 3 can be seen by using (8) in theorem 2, and noting that  $C(|\psi\rangle_{A|BC}) = N(|\psi\rangle_{A|BC})$  for  $2 \otimes t \otimes s$  ( $t \geq 2$ ,  $s \geq 2$ ) systems and  $N(\rho_{AB}) \leq C(\rho_{AB})$  for  $2 \otimes m$  systems.

Given a bipartite state  $\rho_{AB}$ , the CREN is defined as the convex roof extended negativity of pure states [14, 21]

$$\tilde{N}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle),$$

with the infimum taking over all possible decompositions of  $\rho_{AB}$  in a mixture of pure states,  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ ,  $p_i \geq 0$ ,  $\sum_i p_i = 1$ .

For a mixed state  $\rho_{ABC}$  in  $2 \otimes 2 \otimes 2^{n-2}$  systems, the following monogamy inequality holds [14],  $\tilde{N}_{A|BC}^\alpha(\rho) \geq \tilde{N}_{AB}^\alpha + \tilde{N}_{AC}^\alpha$  for  $\alpha \geq 2$ , and the following polygamy inequality holds,  $\tilde{N}_{A|BC}^\alpha(\rho) < \tilde{N}_{AB}^\alpha + \tilde{N}_{AC}^\alpha$  for  $\alpha \leq 0$ . For multiqubit mixed states  $\rho_{AB_1 \dots B_{n-1}}$ , one has the following monogamy inequality for the  $\alpha$ th power of CREN [14]:

$$\tilde{N}_{A|B_1 \dots B_{n-1}}^\alpha(\rho) \geq \tilde{N}_{AB_1}^\alpha + \dots + \tilde{N}_{AB_{n-1}}^\alpha$$

for  $\alpha \geq 2$ , and the following polygamy inequality:

$$\tilde{N}_{A|B_1 \dots B_{n-1}}^\alpha(\rho) < \tilde{N}_{AB_1}^\alpha + \dots + \tilde{N}_{AB_{n-1}}^\alpha$$

for  $\alpha \leq 0$ , where  $\tilde{N}_{A|B_1 \dots B_{n-1}}(\rho)$  is the negativity of  $\rho$  under bipartite partition  $A|B_1 \dots B_{n-1}$ , and  $\tilde{N}_{AB_i}$  is the negativity of the quantum state  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{n-1}}(\rho)$ .

With a similar consideration to concurrence, we obtain the following result.

**Corollary 1** For any  $2 \otimes 2 \otimes 2^{n-2}$  mixed state  $\rho_{ABC}$ , and  $0 \leq \beta \leq \alpha$ ,  $\alpha \geq 2$ :

(1) if  $\tilde{N}_{AB} \leq \tilde{N}_{AC}$ , the concurrence satisfies

$$\tilde{N}_{A|BC}^\beta \geq \tilde{N}_{AB}^\beta + (2^{\frac{\beta}{\alpha}} - 1) \tilde{N}_{AC}^\beta; \quad (10)$$

(2) if  $\tilde{N}_{AB} \geq \tilde{N}_{AC}$ , the concurrence satisfies

$$\tilde{N}_{A|BC}^\beta \geq (2^{\frac{\beta}{\alpha}} - 1) \tilde{N}_{AB}^\beta + \tilde{N}_{AC}^\beta. \quad (11)$$

**Corollary 2** For any  $n$ -qubit quantum state  $\rho_{AB_1 \dots B_{n-1}}$  such that  $\tilde{N}_{AB_i} \leq \tilde{N}_{A|B_{i+1} \dots B_{n-1}}$  ( $i = 1, \dots, m$ ) and  $\tilde{N}_{AB_j} \geq \tilde{N}_{A|B_{j+1} \dots B_{n-1}}$  ( $j = m+1, \dots, n-2$ ),  $\forall 1 \leq m \leq n-3$ ,  $n \geq 4$ , we have

$$\begin{aligned} \tilde{N}^\beta(\rho_{A|B_1 B_2 \dots B_{n-1}}) &\geq \sum_{i=1}^m (2^{\frac{\beta}{\alpha}} - 1)^{i-1} \tilde{N}^\beta(\rho_{AB_i}) \quad (12) \\ &+ (2^{\frac{\beta}{\alpha}} - 1)^{m+1} \sum_{i=m+1}^{n-2} \tilde{N}^\beta(\rho_{AB_i}) \\ &+ (2^{\frac{\beta}{\alpha}} - 1)^m \tilde{N}^\beta(\rho_{AB_{n-1}}), \end{aligned}$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ .

#### IV. MONOGAMY INEQUALITY FOR EOF

The entanglement of formation (EoF) [22, 23] is a well-defined and important measure of quantum entanglement for bipartite systems. Let  $H_A$  and  $H_B$  be  $m$ - and  $n$ -dimensional ( $m \leq n$ ) vector spaces, respectively. The EoF of a pure state  $|\psi\rangle \in H_A \otimes H_B$  is defined by  $E(|\psi\rangle) = S(\rho_A)$ , where  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$  and  $S(\rho) = \text{Tr}(\rho \log_2 \rho)$ . For a bipartite mixed state  $\rho_{AB} \in H_A \otimes H_B$ , the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$

with the infimum taking over all possible decompositions of  $\rho_{AB}$  in a mixture of pure states  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , where  $p_i \geq 0$  and  $\sum_i p_i = 1$ .

Denote  $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$ , where  $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ . One has [15]

$$E(\rho_{AB}) \geq f(C_{AB}^2). \quad (13)$$

**Lemma 2** If  $0 \leq x \leq y \leq 1$ , we have

$$f^\beta(x^2 + y^2) \geq f^\beta(x^2) + (2^{\frac{\beta}{\alpha}} - 1) f^\beta(y^2), \quad (14)$$

where  $f^\beta(x^2 + y^2) = (f(x^2 + y^2))^\beta$ ,  $0 \leq \beta \leq \alpha$  and  $\alpha \geq \sqrt{2}$ .

[Proof] Since  $0 \leq x \leq y \leq 1$  and  $f(x)$  is a monotonically increasing function for  $0 \leq x \leq 1$ , one has  $f(x^2) \leq f(y^2)$  and  $f^\alpha(x^2 + y^2) \geq f^\alpha(x^2) + f^\alpha(y^2)$  for  $\alpha \geq \sqrt{2}$  [13]. Let  $z \in [0, 1]$ .

If  $x = 0$ , i.e.,  $f(x^2) = 0$ , we have

$$\begin{aligned} f^{\alpha z}(x^2 + y^2) &\geq (f^\alpha(x^2) + f^\alpha(y^2))^z \\ &= (f^\alpha(y^2))^z \\ &\geq (2^z - 1) f^{\alpha z}(y^2). \end{aligned}$$

If  $x \neq 0$ , i.e.,  $f(x^2) \neq 0$ , we have

$$\begin{aligned} f^{\alpha z}(x^2 + y^2) &\geq (f^\alpha(x^2) + f^\alpha(y^2))^z \\ &= f^{\alpha z}(x^2) \left(1 + \frac{f^\alpha(y^2)}{f^\alpha(x^2)}\right)^z \\ &\geq f^{\alpha z}(x^2) + (2^z - 1) f^{\alpha z}(y^2), \end{aligned}$$

where the last inequality is obtained by using lemma 1. The Lemma 2 is proved by setting  $\alpha z = \beta$ .  $\square$

It has been shown that the entanglement of formation does not satisfy monogamy inequality such as  $E_{AB} + E_{AC} \leq E_{A|BC}$  [12]. In [13] the authors showed that  $E^\alpha(\rho_{A|B_1 B_2 \dots B_{n-1}}) \geq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i})$  for  $\alpha \geq \sqrt{2}$ , and  $E^\alpha(\rho_{A|B_1 B_2 \dots B_{n-1}}) \leq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i})$  for  $\alpha \leq 0$ . In Ref. [15] tighter monogamy relation for  $E^\alpha(\alpha \geq \sqrt{2})$  has been derived for  $n$ -qubit states.

In fact, applying the same approach to the theorems 1 and 2, we can prove the following results generally:

**Theorem 4** For any  $2 \otimes 2 \otimes 2$  mixed state  $\rho \in H_A \otimes H_B \otimes H_C$ , and  $0 \leq \beta \leq \alpha$ ,  $\alpha \geq \sqrt{2}$ .

(1) If  $C_{AB_1} \leq C_{AC}$ , we have

$$E_{A|BC}^\beta \geq E_{AB}^\beta + (2^{\frac{\beta}{\alpha}} - 1)E_{AC}^\beta; \quad (15)$$

(2) If  $C_{AB} \geq C_{AC}$ , we have

$$E_{A|BC}^\beta \geq (2^{\frac{\beta}{\alpha}} - 1)E_{AB}^\beta + C_{AC}^\beta. \quad (16)$$

[Proof] Let  $\alpha \geq \sqrt{2}$  and  $\beta \in [0, \alpha]$ . If  $C_{AB} \leq C_{AC}$ , we have

$$\begin{aligned} E_{A|BC}^\beta &\geq f^\beta(C_{A|BC}^2) \\ &\geq f^\beta(C_{AB}^2 + C_{AC}^2) \\ &\geq f^\beta(C_{AB}^2) + (2^{\frac{\beta}{\alpha}} - 1)f^\beta(C_{AC}^2) \\ &= E_{AB}^\beta + (2^{\frac{\beta}{\alpha}} - 1)E_{AC}^\beta, \end{aligned}$$

where the first inequality is due to the inequality (13), the second inequality is obtained from the inequality  $C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2$ , the third inequality holds because of Lemma 2 and the last equality is obtained from  $E(\rho) = f(C^2(\rho))$  for two qubit states. The result for the case 2 can be similarly proved.  $\square$

*Example 3.* Consider the  $W$  state,  $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ . We have  $E_{A|BC} = 0.918296$ ,  $E_{AB} = E_{AC} = 0.550048$ . Therefore,  $E_{A|BC} < E_{AB} + E_{AC}$ . It is easily verified that  $E_{A|BC} > 0.897968 = \max_{\alpha \geq \sqrt{2}} (E_{AB} + (2^{\frac{1}{\alpha}} - 1)E_{AC})$ . Denote  $u(\beta, \alpha) = E_{A|BC}^\beta - E_{AB}^\beta - (2^{\frac{\beta}{\alpha}} - 1)E_{AC}^\beta = 0.918296^\beta - 2^{\frac{\beta}{\alpha}} \times 0.550048^\beta$ . For  $0 \leq \beta \leq 1$  and  $\alpha \geq \sqrt{2}$ , we have  $u(\beta, \alpha) \geq 0$ , see Fig.2.

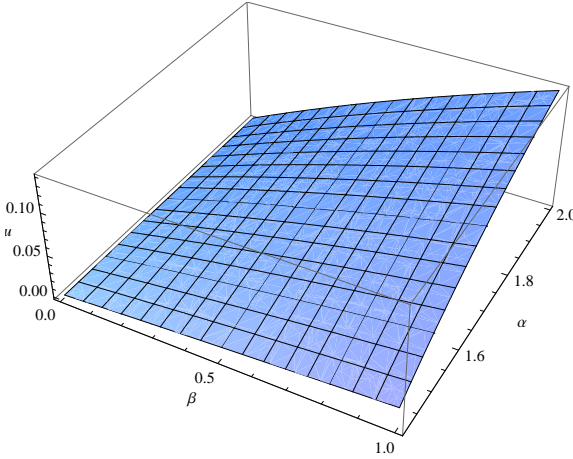


FIG. 2:  $u(\beta, \alpha)$  vs.  $0 \leq \beta \leq 1$  and  $\alpha \geq \sqrt{2}$ .

For  $n$ -qubit quantum states, we have follow theorem.

**Theorem 5** For any  $n$ -qubit mixed state  $\rho_{AB_1 \dots B_{n-1}}$  such that  $C_{AB_i} \leq C_{A|B_{i+1} \dots B_{n-1}}$  ( $i = 1, \dots, m$ ) and

$C_{AB_j} \geq C_{A|B_{j+1} \dots B_{n-1}}$  ( $j = m+1, \dots, n-2$ ),  $\forall 1 \leq m \leq n-3$  and  $n \geq 4$ , we have

$$\begin{aligned} E_{A|B_1 B_2 \dots B_{n-1}}^\beta &\geq \sum_{i=1}^m (2^{\frac{\beta}{\alpha}} - 1)^{i-1} E^\beta(\rho_{AB_i}) \\ &\quad + (2^{\frac{\beta}{\alpha}} - 1)^{m+1} \sum_{i=m+1}^{n-2} E^\beta(\rho_{AB_i}) \\ &\quad + (2^{\frac{\beta}{\alpha}} - 1)^m E^\beta(\rho_{AB_{n-1}}), \end{aligned} \quad (17)$$

where  $0 \leq \beta \leq \alpha$  and  $\alpha \geq 2$ ,  $E_{A|B_1 B_2 \dots B_{n-1}}$  is the entanglement of formation of  $\rho$  under bipartite partition  $A|B_1 B_2 \dots B_{n-1}$ , and  $E_{AB_i}$ ,  $i = 1, 2, \dots, n-1$ , is the entanglement of formation of the mixed state  $\rho_{AB_i} = \text{Tr}_{B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_{n-1}}(\rho)$ .

[Proof] Denote  $k = 2^{\frac{\beta}{\alpha}} - 1$ . For  $\alpha \geq \sqrt{2}$  and  $\beta \in [0, \alpha]$ , we have

$$\begin{aligned} E_{A|B_1 B_2 \dots B_{n-1}}^\beta &\geq f^\beta(C_{A|B_1 B_2 \dots B_{n-1}}^2) \\ &\geq f^\beta(C_{AB_1}^2 + C_{A|B_2 \dots B_{n-1}}^2) \\ &\geq f^\beta(C_{AB_1}^2) + k f^\beta(C_{A|B_2 \dots B_{n-1}}^2) \\ &\geq \dots \\ &\geq \sum_{i=1}^m k^{i-1} f^\beta(C_{AB_i}^2) + k^m f^\beta(C_{A|B_{m+1} \dots B_{n-1}}^2) \\ &= \sum_{i=1}^m k^{i-1} E^\beta(\rho_{AB_i}) + k^m f^\beta(C_{A|B_{m+1} \dots B_{n-1}}^2), \end{aligned} \quad (18)$$

where the first inequality is due to (13), the third to the fifth inequalities are due to  $C_{A|B_i} \leq C_{A|B_{i+1} \dots B_{n-1}}$  ( $i = 1, \dots, m$ ) and Lemma 2. Moreover,

$$\begin{aligned} & f^\beta(C_{A|B_{m+1} \dots B_{n-1}}^2) \\ &\geq f^\beta(C_{AB_{m+1}}^2 + C_{A|B_{m+2} \dots B_{n-1}}^2) \\ &\geq k f^\beta(C_{AB_{m+1}}^2) + f^\beta(C_{A|B_{m+2} \dots B_{n-1}}^2) \\ &\geq \dots \\ &\geq k \sum_{i=m+1}^{n-2} f^\beta(C_{A|B_i}^2) + f^\beta(C_{AB_{n-1}}^2) \\ &= k \sum_{i=m+1}^{n-2} E^\beta(\rho_{AB_i}) + E^\beta(\rho_{AB_{n-1}}), \end{aligned} \quad (19)$$

where the second to the fourth inequalities are due to  $C_{AB_i} \geq C_{A|B_{i+1} \dots B_{n-1}}$  ( $i = m+1, \dots, n-2$ ) and Lemma 2.

Combining (18) and (19) we obtain the theorem 5.  $\square$

Theorem 5 gives the monogamy relations satisfied by the  $\beta$ th ( $0 \leq \beta \leq \alpha$ ,  $\alpha \geq \sqrt{2}$ ) power of EoF for  $n$ -qubit states, which is a problem remained unsolved in Ref.[13, 15] for  $\beta \in (0, \sqrt{2})$ . If we take  $\beta = \alpha \geq \sqrt{2}$ , Theorem 5 reduces to the result in Ref.[13]. In addition

if we take  $\beta = 1$  and  $\alpha = \sqrt{2}$ , we have

$$\begin{aligned}
E(|\psi\rangle_{A|B_1B_2\dots B_{n-1}}) &\geq \sum_{i=1}^m (2^{\frac{1}{\sqrt{2}}} - 1)^{i-1} E(\rho_{AB_i}) \quad (20) \\
&+ (2^{\frac{1}{\sqrt{2}}} - 1)^{m+1} \sum_{i=m+1}^{n-2} E(\rho_{AB_i}) \\
&+ (2^{\frac{1}{\sqrt{2}}} - 1)^m E(\rho_{AB_{n-1}}),
\end{aligned}$$

which gives first time the tight monogamy inequality satisfied by the entanglement of formation itself.

## V. CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We have investigated

the monogamy relations related to the concurrence, the negativity, CREN and the entanglement of formation for general  $n$ -qubit states. We have derived the monogamy inequalities satisfied by  $C^\beta$ ,  $N^\beta$ ,  $\tilde{N}^\beta$  for  $\beta \in (0, 2)$ , and  $E^\beta$  for  $\beta \in (0, \sqrt{2})$  for  $n$ -qubit states. These monogamy relations are complementary to the existing ones with different regions of parameter  $\beta$ . Our new monogamy relations also include the existing ones as special cases. Our approach may be used to study further monogamy properties related to other quantum entanglement measures such as Tsallis- $q$  entanglement and to quantum correlations such as quantum discord.

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