Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Tighter constraints of multiqubit entanglement

by

Long-Mei Yang, Bin Chen, Shao-Ming Fei, and Zhi-Xi Wang

Preprint no.: 17 2019

Tighter constraints of multiqubit entanglement

Long-Mei Yang[1], Bin Chen[2],* Shao-Ming Fei[1,3], and Zhi-Xi Wang[1†]
[1] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[2] *College of Mathematical Science, Tianjin Normal University, Tianjin 300387, China*
[3] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

Monogamy and polygamy relations characterize the distributions of entanglement in multipartite systems. We provide classes of monogamy and polygamy inequalities of multiqubit entanglement in terms of concurrence, entanglement of formation, negativity, Tsallis-q entanglement and Rényi-α entanglement, respectively. We show that these inequalities are tighter than the existing ones for some classes of quantum states.

I. INTRODUCTION

Quantum entanglement is an essential feature of quantum mechanics which distinguishes the quantum from the classical world and plays a very important role in quantum information processing [1–4]. One singular property of quantum entanglement is that a quantum system entangled with one of the other subsystems limits its entanglement with the remaining ones, known as the monogamy of entanglement (MoE) [5, 6]. MoE plays a key role in many quantum information and communication processing tasks such as the security proof in quantum cryptographic scheme [7] and the security analysis of quantum key distribution [8].

For a tripartite quantum state ρ_{ABC}, MoE can be described as the following inequality

$$\mathcal{E}(\rho_{A|BC}) \geq \mathcal{E}(\rho_{AB}) + \mathcal{E}(\rho_{AC}), \tag{1}$$

where $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$ and $\rho_{AC} = \mathrm{tr}_B(\rho_{ABC})$ are reduced density matrices, and $\mathcal{E}$ is an entanglement measure. However, it has been shown that not all entanglement measures satisfy such monogamy relations. It has been shown that the squared concurrence $\mathcal{C}^2$ [9, 10], the squared entanglement of formation (EoF) E^2 [11] and the squared convex-roof extended negativity (CREN) $\mathcal{N}_c^2$ [12, 13] satisfy the monogamy relations for multiqubit states.

Another important concept is the assisted entanglement, which is a dual amount to bipartite entanglement measure. It has a dually monogamous property in multipartite quantum systems and gives rise to polygamy relations. For a tripartite state ρ_{ABC}, the usual polygamy relation is of the form,

$$\mathcal{E}^a(\rho_{A|BC}) \leq \mathcal{E}^a(\rho_{AB}) + \mathcal{E}^a(\rho_{AC}), \tag{2}$$

where $\mathcal{E}^a$ is the corresponding entanglement measure of assistance associated to $\mathcal{E}$. Such polygamy inequality has been deeply investigated in recent years, and was generalized to multiqubit systems and classes of higher-dimensional quantum systems [12, 14–20].

Recently, generalized classes of monogamy inequalities related to the βth power of entanglement measures were proposed. In Ref. [21, 22], the authors proved that the squared concurrence and CREN satisfy the monogamy inequalities in multiqubit systems for $\beta \geq 2$. It has also been shown that the EoF satisfies monogamy relations when $\beta \geq \sqrt{2}$ [21, 22, 24]. Besides, the Tsallis-q entanglement and Rényi-α entanglement satisfy monogamy relations when $\beta \geq 1$ [14, 22–24] for some cases. Moreover, the corresponding polygamy relations have also been established [16–18, 20, 25, 26].

*Corresponding author: chenbin5134@163.com
†Corresponding author: wangzhx@cnu.edu.cn

In this paper, we investigate monogamy relations and polygamy relations in multiqubit systems. We provide tighter constraints of multiqubit entanglement than all the existing ones, thus give rise to finer characterizations of the entanglement distributions among the multiqubit systems.

II. TIGHTER CONSTRAINTS RELATED TO CONCURRENCE

We first consider the monogamy inequalities and polygamy inequalities for concurrence. For a bipartite pure state $|\psi\rangle_{AB}$ in Hilbert space $H_A\otimes H_B$, the concurrence is defined as [27**?** , 28] $\mathcal{C}(|\psi\rangle_{AB})=\sqrt{2(1-\mathrm{tr}\rho_A^2)}$ with $\rho_A=\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$. The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex roof extension, $\mathcal{C}(\rho_{AB})=\min\limits_{\{p_i,|\psi_i\rangle\}}\sum\limits_i p_i\mathcal{C}(|\psi_i\rangle)$, where the minimum is taken over all possible decompositions of $\rho_{AB}=\sum\limits_i p_i|\psi_i\rangle\langle\psi_i|$ with $\sum p_i=1$ and $p_i\geq 0$. For an N-qubit state $\rho_{AB_1\cdots B_{N-1}}\in H_A\otimes H_{B_1}\otimes\cdots\otimes H_{B_{N-1}}$, the concurrence $\mathcal{C}(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$ under bipartite partition A and $B_1\cdots B_{N-1}$ satisfies [21]

$$\begin{aligned}&\mathcal{C}^\beta(\rho_{A|B_1\cdots B_{N-1}})\\&\quad\geq\mathcal{C}^\beta(\rho_{AB_1})+\mathcal{C}^\beta(\rho_{AB_2})+\cdots+\mathcal{C}^\beta(\rho_{AB_{N-1}}),\end{aligned}\tag{3}$$

for $\beta\geq 2$, where ρ_{AB_j} denote two-qubit reduced density matrices of subsystems AB_j for $j=1,2,\ldots,N-1$. Later, the relation (3) is improved for the case $\beta\geq 2$ [24] as

$$\begin{aligned}&\mathcal{C}^\beta(\rho_{A|B_1\cdots B_{N-1}})\\&\quad\geq\mathcal{C}^\beta(\rho_{AB_1})+\tfrac{\beta}{2}\mathcal{C}^\beta(\rho_{AB_2})+\cdots\\&\qquad+\left(\tfrac{\beta}{2}\right)^{m-1}\mathcal{C}^\beta(\rho_{AB_m})\\&\qquad+\left(\tfrac{\beta}{2}\right)^{m+1}[\mathcal{C}^\beta(\rho_{AB_{m+1}})+\cdots+\mathcal{C}^\beta(\rho_{AB_{N-2}})]\\&\qquad+\left(\tfrac{\beta}{2}\right)^{m}\mathcal{C}^\beta(\rho_{AB_{N-1}})\end{aligned}\tag{4}$$

conditioned that $\mathcal{C}(\rho_{AB_i})\geq\mathcal{C}(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i=1,2,\ldots,m$, and $\mathcal{C}(\rho_{AB_j})\leq\mathcal{C}(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j=m+1,\ldots,N-2$. The relation (4) is further improved for $\beta\geq 2$ as [22]

$$\begin{aligned}&\mathcal{C}^\beta(\rho_{A|B_1\cdots B_{N-1}})\\&\quad\geq\mathcal{C}^\beta(\rho_{AB_1})+\left(2^{\frac{\beta}{2}}-1\right)\mathcal{C}^\beta(\rho_{AB_2})+\cdots\\&\qquad+\left(2^{\frac{\beta}{2}}-1\right)^{m-1}\mathcal{C}^\beta(\rho_{AB_m})\\&\qquad+\left(2^{\frac{\beta}{2}}-1\right)^{m+1}[\mathcal{C}^\beta(\rho_{AB_{m+1}})+\cdots+\mathcal{C}^\beta(\rho_{AB_{N-2}})]\\&\qquad+\left(2^{\frac{\beta}{2}}-1\right)^{m}\mathcal{C}^\beta(\rho_{AB_{N-1}})\end{aligned}\tag{5}$$

with the same conditions as in (4).

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by [29, 30]

$$\mathcal{C}_a(\rho_{AB})=\max_{\{p_i,|\psi_i\rangle\}}\sum_i p_i\mathcal{C}(|\psi_i\rangle),\tag{6}$$

where the maximun is taken over all possible pure state decompositions of ρ_{AB}, and $\mathcal{C}(|\psi\rangle_{AB})=\mathcal{C}_a(|\psi\rangle_{AB})$. The generalized polygamy relation based on the concurrence of assistance was established in [16, 17]

$$\begin{aligned}&\mathcal{C}^2(|\psi\rangle_{A|B_1\cdots B_{N-1}})\\&\quad=\mathcal{C}_a^2(|\psi\rangle_{A|B_1\cdots B_{N-1}})\\&\quad\leq\mathcal{C}_a^2(\rho_{AB_1})+\mathcal{C}_a^2(\rho_{AB_2})+\cdots+\mathcal{C}_a^2(\rho_{AB_{N-1}}).\end{aligned}\tag{7}$$

These monogamy and polygamy relations for concurrence can be further tightened under some conditions. To this end, we first introduce the following lemma.

Lemma 1. *Suppose that k is a real number satisfying $0<k\leq 1$, then for any $0\leq t\leq k$ and non-negative real numbers m,n, we have*

$$(1+t)^m\geq 1+\frac{(1+k)^m-1}{k^m}t^m\tag{8}$$

for $m\geq 1$, and

$$(1+t)^n\leq 1+\frac{(1+k)^n-1}{k^n}t^n\tag{9}$$

for $0\leq n\leq 1$.

Proof: We first consider the function $f(m,x)=(1+x)^m-x^m$ with $x\geq\frac{1}{k}$ and $m\geq 1$. Then $f(m,x)$ is an increasing function of x, since $\frac{\partial f(m,x)}{\partial x}=m[(1+x)^{m-1}-x^{m-1}]\geq 0$. Thus,

$$f(m,x)\geq f(m,\frac{1}{k})=\left(1+\frac{1}{k}\right)^m-\left(\frac{1}{k}\right)^m=\frac{(k+1)^m-1}{k^m}.\tag{10}$$

Set $x=\frac{1}{t}$ in (10), we get the inequality (8).

Similar to the proof of inequality (8), we can obtain the inequality (9), since in this case $f(n,x)$ is a decreasing function of x for $x\geq\frac{1}{k}$ and $0\leq n\leq 1$. ■

In the next, we denote $\mathcal{C}_{AB_i}=\mathcal{C}(\rho_{AB_i})$ the concurrence of ρ_{AB_i} and $\mathcal{C}_{A|B_1\cdots B_{N-1}}=\mathcal{C}(\rho_{A|B_1\cdots B_{N-1}})$ for convenience.

Lemma 2. *Suppose that k is a real number satisfying $0<k\leq 1$. Then for any $2\otimes 2\otimes 2^{n-2}$ mixed state $\rho\in H_A\otimes H_B\otimes H_C$, if $\mathcal{C}^2_{AC}\leq k\mathcal{C}^2_{AB}$, we have*

$$\mathcal{C}^{\beta}_{A|BC}\geq\mathcal{C}^{\beta}_{AB}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{AC}, \tag{11}$$

for all $\beta\geq 2$.

Proof: Since $\mathcal{C}^2_{AC}\leq k\mathcal{C}^2_{AB}$ and $\mathcal{C}_{AB}>0$, we obtain

$$\begin{aligned}\mathcal{C}^{\beta}_{A|BC}&\geq(\mathcal{C}^2_{AB}+\mathcal{C}^2_{AC})^{\frac{\beta}{2}}\\&=\mathcal{C}^{\beta}_{AB}\Big(1+\frac{\mathcal{C}^2_{AC}}{\mathcal{C}^2_{AB}}\Big)^{\frac{\beta}{2}}\\&\geq\mathcal{C}^{\beta}_{AB}\Big[1+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big(\frac{\mathcal{C}^2_{AC}}{\mathcal{C}^2_{AB}}\Big)^{\frac{\beta}{2}}\Big]\\&=\mathcal{C}^{\beta}_{AB}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{AC},\end{aligned} \tag{12}$$

where the first inequality is due to the fact, $\mathcal{C}^2_{A|BC}\geq\mathcal{C}^2_{AB}+\mathcal{C}^2_{AC}$ for arbitrary $2\otimes 2\otimes 2^{n-2}$ tripartite state ρ_{ABC} [9, 31] and the second is due to Lemma 1. We can also see that if $\mathcal{C}_{AB}=0$, then $\mathcal{C}_{AC}=0$, and the lower bound becomes trivially zero. ■

For multiqubit systems, we have the following Theorems.

Theorem 1. *Suppose k is a real number satisfying $0<k\leq 1$. For an N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $k\mathcal{C}^2_{AB_i}\geq\mathcal{C}^2_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $\mathcal{C}^2_{AB_j}\leq k\mathcal{C}^2_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, $\forall 1\leq m\leq N-3$, $N\geq 4$, then we have*

$$\begin{aligned}&\mathcal{C}^{\beta}_{A|B_1\cdots B_{N-1}}\\&\geq\mathcal{C}^{\beta}_{AB_1}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{AB_2}+\cdots\\&\quad+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{m-1}\mathcal{C}^{\beta}_{AB_m}\\&\quad+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{m+1}\Big(\mathcal{C}^{\beta}_{AB_{m+1}}+\cdots+\mathcal{C}^{\beta}_{AB_{N-2}}\Big)\\&\quad+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{m}\mathcal{C}^{\beta}_{AB_{N-1}}\end{aligned} \tag{13}$$

for all $\beta\geq 2$.

Proof: From the inequality (11), we have

$$\begin{aligned}&\mathcal{C}_{A|B_1B_2\cdots B_{N-1}}\\&\geq\mathcal{C}^{\beta}_{AB_1}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{A|B_2\cdots B_{N-1}}\\&\geq\mathcal{C}^{\beta}_{AB_1}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{AB_2}\\&\quad+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{2}\mathcal{C}^{\beta}_{A|B_3\cdots B_{N-1}}\\&\geq\cdots\\&\geq\mathcal{C}^{\beta}_{AB_1}+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)\mathcal{C}^{\beta}_{AB_2}\\&\quad+\cdots+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{m-1}\mathcal{C}^{\beta}_{AB_m}\\&\quad+\Big(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big)^{m}\mathcal{C}^{\beta}_{A|B_{m+1}\cdots B_{N-1}}.\end{aligned} \tag{14}$$

Since $\mathcal{C}^2_{AB_j}\leq k\mathcal{C}^2_{A|B_{j+1}\cdots B_{N-1}}$, for $j=m+1,\ldots,N-2$, we get

$$\begin{aligned}&\mathcal{C}^{\beta}_{A|B_{m+1}\cdots B_{N-1}}\\&\geq\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^{\beta}_{AB_{m+1}}+\mathcal{C}^{\beta}_{A|B_{m+2}\cdots B_{N-1}}\\&\geq\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\Big(\mathcal{C}^{\beta}_{AB_{m+1}}+\cdots+\mathcal{C}^{\beta}_{AB_{N-2}}\Big)+\mathcal{C}^{\beta}_{AB_{N-1}}.\end{aligned} \tag{15}$$

Combining (14) and (15), we get the inequality (13). ■

If we replace the conditions $k\mathcal{C}_{AB_i}\geq\mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $\mathcal{C}^2_{AB_j}\leq k\mathcal{C}^2_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, $\forall 1\leq m\leq N-3$, $N\geq 4$, in Theorem 1 by $k\mathcal{C}^2_{AB_i}\geq\mathcal{C}^2_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,N-2$, then we have the following theorem.

Theorem 2. *Suppose k is a real number satisfying $0<k\leq 1$. For an N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$,*

if $k\mathcal{C}^2_{AB_i} \geq \mathcal{C}^2_{A|B_{i+1}\cdots B_{N-1}}$ *for all* $i = 1, 2, \ldots, N-2$*, then we have*

$$\begin{aligned}\mathcal{C}^\beta_{A|B_1\cdots B_{N-1}} &\geq \mathcal{C}^\beta_{AB_1} + \tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^\beta_{AB_2} + \cdots \\ &\quad + \left(\tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{N-2}\mathcal{C}^\beta_{AB_{N-1}}\end{aligned} \tag{16}$$

for $\beta \geq 2$.

It can be seen that the inequalities (13) and (16) are tighter than the ones given in Ref. [22], since

$$\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}} \geq 2^{\frac{\beta}{2}} - 1$$

for $\beta \geq 2$ and $0 < k \leq 1$. The equality holds when $k = 1$. Namely, the result (5) given in [22] are just special cases of ours for $k = 1$. As $\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}$ is a decreasing function with respect to k for $0 < k \leq 1$ and $\beta \geq 2$, we find that the smaller k is, the tighter the inequalities (11), (13) and (16) are.

Example 1 Consider the three-qubit state $|\psi\rangle_{ABC}$ in generalized Schmidt decomposition form [32, 33],

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{17}$$

where $\lambda_i \geq 0$, $i =, 1, 2..., 4$, and $\sum\limits_{i=0}^{4}\lambda_i^2 = 1$. Then we get $\mathcal{C}_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $\mathcal{C}_{AB} = 2\lambda_0\lambda_2$ and $\mathcal{C}_{AC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_3 = \frac{1}{2}$, $\lambda_2 = \frac{\sqrt{2}}{2}$ and $\lambda_1 = \lambda_4 = 0$. We have $\mathcal{C}_{A|BC} = \frac{\sqrt{3}}{2}$, $\mathcal{C}_{AB} = \frac{\sqrt{2}}{2}$ and $\mathcal{C}_{AC} = \frac{1}{2}$. Then $\mathcal{C}^\beta_{AB} + (2^{\frac{\beta}{2}} - 1)\mathcal{C}^\beta_{AC} = \left(\frac{\sqrt{2}}{2}\right)^\beta + (2^{\frac{\beta}{2}} - 1)\left(\frac{1}{2}\right)^\beta$ and $\mathcal{C}^\beta_{AB} + \frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^\beta_{AC} = \left(\frac{\sqrt{2}}{2}\right)^\beta + \frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\left(\frac{1}{2}\right)^\beta$. One can see that our result is better than the result (5) in [22] for $\beta \geq 2$, hence better than (3) and (4) given in [21, 24], see Fig. 1.

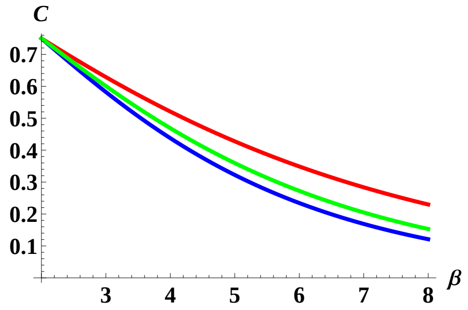

FIG. 1. The y axis is the lower bound of the concurrence $\mathcal{C}^\beta_{A|BC}$. The red (green) line represents the lower bound from our result for $k = 0.6$ ($k = 0.8$), and the blue line represents the lower bound of (5) from [22].

We now discuss the polygamy relations for the CoA of $\mathcal{C}_a(|\psi\rangle_{A|B_1\cdots B_{N-1}})$ for $0 \leq \beta \leq 2$. We have the following Theorem.

Theorem 3. *Suppose* k *is a real number satisfying* $0 < k \leq 1$. *For an* N*-qubit pure state* $|\psi\rangle_{AB_1\cdots B_{N-1}}$, *if* $k\mathcal{C}^2_{aA|B_i} \geq \mathcal{C}^2_{aA|B_{i+1}\cdots B_{N-1}}$ *for* $i = 1, 2, \ldots, m$, *and* $\mathcal{C}^2_{aA|B_j} \leq k\mathcal{C}^2_{aA|B_{j+1}\cdots B_{N-1}}$ *for* $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, *then we have*

$$\begin{aligned}&\mathcal{C}^\beta_a(|\psi\rangle_{A|B_1\cdots B_{N-1}}) \\ &\leq \mathcal{C}^\beta_{aAB_1} + \tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^\beta_{aAB_2} + \cdots \\ &\quad + \left(\tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m-1}\mathcal{C}^\beta_{aAB_m} \\ &\quad + \left(\tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m+1}\left(\mathcal{C}^\beta_{aAB_{m+1}} + \cdots + \mathcal{C}^\beta_{aAB_{N-2}}\right) \\ &\quad + \left(\tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m}\mathcal{C}^\beta_{aAB_{N-1}}\end{aligned} \tag{18}$$

for all $0 \leq \beta \leq 2$.

Proof: The proof is similar to the proof of Theorem 1 by using inequality (9). ■

Theorem 4. *Suppose* k *is a real number satisfying* $0 < k \leq 1$. *For an* N*-qubit pure state* $|\psi\rangle_{AB_1\cdots B_{N-1}}$, *if* $k\mathcal{C}^2_{aAB_i} \geq \mathcal{C}^2_{aA|B_{i+1}\cdots B_{N-1}}$ *for all* $i = 1, 2, \ldots, N-2$, *then we have*

$$\begin{aligned}\mathcal{C}^\beta_a|\psi\rangle_{A|B_1\cdots B_{N-1}} &\leq \mathcal{C}^\beta_{aAB_1} + \tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{C}^\beta_{aAB_2} + \cdots \\ &\quad + \left(\tfrac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{N-2}\mathcal{C}^\beta_{aAB_{N-1}}\end{aligned} \tag{19}$$

for $0 \leq \beta \leq 2$.

The inequalities (18) and (19) are also upper bounds of $\mathcal{C}(|\psi\rangle_{A|B_1\cdots B_{N-1}})$ for pure state $|\psi\rangle_{AB_1\cdots B_{N-1}}$ since $\mathcal{C}(|\psi\rangle_{A|B_1\cdots B_{N-1}}) = \mathcal{C}_a(|\psi\rangle_{A|B_1\cdots B_{N-1}})$.

III. TIGHTER CONSTRAINTS RELATE TO EOF

Let H_A and H_B be two Hilbert spaces with dimension m and n ($m \leq n$), respectively. Then the entanglement of formation (EoF) [34, 35] is defined as follows: for a pure state $|\psi\rangle_{AB} \in H_A \otimes H_B$, the EoF is given by

$$E(|\psi\rangle_{AB}) = \mathcal{S}(\rho_A), \tag{20}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ and $\mathcal{S}(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the EoF is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{21}$$

with the minimum taking over all possible pure state decomposition of ρ_{AB}.

In Ref. [36], Wootters showed that $E(|\psi\rangle) = f(\mathcal{C}^2(|\psi\rangle))$ for $2 \otimes m$ ($m \geq 2$) pure state $|\psi\rangle$, and $E(\rho) = f(\mathcal{C}^2(\rho))$ for two-qubit mixed state ρ, where $f(x) = H\big(\frac{1+\sqrt{1-x}}{2}\big)$ and $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$. $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$, and satisfies the following relations:

$$f^{\sqrt{2}}(x^2 + y^2) \geq f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{22}$$

where $f^{\sqrt{2}}(x^2 + y^2) = [f(x^2 + y^2)]^{\sqrt{2}}$.

Although EoF does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{A|BC}$ [37], the authors in [38] showed that EoF is a monotonic function satisfying $E^2(\rho_{A|B_1B_2\cdots B_{N-1}}) \geq \sum_{i=1}^{N-1} E^2(\rho_{AB_i})$. For N-qubit systems, one has [21]

$$E^\beta_{A|B_1B_2\cdots B_{N-1}} \geq E^\beta_{AB_1} + E^\beta_{AB_2} + \cdots + E^\beta_{AB_{N-1}}, \tag{23}$$

for $\beta \geq \sqrt{2}$, where $E_{A|B_1B_2\cdots B_{N-1}}$ is the EoF of ρ under bipartite partition $A|B_1B_2\cdots B_{N-1}$, and E_{AB_i} is the EoF of the mixed state $\rho_{AB_i} = \mathrm{Tr}_{B_1\cdots B_{i-1}, B_{i+1}\cdots B_{N-1}}(\rho)$ for $i = 1, 2, \ldots, N-1$. Recently, the authors in Ref. [24] proposed a monogamy relation that is tighter than the inequality (23),

$$\begin{aligned} &E^\beta_{A|B_1B_2\cdots B_{N-1}} \\ &\geq E^\beta_{AB_1} + \tfrac{\beta}{\sqrt{2}} E^\beta_{AB_2} + \cdots + \big(\tfrac{\beta}{\sqrt{2}}\big)^{m-1} E^\beta_{AB_m} \\ &\quad + \big(\tfrac{\beta}{\sqrt{2}}\big)^{m+1} \big(E^\beta_{AB_{m+1}} + \cdots + E^\beta_{AB_{N-2}}\big) \\ &\quad + \big(\tfrac{\beta}{\sqrt{2}}\big)^{m} E^\beta_{AB_{N-1}}, \end{aligned} \tag{24}$$

if $\mathcal{C}_{AB_i} \geq \mathcal{C}_{A|B_{j+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $\mathcal{C}_{AB_j} \leq \mathcal{C}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$ for $\beta \geq \sqrt{2}$. The inequality (24) is also improved to

$$\begin{aligned} &E^\beta_{A|B_1B_2\cdots B_{N-1}} \\ &\geq E^\beta_{AB_1} + \Big(2^{\frac{\beta}{\sqrt{2}}} - 1\Big) E^\beta_{AB_2} + \cdots + \Big(2^{\frac{\beta}{\sqrt{2}}} - 1\Big)^{m-1} \\ &\quad \times E^\beta_{AB_m} + \Big(2^{\frac{\beta}{\sqrt{2}}} - 1\Big)^{m+1} \big(E^\beta_{AB_{m+1}} + \cdots + E^\beta_{AB_{N-2}}\big) \\ &\quad + \Big(2^{\frac{\beta}{\sqrt{2}}} - 1\Big)^{m} E^\beta_{AB_{N-1}}, \end{aligned} \tag{25}$$

under the same conditions as that of inequality (24).

In fact, these inequalities can be further improved to even tighter monogamy relations.

Theorem 5. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kE^{\sqrt{2}}_{AB_i} \geq E^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E^{\sqrt{2}}_{AB_j} \leq kE^{\sqrt{2}}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, the entanglement of formation $E(\rho)$ satisfies*

$$\begin{aligned} &E^\beta_{A|B_1B_2\cdots B_{N-1}} \\ &\geq E^\beta_{AB_1} + \tfrac{(1+k)^t - 1}{k^t} E^\beta_{AB_2} + \cdots + \Big(\tfrac{(1+k)^t - 1}{k^t}\Big)^{m-1} E^\beta_{AB_m} \\ &\quad + \Big(\tfrac{(1+k)^t - 1}{k^t}\Big)^{m+1} \big(E^\beta_{AB_{m+1}} + \cdots + E^\beta_{AB_{N-2}}\big) \\ &\quad + \Big(\tfrac{(1+k)^t - 1}{k^t}\Big)^{m} E^\beta_{AB_{N-1}}, \end{aligned} \tag{26}$$

for $\beta \geq \sqrt{2}$, where $t = \frac{\beta}{\sqrt{2}}$.

Proof: For $\beta \geq \sqrt{2}$ and $kf^{\sqrt{2}}(x^2) \geq f^{\sqrt{2}}(y^2)$, we find

$$\begin{aligned} f^{\beta}(x^2+y^2) &= [f^{\sqrt{2}}(x^2+y^2)]^t \\ &\geq [f^{\sqrt{2}}(x^2)+f^{\sqrt{2}}(y^2)]^t \\ &\geq [f^{\sqrt{2}}(x^2)]^t + \tfrac{(1+k)^t-1}{k^t}[f^{\sqrt{2}}(y^2)]^t \\ &= f^{\beta}(x^2) + \tfrac{(1+k)^t-1}{k^t} f^{\beta}(y^2), \end{aligned} \tag{27}$$

where the first inequality is due to the inequality (22), and the second inequality can be obtained from inequality (8).

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in H_A \otimes H_{B_1} \otimes \cdots H_{B_{N-1}}$ be the optimal decomposition of $E_{A|B_1B_2\cdots B_{N-1}}(\rho)$ for the N-qubit mixed state ρ. Then [22]

$$E_{A|B_1B_2\cdots B_{N-1}} \geq f(\mathcal{C}^2_{A|B_1B_2\cdots B_{N-1}}). \tag{28}$$

Thus,

$$\begin{aligned} & E^{\beta}_{A|B_1B_2\cdots B_{N-1}} \\ & \geq f^{\beta}(\mathcal{C}^2_{A|B_1B_2\cdots B_{N-1}}) \\ & \geq f^{\beta}(\mathcal{C}^2_{A|B_1}) + \tfrac{(1+k)^t-1}{k^t} f^{\beta}(\mathcal{C}^2_{A|B_2}) + \cdots \\ & \quad + \left(\tfrac{(1+k)^t-1}{k^t}\right)^{m-1} f^{\beta}(\mathcal{C}^2_{A|B_m}) + \left(\tfrac{(1+k)^t-1}{k^t}\right)^{m+1} \\ & \quad [f^{\beta}(\mathcal{C}^2_{AB_{m+1}}) + \cdots + f^{\beta}(\mathcal{C}^2_{AB_{N-2}})] \\ & \quad + \left(\tfrac{(1+k)^t-1}{k^t}\right) f^{\beta}(\mathcal{C}^2_{AB_{N-1}}) \\ & = E^{\beta}_{AB_1} + \left(\tfrac{(1+k)^t-1}{k^t}\right) E^{\beta}_{AB_2} + \cdots + \left(\tfrac{(1+k)^t-1}{k^t}\right)^{m-1} \\ & \quad \times E^{m-1}_{AB_m} + \left(\tfrac{(1+k)^t-1}{k^t}\right)^{m+1} (E^{\beta}_{AB_{m+1}} + \cdots + E^{\beta}_{AB_{N-2}}) \\ & \quad + \left(\tfrac{(1+k)^t-1}{k^t}\right) E^{\beta}_{AB_{N-1}}, \end{aligned} \tag{29}$$

where the first inequality holds due to (28), the second inequality is similar to the proof of Theorem 1 by using inequality (27), and the last equality holds since for any $2 \otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i}) = f[\mathcal{C}^2(\rho_{AB_i})]$. ■

Similar to the case of concurrence, we have also the following tighter monogamy relation for EoF.

Theorem 6. *Suppose k is a real number satisfying $0 < k \leq 1$. For an N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kE^{\sqrt{2}}_{AB_i} \geq E^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \ldots, N-2$, we have*

$$\begin{aligned} E^{\beta}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\beta}_{AB_1} + \tfrac{(1+k)^t-1}{k^t} E^{\beta}_{AB_2} + \cdots \\ + \left(\tfrac{(1+k)^t-1}{k^t}\right)^{N-2} E^{\beta}_{AB_{N-1}}, \end{aligned} \tag{30}$$

for $\beta \geq \sqrt{2}$ and $t = \frac{\beta}{\sqrt{2}}$.

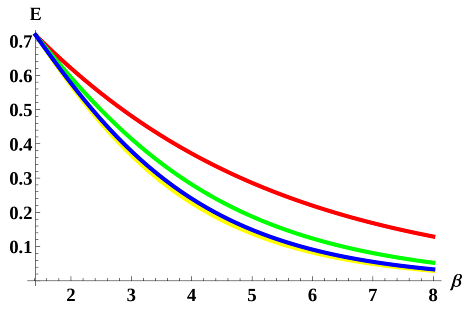

FIG. 2. The y axis is the lower bound of the EoF $E^{\beta}_{A|BC}$. The red (green resp. blue) line represents the lower bound from our result for $k = 0.5$ ($k = 0.7$ resp. $k = 0.9$), and the yellow line represents the lower bound from the result in [22].

As $\frac{(1+k)^t-1}{k^t} \geq 2^t - 1$ for $t \geq 1$ and $0 < k \leq 1$, our new monogamy relations (26) and (30) are tighter than the ones given in [21, 22, 24]. Also, for $0 < k \leq 1$ and $\beta \geq 2$, the smaller k is, the tighter inequalities (26) and (30) are.

Example 2 Let us again consider the three-qubit state $|\psi\rangle_{ABC}$ defined in (17) with $\lambda_0 = \lambda_3 = \frac{1}{2}$, $\lambda_2 = \frac{\sqrt{2}}{2}$ and $\lambda_1 = \lambda_4 = 0$. Then $E_{A|BC} = 2 - \log_2 3 \approx 0.811278$, $E_{AB} = -\frac{2+\sqrt{2}}{4}\log_2\frac{2+\sqrt{2}}{4} - \frac{2-\sqrt{2}}{4}\log_2\frac{2-\sqrt{2}}{4} \approx 0.600876$ and $E_{AB} = -\frac{2+\sqrt{3}}{4}\log_2\frac{2+\sqrt{3}}{4} - \frac{2-\sqrt{3}}{4}\log_2\frac{2-\sqrt{3}}{4} \approx 0.354579$. Thus, $E^{\beta}_{AB} + (2^{\frac{\beta}{2}} - 1)E^{\beta}_{AC} = (0.600876)^{\beta} + (2^{\frac{\beta}{2}} - 1)0.354579^{\beta}$, $E^{\beta}_{AB} + \frac{1.5^{\frac{\beta}{2}}-1}{0.5^{\frac{\beta}{2}}} E^{\beta}_{AC} = (0.600876)^{\beta} + \frac{1.5^{\frac{\beta}{2}}-1}{0.5^{\frac{\beta}{2}}} 0.354579^{\beta}$, $E^{\beta}_{AB} + \frac{1.7^{\frac{\beta}{2}}-1}{0.7^{\frac{\beta}{2}}} E^{\beta}_{AC} = (0.600876)^{\beta} + \frac{1.7^{\frac{\beta}{2}}-1}{0.7^{\frac{\beta}{2}}} 0.354579^{\beta}$ and $E^{\beta}_{AB} + \frac{1.9^{\frac{\beta}{2}}-1}{0.9^{\frac{\beta}{2}}} E^{\beta}_{AC} = (0.600876)^{\beta} + \frac{1.9^{\frac{\beta}{2}}-1}{0.9^{\frac{\beta}{2}}} 0.354579^{\beta}$. One can see that our result is better than the one in [22] for $\beta \geq \sqrt{2}$, hencee better than the ones in [21, 24], see Fig. 2.

We can also provide tighter polygamy relations for the entanglement of assistance. The entanglement of assistance (EoA) of ρ_{AB} is defined as [39],

$$E^a_{\rho_{AB}} = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{31}$$

with the maximization taking over all possible pure decompositions of ρ_{AB}. For any dimensional multipartite quantum state $\rho_{AB_1B_2\cdots B_{N-1}}$, a general polygamy inequality of multipartite quantum entanglement was established as [18],

$$E^a(\rho_{A|B_1B_2\cdots B_{N-1}}) \leq \sum_{i=1}^{N-1} E^a(\rho_{A|B_i}). \tag{32}$$

Using the same approach as for concurrence, we have the following Theorems.

Theorem 7. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kE^a_{AB_i} \geq E^a_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E^a_{AB_j} \leq kE^a_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, we have*

$$\begin{aligned}
&(E^a_{A|B_1B_2\cdots B_{N-1}})^\beta \\
&\quad \leq (E^a_{AB_1})^\beta + \tfrac{(1+k)^\beta-1}{k^\beta}(E^a_{AB_2})^\beta + \cdots \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m-1}(E^a_{AB_m})^\beta \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m+1}[(E^a_{AB_{m+1}})^\beta + \cdots + (E^a_{AB_{N-2}})^\beta] \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m}(E^a_{AB_{N-1}})^\beta,
\end{aligned} \tag{33}$$

for $0 \leq \beta \leq 1$.

Theorem 8. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kE^a_{AB_i} \geq E^a_{A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \ldots, N-2$, we have*

$$\begin{aligned}
(E^a_{A|B_1B_2\cdots B_{N-1}})^\beta \leq{}& (E^a_{AB_1})^\beta + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)(E^a_{AB_2})^\beta \\
&+ \cdots \\
&+ \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{N-2}(E^a_{AB_{N-1}})^\beta,
\end{aligned} \tag{34}$$

for $0 \leq \beta \leq 1$.

IV. TIGHTER CONSTRAINTS RELATED TO NEGATIVITY

The negativity, a well-known quantifier of bipartite entanglement, is defined as $\mathcal{N}(\rho_{AB}) = \big(\|\rho_{AB}^{T_A}\| - 1\big)/2$ [41], where $\rho_{AB}^{T_A}$ is the partial transposed matrix of ρ_{AB} with respect to the subsystem A, and $\|X\|$ denotes the trace norm of X, i.e., $\|X\| = \mathrm{Tr}\sqrt{XX^\dagger}$. For convenient, we use the definition of negativity as $\|\rho_{AB}^{T_A}\| - 1$. Particularly, for any bipartite pure state $|\psi\rangle_{AB}$, $\mathcal{N}(|\psi\rangle_{AB}) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} = (\mathrm{Tr}\sqrt{\rho_A})^2 - 1$, where $\lambda_i s$ are the eigenvalues of the reduced density matrix $\rho_A = \mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi|$. The convex-roof extended negativity (CREN) of a mixed state ρ_{AB} is defined by

$$\mathcal{N}_c(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle), \tag{35}$$

where the minimum is taken over all possible pure state decomposition of ρ_{AB}. Thus $\mathcal{N}_c(\rho_{AB}) = \mathcal{C}(\rho_{AB})$ for any two-qubit mixed state ρ_{AB}. The dual to the CREN of a mixed state ρ_{AB} is defined as

$$\mathcal{N}^a_c(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle), \tag{36}$$

with the maximum taking over all possible pure state decomposition of ρ_{AB}. Furthermore, $\mathcal{N}^a_c(\rho_{AB}) = \mathcal{C}^a(\rho_{AB})$ for any two-qubit mixed state ρ_{AB}[12].

Similar to the concurrence and EoF, we have the following Theorems.

Theorem 9. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $k\mathcal{N}^2_{cAB_i} \geq \mathcal{N}^2_{cA|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $\mathcal{N}^2_{cAB_j} \leq k\mathcal{N}^2_{cA|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$,*

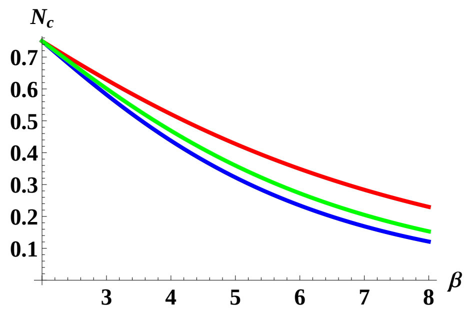

FIG. 3. The y axis is the lower bound of the negativity $\mathcal{N}_c(|\psi\rangle_{A|BC})$, which are functions of β. The red (green) line represents the lower bound from our result for $k=0.6$ ($k=0.8$), and the blue line represents the lower bound from the result in [22].

$\forall 1 \le m \le N-3$, $N \ge 4$, then we have

$$\begin{aligned}&\mathcal{N}^{\beta}_{cA|B_1\cdots B_{N-1}}\\&\ge \mathcal{N}^{\beta}_{cAB_1}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{N}^{\beta}_{cAB_2}+\cdots\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m-1}\mathcal{N}^{\beta}_{cAB_m}\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m+1}(\mathcal{N}^{\beta}_{cAB_{m+1}}+\cdots+\mathcal{N}^{\beta}_{cAB_{N-2}})\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m}\mathcal{N}^{\beta}_{cAB_{N-1}}\end{aligned} \tag{37}$$

for all $\beta \ge 2$.

Theorem 10. *Suppose k is a real number satisfying $0<k\le 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if all $k\mathcal{N}^2_{cAB_i} \ge \mathcal{N}^2_{cA|B_{i+1}\cdots B_{N-1}}$ for all $i=1,2,\ldots,N-2$, then*

$$\begin{aligned}\mathcal{N}^{\beta}_{cA|B_1\cdots B_{N-1}} \ \ge\ & \mathcal{N}^{\beta}_{cAB_1}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{N}^{\beta}_{cAB_2}+\cdots\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{N-2}\mathcal{N}^{\beta}_{cAB_{N-1}}\end{aligned} \tag{38}$$

for $\beta \ge 2$.

Example 3 Consider the state in Example 1 with $\lambda_0=\lambda_3=\frac{1}{2}$, $\lambda_2=\frac{\sqrt{2}}{2}$ and $\lambda_1=\lambda_4=0$. We have $\mathcal{N}_{cA|BC}=\frac{\sqrt{3}}{2}$, $\mathcal{N}_{cAB}=\frac{\sqrt{2}}{2}$ and $\mathcal{C}_{cAC}=\frac{1}{2}$. Then $\mathcal{N}^{\beta}_{cAB}+(2^{\frac{\beta}{2}}-1)\mathcal{N}^{\beta}_{cAC}=\left(\frac{\sqrt{2}}{2}\right)^{\beta}+\left(2^{\frac{\beta}{2}}-1\right)\left(\frac{1}{2}\right)^{\beta}$ and $\mathcal{N}^{\beta}_{cAB}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\mathcal{N}^{\beta}_{cAC}=\left(\frac{\sqrt{2}}{2}\right)^{\beta}+\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\left(\frac{1}{2}\right)^{\beta}$. One can see that our result is better than the one in [22] for $\beta \ge 2$, thus also better than the ones in [21, 24], see Fig. 3.

For the negativity of assistance $\mathcal{N}^a_c$, we have the following results.

Theorem 11. *Suppose k is a real number satisfying $0<k\le 1$. For an N-qubit pure state $|\psi\rangle_{A|B_1\cdots B_{N-1}}$, if $k(\mathcal{N}^a_{cA|B_i})^2 \ge (\mathcal{N}^a_{cA|B_{i+1}\cdots B_{N-1}})^2$ for $i=1,2,\ldots,m$, and $(\mathcal{N}^a_{cAB_j})^2 \le k(\mathcal{N}^a_{cA|B_{j+1}\cdots B_{N-1}})^2$ for $j=m+1,\ldots,N-2$, $\forall 1\le m\le N-3$, $N\ge 4$, then we have*

$$\begin{aligned}&[\mathcal{N}^a_c(|\psi\rangle_{A|B_1\cdots B_{N-1}})]^{\beta}\\&\le (\mathcal{N}^a_{cAB_1})^{\beta}+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)(\mathcal{N}^a_{cAB_2})^2+\cdots\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m-1}(\mathcal{N}^a_{cAB_m})^{\beta}\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m+1}[(\mathcal{N}^a_{cAB_{m+1}})^{\beta}+\cdots\\&+(\mathcal{N}^a_{cAB_{N-2}})^{\beta}]\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{m}(\mathcal{N}^a_{cAB_{N-1}})^{\beta}\end{aligned} \tag{39}$$

for all $0\le\beta\le 2$.

Theorem 12. *Suppose k is a real number satisfying $0<k\le 1$. For any N-qubit mixed state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $k(\mathcal{N}^a_{cAB_i})^2 \ge (\mathcal{N}^a_{cA|B_{i+1}\cdots B_{N-1}})^2$ for all $i=1,2,\ldots,N-2$, then*

$$\begin{aligned}&[\mathcal{N}^a_c(|\psi\rangle_{A|B_1\cdots B_{N-1}})]^{\beta}\\&\le (\mathcal{N}^a_{cAB_1})^{\beta}+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)(\mathcal{N}^a_{cAB_2})^{\beta}+\cdots\\&+\left(\frac{(1+k)^{\frac{\beta}{2}}-1}{k^{\frac{\beta}{2}}}\right)^{N-2}(\mathcal{N}^a_{cAB_{N-1}})^{\beta}\end{aligned} \tag{40}$$

for $0\le\beta\le 2$.

V. TIGHTER MONOGAMY RELATIONS FOR TSALLIS-q ENTANGLEMENT AND RÉNYI-α ENTANGLEMENT

In this section, we study the Tsallis-q entanglement and Rényi-α entanglement, and establish the corresponding monogamy and polygamy relations for the two entanglement measures, respectively.

A. Tighter monogamy and polygamy relations for Tsallis-q entanglement

The Tsallis-q entanglement of a bipartite pure state $|\psi\rangle_{AB}$ is defined as [14]

$$T_q(|\psi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1-\mathrm{Tr}\rho_A^q), \tag{41}$$

where $q>0$ and $q\neq 1$. For the case q tends to 1, $T_q(\rho)$ is just the von Neumann entropy, $\lim_{q\to 1} T_q(\rho) = -\mathrm{Tr}\rho\log_2\rho = S(\rho)$. The Tsallis-$q$ entanglement of a bipartite mixed state ρ_{AB} is given by $T_q(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i T_q(|\psi_i\rangle)$ with the minimum taken over all possible pure state decompositions of ρ_{AB}. For $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, Yuan *et al.* proposed an analytic relationship between the Tsallis-q entanglement and concurrence,

$$T_q(|\psi\rangle_{AB}) = g_q(\mathcal{C}^2(|\psi\rangle_{AB})), \tag{42}$$

where

$$g_q(x) = \frac{1}{q-1}\Big[1-\Big(\frac{1+\sqrt{1-x}}{2}\Big)^q - \Big(\frac{1-\sqrt{1-x}}{2}\Big)^q\Big] \tag{43}$$

with $0\leq x\leq 1$ [40]. It has also been proved that $T_q(|\psi\rangle) = g_q(\mathcal{C}^2(|\psi\rangle))$ if $|\psi\rangle$ is a $2\otimes m$ pure state, and $T_q(\rho) = g_q(\mathcal{C}^2(\rho))$ if ρ is a two-qubit mixed state. Hence, (42) holds for any q such that $g_q(x)$ in (43) is monotonically increasing and convex. Particularly, one has that

$$g_q(x^2+y^2) \geq g_q(x^2) + g_q(y^2) \tag{44}$$

for $2\leq q\leq 3$. In Ref. [14], Kim provided a monogamy relation for the Tsallis-q entanglement,

$$T_{qA|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T_{qA|B_i}, \tag{45}$$

where $i=1,2,\ldots,N-1$ and $2\leq q\leq 3$. Later, this relation was improved as follows: if $\mathcal{C}_{AB_i} \geq \mathcal{C}_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $\mathcal{C}_{AB_j} \leq \mathcal{C}_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, $\forall 1\leq m\leq N-3$, $N\geq 4$, then

$$\begin{aligned} &T^\beta_{qA|B_1B_2\cdots B_{N-1}} \\ &\geq T^\beta_{qA|B_1} + (2^\beta-1)T^\beta_{qA|B_2} + \cdots + (2^\beta-1)^{m-1}T^\beta_{qA|B_m} \\ &\quad +(2^\beta-1)^{m+1}(T^\beta_{qA|B_{m+1}} + \cdots + T^\beta_{qA|B_{N-2}}) \\ &\quad +(2^\beta-1)^m T^\beta_{qA|B_{N-1}}, \end{aligned} \tag{46}$$

where $\beta\geq 1$ and $T^\beta_{qA|B_1B_2\cdots B_{N-1}}$ quantifies the Tsallis-q entanglement under partition $A|B_1B_2\cdots B_{N-1}$, and $T^\beta_{qA|B_i}$ quantifies that of the two-qubit subsystem AB_i with $2\leq q\leq 3$. Moreover, for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, one has

$$T^2_{qA|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T^2_{qA|B_i}. \tag{47}$$

We now provide monogamy relations which are tighter than (45) and (46).

Theorem 13. *Suppose k is a real number satisfying $0<k\leq 1$. For an arbitrary N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $kT_{qAB_i} \geq T_{qA|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $T_{qAB_j} \leq kT_{qA|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, $\forall 1\leq m\leq N-3$, $N\geq 4$, then we have*

$$\begin{aligned} &T^\beta_{qA|B_1B_2\cdots B_{N-1}} \\ &\geq T^\beta_{qA|B_1} + \tfrac{(1+k)^\beta-1}{k^\beta}T^\beta_{qA|B_2} + \cdots \\ &\quad +\Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m-1} T^\beta_{qA|B_m} \\ &\quad +\Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m+1} (T^\beta_{qA|B_{m+1}} + \cdots + T^\beta_{qA|B_{N-2}}) \\ &\quad +\Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m} T^\beta_{qA|B_{N-1}}, \end{aligned} \tag{48}$$

for $\beta\geq 1$ and $2\leq q\leq 3$.

Theorem 14. *Suppose k is a real number satisfying $0<k\leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if all $kT_{qAB_i} \geq T_{qA|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,N-2$, then we have*

$$\begin{aligned} T^\beta_{qA|B_1\cdots B_{N-1}} \geq\ & T^\beta_{qAB_1} + \tfrac{(1+k)^\beta-1}{k^\beta}T^\beta_{qAB_2} + \cdots \\ & +\Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{N-2} T^\beta_{qAB_{N-1}}, \end{aligned} \tag{49}$$

for $\beta\geq 1$ and $2\leq q\leq 3$.

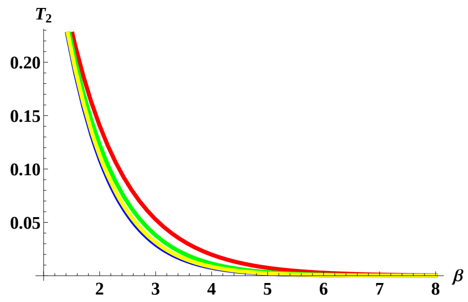

FIG. 4. The y axis is the lower bound of the Tsallis-q entanglement $T_q^\beta(|\psi\rangle_{A|BC})$. The red (green resp. yellow) line represents the lower bound from our result for $k = 0.5$ ($k = 0.7$ resp. $k = 0.9$), and the blue line represents the lower bound from the result in [22].

Example 4 Consider the quantum state given in Example 1 with $\lambda_0 = \lambda_3 = \frac{1}{2}$, $\lambda_2 = \frac{\sqrt{2}}{2}$ and $\lambda_1 = \lambda_4 = 0$. For $q = 2$, one has $T_{2A|BC} = \frac{3}{8}$, $T_{2AB} = \frac{1}{4}$ and $T_{2AC} = \frac{1}{8}$. Then $T_{2AB}^\beta + (2^\beta - 1)T_{2AC}^\beta = (\frac{1}{4})^\beta + (2^\beta - 1)(\frac{1}{8})^\beta$ and $T_{2AB}^\beta + \frac{(1+k)^\beta - 1}{k^\beta} T_{2AC}^\beta = \left(\frac{1}{4}\right)^\beta + \frac{(1+k)^\beta-1}{k^\beta}\left(\frac{1}{8}\right)^\beta$. It can be seen that our result is better than the one in [22] for $\beta \geq 1$, and also better than the ones given in [21, 24], see Fig. 4.

As a dual quantity to Tsallis-q entanglement, the Tsallis-q entanglement of assistance (TEoA) is defined by [14], $T_q^a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i T_q(|\psi_i\rangle)$, where the maximum is taken over all possible pure state decompositions of ρ_{AB}. If $1 \leq q \leq 2$ or $3 \leq q \leq 4$, the function g_q defined in (43) satisfies

$$g_q(\sqrt{x^2 + y^2}) \leq g_q(x) + g_q(y), \tag{50}$$

which leads to the Tsallis polygamy inequality

$$T_{qA|B_1B_2\cdots B_{N-1}}^a \leq \sum_{i=1}^{N-1} T_{qA|B_i}^a \tag{51}$$

for any multi-qubit state $\rho_{A|B_1B_2\cdots B_{N-1}}$ [25]. Here we provide tighter polygamy relations related to Tsallis-q entanglement. We have the following results.

Theorem 15. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kT_{qAB_i}^a \geq T_{qA|B_{i+1}\cdots B_{N-1}}^a$ for $i = 1, 2, \ldots, m$, and $T_{qAB_j}^a \leq kT_{qA|B_{j+1}\cdots B_{N-1}}^a$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, then*

$$\begin{aligned}
&(T_{qA|B_1B_2\cdots B_{N-1}}^a)^\beta \\
&\leq (T_{qAB_1}^a)^\beta + \tfrac{(1+k)^\beta-1}{k^\beta}(T_{qAB_2}^a)^\beta + \cdots \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m-1}(T_{qAB_m}^a)^\beta \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m+1}[(T_{qAB_{m+1}}^a)^\beta + \cdots + (T_{qAB_{N-2}}^a)^\beta] \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m}(T_{qAB_{N-1}}^a)^\beta,
\end{aligned} \tag{52}$$

for $0 \leq \beta \leq 1$ with $1 \leq q \leq 2$ or $3 \leq q \leq 4$.

Theorem 16. *Suppose k is a real number satisfying $0 < k \leq 1$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kT_{qAB_i}^a \geq T_{qA|B_{i+1}\cdots B_{N-1}}^a$ for all $i = 1, 2, \ldots, N-2$, we have*

$$\begin{aligned}
T_{qA|B_1B_2\cdots B_{N-1}}^\beta &\leq T_{qAB_1}^\beta + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)T_{qAB_2}^\beta + \cdots \\
&\quad + \left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{N-2}T_{qAB_{N-1}}^\beta,
\end{aligned} \tag{53}$$

for $0 \leq \beta \leq 1$ with $1 \leq q \leq 2$ or $3 \leq q \leq 4$.

B. Tighter monogamy and polygamy relations for Rényi-α entanglement

For a bipartite pure state $|\psi\rangle_{AB}$, the Rényi-α entanglement is defined as [42] $E(|\psi\rangle_{AB}) = S_\alpha(\rho_A)$, where $S_\alpha(\rho) = \frac{1}{1-\alpha}\log_2 \mathrm{Tr}\rho^\alpha$ for any $\alpha > 0$ and $\alpha \neq 1$, and $\lim_{\alpha\to 1} S_\alpha(\rho) = S(\rho) = -\mathrm{Tr}\rho\log_2\rho$. For a bipartite mixed state ρ_{AB}, the Rényi-α entanglement is given by

$$E_\alpha(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E_\alpha(|\psi_i\rangle), \tag{54}$$

where the minimum is taken over all possible pure-state decompositions of ρ_{AB}. For each $\alpha > 0$, one has $E_\alpha(\rho_{AB}) = f_\alpha(\mathcal{C}(\rho_{AB}))$, where $f_\alpha(x) = \frac{1}{1-\alpha}\log\left[\left(\frac{1-\sqrt{1-x^2}}{2}\right)^2 + \left(\frac{1+\sqrt{1-x^2}}{2}\right)^2\right]$ is a monotonically increasing and convex function [23]. For $\alpha \geq 2$ and any

n-qubit state $\rho_{A|B_1B_2\cdots B_{N-1}}$, one has [14]

$$E_{\alpha A|B_1B_2\cdots B_{N-1}} \geq E_{\alpha A|B_1} + E_{\alpha A|B_2} + \cdots + E_{\alpha A|B_{N-1}}. \quad (55)$$

We propose the following two monogamy relations for the Rényi-α entanglement, which are tighter than the previous results.

Theorem 17. *Suppose k is a real number satisfying $0 < k \leq 1$. For an arbitrary N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $kE_{\alpha AB_i} \geq E_{\alpha A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E_{\alpha AB_j} \leq kT_{\alpha A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, then*

$$\begin{aligned}(E_{\alpha A|B_1B_2\cdots B_{N-1}})^\beta &\geq (E_{\alpha A|B_1})^\beta + \tfrac{(1+k)^\beta-1}{k^\beta}(E_{\alpha A|B_2})^\beta + \cdots \\ &+\left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m-1}(E_{\alpha A|B_m})^\beta \\ &+\left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m+1}[(E_{\alpha A|B_{m+1}})^\beta + \cdots + (E_{\alpha A|B_{N-2}})^\beta] \\ &+\left(\tfrac{(1+k)^\beta-1}{k^\beta}\right)^{m}(E_{\alpha A|B_{N-1}})^\beta,\end{aligned} \quad (56)$$

for $\beta \geq 1$ and $\alpha \geq 2$.

Theorem 18. *Suppose k is a real number satisfying $0 < k \leq 1$. For an arbitrary N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $kE_{\alpha AB_i} \geq E_{\alpha A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \ldots, N-2$, then*

$$\begin{aligned}(E_{\alpha A|B_1\cdots B_{N-1}})^\beta &\geq (E_{\alpha AB_1})^\beta + \left(\tfrac{(1+k)^\alpha-1}{k^\alpha}\right)(E_{\alpha AB_2})^\beta + \cdots \\ &+\left(\tfrac{(1+k)^\alpha-1}{k^\alpha}\right)^{N-2}(E_{\alpha AB_{N-1}})^\beta\end{aligned} \quad (57)$$

for $\beta \geq 1$ and $\alpha \geq 2$.

Example 5 Consider again the state given in Example 1 with $\lambda_0 = \lambda_3 = \frac{1}{2}$, $\lambda_2 = \frac{\sqrt{2}}{2}$ and $\lambda_1 = \lambda_4 = 0$. For $\alpha = 2$, we find $E_{2A|BC} = \log_2 \frac{8}{5} \approx 678072$, $E_{2AB} = \log_2 \frac{8}{7} \approx 0.415037$ and $E_{2AC} = \log_2 \frac{4}{3} \approx 0.192645$. Then $E^\alpha_{2AB} + E^\alpha_{2AC} = 0.415037^\alpha + 0.192645^\alpha$ and $E^\alpha_{2AB} + \frac{(1+k)^\alpha-1}{k^\alpha}E^\alpha_{2AC} = 0.415037^\alpha + \frac{(1+k)^\alpha-1}{k^\alpha}0.192645^\alpha$. One can see that our result is better than the result in [14], and the smaller k is, the tighter relation is, see Fig. 5.

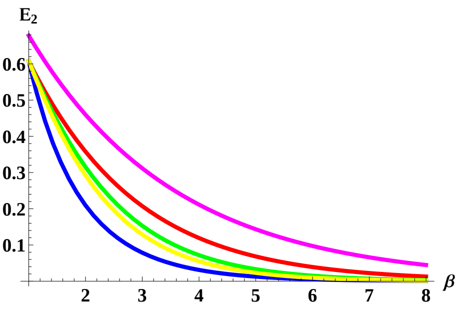

FIG. 5. The y axis is the lower bound of the Rényi entropy entanglement $E_2^\beta(|\psi\rangle_{A|BC})$. The purple line represents the value of $E_2^\beta(|\psi\rangle_{A|BC})$, the red (green resp. yellow) line represents the lower bound from our result for $k = 0.5$ ($k = 0.7$ resp. $k = 0.9$), and the blue line represents the lower bound from the result (55) in [14].

The Rényi-α entanglement of assistance (REoA), a dual quantity to Rényi-α entanglement, is defined as $E^a_\alpha(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E_\alpha(|\psi_i\rangle)$, where the maximum is taken over all possible pure state decompositions of ρ_{AB}. For $\alpha \in [\frac{\sqrt{7}-1}{2}, \frac{\sqrt{13}-1}{2}]$ and any n-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$, a polygamy relation of multi-partite quantum entanglement in terms of REoA has been given by [20]:

$$E^a_{\alpha A|B_1B_2\cdots B_{N-1}} \leq E^a_{\alpha A|B_1} + E^a_{\alpha A|B_2} + \cdots + E^a_{\alpha A|B_{N-1}}. \quad (58)$$

We improve this inequality to be a tighter ones under some netural conditions.

Theorem 19. *Suppose k is a real number satisfying $0 < k \leq 1$. For an arbitrary N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $kE^a_{\alpha AB_i} \geq E^a_{\alpha A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E^a_{\alpha AB_j} \leq kE^a_{\alpha A|B_{j+1}\cdots B_{N-1}}$ for $j =$*

$m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, *then*

$$\begin{aligned}
&(E^a_{\alpha A|B_1B_2\cdots B_{N-1}})^\beta \\
&\leq (E^a_{\alpha A|B_1})^\beta + \tfrac{(1+k)^\beta-1}{k^\beta}(E^a_{\alpha A|B_2})^\beta + \cdots \\
&\quad + \Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m-1}(E^a_{\alpha A|B_m})^\beta \\
&\quad + \Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m+1}\Big[(E^a_{\alpha A|B_{m+1}})^\beta + \cdots + (E^a_{\alpha A|B_{N-2}})^\beta\Big] \\
&\quad + \Big(\tfrac{(1+k)^\beta-1}{k^\beta}\Big)^{m}(E^a_{\alpha A|B_{N-1}})^\beta,
\end{aligned} \tag{59}$$

for $0 \leq \beta \leq 1$ *with* $\frac{\sqrt{7}-1}{2} \leq \alpha \leq \frac{\sqrt{13}-1}{2}$.

Theorem 20. *Suppose k is a real number satisfying $0 < k \leq 1$. For an arbitrary N-qubit mixed state $\rho_{AB_1B_2\cdots B_{N-1}}$, if $kE^a_{\alpha AB_i} \geq E^a_{\alpha A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \ldots, N-2$, then*

$$\begin{aligned}
&(E^a_{\alpha A|B_1\cdots B_{N-1}})^\beta \\
&\leq (E^a_{\alpha AB_1})^\beta + \Big(\tfrac{(1+k)^\alpha-1}{k^\alpha}\Big)(E^a_{\alpha AB_2})^\beta + \cdots \\
&\quad + \Big(\tfrac{(1+k)^\alpha-1}{k^\alpha}\Big)^{N-2}(E^a_{\alpha AB_{N-1}})^\beta
\end{aligned} \tag{60}$$

for $0 \leq \beta \leq 1$, *with* $\frac{\sqrt{7}-1}{2} \leq \alpha \leq \frac{\sqrt{13}-1}{2}$.

VI. CONCLUSION

Both entanglement monogamy and polygamy are fundamental properties of multipartite entangled states. We have presented monogamy relations related to the βth power of concurrence, entanglement of formation, negativity, Tsallis-q and Rényi-α entanglement. We also provide polygamy relations related to these entanglement measures. All the relations we presented in this paper are tighter than the previous results. These tighter monogamy and polygamy inequalities can also provide finer characterizations of the entanglement distributions among the multiqubit systems. Our results provide a rich reference for future work on the study of multiparty quantum entanglement. And our approaches are also useful for further study on the monogamy and polygamy properties related to measures of other quantum correlations and quantum coherence [43, 44].

VII. ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China under Grant Nos. 11805143 and 11675113, and Key Project of Beijing Municipal Commission of Education under No. KZ201810028042.

[1] M. Jafarpour, F. Kazemi Hasanvand and D. Afshar, Dynamics of entanglement and measurement-induced disturbance for a hybrid qubit-qutrit system interacting with a spin-chain environment: a mean field approach, Commun. Theor. Phys. **67**, 27 (2017).

[2] M. Y. Wang, J. Z. Xu, F. L. Yan, and T. Gao, Entanglement concentration for polarization-spatial-time-bin hyperentangled Bell states, Europhys. Lett. **123**, 60002 (2018).

[3] H. L. Huang, A. K. Goswami, W. S. Bao, and P. K. Panigrahi, Demonstration of essentiality of entanglement in a Deutsch-like quantum algorithm, Sci. China-Phys. Mech. Astron. **61**, 060311 (2018).

[4] F. G. Deng, B. C. Ren, and X. H. Li. Quantum hyperentanglement and its applications in quantum information processing, Sci. Bull. **62**, 46 (2017).

[5] B. M. Terhal, Is entanglement monogamous, IBM J. Res. Dev. **48**, 71 (2004).

[6] J. S. Kim, G. Gour, and B. C. Sanders, Limitations to sharing entanglement, Contemp. Phys. **53**, 417 (2012).

[7] C. H. Bennett, Quantum cryptography using any two nonorthogonal states, Phys. Rev. Lett. **68**, 3121 (1992).

[8] M. Pawlowski, Security proof for cryptographic protocols based only on the monogamy of Bells inequality violations, Phys. Rev. A **82**, 032313 (2010).

[9] T. J. Osborne and F. Verstraete, General monogamy inequality for bipartite qubit entanglement, Phys. Rev. Lett. **96**, 220503 (2006).

[10] Y. K. Bai, M. Y. Ye, and Z. D. Wang, Entanglement monogamy and entanglement evolution in multipartite systems, Phys. Rev. A **80**, 044301 (2009).

[11] T. R. de Oliveira, M. F. Cornelio, and F. F. Fanchini, Monogamy of entanglement of formation, Phys. Rev. A **89**, 034303 (2014).

[12] J. S. Kim, A. Das, and B. C. Sanders, Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extended negativity, Phys. Rev. A **79**, 012329 (2009).

[13] F. Liu, Monogamy relations for squared entanglement negativity, Commun. Theor. Phys. **66**, 407 (2016).

[14] J. S. Kim, Tsallis entropy and entanglement constraints in multiqubit systems, Phys. Rev. A **81**, 062328 (2010).

[15] F. Buscemi, G. Gour, and J. S. Kim, Polygamy of distributed entanglement, Phys. Rev. A **80**, 012324 (2009).

[16] G. Gour, D. A. Meyer, and B. C. Sanders, Deterministic entanglement of assistance and monogamy constraints, Phys. Rev. A **72**, 042329 (2005).

[17] G. Gour, S. Bandyopadhay, and B. C. Sanders, Dual monogamy inequality for entanglement, J. Math. Phys. **48**, 012108 (2007).

[18] J. S. Kim, General polygamy inequality of multiparty quantum entanglement, Phys. Rev. A **85**, 062302 (2012).

[19] J. S. Kim, Tsallis entropy and general polygamy of multiparty quantum entanglement in arbitrary dimensions, Phys. Rev. A **94**, 062338 (2016).

[20] W. Song, J. Zhou, M. Yang, J. L. Zhao, D. C. Li, L. H. Zhang and Z. L. Cao, Polygamy relation for the Rényi- entanglement of assistance in multi-qubit systems, arxiv:1703.02858.

[21] X. N. Zhu and S. M. Fei, Entanglement monogamy relations of qubit systems, Phys. Rev. A **90**, 024304 (2014).

[22] Z. X. Jin, J. Li, T. Li and S. M. Fei, Tighter monogamy relations in multiqubit systems, Phys. Rev A **97**, 032336 (2018).

[23] J. S. Kimand B. C. Sanders, Monogamy of multi-qubit entanglement using Rényi entropy, J. Phys.A: Math. Theor. **43**, 445305 (2010).

[24] Z. X. Jin and S. M. Fei, Tighter entanglement monogamy relations of qubit systems, Quantum Inf. Process. **16**. 77 (2017).

[25] J. S. Kim, Generalized entanglement constraints in multiqubit systems in terms of Tsallis entropy, Ann. Phys. **373**, 197 (2016).

[26] Yu. Luo, and Y. M. Li, Hierarchical polygamy inequality for entanglement of Tsallis q-entropy, Commun. Theor. Phys. **69**, 532 (2018).

[27] P. Rungta, V. Buzek, C. M. Caves, M. Hillery, and G. J. Milburn, Universal state inversion and concurrence in arbitrary dimensions, Phys. Rev. A **64**, 042315 (2001).

[28] S. Albeverio and S. M. Fei, A note on invariants and entanglements, J. Opt. B: Quantum Semiclass Opt. **3**, 223 (2001).

[29] T. Laustsen, F. Verstraete, and S. J. Van Enk, Local vs. joint measurements for the entanglement of assistance, Quantum Inf. Comput. **3**, 64 (2003).

[30] C. S. Yu and H. S. Song, Entanglement monogamy of tripartite quantum states, Phys. Rev. A **77**, 032329 (2008).

[31] X. J. Ren and W. Jiang, Entanglement monogamy inequality in a $2 \otimes 2 \otimes 4$ system, Phys. Rev. A **81**, 024305 (2010).

[32] A. Acin, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Generalized Schmidt decomposition and classification of three-quantum-bit states, Phys. Rev. Lett. **85**, 1560 (2000).

[33] X. H. Gao and S. M. Fei, Estimation of concurrence for multipartite mixed states, Eur. Phys. J. Spec. Topics **159**, 71 (2008).

[34] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).

[35] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K.Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).

[36] W. K.Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[37] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).

[38] Y. K. Bai, N. Zhang, M. Y. Ye, and Z. D. Wang, Exploring multipartite quantum correlations with the square of quantum discord, Phys. Rev. A **88**, 012123 (2013).

[39] O. Cohen, Unlocking hidden entanglement with classical information, Phys. Rev. Lett. **80**, 2493 (1998).

[40] G. M. Yuan, W. Song, M. Yang, D. C. Li, J. L. Zhao, and Z. L. Cao, Monogamy relation of multi-qubit systems for squared Tsallis-q entanglement, Sci. Rep. **6**, 28719 (2016).

[41] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[42] G. Vidal, Entanglement monotones, J. Mod. Opt. **47**, 355 (2000).

[43] X. D. Yu, D. J. Zhang, G. F. Xu, and D. M. Tong, Alternative framework for quantifying coherence, Phys. Rev A **94**, 060302(R) (2016).

[44] H. J. Zhang, B. Chen, M. Li, S. M. Fei, and G. L. Long, Estimation on geometric measure of quantum coherence, Commun. Theor. Phys. **67**, 166 (2017).