Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The norms of Bloch vectors and classification of four qudits quantum states

by

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Preprint no.: 20 2019

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Abstract

We investigate the norms of the Bloch vectors for any quantum state with subsystems less than or equal to four. Tight upper bounds of the norms are obtained, which can be used to derive tight upper bounds for entanglement measure defined by the norms of Bloch vectors. By using these bounds a trade-off relation of the norms of Bloch vectors is discussed. Theses upper bounds are then applied on separability. Necessary conditions are presented for different kinds of separable states in four-partite quantum systems. We further present a complete classification of quantum states for four qudits quantum systems.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

I. INTRODUCTION

Quantum entanglement, as the remarkable nonlocal feature of quantum mechanics, is recognized as a valuable resource in the rapidly expanding field of quantum information science, with various applications [1, 2] such as quantum computation , quantum teleportation, dense coding, quantum cryptographic schemes, quantum radar, entanglement swapping and remote states preparation. Tr

It is known that the Bloch vectors give one of the possible descriptions of qudit states. The Bloch vectors are then generalized to composite quantum systems with many subsystems. From the norms of the Bloch vectors in the generalized Bloch representation of a quantum state, separable conditions for both bi- and multi-partite quantum states have been presented in [3–6]. Two multipartite entanglement measures for N-qubit and N-qudit pure states are given in [7, 8]. A general framework for detecting genuine multipartite entanglement and non full separability in multipartite quantum systems of arbitrary dimensions has been introduced in [9]. In [10, 11] it has been shown that the norms of the Bloch vectors have a close relationship to the maximal violation of a kind of multi Bell inequalities and to the concurrence [12, 13]. However, with the increasing of the dimensions of the subsystems, the norms of Bloch vectors for density matrices become hard to describe[14–17].

In this paper, we study the Bloch representations of quantum states with the number of subsystems less than or equal to four. We present tight upper bounds for the norms of Bloch vectors. These upper bounds are then used to derive tight upper bounds for entanglement measure in [7, 8]. A trade-off relation of the norms of Bloch vectors is also discussed by these bounds. Then we investigate different subclasses of bi-separable states in four-partite systems. Necessary conditions are presented for these kinds of separable states. By these analyses we present a complete classification of four qudits quantum states.

II. UPPER BOUNDS OF THE NORMS OF BLOCH VECTORS

Let λ_is $(i=1,...,d^2-1)$ be orthogonal generators of $SU(d)$ which satisfy $\lambda_i^+ = \lambda_i, \mathrm{Tr}(\lambda_i)=0, \mathrm{Tr}(\lambda_i\lambda_j)=2\delta_{ij}$. Denote the identity operator by I_d. One finds that I_d and λ_is compose an orthogonal basis of the linear space consisting of all $d\times d$ Hermitian matrices with respect to the Hilbert-Schmidt inner product. By using $\mathrm{Tr}\rho=1$ and $\langle\lambda_i\rangle=\mathrm{Tr}(\rho\lambda_i)$, we get that any density operator ρ can be written in the form:

$$\rho=\frac{1}{d}\mathrm{I}_d+\frac{1}{2}\sum_{i=1}^{d^2-1}\langle\lambda_i\rangle\lambda_i. \tag{1}$$

The Bloch vector[14–22]is defined by $\mathbf{b}=(b_1,...,b_{d^2-1})\equiv(\langle\lambda_1\rangle,...,\langle\lambda_{d^2-1}\rangle)$. The state can be determined by measuring values of λ_is, the state ρ can also be given by the map $\mathbf{b}\to\rho=\frac{1}{d}\mathrm{I}_d+\frac{1}{2}\sum\limits_{i=1}^{d^2-1}b_i\lambda_i$. The set of all the Bloch vectors that constitute a density operator is known as the Bloch vector space $B(\mathbb{R}^{d^2-1})$.

A matrix of the form (1) is of unit trace and Hermitian, but it might not be positive. To guarantee the positivity restrictions must be imposed on the Bloch vector. It is shown that $B(\mathbb{R}^{d^2-1})$ is a subset of the ball $D_R(\mathbb{R}^{d^2-1})$ of radius $R=\sqrt{2(1-\frac{1}{d})}$, which is the minimum ball containing it, and that the ball $D_r(\mathbb{R}^{d^2-1})$ of radius $r=\sqrt{\frac{2}{d(d-1)}}$ is included in $B(\mathbb{R}^{d^2-1})$ [23], that is, $D_r(\mathbb{R}^{d^2-1})\subseteq B(\mathbb{R}^{d^2-1})\subseteq D_R(\mathbb{R}^{d^2-1})$.

Using the generators of $SU(d)$, any quantum state $\rho \in H_1^d \otimes H_2^d$ can be writing as:

$$\rho = \frac{1}{d^2} I \otimes I + \frac{1}{2d} \sum_{k=1}^{d^2-1} r_k \lambda_k \otimes I + \frac{1}{2d} \sum_{l=1}^{d^2-1} s_l I \otimes \lambda_l + \frac{1}{4} \sum_{k=1}^{d^2-1} \sum_{l=1}^{d^2-1} t_{kl} \lambda_k \otimes \lambda_l, \tag{2}$$

where $r_k = \mathrm{Tr}(\rho \lambda_k \otimes I)$, $s_l = \mathrm{Tr}(\rho I \otimes \lambda_l)$ and $t_{kl} = \mathrm{Tr}(\rho \lambda_k \otimes \lambda_l)$. We denote by $T^{(12)}$ a vector with entries t_{kl}. By using $\mathrm{Tr}(\rho^2) \leq 1$, one obtains that

$$\|T^{(12)}\|^2 \leq \frac{4(d^2-1)}{d^2}, \tag{3}$$

where $||\cdot||$ stands for the Hilbert-Schmidt norm or Frobenius norm.

We then consider the upper bounds of the Hilbert-Schmidt norm of the Bloch vectors for tripartite quantum systems. Let $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d$ be a quantum state, which can be represented by Bloch vectors as follow:

$$\begin{aligned}\rho = &\frac{1}{d^3} I \otimes I \otimes I + \frac{1}{2d^2}\left(\sum t_i^1 \lambda_i \otimes I \otimes I + \sum t_j^2 I \otimes \lambda_j \otimes I + \sum t_k^3 I \otimes I \otimes \lambda_k\right) \\ &+ \frac{1}{4d}\left(\sum t_{ij}^{12} \lambda_i \otimes \lambda_j \otimes I + \sum t_{ik}^{13} \lambda_i \otimes I \otimes \lambda_k + \sum t_{jk}^{23} I \otimes \lambda_j \otimes \lambda_k\right) \\ &+ \frac{1}{8} \sum t_{ijk}^{123} \lambda_i \otimes \lambda_j \otimes \lambda_k,\end{aligned} \tag{4}$$

where $t_i^1 = \mathrm{Tr}(\rho \lambda_i \otimes I \otimes I), t_j^2 = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes I), t_k^3 = \mathrm{Tr}(\rho I \otimes I \otimes \lambda_k), t_{ij}^{12} = \mathrm{Tr}(\rho \lambda_i \otimes \lambda_j \otimes I)$, $t_{ik}^{13} = \mathrm{Tr}(\rho \lambda_i \otimes I \otimes \lambda_k), t_{jk}^{23} = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes \lambda_k), t_{ijk}^{123} = \mathrm{Tr}(\rho \lambda_i \otimes \lambda_j \otimes \lambda_k)$ in the above representation. Define further $T^{(x)}, T^{(xy)}, T^{(123)}$ be the vectors with entries $t_i^x, t_{ij}^{xy}, t_{ijk}^{123}$, and $1 \leq x < y < z \leq 3$.

Theorem 1: For $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d$ with Bloch representation (4), we have:

$$\|T^{(123)}\|^2 \leq \frac{1}{d^3}(8d^3 - 24d + 16) \tag{5}$$

See supplemental material for the proof of the theorem.

We further consider four-partite quantum states. Let $\rho \in \mathrm{H}_{1234} = \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ be a mixed quantum state with the Bloch representation

$$\rho = \frac{1}{d^4} I \otimes I \otimes I \otimes I + \frac{1}{2d^3} M_1 + \frac{1}{4d^2} M_2 + \frac{1}{8d} M_3 + \frac{1}{16} M_4, \tag{6}$$

where

$$
\begin{aligned}
M_1 &= \sum_i t_i^1 \lambda_i \otimes I \otimes I \otimes I + \sum_j t_j^2 I \otimes \lambda_j \otimes I \otimes I + \sum_k t_k^3 I \otimes I \otimes \lambda_k \otimes I \\
&\quad + \sum_l t_l^4 I \otimes I \otimes I \otimes \lambda_l, \\
M_2 &= \sum_{i,j} t_{ij}^{12} \lambda_i \otimes \lambda_j \otimes I \otimes I + \sum_{i,k} t_{ik}^{13} \lambda_i \otimes I \otimes \lambda_k \otimes I + \sum_{i,l} t_{il}^{14} \lambda_i \otimes I \otimes I \otimes \lambda_l \\
&\quad + \sum_{j,k} t_{jk}^{23} I \otimes \lambda_j \otimes \lambda_k \otimes I + \sum_{j,l} t_{jl}^{24} I \otimes \lambda_j \otimes I \otimes \lambda_l + \sum_{k,l} t_{kl}^{34} I \otimes I \otimes \lambda_k \otimes \lambda_l, \\
M_3 &= \sum_{i,j,k} t_{ijk}^{123} \lambda_i \otimes \lambda_j \otimes \lambda_k \otimes I + \sum_{i,j,l} t_{ijl}^{124} \lambda_i \otimes \lambda_j \otimes I \otimes \lambda_l + \sum_{j,k,l} t_{jkl}^{234} I \otimes \lambda_j \otimes \lambda_k \otimes \lambda_l, \\
M_4 &= \sum_{i,j,k,l} t_{ijkl}^{1234} \lambda_i \otimes \lambda_j \otimes \lambda_k \otimes \lambda_l.
\end{aligned}
$$

We have defined $t_i^1 = \mathrm{Tr}(\rho\lambda_i \otimes I \otimes I \otimes I), t_j^2 = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes I \otimes I), t_k^3 = \mathrm{Tr}(\rho I \otimes I \otimes \lambda_k \otimes I)$, $t_l^4 = \mathrm{Tr}(\rho I \otimes I \otimes I \otimes \lambda_l)$, $t_{ij}^{12} = \mathrm{Tr}(\rho\lambda_i \otimes \lambda_j \otimes I \otimes I), t_{ik}^{13} = \mathrm{Tr}(\rho\lambda_i \otimes I \otimes \lambda_k \otimes I), t_{il}^{14} = \mathrm{Tr}(\rho\lambda_i \otimes I \otimes I \otimes \lambda_l), t_{jk}^{23} = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes \lambda_k \otimes I), t_{jl}^{24} = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes I \otimes \lambda_l), t_{kl}^{34} = \mathrm{Tr}(\rho I \otimes I \otimes \lambda_k \otimes \lambda_l)$, $t_{ijk}^{123} = \mathrm{Tr}(\rho\lambda_i \otimes \lambda_j \otimes \lambda_k \otimes I), t_{ijl}^{124} = \mathrm{Tr}(\rho\lambda_i \otimes \lambda_j \otimes I \otimes \lambda_l), t_{ikl}^{134} = \mathrm{Tr}(\rho\lambda_i \otimes I \otimes \lambda_k \otimes \lambda_l), t_{jkl}^{234} = \mathrm{Tr}(\rho I \otimes \lambda_j \otimes \lambda_k \otimes \lambda_l)$, $t_{ijkl}^{1234} = \mathrm{Tr}(\rho\lambda_i \otimes \lambda_j \otimes \lambda_k \otimes \lambda_l)$ in the above representation. Define further $T^{(x)}, T^{(xy)}, T^{(xyz)}, T^{(1234)}$ be the vectors with entries $t_i^x, t_{ij}^{xy}, t_{ijk}^{xyz}, t_{ijkl}^{1234}$, and $1 \le x < y < z \le 4$.

Theorem 2: For $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ with Bloch representation (6), we have:

$$
\|T^{(1234)}\|^2 \le \frac{16(d^2-1)^2}{d^4}. \tag{7}
$$

See supplemental material for the proof of the theorem.

The two upper bounds for norms of Bloch vectors are tight and useful as that will be shown in the following remarks.

Remark 1: The Bloch vectors are used to define a valid entanglement measure in [7, 8] as follows. For a N-qudit pure state, the entanglement measure is defined as

$$
E_T(|\psi\rangle) = \frac{d^N}{2^N} ||T^{(N)}|| - (\frac{d(d-1)}{2})^{\frac{N}{2}}, \tag{8}
$$

where $T^{(N)}$ is defined as a tensor with elements $t_{i_1 i_2 \cdots i_N}^{12\cdots \mathrm{N}} = \mathrm{Tr}(\rho\lambda_{i_1} \otimes \lambda_{i_2} \otimes \cdots \otimes \lambda_{i_N})$.

By Theorem 1 and 2, one obtains the upper bounds of $E_T(|\psi\rangle)$ for $N=3$ and $N=4$ as follows.

$$
E_T(|\psi\rangle) \le \begin{cases} \sqrt{\dfrac{d^3(d-1)^2}{8}}(\sqrt{d+2} - \sqrt{d-1}), & \text{N=3;} \\ \dfrac{d^2(d-1)}{2}, & \text{N=4.} \end{cases} \tag{9}
$$

By considering the the tripartite-qutrit state $|\psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ and the four-qubit state $|\phi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, one computes the upper bounds of $E_T(|\psi\rangle)$ are 3.01969 and 2 respectively(coincide with that in [8]). Thus the upper bounds of $E_T(|\psi\rangle)$ are tight.

Remark 2: We consider four-partite quantum systems. In [24] we have shown that for the state ρ with representation (6), we have:

$$\sum_{1\leq x<y<z\leq 4} \|\mathrm{T}^{(xyz)}\|^2 \leq \frac{8(d^2-1)^3}{d^3(d^2-2)}, \tag{10}$$

where $||\cdot||$ stands for the l_2 norm of a vector.

Set $d = 2$, we get $\sum\limits_{1\leq x<y<z\leq 4} \|\mathrm{T}^{(\mathrm{xyz})}\|^2 \leq 13.5$. By theorem 1 one has $\|\mathrm{T}^{(\mathrm{xyz})}\|^2 \leq 4$. Thus we obtain that it is impossible for $\|\mathrm{T}^{(123)}\|, \|\mathrm{T}^{(124)}\|, \|\mathrm{T}^{(134)}\|$, and $\|\mathrm{T}^{(234)}\|$ attaining 4 simultaneously.

III. NECESSARY CONDITIONS FOR BI-SEPARABLE STATES.

In this section, we investigate subclasses of the bi-separable states in four partite quantum systems by the upper bounds of norms for Bloch vectors. Let's start with the following definition.

Definition: Let $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ be a quantum state with d being the dimension of the subsystems $\mathrm{H}_i, i = 1, 2, 3, 4$. If ρ can be written as $\rho = \sum_k p_k |x_k\rangle\langle x_k|$, where $\sum_k p_k = 1, |x_k\rangle$ is in one of the following sets: $\{|\phi_1\rangle \otimes |\phi_{234}\rangle, |\phi_2\rangle \otimes |\phi_{134}\rangle, |\phi_3\rangle \otimes |\phi_{124}\rangle, |\phi_4\rangle \otimes |\phi_{123}\rangle\}$, $\{|\psi_{12}\rangle \otimes |\psi_{34}\rangle, |\psi_{13}\rangle \otimes |\psi_{24}\rangle, |\psi_{14}\rangle \otimes |\psi_{23}\rangle\}$, $\{|\xi_1\rangle \otimes |\xi_2\rangle \otimes |\xi_{34}\rangle, |\xi_1\rangle \otimes |\xi_3\rangle \otimes |\xi_{24}\rangle, |\xi_1\rangle \otimes |\xi_4\rangle \otimes |\xi_{23}\rangle, |\xi_{14}\rangle \otimes |\xi_2\rangle \otimes |\xi_3\rangle, |\xi_{13}\rangle \otimes |\xi_2\rangle \otimes |\xi_4\rangle, |\xi_{12}\rangle \otimes |\xi_3\rangle \otimes |\xi_4\rangle\}$ and $\{|\chi_1\rangle \otimes |\chi_2\rangle \otimes |\chi_3\rangle \otimes |\chi_4\rangle\}$, then ρ is called $1-3$ separable, $2-2$ separable, $1-1-2$ separable, and $1-1-1-1$ separable respectively.

The following theorem gives necessary conditions of these kinds of separable states.

Theorem 3: Let $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ be a four-qudit quantum state. We have

$$\|T^{(1234)}\|^2 \leq \begin{cases} \frac{16}{d^4}((d-1)(d^3-3d+2)), & \text{if } \rho \text{ is 1-3 separable;} \\ \frac{16}{d^4}(d^2-1)^2, & \text{if } \rho \text{ is 2-2 separable;} \\ \frac{16}{d^4}(d^2-1)(d-1)^2, & \text{if } \rho \text{ is 1-1-2 separable;} \\ \frac{16}{d^4}(d-1)^4, & \text{if } \rho \text{ is 1-1-1-1 separable.} \end{cases}$$

See supplemental material for the proof of the theorem.

The following two examples show that the upper bounds in theorem 3 are nontrivial and are tight.

Example 1: Consider the quantum state $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$,

$$\rho = x|\psi\rangle\langle\psi| + \frac{1-x}{16} I, \tag{11}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ and I stands for the identity operator. By theorem 3, we compute that $\|T^{(1234)}\|^2 = 9x^2$. Thus for $\frac{2}{3} < x \leq 1$ and $\frac{1}{\sqrt{3}} < x \leq \frac{2}{3}$, ρ will be not 1-3 separable and not 1-1-2 separable respectively. While for $\frac{1}{3} < x \leq \frac{1}{\sqrt{3}}$, ρ is not 1-1-1-1 separable.

Example 2: Consider bi-separable state $\rho^d = |\psi_+^d\rangle\langle\psi_+^d|$ with $|\psi_+^d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |ii\rangle \otimes \frac{1}{\sqrt{d}}\sum_{i=1}^d |ii\rangle$. One computes that $\|T^{(1234)}\|^2 = \frac{16}{d^4}(d^2-1)^2$ which means that the upper bound for 2-2 separable states in theorem 3 is saturated. Actually, the upper bound can be also attained by considering the maximal entangled states as shown in remark 1.

Remark 3: With above theorems and examples, we are ready to classify the four-partite quantum states by using the norms of the Bloch vector $\|T^{(1234)}\|$, as shown in Fig.1. It is worth mentioning that the 1-3 separable quantum states are always in the interior of the bi-separable set, while for some 2-2 separable quantum states the boundary of the bi-separable set is attainable. Since the upper bound for 2-2 separable states is just the upper bound for any four qudits states, we conclude that it is possible that the 2-2 separable state is on the boundary of the set of states(see Fig. 1).

IV. CONCLUSIONS AND REMARKS

It is a basic and fundamental question in quantum entanglement theory to classify and detect entanglement states. In this paper, we have investigated the norms of the Bloch vectors for any quantum state with subsystems less than or equal to four. Tight upper bounds of the norms have been derived, which are used to derive tight upper bounds for entanglement measure defined by the norms of Bloch vectors. A trade-off relation of the norms of Bloch vectors is also discussed by these bounds. Then these upper bounds have been applied on the separability. Necessary conditions have been presented for 1-3, 2-2, 1-1-2 and 1-1-1-1 separable quantum states in four-partite quantum systems. With these bounds a complete classification of four qudits quantum states is presented.

Acknowledgments This work is supported by the NSFC No. 11775306, 11701568 and 11675113; the Fundamental Research Funds for the Central Universities Grants No. 16CX02049A, 17CX02033A and 18CX02023A; the Shandong Provincial Natural Science

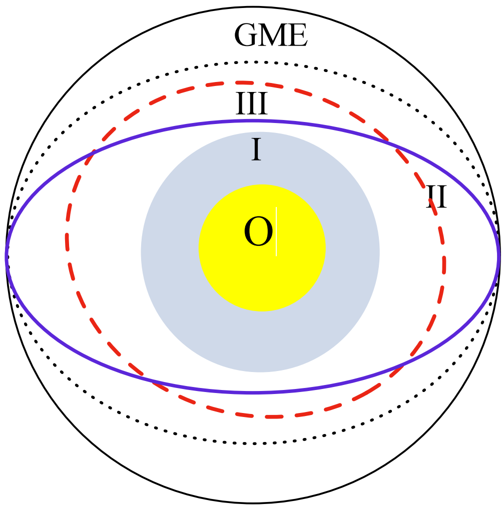

FIG. 1: We put all the four-partite quantum states into a set(the largest circle). Quantum states located in the area between the largest circle and the dotted oval are genuine multipartite entangled(GME). The bipartite separable states are classified to 1-1-1-1 separable part(O, minimal and yellow circle), 1-1-2 separable part(I, blue area), 2-2 separable part(II, blue and solid oval), 1-3 separable part(III, red and dashed circle) and the rest part of the dotted oval.

Foundation No. ZR2016AQ06, ZR2017BA019, and Key Project of Beijing Municipal Commission of Education under No. KZ201810028042.

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Supplemental material for "The norms of Bloch vectors and classification of four qudits quantum states"

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PACS numbers:

A. Proof of Theorem 1

Proof: We start with the pure state. For an arbitrary pure state $\rho = |\psi\rangle\langle\psi|$ one has $\mathrm{Tr}\rho^2 = 1$, which means

$$\begin{aligned}\mathrm{Tr}\rho^2 \;=\; & \frac{1}{d^3} + \frac{1}{2d^2}\left[\sum (t_i^1)^2 + \sum (t_j^2)^2 + \sum (t_k^3)^2\right] \\ & + \frac{1}{4d}\left[\sum (t_{ij}^{12})^2 + \sum (t_{ik}^{13})^2 + \sum (t_{jk}^{23})^2\right] + \frac{1}{8}\sum (t_{ijk}^{123})^2 = 1. \end{aligned} \tag{S1}$$

Set $\|T^{(1)}\|^2 = \sum (t_i^1)^2$, $\|T^{(2)}\|^2 = \sum (t_j^2)^2$, $\|T^{(3)}\|^2 = \sum (t_k^3)^2$, $\|T^{(12)}\|^2 = \sum (t_{ij}^{12})^2$, $\|T^{(13)}\|^2 = \sum (t_{ij}^{13})^2$, $\|T^{(23)}\|^2 = \sum (t_{jk}^{23})^2$ and $\|T^{(123)}\|^2 = \sum (t_{ijk}^{123})^2$.

Then we have:

$$\begin{aligned}\frac{1}{d^3} + & \frac{1}{2d^2}(\|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2) \\ & + \frac{1}{4d}(\|T^{(12)}\|^2 + \|T^{(13)}\|^2 + \|T^{(23)}\|^2) + \frac{1}{8}\|T^{(123)}\|^2 = 1. \end{aligned} \tag{S2}$$

One computes that:

$$\rho_1 = \frac{1}{d}I + \frac{1}{2}\sum t_i^1\lambda_i^1, \rho_{23} = \frac{1}{d^2}I\otimes I + \frac{1}{2d}(\sum t_j^2\lambda_j^2\otimes I + \sum t_k^3 I\otimes\lambda_k^3) + \frac{1}{4}\sum t_{jk}^{23}\lambda_j\otimes\lambda_k.$$

Thus we have:

$$\mathrm{Tr}\rho_1^2 = \frac{1}{d} + \frac{1}{2}\|T^{(1)}\|^2$$

and

$$\mathrm{Tr}\rho_{23}^2 = \frac{1}{d^2} + \frac{1}{2d}(\|T^{(2)}\|^2 + \|T^{(3)}\|^2) + \frac{1}{4}\|T^{(23)}\|^2.$$

Similarly we get:

$$\mathrm{Tr}\rho_2^2 = \frac{1}{d} + \frac{1}{2}\|T^{(2)}\|^2,$$

$$\mathrm{Tr}\rho_{13}^2 = \frac{1}{d^2} + \frac{1}{2d}(\|T^{(1)}\|^2 + \|T^{(3)}\|^2) + \frac{1}{4}\|T^{(13)}\|^2;$$

and

$$\mathrm{Tr}\rho_3^2 = \frac{1}{d} + \frac{1}{2}\|T^{(3)}\|^2,$$

$$\mathrm{Tr}\rho_{12}^2 = \frac{1}{d^2} + \frac{1}{2d}(\|T^{(1)}\|^2 + \|T^{(2)}\|^2) + \frac{1}{4}\|T^{(12)}\|^2.$$

By noticing that we are now considering pure state $\rho = |\psi\rangle\langle\psi|$, one has

$$\mathrm{Tr}\rho_i^2 = \mathrm{Tr}\rho_{jk}^2 \tag{S3}$$

for $ijk \in \{123, 213, 312\}$. Then we get:

$$\begin{aligned}& \frac{3}{d} + \frac{1}{2}(\|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2) \\ & = \frac{3}{d^2} + \frac{1}{d}(\|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2) + \frac{1}{4}(\|T^{(12)}\|^2 + \|T^{(13)}\|^2 + \|T^{(23)}\|^2). \end{aligned}$$

Set

$$A = \left\|T^{(1)}\right\|^2 + \left\|T^{(2)}\right\|^2 + \left\|T^{(3)}\right\|^2$$

and

$$B = \left\|T^{(12)}\right\|^2 + \left\|T^{(13)}\right\|^2 + \left\|T^{(23)}\right\|^2.$$

We have:

$$\frac{1}{4}B = (\frac{1}{2} - \frac{1}{d})A + \frac{3}{d} - \frac{2}{d^2}. \tag{S4}$$

Substitute (S4) into (S2), one has

$$\frac{1}{d^3} + \frac{1}{2d^2}A + \frac{1}{d}\left[(\frac{1}{2} - \frac{1}{d})A + \frac{3}{d} - \frac{3}{d^2}\right] + \frac{1}{8}\left\|T^{(123)}\right\|^2 = 1.$$

Furthermore, we have

$$\frac{1}{8}\left\|T^{(123)}\right\|^2 = 1 - \frac{1}{d^3} - \frac{3}{d^2} + \frac{3}{d^3} - \frac{d-1}{2d^2}A \leq 1 + \frac{2}{d^3} - \frac{3}{d^2}.$$

Thus one has:

$$\left\|T^{(123)}\right\|^2 \leq \frac{1}{d^3}(8d^3 - 24d + 16).$$

Let $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d$ be an arbitrary mixed state with ensemble decomposition $\rho = \sum p_\alpha \left|\psi_\alpha\right\rangle \left\langle\psi_\alpha\right|$. We have:

$$\left\|T^{(123)}(\rho)\right\|^2 = \left\|\sum p_\alpha T^{(123)}(|\psi_\alpha\rangle)\right\|^2 \leq \sum p_\alpha \left\|T^{(123)}(|\psi_\alpha\rangle)\right\|^2 \leq \frac{1}{d^3}(8d^3 - 24d + 16). \tag{S5}$$

■

B. Proof of Theorem 2

Proof: We start the proof with pure state situation. Let $\rho = |\psi\rangle\langle\psi|$ be a pure quantum state in $\mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ with Bloch representation (6) in the main tex. By setting $A = \sum_{i=1}^4 ||T^{(i)}||^2, B = \sum_{1\leq i<j\leq 4} ||T^{(ij)}||^2, C = \sum_{1\leq i<j<k\leq 4} ||T^{ijk}||^2$, and $D = ||T^{(1234)}||^2$, one gets

$$\mathrm{Tr}(\rho^2) = \frac{1}{d^4} + \frac{1}{2d^3}A + \frac{1}{4d^2}B + \frac{1}{8d}C + \frac{1}{16}D = 1. \tag{S6}$$

One can further computes for any $1 \leq i \leq 4, 1 \leq s < t \leq 4$ and $1 \leq x < y < z \leq 4$ that

$$\begin{aligned}
\mathrm{Tr}(\rho_i^2) &= \frac{1}{d} + \frac{1}{2}||T^{(i)}||^2; \\
\mathrm{Tr}(\rho_{st}^2) &= \frac{1}{d^2} + \frac{1}{2d}(||T^{(s)}||^2 + ||T^{(t)}||^2) + \frac{1}{4}||T^{(st)}||^2; \\
\mathrm{Tr}(\rho_{xyz}^2) &= \frac{1}{d^3} + \frac{1}{2d^2}(||T^{(x)}||^2 + ||T^{(y)}||^2 + ||T^{(z)}||^2) \\
&\quad + \frac{1}{4d}(||T^{(xy)}||^2 + ||T^{(xz)}||^2 + ||T^{(yz)}||^2) + \frac{1}{8}||T^{(xyz)}||^2.
\end{aligned}$$

Since we are considering the pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\mathrm{Tr}(\rho_i^2) = \mathrm{Tr}(\rho_{jkl}^2) \tag{S7}$$

holds for any $ijkl \in \{1234, 2134, 3124, 4123\}$. Then by summing the equations in (S7), we obtain

$$\frac{1}{4d^2}B = \frac{2d^2-2}{d^4} + \frac{d^2-3}{4d^3}A - \frac{1}{8}C. \tag{S8}$$

Substituting (S8) into (S6), we get

$$\frac{1}{16}D = 1 - \frac{1}{d^4} - \frac{2d^2-2}{d^4} - \frac{d^2-1}{4d^3}A - \frac{1}{16d}C \leq 1 - \frac{1}{d^4} - \frac{2d^2-2}{d^4} = \frac{(d^2-1)^2}{d^4},$$

which is just $D \leq \frac{16(d^2-1)^2}{d^4}$.

Then we consider a mixed state ρ with ensemble representation $\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$, where $\sum_\alpha p_\alpha = 1$. By the convexity of the Frobenius norm, one derives $D \leq \sum_\alpha p_\alpha D_\alpha \leq \frac{16(d^2-1)^2}{d^4}$, which ends the proof. ■

C. Proof of Theorem 3

Proof: Let $|\psi\rangle \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$ be pure state. Without lose of generality, one sets

$$|\psi\rangle = \begin{cases} |\phi_1\rangle \otimes |\phi_{234}\rangle, & \text{if } |\psi\rangle \text{ is 1-3 separable;} \\ |\psi_{12}\rangle \otimes |\psi_{34}\rangle, & \text{if } |\psi\rangle \text{ is 2-2 separable;} \\ |\xi_1\rangle \otimes |\xi_2\rangle \otimes |\xi_{34}\rangle, & \text{if } |\psi\rangle \text{ is 1-1-2 separable;} \\ |\chi_1\rangle \otimes |\chi_2\rangle \otimes |\chi_3\rangle \otimes |\chi_4\rangle, & \text{if } |\psi\rangle \text{ is 1-1-1-1 separable.} \end{cases}$$

We have

$$\begin{aligned} t_{ijkl}^{1234} &= \mathrm{Tr}(|\psi\rangle\langle\psi|\lambda_i \otimes \lambda_j \otimes \lambda_k \otimes \lambda_l) \\ &= \begin{cases} \mathrm{Tr}(|\phi_1\rangle\langle\phi_1|\lambda_i)\mathrm{Tr}(|\phi_{234}\rangle\langle\phi_{234}|\lambda_j \otimes \lambda_k \otimes \lambda_l), & \text{if } |\psi\rangle \text{ is 1-3 separable;} \\ \mathrm{Tr}(|\psi_{12}\rangle\langle\psi_{12}|\lambda_i \otimes \lambda_j)\mathrm{Tr}(|\psi_{34}\rangle\langle\psi_{34}|\lambda_k \otimes \lambda_l), & \text{if } |\psi\rangle \text{ is 2-2 separable;} \\ \mathrm{Tr}(|\xi_1\rangle\langle\xi_1|\lambda_i)\mathrm{Tr}(|\xi_2\rangle\langle\xi_2|\lambda_j)\mathrm{Tr}(|\xi_{34}\rangle\langle\xi_{34}|\lambda_k \otimes \lambda_l), & \text{if } |\psi\rangle \text{ is 1-1-2 separable;} \\ \mathrm{Tr}(|\chi_1\rangle\langle\chi_1|\lambda_i)\mathrm{Tr}(|\chi_2\rangle\langle\chi_2|\lambda_j)\mathrm{Tr}(|\chi_3\rangle\langle\chi_3|\lambda_k)\mathrm{Tr}(|\chi_4\rangle\langle\chi_4|\lambda_l), & \text{if } |\psi\rangle \text{ is 1-1-1-1 separable.} \end{cases} \\ &= \begin{cases} t_i^1 t_{jkl}^{234}, & \text{if } |\psi\rangle \text{ is 1-3 separable;} \\ t_{ij}^{12} t_{kl}^{34}, & \text{if } |\psi\rangle \text{ is 2-2 separable;} \\ t_i^1 t_j^2 t_{kl}^{34}, & \text{if } |\psi\rangle \text{ is 1-1-2 separable;} \\ t_i^1 t_j^2 t_k^3 t_l^4, & \text{if } |\psi\rangle \text{ is 1-1-1-1 separable.} \end{cases} \end{aligned}$$

Thus

$$||T^{1234}||^2 \ = \begin{cases} ||T^1||^2||T^{234}||^2, & \text{if } |\psi\rangle \text{ is 1-3 separable;} \\ ||T^{12}||^2||T^{34}||^2, & \text{if } |\psi\rangle \text{ is 2-2 separable;} \\ ||T^1||^2||T^2||^2||T^{34}||^2, & \text{if } |\psi\rangle \text{ is 1-1-2 separable;} \\ ||T^1||^2||T^2||^2||T^3||^2||T^4||^2, & \text{if } |\psi\rangle \text{ is 1-1-1-1 separable.} \end{cases}$$

$$\leq \begin{cases} \frac{16}{d^4}((d-1)(d^3-3d+2)), & \text{if } |\psi\rangle \text{ is 1-3 separable;} \\ \frac{16}{d^4}(d^2-1)^2, & \text{if } |\psi\rangle \text{ is 2-2 separable;} \\ \frac{16}{d^4}(d^2-1)(d-1)^2, & \text{if } |\psi\rangle \text{ is 1-1-2 separable;} \\ \frac{16}{d^4}(d-1)^4, & \text{if } |\psi\rangle \text{ is 1-1-1-1 separable.} \end{cases}$$

Then for any mixed state $\rho \in \mathrm{H}_1^d \otimes \mathrm{H}_2^d \otimes \mathrm{H}_3^d \otimes \mathrm{H}_4^d$, one has

$$\begin{aligned} \|T^{(1234)}\|^2 \ &= \ \|\sum_k p_k T_k^{(1234)}\|^2 \leq \sum_k p_k \|T_k^{(1234)}\|^2 \\ &\leq \begin{cases} \frac{16}{d^4}((d-1)(d^3-3d+2)), & \text{if } \rho \text{ is 1-3 separable;} \\ \frac{16}{d^4}(d^2-1)^2, & \text{if } \rho \text{ is 2-2 separable;} \\ \frac{16}{d^4}(d^2-1)(d-1)^2, & \text{if } \rho \text{ is 1-1-2 separable;} \\ \frac{16}{d^4}(d-1)^4, & \text{if } \rho \text{ is 1-1-1-1 separable,} \end{cases} \end{aligned}$$

which ends the proof. ■