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Abstract

Uncertainty relations satisfied by the product of variances of arbitrary n observables have attracted much attention. In a recent article [Phys. Rev. Lett. 120, 230402 (2018)], the authors provided so called strong unitary uncertainty relations for a set of unitary matrices by using the positivity property of the Gram matrix, from which uncertainty relations satisfied by two quantum mechanical observables are obtained. We derive the explicit uncertainty relations satisfied by n quantum mechanical observables from such Gram matrix approach. By some algebraic transformations, we show that these uncertainty relations are just the same as the ones derived from a positive semi-definite Hermitian matrix generated by the mean values of n observables in [Scientific Reports 6, 31192(2016)].

Introduction

Uncertainty relations are of profound significance in quantum mechanics and also in quantum information theory such as entanglement detection [1, 2], security analysis of quantum key distribution in quantum cryptography [3], nonlocality [4]. There are many ways to quantify the uncertainty of measurement outcomes. For instance, the uncertainty relations are expressed in terms of variances of the measurement results, in terms of entropies [5–7], and by means of majorization technique [8–10].

In 1927, Heisenberg [11] introduced his famous uncertainty principle, which says that

$$(\Delta P)^2(\Delta Q)^2 \geq (\frac{\hbar}{2})^2, \tag{1}$$

where $(\Delta P)^2$ and $(\Delta Q)^2$ denote variances of the position P and the momentum Q, respectively. The variance of observable A with respect to the state ρ is defined by $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$, and $\langle A\rangle = tr(\rho A)$ is the mean value of observable A respect to state ρ .

Later Robertson[12] presented the uncertainty relations for arbitrary pairs of non-commuting observables A and B,

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle[A, B]\rangle|^2. \tag{2}$$

where $[A, B] = AB-BA$ is the commutator. The above inequality employs the commutator, a characteristic quantity in quantum mechanics, to set a limit on the measurement precision.

The uncertainty relation (2) is further improved by Schrödinger [13],

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle|^2. \quad (3)$$

The commutator encodes the incompatibility, while the anticommutator encodes the correlations between the observables A and B. The Schrödinger uncertainty relation (3) includes both the commutator and anticommutator, and provides a better lower bound than the uncertainty relation (2).

Recently in Ref. [14], instead of quantum mechanical observables, the authors present a strong unitary uncertainty relation for n unitary operators, by using the Gram matrices. For two unitary matrices case, they prove that the strong unitary uncertainty relation implies the uncertainty relation (3) for two observables. An interesting question is what will be the uncertainty relation for arbitrary n observables. We derive the explicit uncertainty relations satisfied by n quantum mechanical observables from such Gram matrix approach used in [14].

In terms of the covariance matrices of the mean values of Hermitian operators, in [15] uncertainty relations for n observables have been also derived. It would be interesting to compare the n-observable relations from the approach used in [14] to the ones from the approach used in [15]. By a bijection map between the unitary operators and Hermitian operators, we show here that the strong unitary uncertainty relations given in [14] are e-quivalent to the Hermitian uncertainty relations presented in [15], although these two kinds of uncertainty relations are derived from quite different approaches. In proving this equivalence the general representations of n-observable uncertainty relations for both approaches are provided.

Strong unitary uncertainty relations of n unitary operators

Recently the authors in Ref. [14] presented uncertainty relations for unitary operators $U_0 = I, U_1, U_2, ..., U_n$, based on the Gram matrix G given by entries $G_{jk} = (v^{(j)}, v^{(k)}) = Tr[\rho U_j^\dagger U_k] = \langle U_j^\dagger U_k\rangle$, $j,k = 0,1,...,n$, where $v^j = U_j\rho^{1/2}$, ρ is a quantum state, the inner product is defined by $(A,B) = Tr(A^\dagger B)$. As G is positive semi-definite, $det\, G \geq 0$, one obtains the unitary uncertainty relations satisfied by the product of the variances

$$\Delta U_i^2 = 1 - |\langle U_i\rangle|^2 = \langle U_i^\dagger U_i\rangle - \langle U_i^\dagger\rangle\langle U_i\rangle.$$

In particular, for $n = 2$, one gets

$$\Delta U^2 \Delta V^2 \geq |\langle U^\dagger V\rangle - \langle U^\dagger\rangle\langle V\rangle|^2. \quad (4)$$

By writing $U = e^{i\varepsilon A}$ and $V = e^{i\varepsilon B}$ for some Hermitian matrices A and B and small parameter ε, the above relation gives rise to the standard Robertson-Schrödinger uncertainty relation [12, 13],

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}|\langle[A, B]\rangle|^2 + \frac{1}{4}|\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle|^2.$$

One may expect that strong uncertainty relations for general n observables can be similarly derived. Indeed for three unitary operators U, V and W, by direct calculation one obtains

$$\begin{aligned} &\Delta U^2 \Delta V^2 \Delta W^2 \geq \\ &|\langle U^\dagger V\rangle - \langle U^\dagger\rangle\langle V\rangle|^2 + |\langle V^\dagger W\rangle - \langle V^\dagger\rangle\langle W\rangle|^2 \\ &+|\langle W^\dagger U\rangle - \langle W^\dagger\rangle\langle U\rangle|^2 - |\langle U\rangle\langle V^\dagger W\rangle - \langle W\rangle\langle U^\dagger V\rangle| \\ &-|\langle U\rangle\langle V^\dagger W\rangle - \langle V\rangle\langle U^\dagger W\rangle| - |\langle V\rangle\langle U^\dagger W\rangle - \langle W\rangle\langle V^\dagger U\rangle| \\ &+|\langle U\rangle|^2|\langle V^\dagger W\rangle|^2 + |\langle V\rangle|^2|\langle U^\dagger W\rangle|^2 + |\langle W\rangle|^2|\langle U^\dagger V\rangle|^2 \\ &-2Re\langle U^\dagger V\rangle\langle V^\dagger W\rangle\langle W^\dagger U\rangle, \end{aligned} \quad (5)$$

where $Re\{S\}$ stands for the real part of S.

By setting $U = e^{i\varepsilon A}$, $V = e^{i\varepsilon B}$ and $W = e^{i\varepsilon C}$, we can derive the following uncertainty relation,

$$\begin{aligned} &\Delta A^2 \Delta B^2 \Delta C^2 \quad (6) \\ \geq\ &\Delta A^2|\langle BC\rangle - \langle B\rangle\langle C\rangle| + \Delta B^2|\langle CA\rangle - \langle C\rangle\langle A\rangle| \\ &+\Delta C^2|\langle AB\rangle - \langle A\rangle\langle B\rangle| - 2Re\{(\langle AB\rangle - \langle A\rangle\langle B\rangle) \\ &(\langle BC\rangle - \langle B\rangle\langle C\rangle)(\langle CA\rangle - \langle C\rangle\langle A\rangle). \end{aligned}$$

which is just the one given in [15].

In order to compare the main results from [14] with the ones in [15], we first derive the explicit expressions of the unitary uncertainty relations for arbitrary n unitary operators by using the approach in [14].

With respect to the unitary matrices $U_0, U_1, U_2, ..., U_n$, the $(n+1)\times(n+1)$ Gram matrix G is given by the entries $G_{jk} = \langle U_j^\dagger U_k\rangle$, $j, k = 0, 1, .., n$. Let us construct an $n \times n$ Hermitian matrix $\bar{G}$ with entries given by $g_{lm} = \langle U_l^\dagger U_m\rangle - \langle U_l^\dagger\rangle\langle U_m\rangle$, $l, m = 1, 2, ..., n$. Geometrically, the determinant of Gram matrix is the square of the volume of the parallelotope formed by the vectors. In particular, the vectors are linearly independent if and only if the determinant is nonzero. It can be shown that

Proposition 1. $det\, G = det\, \bar{G}$.

Proof: By using property of determinant, we have

$$detG = \begin{vmatrix} 1 & \langle U_1\rangle & \dots & \langle U_n\rangle \\ \langle U_1^\dagger\rangle & \langle U_1^\dagger U_1\rangle & \dots & \langle U_1^\dagger U_n\rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle U_n^\dagger\rangle & \langle U_n^\dagger U_1\rangle & \dots & \langle U_n^\dagger U_n\rangle \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \langle U_1\rangle & \dots & \langle U_n\rangle \\ 0 & \langle U_1^\dagger U_1\rangle - \langle U_1^\dagger\rangle\langle U_1\rangle & \dots & \langle U_1^\dagger U_n\rangle - \langle U_1^\dagger\rangle\langle U_n\rangle \\ \vdots & \vdots & \dots & \vdots \\ 0 & \langle U_n^\dagger U_1\rangle - \langle U_n^\dagger\rangle\langle U_1\rangle & \dots & \langle U_n^\dagger U_n\rangle - \langle U_n^\dagger\rangle\langle U_n\rangle \end{vmatrix}.$$

Then using Laplace expansion along the first column, we have $detG = det\bar{G}$.

In the following we denote $perm(n)$ any permutations of a list $(j_1...j_n)$ with n different elements. The $\mathbf{sign}(j_1...j_n)$ of a permutation $(j_1...j_n)$ is defined to be 1 if the number of pairs of integers (j, k), with $1 \le j \le k \le n$, such that j appears after k in the list $(j_1...j_n)$ is even, and -1 if the number of such pairs is odd. In other words, $\mathbf{sign}(j_1...j_n)$ equals to 1 (-1) if the natural order has been changed even (odd) times. For example, the $\mathbf{sign}(12...n) = 1$.

By the definition of determinant, the determinant of the matrix $\bar{G}$ can be expressed as

$$\begin{aligned} & det\, \bar{G} \\ &= \sum_{(j_1 j_2 ... j_n)\in perm(n)} \mathbf{sign}(j_1 j_2 ... j_n)\, g_{1j_1} \cdots g_{nj_n} \\ &= g_{11} g_{22} \cdots g_{nn} \\ &+ \sum_{(j_1 j_2 ... j_n)\neq(123...n)} \mathbf{sign}(j_1 j_2 ... j_n)\, g_{1j_1}\, g_{2j_2} \cdots g_{nj_n}, \end{aligned} \tag{7}$$

where $(j_1 j_2 ... j_n)$ denotes the permutation of $(1, ..., n)$, $j_1, j_2, ..., j_n = 1, ..., n$.

The matrix $\bar{G}$ is the covariance matrix of vectors $U_1, U_2, ..., U_n$ and is positive semi-definite. Noting that ΔU_i^2 $(i = 1, \dots, n)$ are just the diagonal entries of the matrix $\bar{G}$, we have the following unitary uncertainty relations from (7),

Theorem 1.

$$\begin{aligned}&\triangle U_1^2 \triangle U_2^2 ... \triangle U_n^2 = g_{11} g_{22} ... g_{nn} \\ &\geq - \sum_{(j_1 j_2 ... j_n) \neq (12...n)} \mathbf{sign}(j_1 j_2 ... j_n)\, g_{1j_1} g_{2j_2} ... g_{nj_n}.\end{aligned} \tag{8}$$

Remark The inequalities (4) and (5) can be directly derived from (8) by substituting $g_{lm} = \langle U_l^\dagger U_m \rangle - \langle U_l^\dagger \rangle \langle U_m \rangle$ into (8).

Uncertainty relations of n observables from covariance matrix approach

In quantum mechanics one measures quantum mechanical observables which are Hermitian operators. In Ref. [15] the authors considered the positive semi-definite Hermitian matrix $\bar{M}$ generated by n observables $A_1, ..., A_n$, with entries given by $m_{ls} = \langle A_l^\dagger A_s \rangle - \langle A_l^\dagger \rangle \langle A_s \rangle = \langle A_l A_s \rangle - \langle A_l \rangle \langle A_s \rangle$, $l, s = 1, 2, ..., n$. To compare the relation (8) with the one in [15], we rewrite the n-observable uncertainty relation in [15] first.

By straightforward computation, we have the following result,

$$\begin{aligned} det\bar{M} &= \sum_{(j_1 j_2 ... j_n) \in perm(n)} \mathbf{sign}(j_1 j_2 ... j_n)\, m_{1j_1} m_{2j_2} ... m_{nj_n} \\ &= m_{11} m_{22} ... m_{nn} \\ &\quad + \sum_{(j_1 j_2 ... j_n) \neq (123...n)} \mathbf{sign}(j_1 j_2 ... j_n)\, m_{1j_1} m_{2j_2} ... m_{nj_n}. \end{aligned} \tag{9}$$

Since the diagonal entries of $\bar{M}$ are just the variances of A_j defined by $\triangle A_j^2 = \langle A_j^\dagger A_j \rangle - \langle A_j^\dagger \rangle \langle A_j \rangle = \langle A_j^2 \rangle - \langle A_j \rangle^2$, from the fact that $\bar{M}$ is positive semi-definite, one gets from (9) the following uncertainty relations,

Theorem 2.

$$\begin{aligned}&\triangle A_1^2 \triangle A_2^2 ... \triangle A_n^2 \geq \\ &- \sum_{(j_1 j_2 ... j_n) \neq (123...n)} \mathbf{sign}(j_1 j_2 ... j_n)\, m_{1j_1} m_{2j_2} ... m_{nj_n}.\end{aligned} \tag{10}$$

Relations between (8) and (10)

Next we show that the relations (8) in Theorem 1 from [14] and the relations (10) in Theorem 2 from [15] are equivalent.

Concerning the one to one map between the unitary operators and the Hermitian operators, for an unitary operator U_j, there is a Hermitian operator A_j satisfying $U_j = e^{i\varepsilon A_j}$, $j = 1, 2, ..., n$. Taking Taylor expansions of $U_j = e^{i\epsilon A_j}$, $j = 1, 2, ..., n$,

$$U_j = 1 + i\epsilon A_j - \frac{1}{2}\epsilon^2 A_j^2 + O(\epsilon^3), \tag{11}$$

we have

$$\begin{aligned} \triangle U_j^2 &= \epsilon^2 \triangle A_j^2 + O(\epsilon^3), \\ g_{ij} &= \epsilon^2 m_{ij} + O(\epsilon^3). \end{aligned} \tag{12}$$

Combining (8) and (12), we get

$$\begin{aligned} &\epsilon^{2n} \triangle A_1^2 \triangle A_2^2 ... \triangle A_n^2 + O(\epsilon^{2n+1}) \geq \\ &-\epsilon^{2n} \sum_{(j_1 j_2 ... j_n) \neq (123...n)} \mathbf{sign}(j_1 j_2 ... j_n)\, m_{1j_1} m_{2j_2} ... m_{nj_n} \\ &+ O(\epsilon^{2n+1}), \end{aligned} \tag{13}$$

dividing by ϵ^{2n}, and taking the limit $\epsilon \to 0$, which gives rise to the inequality (10).

Conversely, from (10) and taking into account that $\triangle A_j^2 = \epsilon^{-2} \triangle U_j^2 + O(\epsilon)$ and $m_{ij} = \epsilon^{-2} g_{ij} + O(\epsilon)$ one gets the inequality (8).

Therefore, according to the bijection map $U = e^{i\epsilon A}$ between unitary operators and Hermitian operators, the unitary uncertainty relation (8) and the Hermitian uncertainty relation (10) are equivalent.

Conclusion

In Ref. [14], the authors presented the unitary uncertainty relations based on the Gram matrices of multi unitary operators. In [15] the Hermitian uncertainty relations are given based on the covariance matrices of multi Hermitian operators. By deriving the explicit uncertainty relations satisfied by n quantum mechanical observables from such Gram matrix approach, we have shown that the two kinds of uncertainty relations are equivalent. The related derivations and proofs may highlight further investigations on the uncertainty relations satisfied by the product of variances of n quantum mechanical observables.

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