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GLOBAL MINIMIZERS FOR ANISOTROPIC ATTRACTIVE-REPULSIVE INTERACTIONS

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Abstract. We prove existence of global minimizers for a class of attractive-repulsive interaction potentials that are in general not radially symmetric. The global minimizers have compact support. For potentials including degenerate power-law diffusion the interaction potential can be unbounded from below. Further, a formal calculation indicates that for non-symmetric potentials global minimizers may neither be radial symmetric nor unique.

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1. Introduction

Energy functionals with attractive-repulsive interaction potentials have received a lot of attention in the recent years. In most cases, however, the interaction potential is assumed to be symmetric. To the best of our knowledge there are only few results available so far for non-symmetric potentials. Examples are [14] and [7]. In this paper we analyze minimizers for not necessarily symmetric attractive-repulsive potentials and consider the following energy functional

$$\mathcal{E}(\rho) = \frac{\varepsilon}{m}\int_{\mathbb{R}^d} \rho^m(x)dx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(x)\rho(y)dxdy \ , \ \varepsilon \geq 0, m > 1, d \geq 1. \tag{1.1}$$

Here $\rho \in L^m \cap \mathcal{P}(\mathbb{R}^d)$ for $\varepsilon > 0$, and $\rho \in \mathcal{P}(\mathbb{R}^d)$ for $\varepsilon = 0$, where $\mathcal{P}(\mathbb{R}^d)$ is the set of all Borel probability measures on $\mathbb{R}^d$.
The potential W may be non-symmetric and is not necessarily negative, and therefore may contribute to repulsive effects. For symmetric potentials many results on the existence of minimizers have been obtained.

For $\varepsilon = 0$, it was shown in [3] that a minimizer of (1.1) exists if

(H1) W is bounded from below, i.e. $W(x) > -C_1$ for a suitable constant $C_1 > 0$.
(H2) $W \in L^1_{\text{loc}}(\mathbb{R}^d)$.
(H3) W is symmetric, i.e. $W(x) = W(-x)$ for all $x \in \mathbb{R}^d$.
(H4) $\lim_{|x|\to\infty} W_-(x) = W_\infty$ exists, where W_- denotes the negative part of W, and w.l.o.g. $W_\infty = 0$. Furthermore, W is *unstable*, i.e. there exists $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathcal{E}(\rho) < 0$.
(H5) W is lower semicontinuous.
(H6) There exists an $R_6 > 0$ such that W is strictly increasing on $R^{k-1}\times[R_6,\infty)\times\mathbb{R}^{d-k}$ as function of the $k-$th variable, for all $k \in \{1, 2, \cdots, d\}$.

Remark 1. *(H6) is needed to obtain compactness of minimizers. (H6) is weaker than assuming that there exists an* $R_6 > 0$ *such that*
$W(x) > W(y)$*, for all* $|x| > |y| \geq R_6$.

For $\varepsilon > 0$, minimizers of (1.1) are well studied when W is radially symmetric and purely attractive (see e.g. [11], [10] for details and further related references, [15] for $m = 1$ and not necessarily negative W in the periodic setting, and some more recent results in [6], [4]).
Our main aim in this paper is to prove existence of minimizers for non-symmetric and not necessarily negative W, and to analyze their characteristics. Thus condition (H3) is not needed for our considerations, and we have to replace (H6) as indicated in [3, Rem. 2.8] by e.g.

- There exists an $R_6 > 0$ and an $0 < \delta_k \leq R_6$ for every $k \in \{1, \dots, d\}$ such that $W(x - \delta) < W(x)$, for all $x \in \mathbb{R}^d$ with $x_k \geq R_6$.
  Further $W(x + \delta) < W(x)$ for all $x \in \mathbb{R}^d$ with $x_k \leq -R_6$.
  Here $\delta := (0, \dots, 0, \delta_k, 0, \dots, 0) \in \mathbb{R}^d$ with the k-th coordinate being δ_k.

We assume even more generally that

($\mathcal{H}6$) There exists an $R_6 > 0$ and an $0 < \delta_k \leq R_6$ for every $k \in \{1, ..., d\}$ such that $\frac{1}{2}(W + W^-)(x - \delta) < \frac{1}{2}(W + W^-)(x)$ for all $x \in \mathbb{R}^d$ with $x_k \geq R_6$, where $\delta := (0, \dots, 0, \delta_k, 0, \dots, 0) \in \mathbb{R}^d$ with the k-th coordinate being δ_k.

Remark 2. *An equivalent statement to ($\mathcal{H}$6) would be, that there exists an $R_6 > 0$ and an $0 < \delta_k \leq R_6$ for every $k \in \{1, \dots, d\}$ such that $\frac{1}{2}(W + W^-)(x + \delta) < \frac{1}{2}(W + W^-)(x)$ for all $x \in R^d$ with $x_k \leq -R_6$ where $\delta = (0, \dots, \delta_k, \dots, 0)$.*

Condition ($\mathcal{H}6$) is needed to prove that two separated parts of the support of a minimizer of $\mathcal{E}$ cannot be arbitrarily far from each other.
For $\varepsilon > 0$, we can relax some of the previous conditions. More precisely, we consider a not necessarily symmetric potential W which fulfills (H4), ($\mathcal{H}6$) and

($\mathcal{H}2$) $W_+ \in L^1_{loc}(\mathbb{R}^d)$ and $W_- \in L^{p,\infty}(\mathbb{R}^d)$, where $p > \max\left\{1, \frac{1}{m-1}\right\}$,

instead of (H2). Here W_+ denotes the positive part and W_- the negative part of W.

Remark 3. *An example satisfying (H1), ($\mathcal{H}$2), (H4), (H5) and ($\mathcal{H}$6) is*

$$W(x) = \frac{C_1}{|x|^\alpha + 1} - \frac{C_2}{|x - x_*|^\beta + 1}, \quad 0 < \beta < \alpha, 0 \neq x_* \in \mathbb{R}^d, \text{ and constants } 0 < C_2 < C_1.$$

First, we briefly review some known results regarding minimizers of $\mathcal{E}$.
As already mentioned, for $\varepsilon = 0$, the existence of a compactly supported minimizer of (1.1) was proved in [3], assuming (H1)-(H6). See also [16] and further references therein.

The case $\varepsilon > 0$:
For $d = 3$, and $m > \frac{4}{3}$ it was proved in [1] that a minimizer exists, when $(-W)$ is the Newtonian potential.
In [13, Prop. IV.1, IV.3 and Rem. IV.8] existence of minimizers was proved for $0 \geq W \in L^{p,\infty}(\mathbb{R}^d)$ being radially symmetric, $1 < p < \infty$, $m > \frac{p+1}{p}$, $\inf_{\rho \in L^m \cap \mathcal{P}} \mathcal{E}(\rho) < 0$ and either $m \leq 2$ or $W(r)d + W'(r)r \geq 0$ for almost every $r \geq 0$.
In [14] existence of minimizers for $\mathcal{E}(\rho)$ with non-symmetric potential W was proved by showing that every minimizing sequence is relatively compact if and only if strict subadditivity holds for $\mathcal{E}(\rho)$. This is the case for $m \leq 2$, when there exists a ρ with negative energy. For $m > 2$ the result was shown under suitable growth conditions on

W in [14], Cor. II.1 and Rem. II.5.
In [2], existence of a minimizer was ensured for $0 \geq W \in L^{p,\infty}(B_1(0)) \cap L^q(\mathbb{R}^d \setminus B_1(0))$ being radially symmetric and monotonically decreasing, $1 < p \leq \infty$, $1 \leq q < \infty$, and either $m = 2$ and $\varepsilon < \|W\|_{L^1}$, or $m > 2$.

One could consider the critical case $m = \frac{p}{p+1}$ too, in case $\varepsilon > C$ for a suitable $C = C(W) > 0$, as it was done in [1], [13], and [10]. This is quite technical, since one roughly has to specify the constant C. Therefore we do not deal with this case here.

Our main result for possibly non-symmetric potentials W reads as follows:

Theorem 1. (a) $\varepsilon = 0$:
Let $W : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfy (H1), (H2), (H4), (H5), ($\mathcal{H}6$). Then, there exists a global minimizer $\rho \in \mathcal{P}(\mathbb{R}^d)$ of $\mathcal{E}$ in (1.1). *Furthermore, there exists a $K = K(W, d) > 0$, such that every minimizer of $\mathcal{E}$ has compact support with diameter $\leq K$.*
(b) $\varepsilon > 0$ and $m > 2$:
Let ($\mathcal{H}2$), (H4) and ($\mathcal{H}6$) be satisfied. Then, there exists a global minimizer $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ of $\mathcal{E}$. Furthermore, there exists a constant $K = K(W, d) > 0$, such that every minimizer of $\mathcal{E}$ has compact support with diameter $\leq K$.
(c) $\varepsilon > 0$ and $1 < m \leq 2$:
Let ($\mathcal{H}2$) and (H4) be satisfied. Then, there exists a global minimizer $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ of $\mathcal{E}$. If additionally ($\mathcal{H}6$) is satisfied, then there exists a constant $K = K(W, d) > 0$, such that every minimizer of $\mathcal{E}$ has compact support with diameter $\leq K$.

Remark 4. (i) *For $\varepsilon > 0$ and $1 < m \leq 2$, one can show that every global minimizer has compact support by assuming ($\mathcal{H}2$) and (H4). A uniformly bounded size of the support is, however, not clear, unless ($\mathcal{H}6$) is assumed (see* [3, Rem. 1.6]*).*
(ii) *Suppose that $W_- \in L^{p,\infty}(\mathbb{R}^d)$, $W_+ \in L^1_{loc}(\mathbb{R}^d)$, $1 < p \leq \infty$, $W(x) \to 0$ as $|x| \to \infty$, $\inf_{\rho \in L^m \cap \mathcal{P}} \mathcal{E}(\rho) < 0$, and either $m > 2$ and ($\mathcal{H}6$), or $m \leq 2$. Our results do not need radial symmetry as was assumed in* [2]*. Further, our potential W is not necessarily negative and may not fulfill all conditions assumed in* [14]*. We generalize the results for $m > 2$ here and recover the cases for $m \leq 2$.*
(iii) *The method of our proof is based on the strategy given in* [3]*, where an attractive-repulsive energy without a diffusive part is considered. We extend the approach in* [11] *where, among others, the method from* [3] *is used to prove existence of minimizers for $\varepsilon > 0$ and W being radially symmetric, bounded and purely attractive. The ideas in* [3] *share similarities with the approach in* [1]*, by first reducing the problem in $\mathbb{R}^d$ to $B_R(0)$, the ball with radius R around* 0.
(iv) *In (H4) it is assumed that W is unstable. For general potentials, this is certainly not true. We give some conditions for m, ε and W such that $\inf_{\rho \in L^m \cap \mathcal{P}} \mathcal{E}(\rho) < 0$ (see Lem. 6 and 7).*

Our paper is organized as follows. In Section 2, we present the proof of Theorem 1. Section 3 provides some conditions for unstable potentials. Further, we give a formal argument for a non-radial minimizer in case of a non-symmetric potential.

2. Proof of Theorem 1

For convenience define
$f^-(x) := f(-x)$ and $\mathcal{P}_R(\mathbb{R}^d) := \{\rho \in \mathcal{P}(\mathbb{R}^d) : \operatorname{supp}\rho \subset \overline{B_R}\}$ for all $R > 0$.
Case (a) $\varepsilon = 0$ in Theorem 1 is the simplest one to prove.

Proof of (a), Theorem 1: First note that a minimizer of the energy $\mathcal{E}$ in (1.1) with potential W is also a minimizer of the same energy with potential W^-, since

$$\int_{\mathbb{R}^d} (W*\rho)(x)\rho(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(y)\rho(x)dydx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(y)\rho(x)dxdy$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W^-(y-x)\rho(y)\rho(x)\,dxdy = \int_{\mathbb{R}^d} (W^- * \rho)(y)\rho(y)\,dy.$$

Thus assuming (H6) would not be sufficient. We need to control W^- at infinity, therefore we assume $(\mathcal{H}6)$ instead of (H6). The problem can now be reduced to the symmetric case via the symmetrized potential $\frac{1}{2}(W+W^-)$, and thus (following the same procedure as in [3]) the case $\varepsilon = 0$ is also valid for non-symmetric potentials. This completes the proof. □

Now consider $\varepsilon > 0$. For any $m > 1$ a minimizer ρ_m of $\mathcal{E}$ satisfies

$$\varepsilon\rho_m^{m-1} + \frac{1}{2}(W+W^-)*\rho_m = 2\mathcal{E}[\rho_m] - \int_{\mathbb{R}^d}\varepsilon\Big(\frac{2}{m}-1\Big)\rho_m^m(y)\,dy \tag{2.1}$$

in $\operatorname{supp}\rho_{\mathrm{m}}$ (see e.g. [10], [11]). Since in our situation $W+W^-$ is in general not purely attractive, we have to modify existing techniques in a subtle way. From now on, we assume any of the hypotheses (H1)-(H6) and $(\mathcal{H}2)$, $(\mathcal{H}6)$ only, if this is explicitely stated. We will prove both cases, (b) and (c), in Theorem 1 via a series of lemmas. The first step is to show that $\mathcal{E}$ is lower semicontinuous. By modifying the proof in [14, Th. 2.1] (also used in [8, Lem. 3.3] and [9]) for our problem in $B_R(0)$, we obtain

Lemma 2. *Let $W_- \in L^{p,\infty}(\mathbb{R}^d)$ with $p > 1$ and $m > \frac{p+1}{p}$. Then the energy $\mathcal{E}$ in (1.1) is weakly lower semicontinuous in $L^m(B_R(0))$.*

Proof. Obviously, the first term of the energy $\mathcal{E}$ is weakly lower semicontinuous. Therefore, it is left to prove that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(y)\rho(x)\,dydx \le \liminf_{n\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_n(y)\rho_n(x)\,dydx$$

for a sequence $(\rho_n)_{n\in\mathbb{N}} \subset L^m(B_R(0))$ converging weakly to some $\rho \in L^m(B_R(0))$.

It is sufficient to consider $W_S := W\chi_{B_S(0)}$ for some $S > 0$, which is large enough, e.g. $S > 2R$. Rewriting $W_S = W_{S,+} - W_{S,-}$ with $W_{S,+}, W_{S,-} \ge 0$, being the positive and negative part of W_S, it holds that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_{S,+}(x-y)\rho_n(y)\rho_n(x)\,dydx \ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (W_{S,+}\wedge M)(x-y)\rho_n(y)\rho_n(x)\,dydx$$

for all $M > 0$. Let us define $W_S^M := (W_S \wedge M) \vee (-M)$, then we have due to Hölder's and Young's inequality for convolutions that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(W_{S,-} - [W_{S,-} \wedge (M)]\right)(x-y)\rho_n(y)\rho_n(x)\,dydx \\
&\quad \le \|\rho_n\|_{L^{\frac{2q}{2q-1}}(B_R(0))} \|(W_{S,-} - [W_{S,-} \wedge (M)]) * \rho_n\|_{L^{2q}} \\
&\quad \le \|\rho_n\|^2_{L^{\frac{2q}{2q-1}}(B_R(0))} \|W_{S,-} - [W_{S,-} \wedge (M)]\,\|_{L^q} \\
&\quad \le C\|W_{S,-} - [W_{S,-} \wedge (M)]\,\|_{L^{q(1+\delta)}} \left|\left\{x \in \mathbb{R}^d \,\middle|\, W_-(x) > M\right\}\right|^{\frac{\delta}{1+\delta}} \le CM^{-\frac{p\delta}{1+\delta}},
\end{aligned}$$

where $\frac{p+1}{2} < q < p$, i.e. $p > 1$, and $\delta > 0$ is sufficiently small such that $q(1+\delta) < p$. The last expression is getting arbitrarily small by choosing $M > 0$ large enough. Hence, it is sufficient to prove for each fixed $M > 0$ that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\rho_n(y)\rho_n(x)\,dydx \to \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\rho(y)\rho(x)dydx$$

for $n \to \infty$ in order to obtain weak lower semicontinuity of $\mathcal{E}$.

Since $W_S^M \in L^1 \cap L^\infty$, we know that by weak convergence

$$(W_S^M * \rho_n)(x) \to (W_S^M * \rho)(x) \;\; \text{almost everywhere in } x \in B_R(0)$$

for $n \to \infty$ and that

$$\|W_S^M * \rho_n\|_{L^{m'}} \le \|W_S^M\|_{L^{\frac{m}{2m-2}}} \|\rho_n\|_{L^m} \le C \;\; \text{for } \frac{1}{m} + \frac{1}{m'} = 1\,.$$

We note that the above integral is uniform, since W_S^M is bounded and compactly supported. By Vitali's convergence theorem, it follows that $W_S^M * \rho_n \to W_S^M * \rho$ in $L^{m'}(B_R(0))$. Thus we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\rho_n(y)\rho_n(x)\,dydx - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\rho(y)\rho(x)\,dydx\right| \\
&\quad \le \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\rho_n(y)\big(\rho_n(x) - \rho(x)\big)\,dydx\right| \\
&\qquad + \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_S^M(x-y)\big(\rho_n(y) - \rho(y)\big)\rho(x)\,dydx\right| \\
&\quad \le \|W_S^M * \rho_n - W_S^M * \rho\|_{L^{m'}(B_R(0))}\|\rho_n\|_{L^m} + \left|\int_{\mathbb{R}^d} \big(\rho_n(y) - \rho(y)\big)(W_S^M * \rho)(y)\,dy\right|
\end{aligned}$$

$\to 0$ for $n \to \infty$. □

Remark 5. *Due to our assumptions on p and m, the energy $\mathcal{E}$ is bounded from below. This follows from Hölder's inequality and Young's inequality for convolutions.*

Let $\rho_R \in \mathcal{P}_R(\mathbb{R}^d)$ be a global minimizer of (1.1), which exists due to Lemma 2 and Remark 5. Now we prove that $(\rho_R)_{R \ge R'}$ is uniformly bounded in L^∞, by using uniform boundedness of $(\rho_R)_{R \ge R'}$ in L^m and

$$\varepsilon\rho_R^{m-1}(x) + \left(\frac{1}{2}(W + W^-) * \rho_R\right)(x) = 2\mathcal{E}[\rho_R] - \int_{\mathbb{R}^d} \varepsilon\Big(\frac{2}{m} - 1\Big)\rho_R^m(y)\,dy \tag{2.2}$$

in $\operatorname{supp}\rho_{\mathrm{R}}$. With assumption (H4), there exists an $R' > 0$ such that $\mathcal{E}[\rho_{R'}] < 0$.

Lemma 3. *Let ($\mathcal{H}2$) and (H4) be satisfied. Let $(\rho_R)_{R\geq R'}$ be a sequence of minimizers in $B_R(0)$ and let $R'>0$ be large enough such that $\mathcal{E}[\rho_{R'}]<\tilde{K}<0$ for some constant $\tilde{K}$. Then, there exists a constant $C>0$ such that $\|\rho_R\|_{L^\infty}\leq C$ for all $R\geq R'$.*

Proof. Since $(\rho_R)_{R\geq R'}$ is a minimizing sequence for $\mathcal{E}$ in $L^m(\mathbb{R}^d)\cap\mathcal{P}(\mathbb{R}^d)$, we have that $\|\rho_R\|_{L^m}$ is uniformly bounded, i.e. $\|\rho_R\|_{L^m}\leq C_m$.

Due to $W_-\in L^{p,\infty}(\mathbb{R}^d)$ and (H4), there exists a constant $S>0$ such that $(W_-)\chi_{B_S(0)}\in L^{q_1}(\mathbb{R}^d)$ for $1\leq q_1<p$, and $W_-(x)\leq 1$ for $x\in\mathbb{R}^d\setminus B_S(0)$. Therefore, for $q_2>p$ we can see that

$$\int_{\mathbb{R}^d\setminus B_S(0)}|W_-|^{q_2}\leq\int_0^1\alpha^{q_2}d\lambda_{W_-}(\alpha)\leq\tilde{C}\int_0^1\alpha^{q_2-p-1}d\alpha<\infty,$$

where $\lambda_{W_-}(\alpha):=|\{x|\,|W_-|>\alpha\}|<\tilde{C}\alpha^{-p}$. Thus $(W_-)\chi_{\mathbb{R}^d\setminus B_S(0)}\in L^{q_2}(\mathbb{R}^d)$.

In order to obtain a uniform bound in L^∞, we extend some ideas of the proofs in [1, Prop. 5 and Th. A] to our more general case. Due to (2.2) we have

$$\begin{aligned}\varepsilon\rho_R^{m-1}(x)&<-\left(\frac{1}{2}(W+W^-)*\rho_R\right)(x)-\varepsilon\Big(\frac{2}{m}-1\Big)\|\rho_R\|_{L^m}^m\\&\leq\left(\frac{1}{2}(W+W^-)_-*\rho_R\right)(x)-\varepsilon\Big(\frac{2}{m}-1\Big)\|\rho_R\|_{L^m}^m\end{aligned}\tag{2.3}$$

in supp ρ_R. Now we consider integrability.

$1<m\leq 2$:
Using $W_-\in L^{p,\infty}(\mathbb{R}^d)$ gives integrability of the convolution term on the right hand side of (2.3) and thus integrability of ρ_R^{m-1} for $1<m\leq 2$. More precisely, $\|\rho_R\|_{L^l}$ is finite for $1\leq l\leq m$, since $\rho_R\in L^1\cap L^m$. In case $p>\frac{m}{m-1}$, choose $q_1=\frac{m}{m-1}$, then Young's inequality for convolutions yields

$$\|[(W+W^-)_-]\chi_{B_S(0)}*\rho_R\|_{L^\infty}\leq\|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}}\|\rho_R\|_{L^m}.$$

Similarly, since for the Hölder conjugate we have $q_2'<m$ for $q_2>p$, we obtain that

$$\|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)}*\rho_R\|_{L^\infty}\leq\|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)}\|_{L^{q_2}}\|\rho_R\|_{L^{q_2'}}.\tag{2.4}$$

In case $p\leq\frac{m}{m-1}$, and for $q_2>\frac{m}{m-1}$, the estimate (2.4) holds as well.

Next we estimate $[(W+W^-)_-]\chi_{B_S(0)}*\rho_R$, for $m=2$ and for $1<m<2$.

$m=2$: There exists a $1<q_1<p$ such that

$$\|[(W+W^-)_-]\chi_{B_S(0)}*\rho_R\|_{L^\alpha}\leq\|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}}\|\rho_R\|_{L^2},$$

for $1/\alpha=1/q_1-1/2$. Now consider the iterative formula

$$\|[(W+W^-)_-]\chi_{B_S(0)}*\rho_R\|_{L^{\alpha_i}}\leq\|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}}\|\rho_R\|_{L^{\alpha_{i-1}}},\tag{2.5}$$

where $\alpha_0=2$,

$$\alpha_k=\frac{q_1}{1-q_1+\frac{q_1}{\alpha_{k-1}}},\qquad k\in\mathbb{N}\setminus\{0\}.$$

The sequence $\{\alpha_k\}$ is strictly increasing as long as its elements are positive. We stop the iteration in (2.5), when $\alpha_{k-1} \geq q_1/(q_1-1)$ and then obtain by Young's convolution inequality that

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^\infty} \leq \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}} \|\rho_R\|_{L^{\frac{q_1}{q_1-1}}}. \tag{2.6}$$

$1 < m < 2$ and $p > \frac{1}{m-1}$: We have

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^{\frac{m}{(m-1)^2}}} \leq \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{\frac{1}{m-1}}} \|\rho_R\|_{L^m},$$

therefore $\rho_R \in L^{\frac{m}{m-1}}$. Define $m_0 = m/(m-1)^2$ and consider the iterative inequality

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^{m_k}} \leq \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{\frac{1}{m-1}}} \|\rho_R\|_{L^{(m-1)m_{k-1}}}, \text{ where} \tag{2.7}$$

$$\frac{1}{m_k} = \frac{1}{(m-1)m_{k-1}} + m - 2, \qquad k = 1, 2, \cdots. \tag{2.8}$$

Repeat the iteration (2.7)as long as the m_k are positive.
Obviously we have $m_0 > 1/(m-1)$. Now suppose that $m_{k-1} > 1/(m-1)$. Then

$$\frac{1}{m_k} = \frac{1}{(m-1)m_{k-1}} + m - 2 < 1 + m - 2 = (m-1)\ , \text{ so } m_k > \frac{1}{m-1}.$$

Further, $m_k > m_{k-1}$ as long as $m_k > 0$, due to

$$\frac{1+(1-m)}{(m-1)m_{k-1}} + m - 2 = \frac{1}{(m-1)m_{k-1}} + m - 2 - \frac{1}{m_{k-1}} < 0.$$

Therefore, also in this case the sequence $\{m_k\}$ is strictly increasing as long as all its elements are positive. Iterating until $m_k \geq \frac{1}{(m-1)(2-m)}$ and then stopping, gives

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^\infty} \leq \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{\frac{1}{m-1}}} \|\rho_R\|_{L^{\frac{1}{2-m}}}. \tag{2.9}$$

Therefore, from (2.6)-(2.9) it follows due to (2.3) that $\|\rho_R\|_{L^\infty(\mathbb{R}^d)}$ is uniformly bounded for $m \leq 2$.

$m > 2$: Here we only consider the set where $\rho_R^{m-1}(x) > -\varepsilon\big(\frac{2}{m} - 1\big)C_m^m$, with $C_m := \|\rho_R\|_{L^m}$, and derive (2.9) via an L^∞-estimate and integrability of the convolution term. We have $\rho_R \in L^1$. If the above set is empty, ρ_R is bounded. Further

$$\begin{aligned} &\|(W+W^-)_- * \rho_R\|_{L^r} \\ &\leq C\Big(\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^r} + \|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)} * \rho_R\|_{L^r}\Big). \end{aligned}$$

Using Young's inequality for convolutions, it follows that

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^r} \leq \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}} \|\rho_R\|_{L^m}$$

with $q_1 < p$ and $r = \left(\frac{1}{m} - \left(1 - \frac{1}{q_1}\right)\right)^{-1} = \frac{q_1 m}{q_1 - (q_1-1)m}$, for $m < \frac{q_1}{q_1-1}$.
For $m = \frac{q_1}{q_1-1}$ we take $r = \infty$. Analogously,

$$\|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)} * \rho_R\|_{L^r} \leq \|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)}\|_{L^{q_2}} \|\rho_R\|_{L^m}$$

with $q_2 > p$,

$$r = \frac{q_2 m}{q_2 - (q_2-1)m} \text{ for } m < \frac{q_2}{q_2-1}\ , \text{ and } r = \infty \text{ for } m = \frac{q_2}{q_2-1}\ .$$

Since $\|\rho_R\|_{L^m} \le C_m$, we have $\|(W+W^-)_- * \rho_R\|_{L^r(\mathbb{R}^d)} \le C_r$ for any r with $p < r < \frac{pm}{p-(p-1)m}$ and $m \le \frac{p}{p-1}$. This is due to the following. Since $q_1 < p < q_2$, we have

$$\frac{q_1 m}{q_1-(q_1-1)m} < \frac{pm}{p-(p-1)m} < \frac{q_2 m}{q_2-(q_2-1)m}.$$

We also observe that

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^{q_1}} \le \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}}\|\rho_R\|_{L^1} < \infty,$$

and thus $[(W+W^-)_-]\chi_{B_S(0)} * \rho_R \in L^{q_1} \cap L^{\frac{q_1 m}{q_1-(q_1-1)m}}$. Similarly,

$$\|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)} * \rho_R\|_{L^{q_2}} \le \|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)}\|_{L^{q_2}}\|\rho_R\|_{L^1} < \infty,$$

and so $[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)} * \rho_R \in L^{q_2} \cap L^{\frac{q_2 m}{q_2-(q_2-1)m}}$.

Thus, $[(W+W^-)_-] * \rho_R \in L^r(\mathbb{R}^d)$ with the following range of r:

$$q_2 = \max\{q_1, q_2\} \le r \le \min\left\{\frac{q_1 m}{q_1-(q_1-1)m}, \frac{q_2 m}{q_2-(q_2-1)m}\right\} = \frac{q_1 m}{q_1-(q_1-1)m}.$$

Since q_1 and q_2 are arbitrary with $q_1 < p < q_2$, we get $p < r < \frac{pm}{p-(p-1)m}$.
If $m > \frac{p}{p-1}$, then we already have $\|(W+W^-)_- * \rho_R\|_{L^\infty(\mathbb{R}^d)} \le C_\infty$, and thus $\|\rho_R\|_{L^\infty(\mathbb{R}^d)}$ is uniformly bounded, due to the following. Take $1 < q_1 < p$, so that $m = q_1/(q_1-1)$. This is always possible since $q_1/(q_1-1) > p/(p-1)$. We then have via Young's inequality.

$$\|[(W+W^-)_-]\chi_{B_S(0)} * \rho_R\|_{L^\infty} \le \|[(W+W^-)_-]\chi_{B_S(0)}\|_{L^{q_1}}\|\rho_R\|_{L^m}.$$

Next, we choose m_2 with $1 < m_2 < p/(p-1)$. We note that $\rho_R \in L^{m_2}(\mathbb{R}^d)$, since $m > \frac{p}{p-1}$. Taking $q_2 > p$ so that $m_2 = q_2/(q_2-1)$, we have

$$\|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)} * \rho_R\|_{L^\infty} \le \|[(W+W^-)_-]\chi_{\mathbb{R}^d\setminus B_S(0)}\|_{L^{q_2}}\|\rho_R\|_{L^{m_2}}.$$

Therefore due to (2.3) we obtain uniform boundedness of ρ_R.

Consider the case $\frac{p+1}{p} < m \le \frac{p}{p-1}$. There exists $\delta > 0$ such that $m = \frac{p+1}{p} + \delta$ and, since $\rho_R \in L^1(\mathbb{R}^d)$ and $\varepsilon\rho_R^{m-1} < -\frac{1}{2}(W+W^-)_- * \rho_R - \varepsilon\big(\frac{2}{m}-1\big)\|\rho_R\|_{L^m}^m$, we have that $\|\rho_R\|_{L^s(\mathbb{R}^d)}$ is uniformly bounded for $s = (m-1)r$ and
$pm/\Big(p-(p-1)\frac{p+1}{p}\Big) = mp^2 < r < pm/(p-(p-1)m)$. This can be seen as follows. Using the previous range of r, i.e. $p < r < \frac{pm}{p-(p-1)m}$, in case that $(p+1)/p < m \le p/(p-1)$, we replace m in the denominator by $(p+1)/p$, and then obtain

$$p < \frac{pm}{p-(p-1)\frac{p+1}{p}} = mp^2 < r < \frac{pm}{p-(p-1)m}.$$

We can estimate

$$(m-1)r > (m-1)mp^2 > \Big(\frac{1}{p}+\delta\Big)\Big(1+\frac{1}{p}+\delta\Big)p > \frac{p+1}{p} + (2+p)\delta > \frac{p+1}{p} + 2\delta.$$

Hence, ρ_R is uniformly bounded in $L^{\tilde m}(\mathbb{R}^d)$ where $\tilde m = \frac{p+1}{p} + 2\delta$. Thus ρ_R is uniformly bounded in $L^{\tilde s}(\mathbb{R}^d)$ where $\tilde s = (m-1)\tilde r$ and $\tilde r = \frac{p\tilde m}{p-(p-1)\tilde m}$. In case $\tilde m > \frac{p}{p-1}$, then, as computed earlier, we have $\|(W+W^-)_- * \rho_R\|_{L^\infty(\mathbb{R}^d)} < \infty$. Therefore $\|\rho_R\|_{L^\infty(\mathbb{R}^d)}$ is uniformly bounded due to (2.3). In case $\tilde m \le \frac{p}{p-1}$, we repeat the above calculations obtain that ρ_R is uniformly bounded in $L^{\tilde{\tilde m}}(\mathbb{R}^d)$ with $\tilde{\tilde m} = \frac{p+1}{p} + 3\delta$. With a bootstrapping argument via (2.3), we obtain, after $k > 1/(\delta p(p-1))$ iterations, that $\rho_R \in L^l(\mathbb{R}^d)$, where $l = \frac{p+1}{p} + k\delta > \frac{p}{p-1}$. This implies that ρ_R is bounded.

□

Now, we prove that the mass cannot become too broadly distributed. This result is analogous to [3, Lemma 2.6], but in our case the potential W is not necessarily bounded from below.

Lemma 4. *Let ($\mathcal{H}2$) and (H4) be satisfied. Let R' be as introduced in Lemma 3. Then there exist constants $r = r(W)$ and $c = c(W)$ such that for all $R \geq R'$ global minimizers ρ_R of (1.1) satisfy*

$$\int_{B_r(x_0)} \rho_R(x)dx \geq c > 0 \quad \textit{for all } \; x_0 \in \operatorname{supp} \rho_R \, .$$

Proof. Since $W_- \in L^{p,\infty}(\mathbb{R}^d)$ with $p > \max\left\{1, \frac{1}{m-1}\right\}$, we have
$M^p \left|\left\{x \in \mathbb{R}^d \, : \, W_-(x) > M\right\}\right| \leq C$ for all $M > 0$.
Define $S_M := \{x \in \mathbb{R}^d \, : \, W_-(x) > M\}$ and $S_M^{\complement} := \mathbb{R}^d \backslash S_M$, then $|S_M| \leq CM^{-p}$. Rewrite

$$(W * \rho_R)(x_0) = \int_{S_M} W(y)\rho_R(x_0 - y)\, dy + \int_{S_M^{\complement}} W(y)\rho_R(x_0 - y)\, dy.$$

For every $\mu > 0$ there exists a sufficiently large $M > 0$, such that for some $q < p$ we have

$$\left|\int_{S_M} W_-(y)\rho_R(x_0 - y)\, dy\right| \leq \|\rho_R\|_{L^\infty} \|W_-\|_{L^q(S_M)} |S_M|^{\frac{q-1}{q}} \leq C\mu,$$

since ρ_R is uniformly bounded in L^∞ and $|S_M| \leq CM^{-p}$. For each $A < 0$ there exists an $r > 0$ such that $W_-(x) < -A$ for all $|x| > r$, due to $(H4)$. Therefore

$$\begin{aligned}
\int_{S_M^{\complement}} W(y)\rho_R(x_0 - y)\, dy &= \int_{S_M^{\complement} \cap B_r(0)} W(y)\rho_R(x_0 - y)\, dy + \int_{S_M^{\complement} \backslash B_r(0)} W(y)\rho_R(x_0 - y)\, dy \\
&\geq -M \int_{S_M^{\complement} \cap B_r(0)} \rho_R(x_0 - y)\, dy + A \int_{S_M^{\complement} \backslash B_r(0)} \rho_R(x_0 - y)\, dy \\
&= -M \int_{S_M^{\complement} \cap B_r(x_0)} \rho_R(y)\, dy + A - A \int_{\mathbb{R}^d \backslash \left\{S_M^{\complement} \backslash B_r(0)\right\}} \rho_R(x_0 - y)\, dy \\
&\geq -(M + A) \int_{B_r(x_0)} \rho_R(y)\, dy + A.
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
&2\mathcal{E}(\rho_R) - \int_{\mathbb{R}^d} \varepsilon \left(\frac{2}{m} - 1\right) \rho_R^m(y)\, dy - \varepsilon \rho_R^{m-1}(x_0) \\
&\quad = \frac{(W + W^-)}{2} * \rho_R(x_0) \geq -(M + A) \int_{B_r(x_0)} \rho_R(y)\, dy + A - C\mu.
\end{aligned}$$

Since $\mathcal{E}(\rho_R) < K < 0$, this implies

$$(M + A) \int_{B_r(x_0)} \rho_R(y)\, dy \geq A - C\mu - 2K + \int_{\mathbb{R}^d} \varepsilon \left(\frac{2}{m} - 1\right) \rho_R^m(y)\, dy + \varepsilon \rho_R^{m-1}(x_0).$$

Testing both sides with $\rho_R(x_0)dx_0$, we obtain

$$(M+A)\int_{B_r(x_0)} \rho_R(y)\,dy \geq A - C\mu - 2K + \int_{\mathbb{R}^d} \varepsilon\left(\frac{2}{m}-1\right)\rho_R^m(y)\,dy + \varepsilon\int_{\mathbb{R}^d}\rho_R^m(x_0)\,dx_0$$
$$= A - C\mu - 2K + \frac{2\varepsilon}{m}\int_{\mathbb{R}^d}\rho_R^m(y)\,dy > A - C\mu - 2K.$$

Thus

$$\int_{B_r(x_0)} \rho_R(y)\,dy \geq \frac{A - C\mu - 2K}{M+A} := c > 0,$$

since $|A|$ can be chosen small compared to K and μ is very small. □

Next, we prove that there exists a uniform bound for the distance between two arbitrary disconnected subsets of the support of the minimizers in $B_R(0)$. For $1 < m \leq 2$, we use the strategy in the proof of [16, Th. 3.2]. We do not need any growth assumption on W. For $m > 2$, we use the strategy in [3, Lem. 2.7]. Here we need the growth assumption ($\mathcal{H}6$).

For $1 < m \leq 2$, we consider one of the disconnected parts, rescale its mass to one, and prove that its respective energy is smaller than the minimizing one. In this case the diffusive part of the energy does not grow faster than the interaction part when considering $\alpha\rho$, $\alpha > 1$ instead of ρ in $\mathcal{E}$.
For $m > 2$, this is not the case, and we need a growth condition for W in order to move the disconnected parts of the support together. It would be interesting to see, whether there is another method of proof, which could do without a growth condition like ($\mathcal{H}6$).

Lemma 5. *Let ($\mathcal{H}2$), (H4) hold, and ρ_R be a minimizer of $\mathcal{E}$ in $L^m(\mathbb{R}^d)\cap\mathcal{P}_R(\mathbb{R}^d)$.*

(i) *Let $1 < m \leq 2$ and $R > 0$ large enough such that $\mathcal{E}[\rho_R] < \frac{1}{2}\inf_{\rho\in L^m\cap\mathcal{P}}\mathcal{E}(\rho)$. If $\rho_R = \rho_{R,1}+\rho_{R,2}$ with $\operatorname{supp}\rho_{R,1}$, $\operatorname{supp}\rho_{R,2} \neq \emptyset$, then there exists a constant $D > 0$ such that $\mathrm{dist}(\operatorname{supp}\rho_{\mathrm{R},1}, \operatorname{supp}\rho_{\mathrm{R},2}) < \mathrm{D}$ for all $R > 0$ and for all possible choices of $\rho_{R,1}$ and $\rho_{R,2}$.*

(ii) *Let $m > 2$ and let additionally ($\mathcal{H}6$) hold. Then, for all $R > 0$ each coordinate of the support of ρ_R cannot have gaps larger than $2R_6$, where R_6 is the constant in ($\mathcal{H}6$).*

Proof. $1 < m \leq 2$:
Suppose there exists a splitting such that $\mathrm{dist}(\operatorname{supp}\rho_{\mathrm{R},1}, \operatorname{supp}\rho_{\mathrm{R},2}) > \mathrm{D}$ for some $R > 0$. Define $|\rho_{R,i}| := \int_{\mathbb{R}^d}\rho_{R,i}(x)\,dx$ for $i \in \{1,2\}$. Due to Lemma 4, for $D > 0$ large enough we have $\tilde{m} \leq |\rho_{R,1}|, |\rho_{R,2}| \leq 1-\tilde{m}$ for some $0 < \tilde{m} \leq \frac{1}{2}$. In order to rule out dichotomy, as in the proof of [16, Th. 3.2], we choose $D > 0$ such that

$$W_- < \frac{\tilde{m}}{8(1-\tilde{m})}\left|\inf_{\rho\in L^m\cap\mathcal{P}}\mathcal{E}(\rho)\right| \text{ for all } |x| > D.$$

This is possible due to (H4). Since $\operatorname{supp}\rho_{R,1}\cap\operatorname{supp}\rho_{R,2} = \emptyset$ it holds that

$$\mathcal{E}[\rho_R] \geq \mathcal{E}[\rho_{R,1}] + \mathcal{E}[\rho_{R,2}] - \frac{\tilde{m}}{8(1-\tilde{m})}|\inf\mathcal{E}|.$$

Now we assume w.l.o.g. that $\dfrac{\mathcal{E}[\rho_{R,1}]}{|\rho_{R,1}|} \leq \dfrac{\mathcal{E}[\rho_{R,2}]}{|\rho_{R,2}|} = \dfrac{\mathcal{E}[\rho_{R,2}]}{1-|\rho_{R,1}|}$.

$$\text{Since } \frac{\mathcal{E}[\rho_{R,2}]}{1-|\rho_{R,1}|} \leq \frac{1}{1-|\rho_{R,1}|}\left(\mathcal{E}[\rho_R] + \frac{1}{8}|\inf\mathcal{E}| - \mathcal{E}[\rho_{R,1}]\right),$$

we have

$$\left(\frac{1}{|\rho_{R,1}|} + \frac{1}{1-|\rho_{R,1}|}\right)\mathcal{E}[\rho_{R,1}] \leq \frac{1}{1-|\rho_{R,1}|}\left(\mathcal{E}[\rho_R] + \frac{1}{8}|\inf\mathcal{E}|\right) \text{ . Thus}$$

$$\frac{1}{|\rho_{R,1}|}\mathcal{E}[\rho_{R,1}] \leq \mathcal{E}[\rho_R] + \frac{1}{8}|\inf\mathcal{E}| \leq \frac{1}{2}\inf\mathcal{E} + \frac{1}{8}|\inf\mathcal{E}| \leq \frac{1}{2}\inf\mathcal{E} - \frac{1}{8}\inf\mathcal{E} < \frac{1}{4}\inf\mathcal{E}.$$

Since $1 < m \leq 2$, we obtain a contradiction, because ρ_R is a minimizer and

$$\begin{aligned}
\mathcal{E}\Big[\frac{\rho_{R,1}}{|\rho_{R,1}|}\Big] - \mathcal{E}[\rho_R] &\leq \Big(\frac{1}{|\rho_{R,1}|^2} - 1 - \frac{1-|\rho_{R,1}|}{|\rho_{R,1}|}\Big)\mathcal{E}[\rho_{R,1}] + \frac{\tilde{m}}{8(1-\tilde{m})}|\inf\mathcal{E}| \\
&\leq \frac{1}{|\rho_{R,1}|}\Big(\frac{1}{|\rho_{R,1}|} - 1\Big)\mathcal{E}[\rho_{R,1}] + \frac{|\rho_{R,1}|}{8(1-|\rho_{R,1}|)}|\inf\mathcal{E}| \\
&\leq \Big(\frac{1}{|\rho_{R,1}|} - 1\Big)\Big(\frac{1}{4}\inf\mathcal{E} + \frac{1}{8}|\inf\mathcal{E}|\Big) < 0,
\end{aligned}$$

where we used in the last inequality that $|\rho_{R,1}| < 1$ and $\inf\mathcal{E} < 0$. And also for $m > 2$ we assume, as in [3, Lem. 2.7], that the claim is not true. Consider $H_R \subset R^d$ and $H_L \subset R^d$ with gap in k-direction of at least R_6. Hereby, H_R denotes the "right side", and H_L the "left side", respectively. Assume that the support of some minimizer is split into two parts such that we can write $\rho = \rho|_{H_L} + \rho|_{H_R}$. Use δ_k from ($\mathcal{H}6$) to move $\rho|_{H_R}$ towards $\rho|_{H_L}$, i.e. consider $\rho_\delta(x) := \rho_\delta|_{H_L}(x) + \rho_\delta|_{H_R}(x)$, where

$$\rho_\delta|_{H_L}(x) = \rho|_{H_L}(x), \quad \rho_\delta|_{H_R}(x) = \rho|_{H_R}(x+\delta), \qquad \delta = (0,\dots,\delta_k,\dots,0).$$

Direct computations show that

$$\begin{aligned}
&\mathcal{E}[\rho_{R,l} + \rho_{R,r-\delta}] - \frac{\varepsilon}{m}\int_{\mathbb{R}^d}\rho_\delta^m \\
= \;& \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_L}(y)\rho_{H_L}(x)\,dydx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_R}(y+\delta)\rho_{H_R}(x+\delta)\,dydx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_L}(y)\rho_{H_R}(x+\delta)\,dydx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_R}(y+\delta)\rho_{H_L}(x)\,dydx \\
= \;& \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_L}(y)\rho_{H_L}(x)\,dydx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_R}(y)\rho_{H_R}(x)\,dydx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y-\delta)\rho_{H_L}(y)\rho_{H_R}(x)\,dydx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y+\delta)\rho_{H_R}(y)\rho_{H_L}(x)\,dydx \\
= \;& \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_L}(y)\rho_{H_L}(x)\,dydx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_{H_R}(y)\rho_{H_R}(x)\,dydx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y-\delta) + W(-x+y+\delta)\rho_{H_L}(y)\rho_{H_R}(x)\,dydx \\
< \;& \mathcal{E}[\rho_R] - \frac{\varepsilon}{m}\int_{\mathbb{R}^d}\rho_R^m
\end{aligned}$$

since $x \in H_R$ and $y \in H_L$, i.e. $x_k - y_k \geq R_6$. Due to $\|\rho_\delta\|_{L^m} = \|\rho_R\|_{L^m}$, we obtain $\mathcal{E}[\rho_\delta] < \mathcal{E}[\rho_R]$. Thus we have a contradiction, since ρ_R is a minimizer. $\square$

The above computations also work for the case $\varepsilon = 0$, which is a part of the proof for Theorem 1 (a). There details were omitted.

Proof of (b) and (c) in Theorem 1 Using Lemma 4 and 5 we conclude that there exists a constant $S > 0$ such that the global minimizers ρ_R in $L^m \cap \mathcal{P}$ are identical for all $R > S$, and thus a minimizer belongs to $L^m \cap \mathcal{P}$ (see [3, Lem. 2.10] or [11, Lem. 16]). Hence, there exists a global minimizer of $\mathcal{E}$ in $L^m \cap \mathcal{P}$ which is compactly supported. Since the support of ρ_R is independent of R we can use the analogous result for ρ in Lemma 4 to conclude that every global minimizer of $\mathcal{E}$ in $L^m \cap \mathcal{P}$ with negative energy has compact support (compare [3, Cor. 1.5]). This completes the proof. $\square$

3. Conditions for unstable potentials

Now we provide some conditions for (H4) to be satisfied, i.e. there exists $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{E}[\rho] < 0$. For $m \geq 2$ we extend the approach in [2, Lem. 1].

Lemma 6. *Let $W_- \in L^{p,\infty}(\mathbb{R}^d)$, $W_+ \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} W(x)\,dx < 0$. Suppose either $m > 2$ and $\varepsilon > 0$, or $m = 2$ and $0 < \varepsilon < -\int_{\mathbb{R}^d} W(x)\,dx$. Then, there exists a function $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{E}[\rho] < 0$.*

Proof. Let $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$, then $\rho_\lambda(x) := \lambda^d \rho(\lambda x)$ is also an admissible function for $\lambda > 0$. It holds that

$$\mathcal{E}[\rho_\lambda] = \lambda^d \Big(\lambda^{(md-2d)} \frac{\varepsilon}{m} \|\rho\|_{L^m}^m + \frac{1}{2} \lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W\Big(\frac{x-y}{\lambda}\Big) \rho(y)\rho(x)\,dydx \Big).$$

Splitting up the second term into its positive and negative part, we consider

$$\lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_+\Big(\frac{x-y}{\lambda}\Big) \rho(y)\rho(x)\,dydx - \lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big) \rho(y)\rho(x)\,dydx,$$

with $W = W_+ - W_-$ and $W_-, W_+ \geq 0$. Then

$$\lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_+\Big(\frac{x-y}{\lambda}\Big) \rho(y)\rho(x)\,dydx \leq \|W_+\|_{L^1} \|\rho\|_{L^2}^2 \ , \text{ and}$$

$$\begin{aligned}
&-\lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big) \rho(y)\rho(x)\,dydx \\
&\leq -\lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big) \chi_{B_{\lambda R}(0)}(|x-y|) \rho^2(x)\,dydx \\
&\qquad -\lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big) \chi_{B_{\lambda R}(0)}(|x-y|) \big(\rho(y) - \rho(x)\big) \rho(x)\,dydx \\
&= -\|W_- \chi_{B_R(0)}\|_{L^1} \|\rho\|_{L^2}^2 - \lambda^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_-\Big(\frac{y}{\lambda}\Big) \chi_{B_{\lambda R}(0)}(|y|) \big(\rho(x-y) - \rho(x)\big) \rho(x)\,dydx \\
&= A_1 + A_2.
\end{aligned}$$

Here the first equality follows from Fubini and since ρ does not depend on y.

Following the strategy in the proof of [12, Th. 2.16], and using Hölder's inequality and a change of variables, we obtain

$$\begin{aligned}
A_2 &\leq \|\rho\|_{L^2}\Big[\lambda^{-d}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} W_-\Big(\frac{y}{\lambda}\Big)\chi_{B_{\lambda R}(0)}(|y|)\big|\rho(x-y)-\rho(x)\big|\,dy\Big)^2 dx\Big]^{\frac{1}{2}}\\
&\leq \|\rho\|_{L^2}\Big[\|W_-\chi_{B_R(0)}\|_{L^1}\int_{\mathbb{R}^d} W_-(y)\chi_{B_R(0)}(|y|)\int_{\mathbb{R}^d}\big(\rho(x)-\rho(x-\lambda y)\big)^2\,dxdy\Big]^{\frac{1}{2}}\\
&\leq \|\rho\|_{L^2}\Big[\|W_-\chi_{B_R(0)}\|_{L^1}\int_{\mathbb{R}^d} W_-(y)\chi_{B_R(0)}(|y|)\Big(\int_{\mathbb{R}^d}\big(\rho(x)-\rho(x-\lambda y)\big)\rho(x)\,dxdy\\
&\qquad+\int_{\mathbb{R}^d}\big(\rho(x)-\rho(x+\lambda y)\big)\rho(x)\,dxdy\Big)\Big]^{\frac{1}{2}}.
\end{aligned}$$

Hence, by dominated convergence, this term converges to zero for $\lambda \to 0$. Therefore, for every $\delta > 0$ there exists $1 >> \lambda > 0$ such that for large $R > 0$ we have

$$\begin{aligned}
\mathcal{E}[\rho_\lambda] &\leq \lambda^d\Big(\lambda^{md-2d}\frac{\varepsilon}{m}\|\rho\|_{L^m}^m + \frac{1}{2}\|W_+\|_{L^1}\|\rho\|_{L^2}^2 - \frac{1}{2}\|W_-\chi_{B_R(0)}\|_{L^1}\|\rho\|_{L^2}^2 + \delta\Big)\\
&\leq \lambda^d\Big(\lambda^{md-2d}\frac{\varepsilon}{m}\|\rho\|_{L^m}^m + \frac{1}{2}\int_{\mathbb{R}^d}(W\chi_{B_R(0)})(x)\,dx\|\rho\|_{L^2}^2 + 2\delta\Big).
\end{aligned}$$

□

Remark 6. *For $m = 2$ and $W_- \notin L^1(\mathbb{R}^d)$, the potential W is unstable for all $\varepsilon > 0$.*

Lemma 7. *Let $1 < m < 2$, $p > 1$, and $W_- \in L^{p,\infty}(\mathbb{R}^d)$, $W_+ \in L^{\frac{1}{m-1}}(\mathbb{R}^d)$. Let $q \leq p$. If $m > \frac{q+1}{q}$ and $W_-(x) \geq C|x|^{-\frac{d}{q}}$ for all $|x| > R \geq 0$, then there exists a function $\rho \in L^m(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{E}[\rho] < 0$ for all $\varepsilon > 0$.*

Proof. As in the proof of Lemma 6, consider the rescaled function ρ_λ and split up W into its positive and negative part. Denoting $W_{+,\lambda}(x) = W_+(x/\lambda)$ and using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_+\Big(\frac{x-y}{\lambda}\Big)\rho(y)\rho^{m-1}(x)\rho^{2-m}(x)\,dydx\\
&\leq \|W_+ * \rho\|_{L^{\frac{m}{(m-1)^2}}}\left\|\rho^{m-1}\right\|_{L^{\frac{m}{m-1}}}\left\|\rho^{2-m}\right\|_{L^{\frac{1}{2-m}}}\\
&\leq \|W_{+,\lambda} * \rho\|_{L^{\frac{m}{(m-1)^2}}}\|\rho\|_{L^m}^{m-1}\|\rho\|_{L^1}^{2-m} \leq \|W_{+,\lambda} * \rho\|_{L^{\frac{m}{(m-1)^2}}}\|\rho\|_{L^m}^{m-1}\\
&\leq \lambda^{(m-1)d}\|W_+\|_{L^{\frac{1}{m-1}}}\|\rho\|_{L^m}^m.
\end{aligned}$$

Here we used Young's inequality and a change of variables in the last inequality. Moreover, for $\lambda < 1$ it holds that

$$\begin{aligned}
&-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big)\rho(y)\rho(x)\,dydx\\
&\qquad\leq -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_-\Big(\frac{x-y}{\lambda}\Big)\chi_{\mathbb{R}^d\setminus B_{\lambda R}(0)}(|x-y|)\rho(y)\rho(x)\,dydx\\
&\qquad\leq -\lambda^{\frac{d}{q}}C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{-\frac{d}{q}}\chi_{\mathbb{R}^d\setminus B_R(0)}(|x-y|)\rho(y)\rho(x)\,dydx.
\end{aligned}$$

Therefore, we have

$$\mathcal{E}[\rho_\lambda] \leq \lambda^{(m-1)d}\Big(\frac{\varepsilon}{m}\|\rho\|_{L^m}^m + \frac{1}{2}\|W_+\|_{L^{\frac{1}{m-1}}}\|\rho\|_{L^m}^m\Big) - \lambda^{\frac{d}{q}}\tilde{C}(\rho, R, q).$$

For $\lambda > 0$ sufficiently small this gives our statement, since $(m-1)d > \frac{d}{q}$ and choosing ρ not only being concentrated in $B_R(0)$. This completes the proof. □

Appendix:
Generally it is an open question, whether or not global minimizers are radial, even for radial symmetric potentials. Some specific cases for existence and non-existence of radial minimizers for radial symmetric potentials are given in [8], [5], and [6]. We expect that global minimizers are not radial for non-symmetric potentials W with non-radially symmetric, symmetrized potential $\frac{1}{2}(W + W^-)$, in two and higher dimensions, but do not have a rigorous proof yet. Below, we give a formal example in the limiting case where the potential is non-symmetric and the interaction is local, i.e. a dirac delta.

Let $\mathbf{e}_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$, $d \geq 2$. Consider

$$\mathcal{E}(\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x - y - \mathbf{e}_1)d\rho(x)d\rho(y), \tag{3.1}$$

where $W(x) = -\delta_0$. First note that for

$$\mathcal{E}_0(\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x - y)d\rho(x)d\rho(y),$$

and with $W(x) = -\delta_0$, the minimizer is the dirac mass at the origin (up to translation), i.e. $\rho_{\min} = \delta_0$ and $\mathcal{E}_0(\rho_{\min}) = -1$.

On the other hand, for the above given asymmetric potential $\mathcal{E}$, we look for minimizers that are compactly supported. It turns out that the minimizers are

$$\rho_{\min} = \alpha\delta_{-\mathbf{e}_1} + \frac{1}{2}\delta_0 + \left(\frac{1}{2} - \alpha\right)\delta_{\mathbf{e}_1}, \tag{3.2}$$

where $\alpha \in [0, 1/2]$ (up to translation), and $\mathcal{E}(\rho_{\min}) = -1/8$. Thus minimizers are neither radially symmetric nor unique.

Indeed, if compactly supported, minimizers must be of the form

$$\rho_{\min} = \sum_{j=1}^{k} m_j\delta_{z_0 + j\mathbf{e}_1} \quad , \quad \sum_{j=1}^{k} m_j = 1 \quad , \quad 0 < m_i < 1.$$

Therefore, we have the following minimizing problem

$$\mathcal{E}(\rho_{\min}) = -\sum_{j=1}^{k-1} m_j m_{j+1} \quad , \quad \sum_{j=1}^{k} m_j = 1 \quad , \quad 0 < m_i < 1.$$

This can be reformulated as a maximizing problem in the following way:

$$\text{maximize} \sum_{j=1}^{k-1} m_j m_{j+1} \quad , \quad \sum_{j=1}^{k} m_j = 1, \quad , \quad 0 < m_i < 1.$$

Due to the method of Lagrange multiplers, there exists a constant λ such that

$$\lambda\langle 1, 1, \cdots, 1\rangle = \langle m_2, (m_1 + m_3), (m_2 + m_4), \cdots, (m_{k-2} + m_k), m_{k-1}\rangle.$$

If $k \geq 4$, then $m_2 = \lambda = m_2 + m_4$, therefore $m_4 = 0$, which contradicts our hypothesis that $m_i > 0$. Therefore, $k \leq 3$. The case $k = 2$ implies

$$\lambda\langle 1, 1\rangle = \langle m_2, m_1\rangle \quad \Longrightarrow \quad m_1 = m_2 = \frac{1}{2}. \tag{3.3}$$

This is a special case of (3.2). For $k = 3$, we obtain

$$\lambda\langle 1, 1, 1\rangle = \langle m_2, m_1 + m_3, m_2\rangle \quad \Longrightarrow \quad m_2 = \frac{1}{2},\ m_1 + m_3 = \frac{1}{2}. \tag{3.4}$$

From (3.3) and (3.4), we obtain (3.2).

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