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Probability, valuations, hyperspaces: Three monads on Top and the support as a morphism

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Probability, valuations, hyperspace: Three monads on Top and the support as a morphism

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We consider three monads on Top, the category of topological spaces, which formalize topological aspects of probability and possibility in categorical terms. The first one is the hyperspace monad H, which assigns to every space its space of closed subsets equipped with the lower Vietoris topology. The second is the monad V of continuous valuations, also known as the extended probabilistic powerdomain. Both monads are constructed in terms of double dualization. This not only reveals a strong similarity between them, but also allows us to prove that the operation of taking the support of a continuous valuation is a morphism of monads $V \to H$. In particular, this implies that every H-algebra (topological complete semilattice) is also a V-algebra. Third, we show that V can be restricted to a submonad of τ-smooth probability measures on Top. By composing these two morphisms of monads, we obtain that taking the support of a τ-smooth probability measure is also given by a morphism of monads.

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1. Introduction

In recent decades, aspects of measure and probability theory have been reformulated in categorical terms using the categorical structure of monads in the sense of Eilenberg and Moore [EM65]. All probability monads are variations on the distribution monad on Set (see, for example, [Jac11]), whose underlying functor assigns to a set the set of its finitely supported probability distributions, or equivalently the set of formal finite convex combinations of its elements. Close relatives of the distribution monad are used to treat probability measures in the sense of measure theory. These monads live on suitable categories of spaces with analytic structure, for example the category of measurable spaces, compact Hausdorff spaces, or complete metric spaces. The monad approach has two core features: conditional probabilities, in the sense of Markov kernels, arise as Kleisli morphisms [Gir82]; and it provides a conceptually simple definition of integration or expectation for all algebras of the monad [Per18, Chapter 1].

In this paper, we consider two monads of this type on Top, the category of all topological spaces and continuous maps. Concretely, we develop the monad of continuous valuations V, and the monad of τ-smooth Borel probability measures P, which is a submonad of V. Our treatment of V, and partly also our treatment of P, is largely a review of known material presented in a systematic fashion. Our exposition shows how to exploit duality theory for continuous valuations to obtain a simple description of V, and how to use the embedding of P as a submonad of V to reason about P in similarly simple terms.

In some situations, one may only be interested in whether an event is possible at all rather than in its likelihood or propensity. Computer scientists call this situation nondeterminism. The distinction between possibility and impossibility can be treated via monads which are similar to probability monads. Instead of assigning to every space X the collection of probability measures or valuations of a certain type on X, one now assigns to X the collection of subsets of a certain type, where one can think of a subset as specifying those outcomes which are possible. The simplest monad of this type is arguably the finite powerset monad on Set [Man03, Example 4.18], which assigns to every set the collection of its finite subsets. In this paper, we consider a close relative of this monad on Top, the hyperspace monad H, which assigns to every topological space the space of closed subsets with the lower Vietoris topology. While this is also mostly known, our systematic exposition is of interest insofar as our treatment of H is perfectly analogous to our treatment of V, which suggests that both of these monads are instances of a general construction which remains to be found.

It is elementary to verify that the finite distribution monad and the finite powerset monad are related by a morphism of monads, namely the natural transformation which assigns to a finitely supported probability measure its support, which is the subset of elements that carry nonzero weight. That this transformation is a morphism of monads comprises the statement that the support of a convex combination of finitely supported probability measures is given by the union of the supports of the contributing measures. The main new result of this paper is that these statements generalize: forming the support is a morphism of monads $V \to H$, which takes a continuous valuation on any space X to a closed subset of X. We believe that this is the most general context in which it is meaningful to talk about supports of (unsigned) measures. From the point of view of denotational semantics, our monads model probabilistic and nondeterministic computation. Our morphism $V \to H$ yields a continuous map from a probabilistic powerspace to the possibilistic Hoare powerspace that respects the respective monad structures. This formalizes the passage from probabilistic computation to nondeterministic computation.

In summary, we study the following three monads on Top, the category of topological spaces and continuous maps: The monad H of closed subsets with the lower Vietoris topology (Section 2), the monad V of continuous valuations (Section 3), and the monad P of τ-smooth Borel probability measures (Section 4). The monad H is a version of the Hoare powerdomain [Sch93]. The monad V is also known as the extended probabilistic powerdomain [AJK04]. In contrast to H and V, the monad P has, to the best of our knowledge, not been considered before in this generality. The first part of our work (Sections 2 and 3) mostly contains known results, but our constructions of these monads seem to be novel: we define them through *double dualization*, a common theme in the theory of monads [Luc17]. The idea is that measures, as well as closely related objects, can be seen as dual to functions, which are themselves dual to points. From the point of view of functional analysis, this amounts to the well-known Markov–Riesz duality. From the point of view of theoretical computer science, this is saying that the monads we consider look like submonads of a sort of continuation monad. This duality theory turns out to be of great utility in the construction of the monad structures and the proofs that the monad axioms are satisfied. Our double dual construction, unlike the standard approaches, does not rely on Cartesian closure, since Top is not Cartesian closed. However, it recovers the exponential objects whenever they exists (see Appendix B). This duality theory turns out to be a powerful tool, revealing the structural similarity between the hyperspace monad H and the valuation monad V, and yielding a simple proof of our main result, that the support is a morphism of monads.

Technically, we prove that Scott-continuous modular functionals from a lattice of open sets $\mathcal{O}(X)$ to $\{0, 1\}$ are in canonical bijection with closed sets in X, and that the topology of pointwise convergence for such functionals corresponds to the lower Vietoris topology on the space of closed sets HX (Proposition 2.14). In particular, a closed set C assigns the truth value 1 to an open set U if an only if $C \cap U$ is nonempty. The valuation monad V has a similar duality theory: continuous valuations on a space X are in canonical bijection with Scott-continuous modular functionals on the space of lower semicontinuous functions on X (Theorem 3.6). We define the monad structure of V using this duality and therefore the monad structure of V parallels the one of the hyperspace monad H.

In Section 4, we show that the functor of τ-smooth Borel probability measures is a submonad of V. This seems to be the most general probability monad on topological spaces appearing in the literature, as it is defined on the entire category Top. Its restriction to the subcategory of compact Hausdorff spaces is the Radon monad [Świ74; Kei08b].

In Section 5, we define the support of a continuous valuation, and prove that the operation of taking the support is a morphism of monads from V to H. This operation can be described in the following way. Given a valuation ν, its support is the unique closed set $\mathrm{supp}(\nu)$ such that, for each open set U, the set $\mathrm{supp}(\nu)$ intersects U if and only if $\nu(U)$ is strictly positive. From the possibilistic point of view, an open set U is possible if an only if it has positive probability. In this way, the support induces a map $\mathrm{supp} : VX \to HX$ from valuations to closed subsets. We prove that this map is continuous, natural, and that it respects the two monad structures.

A feature that closed sets, valuations, and measures share is the possibility of forming products and marginals (or projections). This is encoded in the fact that the monads in question are commutative, or, equivalently, symmetric monoidal (see Appendix C). These standard formal constructions yield the familiar notions of products and projections of closed sets, and of product and marginal probability measures.

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2. The hyperspace monad

The powerset monad on the category of sets is among the most elementary examples of monads. It has an analogue on the category of topological spaces, which we study in this section. But its best-known analogue is on metric spaces, where the Hausdorff metric equips the space of nonempty closed subsets of a bounded metric space with a metric, turning it into a metric space in its own right [Hau14]. For a topological space X which may not carry a metric, one version of the hyperspace of X was introduced by Vietoris [Vie22] who equipped the set of closed subsets of X with the *Vietoris topology*. This construction yields an endofunctor of Top which preserves compactness and connectedness. The deep study of hyperspaces by Michael [Mic51] showed that the Vietoris topology, when restricted to the nonempty closed sets, is induced by the Hausdorff metric whenever the base space is a compact metric space. The results of Michael also equip the Vietoris functor on the category of compact spaces with a monad structure.

The Vietoris topology is the minimal common refinement of the *lower Vietoris topology* and the *upper Vietoris topology*. The functor that assigns to a topological space the set of its closed subsets with the lower Vietoris topology has been introduced by Smyth [Smy83]. That this endofunctor has a monad structure has been shown by Schalk [Sch93] under the almost inconsequential restriction to the category of T_0 spaces. She also studied the algebras of this monad [Sch93]. Several related topological results are due to Clementino and Tholen [CT97].

In this section, we discuss this functor, which assigns to a topological space the space of its closed subsets equipped with the lower Vietoris topology, as well as its monad structure. This relies on a duality between this hyperspace HX and certain maps from the open subsets $\mathcal{O}(X)$ to the Sierpiński space. In Section 5, we show that this hyperspace creates a harmonious connection between probabilistic and possibilistic constructions: There is a morphism of monads to the hyperspace monad which takes every continuous valuation, and in particular every τ-smooth Borel probability measure, to its support.

In working with hyperspaces, there is a choice to make concerning the membership of the empty set. This choice is relatively inconsequential, in the sense that most results hold either way. While most of the works mentioned above have excluded the empty set, we will include it to make the analogy with the extended probabilistic powerdomain of Section 3 as close as possible.

2.1. Hit and miss

The following definitions are well-established [CT97, Section 1.1].

Definition 2.1. *Let X be a topological space and consider a subset $A \subseteq X$. We say that A hits an open set $U \subseteq X$ if and only if $A \cap U \neq \varnothing$. We denote by $\mathrm{Hit}(U)$ the collection of closed subsets of X which hit U,*

$$\mathrm{Hit}(U) = \{C \subseteq X : \mathrm{cl}(C) = C \text{ and } C \cap U \neq \varnothing\}.$$

In particular, C hits the complement of U if and only if it is not a subset of U. The concept of hitting will allow us to think of closed sets as functionals on the open sets (Proposition 2.10).

Lemma 2.2. *Let X be a topological space and consider $A, U \subseteq X$ where U is open. Then U and A are disjoint if and only if U and $\mathrm{cl}(A)$ are disjoint. In other words, A hits U if and only if $\mathrm{cl}(A)$ hits U.*

Proof. Suppose that $x \in U \cap \mathrm{cl}(A)$. Then, by definition of closure, every open neighborhood of x contains a point of A. In particular, U is such a neighborhood. The other direction is trivial. □

Lemma 2.3. *Let X be a topological space and let $C, D \subseteq X$ be closed sets. Then $C = D$ if and only if they hit the same open sets.*

Proof. Suppose that there exists a point $x \in X$ such that $x \in C$ and $x \notin D$. Then the complement of D is an open set hit by C (at x), but not by D. The other direction is trivial. □

We have shown that closed sets are uniquely determined by the open sets that they hit, and that, through being hit, open sets may detect other sets only up to their closure. This is the basis behind the duality theory of Section 2.3.

2.2. Topology on the hyperspace

The following definition is well-established [CT97, Section 1.2].

Definition 2.4 (Hyperspace)**.** *Let X be a topological space. The hyperspace over X, denoted by HX, is the space whose points are the closed subsets of X, including the empty set. HX is equipped with the lower Vietoris topology, the topology on HX generated by subbasic sets of the form*

$$\big\{ \operatorname{Hit}(U) : U \subseteq X \text{ is open} \big\}.$$

The set of closed subsets of a topological space is more commonly equipped with the full Vietoris topology [Vie22]. Intuitively, the lower Vietoris topology relates to the full Vietoris topology as the topology of lower semicontinuity on $\mathbb{R}$, with generating open sets of the form (a, ∞), relates to the usual topology of $\mathbb{R}$.

Proposition 2.5. *Let X be a topological space. Then the specialization preorder on HX is the order of set inclusion: given $C, D \in HX$, we have $C \in \operatorname{cl}(\{D\})$ if and only if $C \subseteq D$. Equivalently, $C \subseteq D$ if and only if*

$$C \in \operatorname{Hit}(U) \implies D \in \operatorname{Hit}(U)$$

for every open set $U \subseteq X$.

Note the difference between $\operatorname{cl}(\{D\})$, the closure of the singleton $\{D\}$ in HX, and $\operatorname{cl}(D)$, the closure of D in X.

Proof of Proposition 2.5. We have $C \in \operatorname{cl}(\{D\})$ if and only if, for every open $U \subseteq X$ such that $C \in \operatorname{Hit}(U)$, we have $D \in \operatorname{Hit}(U)$ as well. Suppose that $C \subseteq D$. If C hits U, then D also hits U. Conversely, suppose that $C \nsubseteq D$. Then C hits the open set $X \setminus D$, but D does not. □

Corollary 2.6. *For any topological space X, its hyperspace HX has the T_0 property.*

Proof. The T_0 property is equivalent to the antisymmetry of the specialization preorder. The order of set inclusion is antisymmetric. □

Since the point $X \in HX$ is contained in every nonempty open set, it is dense in HX.

The following alternative description of the topology is well-known [Hof79, Example 2.3(a)]. We denote by $\downarrow\{C\}$ the principal downset generated by C, the set of closed subsets of C.

Lemma 2.7. *The lower Vietoris topology on HX is generated by complements of sets of the form $\downarrow\{C\}$ for $C \in HX$.*

Proof. The complement of $\downarrow\{C\}$ contains precisely the sets which hit the open set $X \setminus C$. ☐

Thus the lower Vietoris topology is determined by the inclusion order of closed sets, and coincides with the *lower topology* on the set of closed sets, also known as the *weak order topology* [Sch93]. It is the coarsest topology on HX such that, for each $C \in HX$, the closure $\mathrm{cl}(\{C\})$ is the downset $\downarrow\{C\}$, the set of closed subsets of C. Therefore the spaces HX that we consider are the same as the *Hoare powerdomains* (defined, for example, in [Sch93, Section 6.3], with the minor difference that she excludes the empty set).

Remark 2.8. For every topological space X the space HX is sober [Sch93, Proposition 1.7].

2.3. Duality theory

We will show that the closed sets of a topological space X can be identified with join-preserving maps $\mathcal{O}(X) \to \{0,1\}$, where $\mathcal{O}(X)$ is the lattice of open subsets of X. Since $\mathcal{O}(X)$ can itself be identified with the continuous maps from X into the Sierpiński space, the set of closed sets arises from a double-dualization construction [Luc17]. By analogy with functional analysis, we will call certain maps "functionals" to indicate that their domain is a space of functions and that their codomain is a fixed dualizing object.

A map between complete lattices, say $f : (L, \le) \to (K, \le)$, is *join-preserving* if $f(\bigvee A) = \bigvee f(A)$ for all $A \subseteq L$. Every join-preserving map is monotone.

Let $C \subseteq X$ be closed. We assign to C the functional $\Phi_C : \mathcal{O}(X) \to \{0,1\}$ defined by

$$\Phi_C(U) \;\coloneqq\; \begin{cases} 1 & \text{if } C \cap U \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Lemma 2.9. *The functional Φ_C is join-preserving.*

Proof. C hits a union of open sets if and only if it hits some set in that union. ☐

Conversely, let $\phi : \mathcal{O}(X) \to \{0,1\}$ be a join-preserving map. Then the preimage $\phi^{-1}(0)$ contains its supremum, the union of all open sets which ϕ maps to zero. We call the *support* of ϕ the complement of this maximal open set,

$$\mathrm{supp}(\phi) \;\coloneqq\; X \setminus \Big(\bigvee \phi^{-1}(0)\Big). \tag{2.2}$$

Proposition 2.10. *Let X be a topological space. The assignments $C \mapsto \Phi_C$ and $\phi \mapsto$ $\operatorname{supp}(\phi)$ given by* (2.1) *and* (2.2) *are inverse bijections, implementing an isomorphism of complete lattices between the closed subsets, ordered by inclusion, and join-preserving functionals $\mathcal{O}(X) \to \{0,1\}$, ordered pointwise.*

Proof. It is enough to show that $C \mapsto \Phi_C$ and $\phi \mapsto \operatorname{supp}(\phi)$ are inverse to each other, as both are obviously monotone.

We first show $\operatorname{supp}(\Phi_C) = C$ for every closed set C. By definition,

$$\operatorname{supp}(\Phi_C) \;=\; \Big(\bigcup \{U : \Phi_C(U) = 0\}\Big)^c \;=\; \bigcap \{U^c : C \cap U = \varnothing\}.$$

Hence $\operatorname{supp}(\Phi_C)$ is the intersection of all closed sets which contain C, and this is clearly C itself.

We now show $\Phi_{\operatorname{supp}(\phi)} = \phi$ for a join-preserving $\phi : \mathcal{O}(X) \to \{0,1\}$ by proving that $\Phi_{\operatorname{supp}(\phi)}(U) = 0$ for some open set U if and only if $\phi(U) = 0$. The former condition is equivalent to

$$\Big(\bigcup \{V : \phi(V) = 0\}\Big)^c \cap U \;=\; \varnothing,$$

or yet equivalently

$$U \;\subseteq\; \bigcup \{V : \phi(V) = 0\}, \tag{2.3}$$

and hence equivalent to $\phi(U) = 0$, since ϕ is join-preserving. Therefore $\Phi_{\operatorname{supp}(\phi)} = \phi$. ☐

Proposition 2.10 can be used to establish the well-known duality between points of a sober space and principal prime ideals in its frame of open sets [Joh82, Section II.1]. Concretely, the functionals $\mathcal{O}(X) \to \{0,1\}$ which additionally preserve finite meets correspond to irreducible closed subsets. In a sober space, these are exactly the closures of unique points.

We turn to the double-dualization aspect of Proposition 2.10. Consider the following analogy. By the Markov-Riesz representation theorem, measures on a compact Hausdorff space X can be identified with positive linear real-valued functionals on the space of real-valued continuous functions on X. Here we have an analogous phenomenon: closed subsets can be identified with structure-preserving functionals $\mathcal{O}(X) \to \{0,1\}$, while the lattice $\mathcal{O}(X)$ itself plays the role of functions into $\{0,1\}$. We make this precise by using the following well-known topology.

Notation 2.11 (Sierpiński space)**.** *The Sierpiński space S is the set $\{0,1\}$ equipped with the topology $\{\varnothing, \{1\}, \{0,1\}\}$.*

By identifying an open set with its indicator function, we identify the open subsets of a space X with the continuous maps $X \to S$, hence naturally $\mathcal{O}(X) \cong \mathsf{Top}(X, S)$. A useful fact is that the topology on S is the Scott topology with respect to the partial order where $0 \leq 1$.

Definition 2.12. *Let X be a topological space. We equip $\mathcal{O}(X)$ with the Scott topology with respect to the inclusion order, and equivalently* $\mathsf{Top}(X, S)$ *with the Scott topology with respect to the pointwise order on functions.*

Remark 2.13. The frame of open sets $\mathcal{O}(X)$ may also be equipped with the topology of pointwise convergence of the function space S^X, for which a subbase consists of the sets

$$U_x \coloneqq \{U \subseteq X : x \in U\}$$

for $x \in X$. The Scott topology on $\mathcal{O}(X)$ is finer than this topology, often strictly so.

As we recall in Appendix B, the Scott topology coincides with the topology of the exponential object S^X, whenever X is exponentiable.

We now rephrase Proposition 2.10, clarifying how S implements a duality between closed sets and open sets and phrasing it in a way which makes the analogy with continuous valuations (Definition 3.1) explicit.

Proposition 2.14. *Let X be a topological space. The assignments $C \mapsto \Phi_C$ and $\phi \mapsto \mathrm{supp}(\phi)$ given by* (2.1) *and* (2.2) *implement an isomorphism of complete lattices between HX and continuous functionals $\phi : \mathcal{O}(X) \to S$ that satisfy the following two requirements.*

(a) Strictness: $\phi(\varnothing) = 0$.

(b) Modularity: For all $U, V \in \mathcal{O}(X)$ we have $\phi(U \cap V) \vee \phi(U \cup V) = \phi(U) \vee \phi(V)$.

Strictness and modularity are equivalent to the preservation of the empty join and binary joins, respectively. While modularity is equivalent to $\phi(U \cup V) = \phi(U) \vee \phi(V)$ by monotonicity of ϕ (which follows from Scott continuity), we write it in this form to emphasize the analogy with the modularity condition in the definition of continuous valuation (Definition 3.1).

Proof of Proposition 2.14. In light of Proposition 2.10, it is enough to show that the stated conditions on ϕ are equivalent to the join-preservation of Proposition 2.10. Since these conditions amount precisely to the preservation of directed and finite joins, this follows from the well-known equivalence between the preservation of arbitrary joins and the preservation of directed and finite joins. □

Not only are the closed sets continuous functionals on open sets, but the topology of HX, the lower Vietoris topology, is the weak topology of closed sets as functionals.

Proposition 2.15. *Let X be a topological space. The lower Vietoris topology on HX is the initial topology with respect to the family of functionals $HX \to S$ given by $C \mapsto \Phi_C(U)$ for open sets $U \subseteq X$.*

Proof. The only nontrivial open subset of S is $\{1\}$. Its preimage under the map $C \mapsto \Phi_C(U)$ is $\mathrm{Hit}(U)$, and these are precisely the subbasic open sets of HX as U ranges over all open sets. $\square$

In summary, we have the following theorem.

Theorem 2.16. *For any topological space X, its hyperspace HX is homeomorphic to the space of Scott continuous functionals $\mathsf{Top}(X, S) \to S$ that satisfy strictness and modularity, equipped with the weak topology.*

Motivated by this duality result and the similarity with duality pairings in functional analysis, we introduce the following coupling notation to simplify proofs in the remainder of this work.

Notation 2.17 (Truth value and coupling notation)**.** *Let X be a topological space. Consider $x \in X$ and an open subset $U \subseteq X$. We write*

$$[\![x \in U]\!] \coloneqq \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Let $C \subseteq X$ be closed. We write

$$\langle C, U \rangle \;\coloneqq\; [\![C \in \mathrm{Hit}(U)]\!] \;=\; \bigvee_{x \in C} [\![x \in U]\!] \;=\; \begin{cases} 1 & \text{if } C \text{ hits } U \\ 0 & \text{otherwise.} \end{cases}$$

2.4. Functoriality

Definition 2.18. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. For any $C \in HX$, we define*

$$f_\sharp(C) \coloneqq \mathrm{cl}\big(f(C)\big),$$

the closure of the image of C under f.

The fact that the closed set $f_\sharp(C)$ is characterized by the open sets that it hits yields a duality formula which expresses $f_\sharp$ as adjoint to $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ with respect to the duality pairing.

Lemma 2.19. *For every open set $U \subseteq Y$,*

$$\langle f_\sharp(C), U \rangle \;=\; \big\langle C, f^{-1}(U) \big\rangle.$$

Proof. By Lemma 2.2, it is enough to note that $f(C)$ hits U if and only if C hits $f^{-1}(U)$. $\square$

We could have used this characterization to define $f_\sharp(C)$. Since taking the preimage preserves unions, we can apply Proposition 2.10 to see that the functional $U \longmapsto \langle C, f^{-1}(U)\rangle$ uniquely specifies a closed set.

This adjointness immediately implies the following.

Corollary 2.20. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then, for every open $U \subseteq Y$,*

$$f_\sharp^{-1}(\mathrm{Hit}(U)) = \mathrm{Hit}(f^{-1}(U)).$$

In particular, the map $f_\sharp : HX \to HY$ is continuous.

We now treat the functoriality of the assignment $f \mapsto f_\sharp$.

Corollary 2.21. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$.*

Proof. The composition of two adjoints of two monotone maps is the adjoint of their composition. Let $C \in HX$ and let $U \subseteq Z$ be open. Then

$$\begin{aligned}
\langle (g\circ f)_\sharp(C), U\rangle &= \big\langle C, (g\circ f)^{-1}(U)\big\rangle \\
&= \big\langle C, f^{-1}(g^{-1}(U))\big\rangle \\
&= \big\langle f_\sharp(C), g^{-1}(U)\big\rangle \\
&= \langle g_\sharp(f_\sharp(C)), U\rangle,
\end{aligned}$$

which implies the claim. □

We have obtained a functor $H : \mathsf{Top} \to \mathsf{Top}$ which assigns to each topological space X its hyperspace HX, and to each continuous map $f : X \to Y$ the continuous map $f_\sharp : HX \to HY$. We call H the *hyperspace functor*. In fact, H is also a 2-functor in the sense of Appendix A.

Lemma 2.22. *Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps with $f \le g$. Then $f_\sharp \le g_\sharp$. Hence H preserves 2-cells, making H a 2-functor.*

Proof. By Lemma A.3, for every $U \subseteq Y$, we have $f^{-1}(U) \subseteq g^{-1}(U)$. Therefore

$$f_\sharp^{-1}(\mathrm{Hit}(U)) \;=\; \mathrm{Hit}(f^{-1}(U)) \;\subseteq\; \mathrm{Hit}(g^{-1}(U)) \;=\; g_\sharp^{-1}(\mathrm{Hit}(U)).$$

By Lemma A.3 this means that $f_\sharp \le g_\sharp$. □

2.5. Monad structure

We equip the functor H with a monad structure, making it into a topological analog of the covariant powerset monad. Our arguments make crucial use of the duality theory developed in the previous subsection.

2.5.1. Unit

Definition 2.23. *Let X be a topological space. The map $\sigma : X \to HX$ maps points to their topological closure, $x \mapsto \mathrm{cl}(\{x\})$.*

The unit σ may be characterized, or alternatively defined, in terms of hitting.

Lemma 2.24. *For $x \in X$ and every open set $U \subseteq X$,*

$$\langle \sigma(x), U \rangle \;=\; [\![x \in U]\!].$$

The proof is obvious given Lemma 2.2.

Corollary 2.25. *For an open set $U \subseteq X$, we have $\sigma^{-1}(\mathrm{Hit}(U)) = U$. In particular, the map $\sigma : X \to HX$ is continuous.*

Hence σ is a morphism of Top. We now turn to naturality.

Proposition 2.26. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ HX & \xrightarrow{f_\sharp} & HY \end{array}$$

Proof. For $x \in X$ and open $U \subseteq Y$,

$$\begin{aligned} \langle f_\sharp(\sigma(x)), U \rangle \;&=\; \big\langle \sigma(x), f^{-1}(U) \big\rangle \\ &=\; [\![x \in f^{-1}(U)]\!] \\ &=\; [\![f(x) \in U]\!] \\ &=\; \langle \sigma(f(x)), U \rangle. \end{aligned}$$ ◻

Hence we have a natural transformation $\sigma : \mathrm{id} \Rightarrow H$ between endofunctors of Top.

2.5.2. Topological properties of the unit map

We state conditions under which $\sigma : X \to HX$ is a (closed) embedding.

Proposition 2.27. *Let X be a topological space and consider the map $\sigma : X \to HX$. Then the map σ is a subspace embedding if and only if X is T_0.*

Proof. The map σ is injective if and only if no distinct points of X have the same closure, hence if and only if X is T_0. By Corollary 2.25, the topology of X is the initial topology with respect to $\sigma : X \to HX$. Therefore σ is a homeomorphism onto its image if and only if σ is injective. ◻

A similar statement is true even when X is not T_0.

Proposition 2.28. *Let X be a topological space and consider the map $\sigma : X \to HX$. Then σ induces an equivalence in the sense of the 2-categorical structure of Appendix A between X and its image $\sigma(X)$.*

Proof. We know that $\sigma^{-1} : \mathcal{O}(\sigma(X)) \to \mathcal{O}(X)$ is an isomorphism of lattices of open subsets (Corollary 2.25). By Lemma A.4, it is enough to show that every point of $\sigma(X)$ is equivalent to a point in the image of σ, which is clearly the case. □

We state conditions for σ to be a closed embedding. It is in general not true that $\sigma(X)$ is closed in HX, not even if X is T_1 or sober.

Example 2.29. Let X be any topological space in which finite intersections of nonempty open sets are nonempty. Then, since the basic open sets in HX are of the form $\mathrm{Hit}(U_1) \cap \ldots \cap \mathrm{Hit}(U_n)$, and every such set contains $\mathrm{Hit}(U_1 \cap \ldots \cap U_n)$, it follows that every nonempty open set in HX contains the closure of a singleton. Therefore $\sigma(X)$ is dense in HX.

More concretely, this happens whenever X has a dense point $\top$, which is equivalently a greatest element in the specialization preorder. Every space $X = HY$ for any Y is a sober space which has this property, with dense point $Y \in HX$. As a concrete example of a T_1 space with the above finite intersection property of nonempty open subsets, take any infinite set equipped with the cofinite topology.

Proposition 2.30. *Let X be a topological space and consider a closed set $C \subseteq X$. Then the following two conditions are equivalent.*

(a) C is in the closure of $\sigma(X)$.

(b) For every finite collection $U_1, \ldots, U_n$ of open subsets of X hit by C, the intersection $U_1 \cap \cdots \cap U_n$ is nonempty (but not necessarily hit by C).

Proof. C is in the closure of $\sigma(X)$ if and only if, for every open $V \subseteq HX$ containing C, the set $\sigma_\sharp(X)$ hits V. More formally, this condition means that for all open $V \subseteq HX$, we must have

$$[\![C \in V]\!] \overset{!}{\leq} \langle \sigma_\sharp(X), V \rangle = \big\langle X, \sigma^{-1}(V) \big\rangle.$$

This holds if and only if it holds for all basic open sets V, which are the sets of the form

$$V = \mathrm{Hit}(U_1) \cap \cdots \cap \mathrm{Hit}(U_n),$$

for open sets $U_1, \ldots, U_n \subseteq X$. In other words, it holds if and only if, for all open sets $U_1, \ldots, U_n \subseteq X$, we have

$$[\![C \in \mathrm{Hit}(U_1) \cap \cdots \cap \mathrm{Hit}(U_n)]\!] \leq \big\langle X, \sigma^{-1}(\mathrm{Hit}(U_1) \cap \cdots \cap \mathrm{Hit}(U_n)) \big\rangle,$$

or equivalently

$$\bigwedge_i \langle C, U_i \rangle \;\leq\; \langle X, U_1 \cap \cdots \cap U_n \rangle,$$

where on the right-hand side we have applied Corollary 2.25. Since X is the entire space, X hits a set if and only if it is nonempty. Therefore the inequality above states that, if C hits all of the U_i, then the intersection of all the U_i has to be nonempty. ☐

Corollary 2.31. *Let X be a Hausdorff space. Then $\sigma(X)$ is closed in HX.*

Proof. If C contains at least two different points, then C hits disjoint open neighborhoods U_1 and U_2 which separate these points. ☐

2.5.3. Multiplication

Definition 2.32. *Let X be a topological space. The map $\mathcal{U} : HHX \to HX$ assigns to each set of closed sets the closure of their union,*

$$\mathcal{U}(\mathcal{C}) \;:=\; \mathrm{cl}\Big(\bigcup \mathcal{C}\Big) \;=\; \mathrm{cl}\Bigg(\bigcup_{C \in \mathcal{C}} C\Bigg).$$

We characterize $\mathcal{U}$ in terms of hitting, this could also be an alternative definition.

Lemma 2.33. *Let X be a topological space and consider $\mathcal{C} \in HHX$. Then, for every open set $U \subseteq X$,*

$$\langle \mathcal{U}\mathcal{C}, U \rangle \;=\; \bigvee_{C \in \mathcal{C}} \langle C, U \rangle = \langle \mathcal{C}, \mathrm{Hit}(U) \rangle.$$

Proof. The first equation follows from Lemma 2.2 and the fact that a union of sets hits another set if and only if some member hits the other set. The second equation holds because $\mathcal{C}$ hits $\mathrm{Hit}(U)$ if and only if some member $C \in \mathcal{C}$ is in $\mathrm{Hit}(U)$. ☐

Corollary 2.34. *Let X be a topological space and let $U \subseteq X$ be open. Then*

$$\mathcal{U}^{-1}(\mathrm{Hit}(U)) \;=\; \mathrm{Hit}(\mathrm{Hit}(U)).$$

In particular, the map $\mathcal{U} : HHX \to HX$ is continuous.

Therefore $\mathcal{U}$ is a morphism of Top. We turn to naturality.

Proposition 2.35. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then the following diagram commutes.*

$$\begin{array}{ccc} HHX & \xrightarrow{f_{\sharp\sharp}} & HHY \\ \big\downarrow{\scriptstyle \mathcal{U}} & & \big\downarrow{\scriptstyle \mathcal{U}} \\ HX & \xrightarrow{f_\sharp} & HY \end{array}$$

Hence the map $\mathcal{U}$ is natural. In discrete spaces, this amounts to the statement that taking images commutes with unions.

Proof. Let $\mathcal{C} \in HHX$ and let $U \subseteq Y$ be open. Using Corollary 2.20 and Lemma 2.19, we get

$$\begin{aligned}
\langle \mathcal{U}(f_{\sharp\sharp}(\mathcal{C})), U\rangle &= \langle f_{\sharp\sharp}(\mathcal{C}), \mathrm{Hit}(U)\rangle \\
&= \left\langle \mathcal{C}, {f_\sharp}^{-1}(\mathrm{Hit}(U))\right\rangle \\
&= \left\langle \mathcal{C}, \mathrm{Hit}(f^{-1}(U))\right\rangle \\
&= \left\langle \mathcal{U}(\mathcal{C}), f^{-1}(U)\right\rangle \\
&= \langle f_\sharp(\mathcal{U}(\mathcal{C})), U\rangle.
\end{aligned}$$

□

2.5.4. Monad axioms

Proposition 2.36. *Let X be a topological space. Then the following three diagrams commute.*

$$\begin{array}{ccc}
HX & \xrightarrow{\sigma} & HHX \\
& \searrow\!\!\!= & \downarrow \mathcal{U} \\
& & HX
\end{array}
\qquad
\begin{array}{ccc}
HX & \xrightarrow{\sigma_\sharp} & HHX \\
& \searrow\!\!\!= & \downarrow \mathcal{U} \\
& & HX
\end{array}
\qquad
\begin{array}{ccc}
HHHX & \xrightarrow{\mathcal{U}_\sharp} & HHX \\
\downarrow \mathcal{U} & & \downarrow \mathcal{U} \\
HHX & \xrightarrow{\mathcal{U}} & HX
\end{array}
\tag{2.4}$$

These diagrams are the topological analogues of basic facts of set theory. For sets, the first two unitality diagrams state that the union over a singleton set of sets is the set itself, and that the union of singletons is the set whose elements are the respective singletons. The associativity diagram states that taking unions is an associative operation.

Proof of Proposition 2.36. We start with left unitality. Let $C \in HX$ and let $U \subseteq X$ be open. Then

$$\langle \mathcal{U}(\sigma(C)), U\rangle \;=\; \langle \sigma(C), \mathrm{Hit}(U)\rangle \;=\; [\![C \in \mathrm{Hit}(U)]\!] \;=\; \langle C, U\rangle.$$

We turn to right unitality, which works similarly,

$$\langle \mathcal{U}(\sigma_\sharp(C)), U\rangle \;=\; \langle \sigma_\sharp(C), \mathrm{Hit}(U)\rangle \;=\; \left\langle C, \sigma^{-1}(\mathrm{Hit}(U))\right\rangle \;=\; \langle C, U\rangle,$$

since $\sigma^{-1}(\mathrm{Hit}(U)) = U$ by Corollary 2.25.

It remains to consider the associativity diagram. Let $\mathcal{K} \in HHHX$ and let $U \subseteq X$ be open. Then

$$\begin{aligned}
\langle \mathcal{U}(\mathcal{U}_\sharp(\mathcal{K})), U\rangle &= \langle \mathcal{U}_\sharp(\mathcal{K}), \mathrm{Hit}(U)\rangle \\
&= \left\langle \mathcal{K}, \mathcal{U}^{-1}(\mathrm{Hit}(U))\right\rangle
\end{aligned}$$

$$= \langle \mathcal{K}, \mathrm{Hit}(\mathrm{Hit}(U)) \rangle$$

$$= \langle \mathcal{U}(\mathcal{K}), \mathrm{Hit}(U) \rangle$$

$$= \langle \mathcal{U}(\mathcal{U}(\mathcal{K})), U \rangle,$$

since $\mathcal{U}^{-1}(\mathrm{Hit}(U)) = \mathrm{Hit}(\mathrm{Hit}(U))$ by Corollary 2.34. □

We have proven the following statement.

Theorem 2.37. *The triple* $(H, \sigma, \mathcal{U})$ *is a monad on* Top.

We will call $(H, \sigma, \mathcal{U})$, or just H, the *hyperspace monad.* By Lemma 2.22, H is a strict 2-monad for the 2-categorical structure of Top given in Appendix A.

As far as we know, this monad was introduced by Schalk [Sch93, Section 6.3.1], where most of the explicit work is however done for a slightly different monad, the one of nonempty closed subsets of a given space. For example, Proposition 6.7 therein gives the associated adjunction between Top and algebras of the monad.

2.6. Algebras

There is a characterization of the Eilenberg-Moore algebras of the monad H, which is, as far as we know, also due to Schalk [Sch93, Section 6.3.1]. We review the main results. An additional reference, which also discusses related constructions, is Hoffmann's earlier article [Hof79].

It is well-known that the algebras of the powerset monad on Set are the complete join-semilattices. The H-algebras are their topological cousins. An H-algebra is, by definition, a pair (A, a) consisting of a topological space A and a continuous map $a : HA \to A$ such that the following two diagrams commute.

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & HA \\ & \searrow^{=} & \downarrow a \\ & & A \end{array} \qquad \begin{array}{ccc} HHA & \xrightarrow{a_\sharp} & HA \\ \downarrow \mathcal{U} & & \downarrow a \\ HA & \xrightarrow{a} & A \end{array} \tag{2.5}$$

We refer to these diagrams as the unit diagram and the algebra diagram. Every H-algebra A is a T_0 space. To see this, let $x, y \in A$ with $\mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\})$. By the unit triangle of (2.5), we conclude

$$x = a\big(\sigma(x)\big) = a\big(\mathrm{cl}(\{x\})\big) = a\big(\mathrm{cl}(\{y\})\big) = a\big(\sigma(y)\big) = y,$$

which implies that A is T_0. Therefore the algebras of H coincide with the algebras of the restriction of H to the full subcategory of T_0 spaces. (Non-T_0 spaces can still be pseudoalgebras, if we consider H as a 2-monad on a 2-category.)

Definition 2.38 (Topological complete join-semilattice)**.** *A topological complete join-semilattice is a complete lattice L equipped with a sober topology whose specialization preorder coincides with the lattice order and whose join map $\vee : L \times L \to L$ is continuous.*

Since the lattice structure on such a lattice is completely determined by its topology, we may consider these structures as a particular class of topological spaces. Hoffmann [Hof79] calls them *essentially complete T_0 spaces* and also *T_0 topological complete sup-semilattices*, while Schalk [Sch93] calls them *unital inflationary topological semilattices.* These spaces admit several equivalent characterizations [Hof79, Theorem 1.8]. They are precisely the H-algebras.

Theorem 2.39 (Schalk)**.** *The category of H-algebras is equivalent to the subcategory of* Top *whose objects are topological complete join-semilattices, with algebra maps given by the lattice join, and with join-preserving continuous maps as morphisms of algebras.*

This result has been claimed by Hoffmann [Hof79, Theorem 2.6] in the form of a monadicity statement, but he does not seem to state a proof. As far as we know, the proof is essentially due to Schalk [Sch93, Section 6.3], culminating in Theorems 6.9 and 6.10 therein, which state this characterization in the full subcategory of sober spaces. This is not a substantial restriction, since every HA is sober (Remark 2.8), so that the unit diagram in (2.5) makes every algebra A into a retract of a sober space and therefore itself sober. In this way, Schalk's result extends to all of Top, which is why we credit the general result to her.

The monad H, as its metric or Lawvere metric counterpart [ACT10], becomes a Kock-Zöberlein monad [Koc95; Zöb76] upon considering it as a 2-monad on a strict 2-category. This means that whenever a topological space admits an H-algebra structure, then this structure is unique up to isomorphism. This phenomenon is a property-like structure [KL97]. Not every morphism of Top between H-algebras is a morphism of H-algebras, since a continuous map need not preserve joins.

We now present a proof of Theorem 2.39 which is more direct than Schalk's, starting with some auxiliary results.

Lemma 2.40. *Let (A, a) be an H-algebra. Then $a : HA \to A$ is the join of closed sets induced by the specialization preorder of A.*

Proof. The proof only uses the fact that a is a retract of $\sigma : A \to HA$. Let $C \in HA$. Since a is continuous, it is monotone for the specialization preorder. For every $x \in C$ we have $\sigma(x) \subseteq C$ and therefore, by the unit condition for algebras,

$$x \;=\; a(\sigma(x)) \;\leq\; a(C).$$

Hence $a(C)$ is an upper bound for C. Conversely, let u be any upper bound for C. Then $C \subseteq \sigma(u)$, which implies

$$a(C) \;\leq\; a(\sigma(u)) = u.$$

Therefore $a(C)$ is a least upper bound for C, as was to be shown. ☐

The following lemma is immediate from the fact that closed sets are downsets in the specialization preorder.

Lemma 2.41 (Lemma 1.5 in [Sch93])**.** *Let X be a topological space. Then a subset $S \subseteq X$ admits a supremum in the specialization preorder if and only if* $\mathrm{cl}(S)$ *does, in which case they coincide.*

The following lemma is somewhat converse to Lemma 2.40.

Lemma 2.42. *Let A be a space whose specialization preorder is a complete lattice. Suppose that the join map on closed sets $\bigvee : HA \to A$ is continuous. Then $(A, \bigvee)$ is an H-algebra.*

Proof. We need the continuity of $\bigvee$ to ensure that it is a morphism in Top. We have to verify the commutativity of the diagrams (2.5). The unit diagram requires that, for each $x \in A$, we have $\bigvee \mathrm{cl}(\{x\}) = x$, which holds by Lemma 2.41. The algebra diagram requires that, for each $\mathcal{C} \in HHA$, we have $\bigvee \mathcal{U}(\mathcal{C}) = \bigvee \left(\bigvee_\sharp \mathcal{C} \right)$. This holds since the join is associative and by Lemma 2.41. ☐

Lemma 2.43 (Paragraph II.1.9 in [Joh82])**.** *Let X be a sober topological space. Then the specialization preorder of X has directed joins, and every open set $U \subseteq X$ is Scott-open for the specialization preorder.*

Lemma 2.44. *Let L be a sober topological space. Then the specialization preorder of L has binary joins if and only if it has all joins, and the binary join map $\vee : L \times L \to L$ is continuous if and only if the join map for closed sets $\bigvee : HL \to L$ is continuous.*

Proof. If binary (and hence finitary) joins exist, then arbitrary joins exist by Lemma 2.43. The converse is trivial. We thus only need to show that the join map $\bigvee : HL \to L$ is continuous if and only if the binary join map $\vee : L \times L \to L$ is.

Suppose that $\bigvee : HL \to L$ is continuous. It suffices to show that the map $\phi : L \times L \to HL$ where $(x, y) \mapsto \mathrm{cl}(\{x, y\})$ is continuous. Let $U \subseteq L$ be open and consider the basic open set $\mathrm{Hit}(U)$ of HL. We have to prove that the preimage $\phi^{-1}(\mathrm{Hit}(U))$ is open. In fact

$$\begin{aligned} \phi^{-1}(\mathrm{Hit}(U)) &= \{(x, y) \mid \mathrm{cl}(\{x, y\}) \cap U \neq \varnothing\} \\ &= \{(x, y) \mid \{x, y\} \cap U \neq \varnothing\} \\ &= \{(x, y) \mid x \in U\} \cup \{(x, y) \mid y \in U\} \\ &= (U \times L) \cup (L \times U), \end{aligned}$$

which is open in $L \times L$.

Suppose that the binary join map is continuous. Pick an open set $U \subseteq L$. We show that every $C \in \bigvee^{-1}(U)$ has an open neighborhood contained in $\bigvee^{-1}(U)$. Since U is Scott open by Lemma 2.43 and $\bigvee C \in U$, there exists a finite set $\{x_1, ..., x_n\} \subseteq C$ such that $x_1 \vee \ldots \vee x_n \in U$. Since the n-ary join map $L^{\times n} \to L$ is continuous by continuity of the binary one, there exist open neighborhoods $V_i \ni x_i$ such that for all $y_i \in V_i$ we have $y_1 \vee \ldots \vee y_n \in U$ as well. Consider the basic open set $W := \bigcap_{i=1}^{n} \mathrm{Hit}(V_i)$. We have $C \in W$ by construction, and it is easy to see that $W \subseteq \bigvee^{-1}(U)$ since U is an upper set. $\square$

We are now ready to prove the theorem.

Proof of Theorem 2.39. By Lemma 2.40 and Lemma 2.41, we know that for an H-algebra A, every subset of A must have a supremum, that is A is a complete lattice in the specialization preorder. The hyperspace HA is sober (Remark 2.8) and the map $\bigvee : HA \to A$ must be continuous. By the unit condition of (2.5), it follows that A is a retract of a sober space and therefore sober. By Lemma 2.44, the binary join is continuous too. Therefore A is a topological complete join-semilattice and the algebra map is the join of closed sets.

Conversely, suppose that A is a topological complete join-semilattice. By Lemma 2.44, the join map of closed sets $\bigvee : HA \to A$ is continuous. Using Lemma 2.42, we conclude that $(A, \bigvee)$ is an H-algebra.

To complete the proof, suppose that A and B are H-algebras. A morphism m between them is, by definition, a continuous map such that the following diagram commutes.

$$\begin{array}{ccc} HA & \xrightarrow{Hm} & HB \\ \big\downarrow \bigvee & & \big\downarrow \bigvee \\ A & \xrightarrow{m} & B \end{array}$$

Such maps m are precisely those that preserve arbitrary suprema by Lemma 2.41. $\square$

We conclude this section with a remark. In contradiction to a claim by Schalk [Sch93, Sections 6.3 and 6.3.1], not every sober space whose specialization order is a complete lattice is a topological complete join-semilattice. In other words, for a sober space X whose specialization order is a complete lattice, the continuity of the join map of closed sets $\bigvee : HL \to L$, or equivalently of the binary join map $\vee : L \times L \to L$, is not guaranteed. A counterexample seems to be given by Hoffmann [Hof79, Example 5.5 combined with Lemma 1.5], however it is based on what appears to be a faulty reference (reference 5 therein). We give a concrete counterexample, based on Hoffmann's approach.

Example 2.45 (A sober space whose specialization preorder is a complete lattice, but whose binary join map is not continuous)**.** Let X be a T_1 space that is sober but not T_2. For example, X could be the set $\mathbb{N} \cup \{a, b\}$, where the open sets are given by those of $\mathbb{N}$ and those in the form $\{a\} \cup \mathbb{N} \setminus F$ and $\{b\} \cup \mathbb{N} \setminus F$ and $\{a, b\} \cup \mathbb{N} \setminus F$ where $F \subseteq \mathbb{N}$ is finite. Since X is T_1, its specialization preorder is the discrete order.

Consider the set $X^* := X \sqcup \{\bot, \top\}$ with the topology whose nonempty open subsets $W \subseteq X^*$ are those in the form $W = V \cup \{\top\}$ where V is an open subset of X. The specialization preorder of X^* is a complete lattice where

$$x \vee y = \begin{cases} \bot & \text{if } x = y = \bot, \\ x & \text{if } x = y \in X, \\ \top & \text{otherwise,} \end{cases}$$

and similarly for arbitrary joins. To see that X^* is sober, let $K \subseteq X^*$ be a nonempty irreducible closed set. If $\top \in K$, then necessarily $K = X^* = \mathrm{cl}(\{\top\})$. Hence we can assume $K = C \cup \{\bot\}$ for some closed $C \subseteq X$. Suppose that $C = C_1 \cup C_2$ for closed $C_1, C_2 \subseteq X$. Then $K = (C_1 \cup \{\bot\}) \cup (C_2 \cup \{\bot\})$. Since K is irreducible, we must have $C_1 = C$ or $C_2 = C$. Therefore C is also irreducible. Since X is sober, C is the closure of a point in X and so must be K in X^*. We conclude that X^* is sober.

Consider the open subset $\{\top\} \subseteq X^*$. We will show that the preimage

$$\vee^{-1}(\{\top\}) = \big\{(x, y) \in X^* \times X^* \;\big|\; x \vee y = \top\big\}$$

is not open, and therefore that $\vee : X^* \times X^* \to X^*$ is not continuous, by exhibiting $(x, y) \in \vee^{-1}(\{\top\})$ which is not in the interior of this preimage. Since X is not T_2, there are distinct points $x, y \in X$ such that any open neighborhoods $U \ni x$ and $V \ni y$ in X intersect. We then have $x \vee y = \top$ since $x \neq y$. Consider any basic neighborhood $U' \times V'$ of (x, y) in $X^* \times X^*$. We can assume $U' = U \cup \{\top\}$ and $V' = V \cup \{\top\}$ for open sets $U, V \subseteq X$ without loss of generality. Since $U \cap V \neq \varnothing$, we have $z \in X$ such that $(z, z) \in U' \times V'$, but clearly $(z, z) \notin \vee^{-1}(\{\top\})$. Hence (x, y) is not an interior point of $\vee^{-1}(\{\top\})$.

2.7. Products and projections

Taking products of subspaces and projecting subspaces of products to subspaces of the respective factor spaces are basic geometric operations. They are related by the elementary fact that a subspace of a product space is contained in the product of its projections. In this section, we show that H is a commutative monad (Appendix C), or equivalently a symmetric monoidal monad, with respect to the Cartesian monoidal structure on Top. We start by constructing a *strength* transformation $X \times HY \to H(X \times Y)$.

Notation 2.46 (Slice)**.** *Let X and Y be sets or topological spaces. Let $x \in X$ and $A \subseteq X \times Y$. Consider the map $j_x : Y \to X \times Y$ given by $y \mapsto (x, y)$. The slice of A at x is the set*

$$A_x := j_x^{-1}(A) \subseteq Y.$$

It is an elementary fact that j_x is continuous. In other words, if A is open in $X \times Y$, then every slice A_x is open in Y.

Definition 2.47. *Let X and Y be topological spaces. Let $x \in X$ and $C \in HY$. We define $s(x, C) \in H(X \times Y)$ to be the closed set which satisfies*

$$\langle s(x, C), W\rangle \;:=\; \langle C, W_x\rangle$$

for all open $W \subseteq X \times Y$.

Since taking the slice $W \mapsto W_x$ preserves unions and intersections, the duality of Proposition 2.10 applies, hence $s(x, C)$ is well-defined. Since $\langle C, W_x\rangle = \langle C, W \circ j_x\rangle = \langle (j_x)_\sharp C, W\rangle$, we can identify $s(x, C)$ with the closed subset $(j_x)_\sharp C \subseteq X \times Y$.

We have a map $s : X \times HY \to H(X \times Y)$.

Proposition 2.48. *The map $s : X \times HY \to H(X \times Y)$ is continuous.*

Proof. Let $W \subseteq X \times Y$ be open. We have to show that

$$s^{-1}(\mathrm{Hit}(W)) \;=\; \{(x, C) \in X \times HY \mid C \in \mathrm{Hit}(W_x)\}$$

is open as well. Equivalently, that every point $(x, C) \in s^{-1}(\mathrm{Hit}(W))$ is interior. For every $y \in W_x$ we have $(x, y) \in W$. Since W is open in the product topology, we can choose open neighborhoods $U_y \ni x$ and $V_y \ni y$ such that $U_y \times V_y \subseteq W$. Then

$$C \;\in\; \mathrm{Hit}(W_x) \;=\; \mathrm{Hit}\left(\bigcup_{y \in W_x} V_y\right) \;=\; \bigcup_{y \in W_x} \mathrm{Hit}(V_y),$$

which implies that there is a $y \in W_x$ such that $C \in \mathrm{Hit}(V_y)$. Fix such a y, and consider the open set $U_y \times \mathrm{Hit}(V_y) \subseteq X \times HY$. By construction, $(x, C) \in U_y \times \mathrm{Hit}(V_y)$. Consider any $(x', C') \in U_y \times \mathrm{Hit}(V_y)$. Since $U_y \times V_y \subseteq W$, we have $V_y \subseteq W_{x'}$, and hence $C' \in \mathrm{Hit}(V_y) \subseteq \mathrm{Hit}(W_{x'})$ too. Therefore $U_y \times \mathrm{Hit}(V_y) \subseteq s^{-1}(\mathrm{Hit}(W))$. □

The following lemma simplifies some of the upcoming calculations.

Lemma 2.49. *Let X and Y be topological spaces. Then a closed subset $C \subseteq X \times Y$ is uniquely determined by the product sets $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open, that it hits.*

Proof. The collection of sets

$$\{U \times V \,|\, U \subseteq X \text{ and } V \subseteq Y \text{ are open}\}$$

is a basis of the product topology. Hence every open subset of $X \times Y$ can be written as a union of such sets, say

$$\bigcup_{s \in S} (U_s \times V_s)$$

where S is an arbitrary indexing set, and $U_s \subseteq X$ and $V_s \subseteq Y$ are open for every $s \in S$. Since hitting commutes with unions, we have

$$\left\langle C, \bigcup_{s \in S} (U_s \times V_s) \right\rangle \;=\; \bigvee_{s \in S} \langle C, U_s \times V_s\rangle,$$

which proves the claim. □

On open sets in the form $U \times V$, the action of s is particularly simple.

Corollary 2.50. *Let X and Y be topological spaces. Let $x \in X$ and $C \in HY$, and let $U \subseteq X$ and $V \subseteq Y$ be open. Then*

$$\langle s(x,C), U \times V \rangle \;\coloneqq\; [\![x \in U]\!] \wedge \langle C, V \rangle.$$

That is, the set $s(x,C)$ hits $U \times V$ if and only if $x \in U$ and C hits U.

We now turn to naturality of s in both arguments: for all continuous functions $f : X \to Z$ and $g : Y \to W$, the following two diagrams commute.

$$\begin{array}{ccc} X \times HY & \xrightarrow{s} & H(X \times Y) \\ \big\downarrow{\scriptstyle f \times \mathrm{id}} & & \big\downarrow{\scriptstyle (f \times \mathrm{id})_*} \\ Z \times HY & \xrightarrow{s} & H(Z \times Y) \end{array} \qquad \begin{array}{ccc} X \times HY & \xrightarrow{s} & H(X \times Y) \\ \big\downarrow{\scriptstyle \mathrm{id} \times g_*} & & \big\downarrow{\scriptstyle (\mathrm{id} \times g)_*} \\ X \times HW & \xrightarrow{s} & H(X \times W) \end{array}$$

To see why the first diagram commutes, consider open sets $U \subseteq Z$ and $V \subseteq Y$. Then, for each $x \in X$ and every $C \in HY$, we have

$$\begin{aligned} \langle (f \times \mathrm{id})_\sharp s(x,C), U \times V \rangle &= \big\langle s(x,C), f^{-1}(U) \times V \big\rangle \\ &= [\![x \in f^{-1}(U)]\!] \wedge \langle C, V \rangle \\ &= [\![f(x) \in U]\!] \wedge \langle C, V \rangle \\ &= \langle s(f(x), C), U \times V \rangle. \end{aligned}$$

To see why the second diagram commutes, let $U \subseteq X$ and $V \subseteq W$ be open. For each $x \in X$ and every $C \in HY$, we have

$$\begin{aligned} \langle (\mathrm{id} \times g)_\sharp s(x,C), U \times V \rangle &= \big\langle s(x,C), U \times g^{-1}(V) \big\rangle \\ &= [\![x \in U]\!] \cdot \big\langle C, g^{-1}(V) \big\rangle \\ &= k(x) \cdot \langle g_\sharp C, V \rangle \\ &= \langle s(x, g_\sharp C), U \times V \rangle. \end{aligned}$$

Proposition 2.51. *The map s is a strength for the monad H.*

In other words, for all topological spaces X and Y, the following four diagrams commute. The first two involve the unitor u and associator a of the Cartesian monoidal structure of Top, while the other ones involve the structure maps of the monad.

$$\begin{array}{ccc} 1 \times HX & \xrightarrow{s} & H(1 \times X) \\ & {\scriptstyle u}\searrow^{\cong} & \big\downarrow{\scriptstyle u_\sharp}\,{\scriptstyle \cong} \\ & & HX \end{array}$$

$$\begin{array}{ccccc}
(X \times Y) \times HZ & & \xrightarrow{\quad s \quad} & & H((X \times Y) \times Z) \\
{\scriptstyle a}\downarrow{\scriptstyle \cong} & & & & {\scriptstyle a_\sharp}\downarrow{\scriptstyle \cong} \\
X \times (Y \times HZ) & \xrightarrow{\mathrm{id} \times s} & X \times H(Y \times Z) & \xrightarrow{s} & H(X \times (Y \times Z))
\end{array}$$

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\mathrm{id} \times \sigma} & X \times HY \\
& {\scriptstyle \sigma}\searrow & \downarrow{\scriptstyle s} \\
& & H(X \times Y)
\end{array}$$

$$\begin{array}{ccccc}
X \times HHY & \xrightarrow{s} & H(X \times HY) & \xrightarrow{s_\sharp} & HH(X \times Y) \\
\downarrow{\scriptstyle \mathrm{id} \times \mathcal{U}} & & & & \downarrow{\scriptstyle \mathcal{U}} \\
X \times HY & & \xrightarrow{\quad s \quad} & & H(X \times Y)
\end{array}$$

Proof. For the first diagram, let $C \in HX$ and $U \subseteq X$ be open. Denoting by $\bullet$ the unique point of 1, we have

$$\langle u_\sharp(s(\bullet, C)), U\rangle \;=\; \big\langle (s(\bullet, C)), u^{-1}(U)\big\rangle \;=\; \langle C, U\rangle \;=\; \langle u(\bullet, C), U\rangle.$$

For the second diagram, let $U \subseteq X$, $V \subseteq Y$, and $W \subseteq Z$ be open. For each $x \in X$, $y \in Y$, and $C \in HZ$, we have

$$\begin{aligned}
\langle a_\sharp(s((x,y),C)), U \times (V \times W)\rangle &= \big\langle s((x,y),C), a^{-1}(U \times (V \times W))\big\rangle \\
&= \langle s((x,y),C), (U \times V) \times W\rangle \\
&= [\![(x,y) \in U \times W]\!] \wedge \langle C, W\rangle \\
&= [\![x \in U]\!] \wedge ([\![y \in V]\!] \wedge \langle C, W\rangle) \\
&= [\![x \in U]\!] \wedge \langle s(y,C), V \times W\rangle \\
&= \langle s(x, s(y,C)), U \times (V \times W)\rangle.
\end{aligned}$$

For the third diagram, let $U \subseteq X$ and $V \subseteq Y$ be open. For each $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
\langle s(x, \sigma(y)), U \times V\rangle &= [\![x \in U]\!] \wedge \langle \sigma(y), V\rangle \\
&= [\![x \in U]\!] \wedge [\![y \in V]\!] \\
&= [\![(x,y) \in U \times V]\!] \\
&= \langle \sigma((x,y)), U \times V\rangle.
\end{aligned}$$

For the last diagram, let $U \subseteq X$ and $V \subseteq Y$ be open. For each $x \in X$ and $\mathcal{C} \in HHY$, we have

$$\begin{aligned}
\langle \mathcal{U} s_\sharp s(x,\mathcal{C}), U \times V \rangle &= \langle s_\sharp s(x,\mathcal{C}), \mathrm{Hit}(U \times V) \rangle \\
&= \left\langle s(x,\mathcal{C}), s^{-1}(\mathrm{Hit}(U \times V)) \right\rangle \\
&= \langle s(x,\mathcal{C}), U \times \mathrm{Hit}(V) \rangle \\
&= [\![x \in U]\!] \wedge \langle \mathcal{C}, \mathrm{Hit}(V) \rangle \\
&= [\![x \in U]\!] \wedge \langle \mathcal{U}\mathcal{C}, V \rangle \\
&= \langle s(x,\mathcal{U}\mathcal{C}), U \times V \rangle.
\end{aligned}$$

□

We can define the costrength $t : HX \times Y \to H(X \times Y)$ by symmetry, which yields

$$\langle t(C,y), U \times V \rangle = \langle C, U \rangle \wedge [\![y \in V]\!]$$

for all $C \in HX$, $y \in Y$, and open sets $U \subset X$ and $V \subseteq Y$. By symmetry, the costrength satisfies the properties analogous to those of Proposition 2.51.

Proposition 2.52. *The strength and costrength are compatible in the sense that the following diagram commutes.*

$$\begin{array}{ccccc}
HX \times HY & \xrightarrow{\;s\;} & H(HX \times Y) & \xrightarrow{\;t_\sharp\;} & HH(X \times Y) \\
\downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle \mathcal{U}} \\
H(X \times HY) & \xrightarrow{\;s_\sharp\;} & HH(X \times Y) & \xrightarrow{\;\mathcal{U}\;} & H(X \times Y)
\end{array}$$

Proof. Let $C \in HX$ and $D \in HY$, and let $U \subseteq X$ and $V \subseteq Y$ be open. Then

$$\begin{aligned}
\langle \mathcal{U} t_\sharp s(C,D), U \times V \rangle &= \langle t_\sharp s(C,D), \mathrm{Hit}(U \times V) \rangle \\
&= \left\langle s(C,D), t^{-1}(\mathrm{Hit}(U \times V)) \right\rangle \\
&= \langle s(C,D), \mathrm{Hit}(U) \times V \rangle \\
&= [\![C \in \mathrm{Hit}(U)]\!] \wedge \langle D, V \rangle \\
&= \langle C, U \rangle \wedge \langle D, V \rangle.
\end{aligned}$$

An analogous computation shows that $\langle \mathcal{U} s_\sharp t(C,D), U \times V \rangle = \langle C, U \rangle \wedge \langle D, V \rangle$ as well. □

Corollary 2.53. $(H, \sigma, \mathcal{U})$ *is a symmetric monoidal monad.*

Proof. See Proposition C.5. □

The lax monoidal structure is implemented by the multiplication map $HX \times HY \to H(X \times Y)$ given by the product of closed sets $(C, D) \mapsto C \times D$, as the computation in the proof of Proposition 2.52 shows. Due to the universal property of the product in Top, H is an oplax monoidal monad as well, even bilax (see Appendix C). The comultiplication $H(X \times Y) \to HX \times HY$ projects a closed set in the product space $X \times Y$ to the pair of its projections to X and Y.

3. The valuation monad

A valuation is similar to a Borel measure, but is defined only on the open sets of a topological space (see, for example, [AJK04]). Valuations appeared as generalizations of Borel measures better suited to the demands of point-free topology and constructive mathematics. Jones and Plotkin [JP89] defined a monad of subprobability valuations on the category of directed complete partially ordered sets (dcpo's) and Scott-continuous maps. The underlying endofunctor of this monad assigns to a dcpo the set of Scott-continuous subprobability valuations, its *probabilistic powerdomain*. The monad multiplication corresponds to forming the expected valuation by integration. Kirch [Kir93] generalized the construction by working with valuations taking values in $[0, \infty]$, and obtained a monad on the category of continuous domains and Scott-continuous maps [Kir93, Satz 6.1]. He also proved a Markov-Riesz-type duality for valuations [Kir93, Satz 8.1], namely, that on a core-compact space X (for example, a continuous domain) there is a duality of cones between lower semicontinuous functions $X \to [0, \infty]$ and continuous valuations. Heckmann [Hec96] further extended the construction to general topological spaces. He defined VX to be the space of continuous valuations on a topological space X with values in $[0, \infty]$, and proved that the construction forms a monad [Hec95, Section 10]. He also extended the duality result of Kirch, showing that on every topological space X there is a bijection between continuous valuations on X and Isbell-continuous linear functionals from lower semicontinuous functions $X \to [0, \infty]$ to $[0, \infty]$ (see [Hec95, Theorem 9.1]). Alvarez-Manilla, Jung and Keimel [AJK04, Theorem 25] showed that one can view the space of continuous valuations VX as the space of Scott-continuous, monotone, linear functionals from lower semicontinuous functions $X \to [0, \infty]$ to $[0, \infty]$. This duality formula, which is analogous to the one for the monad H (Proposition 2.14), is of crucial importance in the present work. For a detailed history of the monad, see the papers of Alvarez-Manilla et al. [AJK04] and Goubault-Larrecq and Jia [GJ19].

Very recently, and independently of us, Goubault-Larrecq and Jia [GJ19] have studied the algebras of the extended probabilistic powerdomain on the category of T_0 spaces. They show that a T_0 space endowed with a certain structure, called a weakly locally convex sober topological cone, is always an algebra of the extended probabilistic powerdomain. They also prove that under additional assumptions (such as core-compactness of the space), this structure is also sufficient to have an algebra. A full characterization of the algebras of such monads, and a general answer to the question which cones are algebras, is at present lacking (and presumably quite difficult).

We now review the basic constructions and results pertaining to the continuous valuations monad V.

Definition 3.1 (Continuous valuation)**.** *Let X be a topological space. A continuous valuation on X is a map $\nu : \mathcal{O}(X) \to [0, \infty]$ that satisfies the following four conditions.*

(a) Strictness*: $\nu(\varnothing) = 0$.*

(b) Monotonicity*: $U \subseteq V$ implies $\nu(U) \leq \nu(V)$.*

(c) Modularity*: For any $U, V \in \mathcal{O}(X)$, we have $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$.*

(d) Scott continuity*: For any directed net $(U_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{O}(X)$, we have*

$$\nu\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigvee_{\lambda\in A} \nu(U_\lambda).$$

We write VX for the set of continuous valuations on X.

We will equip VX with a topology in Section 3.2 and consider functoriality in X in Section 3.3. We also recall the connection between continuous valuations and Borel measures in Section 4.1.

Remark 3.2. It is clear from the definition that VX only depends on the frame of open subsets of X, or equivalently on the sobrification of X. In particular, VX only depends on the Kolmogorov quotient of X.

3.1. Duality theory

One can define an integration theory for continuous valuations that is analogous to Lebesgue integration for measures, but the role of measurable functions is played by lower semicontinuous functions [Kir93; Jun04]. Thereby continuous valuations are dual to lower semicontinuous functions, just as closed subsets are dual to open subsets (Section 2.3). We recall the definition of integral of a lower semicontinuous function against a continuous valuation, also known as *lower integral*. As far as we know, it was first defined in Kirch's thesis [Kir93], written in German. A reference in English is the later work of Jung [Jun04].

Notation 3.3. *Let X be a topological space. We denote the set of lower semicontinuous functions $X \to [0, \infty]$ by $\mathrm{L}X$ and equip it with the pointwise order.*

We start by recalling some basic properties analogous to those of the Lebesgue integral.

- A lower semicontinuous function $X \to [0,\infty]$ is *simple* if it assumes only finitely many values.
- Every simple lower semicontinuous function $f : X \to [0,\infty]$ can be written as a positive linear combination of indicator functions of open sets, that is in the form
$$f = \sum_{i=1}^{n} r_i \, \mathbb{1}_{U_i}$$
for $r_i \in (0,\infty]$ and $U_i \subseteq X$ open for all i. This can be seen by induction on the number of values that f takes.
- Every lower semicontinuous function $X \to [0,\infty]$ can be expressed as a directed supremum of simple functions.

The integral is defined such that it is continuous with respect to directed suprema.

Definition 3.4. *Let X be a topological space, let $\nu \in VX$, and consider the simple function $f : X \to [0,\infty]$ given by*
$$f := \sum_{i=1}^{n} r_i \, \mathbb{1}_{U_i} .$$
The integral of f with respect to ν, also called pairing of ν and f, is given by
$$\int f \, d\nu := \sum_{i=1}^{n} r_i \, \nu(U_i).$$
Let $g \in \mathrm{L}X$. The integral of g with respect to ν, also called pairing of ν and g, is defined as
$$\int g \, d\nu := \sup \left\{ \int f \, d\nu \;\middle|\; f \le g,\ f \text{ simple} \right\}.$$

It is not immediately obvious that the integral of a simple function is well-defined, since a priori it may depended on the particular representation, which it actually does not [Kir93].

Notation 3.5. *To emphasize the analogy between the lower integral and the coupling notation for hyperspaces (Notation 2.17), we also write $\langle \nu, g \rangle := \int g \, d\nu$.*

This pairing notation is motivated by the following duality result, which seems to be due to Alvarez-Manilla et al. [AJK04, Theorem 25].

Theorem 3.6 (Extension theorem for valuations)**.** *Integration establishes a bijection between continuous valuations on the topological space X and $[0,\infty]$-linear, Scott-continuous functionals $\mathrm{L}X \to [0,\infty]$.*

This is analogous to the situation for closed sets (Proposition 2.14), and can be used to define the monad structure of V analogous to the monad structure of H (Section 3.4).

In applying this duality, a useful fact is that directed suprema in $\mathrm{L}X$ are pointwise suprema, since the pointwise supremum of a directed system of lower semicontinuous functions is again lower semicontinuous.

3.2. Topology on the set of valuations

A continuous valuation is a quantitative analogue of a closed set: a closed set may or may not hit a given open set, while a valuation assigns a number to every open set. Closed subsets of X are equivalent to particular continuous functionals from $\mathcal{O}(X)$ to the Sierpiński space S, and the lower Vietoris topology corresponds to the topology of pointwise convergence of such functionals. Similarly, continuous valuations on X are particular $[0,\infty]$-valued functionals on $\mathcal{O}(X)$ or $\mathrm{L}X$, the former by definition and the latter by Theorem 3.6. Thus there are two ways to equip VX with the topology of pointwise convergence. It turns out that they are equivalent. We may think of this as a version of the Portmanteau theorem for continuous valuations.

Proposition 3.7. *Let X be a topological space. The topology on VX generated by the subbasis of sets of the form*

$$\theta(U,r) \coloneqq \{\nu \,:\, \nu(U) > r\} \tag{3.1}$$

for open $U \subseteq X$ and $r \in [0,\infty)$ is also generated by the subbasis of sets of the form

$$\Theta(f,r) \coloneqq \{\nu \in VX : \langle \nu, f\rangle > r\} \tag{3.2}$$

for $f \in \mathrm{L}X$ and $r \in [0,\infty)$.

Proof. One direction is immediate since $\nu(U) > r$ may be written as $\langle \nu, \mathbb{1}_U\rangle > r$, showing that $\theta(U,r) = \Theta(\mathbb{1}_U, r)$.

For the converse, let $f : X \to [0,\infty]$ be lower semicontinuous and pick $r \in [0,\infty)$. We want to show that, for every $\nu \in \Theta(f,r)$, there is a basic open set of the form

$$\theta(U_1,r_1) \cap \ldots \cap \theta(U_n,r_n)$$

which contains ν and is contained in $\Theta(f,r)$. By definition of integration, we can find a simple lower semicontinuous function

$$g \coloneqq \sum_{i=1}^{n} c_i\, \mathbb{1}_{U_i},$$

with $c_i \in [0,\infty)$ and open $U_i \subseteq X$ for all i, such that $g \le f$ and $\langle \nu, g\rangle > r$. In other words

$$\sum_{i=1}^{n} c_i\, \nu(U_i) \;>\; r,$$

where we assume, without loss of generality, that $\nu(U_i) > 0$ for all i. We choose $\varepsilon > 0$ small enough such that $\nu(U_i) \ge \varepsilon$ for all i and

$$\sum_{i=1}^{n} c_i\,(\nu(U_i) - \varepsilon) \;>\; r.$$

We put $r_i := \nu(U_i) - \varepsilon$. For every $i = 1, \dots, m$ we trivially have $\nu(U_i) > r_i$, and therefore $\nu \in \theta(U_i, r_i)$. Hence the basic open set

$$W := \bigcap_{i=1}^{n} \theta(U_i, r_i)$$

contains ν. We want to show that $W \subseteq \Theta(f, r)$. Indeed for any $\rho \in W$,

$$\langle \rho, f \rangle \geq \langle \rho, g \rangle = \sum_{i=1}^{n} c_i \, \rho(U_i) > \sum_{i=1}^{n} c_i \, r_i > r,$$

and therefore $\rho \in \Theta(f, r)$. □

Definition 3.8. *Let X be a topological space. We define the space VX to be the set of continuous valuations on X, equipped with the topology of Proposition 3.7.*

This weak topology on VX, and the two canonical subbases discussed above, have well-known analogues for Borel measures (Section 4.2).

As in the case of H, the Scott topology on $\mathrm{L}X$ is the topology of the exponential object $[0, \infty]^X$ for all exponentiable X, where $[0, \infty]$ carries the topology generated by the intervals of the form $(a, \infty]$. We elaborate on this in Appendix B.

We give the following well-known caveat. In the case of H, we have seen that the Scott topology on $\mathcal{O}(X)$ is finer than the one of pointwise convergence where we consider $\mathcal{O}(X)$ as the space of continuous functions into the Sierpiński space (Remark 2.13). However, closed sets, as functionals, are continuous, even for the pointwise topology (Proposition 2.14). This is not the case here: The Scott topology on $\mathrm{L}X$ is finer than the topology of pointwise convergence, and integrating against a valuation is Scott-continuous but typically not continuous in the topology of pointwise convergence of functions.

Example 3.9. For typical X and $\nu \in VX$, even the map $\nu : \mathcal{O}(X) \to [0, \infty]$ is not continuous with respect to the topology of pointwise convergence on $\mathcal{O}(X)$, considered as the space of functions $X \to S \subseteq [0, \infty]$. In order for $\nu : \mathcal{O}(X) \to [0, \infty]$ to be continuous for the pointwise topology, the preimage $\mathcal{V} := \nu^{-1}((r, \infty]) = \{U \in O(X) : \nu(U) > r\}$ would have to be open. Hence for any $U \in \mathcal{O}(X)$ in this preimage, there would be a basic open set, say $\mathcal{U} = \mathcal{U}_{x_1} \cap \ldots \cap \mathcal{U}_{x_n}$, where $\mathcal{U}_{x_i} = \{U \in O(X) : x_i \in U\}$ for some $x_1, \dots, x_n \in X$ such that $U \in \mathcal{U} \subseteq \mathcal{V}$. Then the points $x_1, \dots, x_n$ would be such that every open neighborhood of any of these points has ν-mass greater than r. Finding such points is clearly not possible in general; it fails whenever ν is non-atomic, for example for the Lebesgue measure on $[0, 1]$.

Since $\mathcal{O}(X)$ is a subspace of $\mathrm{L}X$, the typical integration map $\nu : \mathrm{L}X \to [0, \infty]$ is also discontinuous with respect to pointwise convergence.

We now consider the specialization preorder on VX.

Lemma 3.10. *For $\nu, \rho \in VX$, the following three statements are equivalent.*

(a) $\nu \leq \rho$ *in the specialization preorder, that is* $\nu \in \mathrm{cl}(\{\rho\})$.

(b) $\nu(U) \leq \rho(U)$ *for all* $U \in \mathcal{O}(X)$.

(c) $\langle \nu, g \rangle \leq \langle \rho, g \rangle$ *for all* $g \in \mathrm{L}X$.

In particular, the specialization order coincides with the pointwise order of valuations as functionals.

The proof is immediate from Proposition 3.7, since it is enough to compare membership of ν and ρ in subbasic open sets. The specialization order is also the canonical order on the space of valuations in the probabilistic powerdomain [JP89].

Corollary 3.11. *For any topological space X, the space VX is T_0.*

Proof. The T_0 property is equivalent to antisymmetry of the specialization preorder, which is immediate by identifying it with either pointwise order from Lemma 3.10. ☐

Similarly to Remark 2.8, it is known that the topology on VX is even sober [Hec96, Proposition 5.1].

We end this subsection with a small excursion to ordered topological spaces and the *stochastic order*. This is of independent interest and will play no further role in this paper. A *preordered topological space* is a topological space X which is at the same time a preordered set $(X, \leq)$ such that the set of all ordered pairs $\{(x, y) \mid x \leq y\}$ is closed in the product topology on $X \times X$.

Definition 3.12 (Stochastic order)**.** *Let X be a preordered topological space. For any two $\nu, \rho \in VX$, we put $\nu \leq \rho$ if and only if $\nu(U) \leq \rho(U)$ for any open upper set $U \subseteq X$.*

The stochastic order can be considered the pointwise order of valuations as functionals on the upper open sets. If the preorder is trivial, the stochastic order degenerates to the specialization preorder on VX, in general it is larger. It can be thought of as an order that asks "how far up does the mass lie".

Example 3.13 (Stochastic dominance on the real line)**.** The stochastic order on $\mathbb{R}$, considered as a preordered topological space with the standard topology and order, is widely used in decision theory, economics, and finance to compare probability measures on the real line (e.g. [RS70]). For two probability measures on $\mathbb{R}$, one has $p \leq q$ if and only if $p((a, \infty)) \leq q((a, \infty))$ for all $a \in \mathbb{R}$. By normalization, this is equivalent to the opposite pointwise order of cumulative distribution functions, $p((-\infty, a]) \geq q((-\infty, a])$ for all $a \in \mathbb{R}$.

The stochastic order makes sense on any preordered topological space, and it has the same interpretation as for the real line. In our setting, the stochastic order can be considered an instance of the specialization order. Let X' be the space X, but carrying the topology where only the upper opens subsets of X are open in X'. Then every continuous valuation on X restricts to a continuous valuation on X'. Moreover, we have $\nu \leq \rho$ in the stochastic order on X if and only if $\nu \leq \rho$ in the specialization preorder on X'.

3.3. Functoriality

We will show that V is a functor $\mathsf{Top} \to \mathsf{Top}$. This, as well as the construction of the monad structure of V, can either be achieved in terms of open sets, or by the duality with lower semicontinuous functions. We use the latter approach, since it is analogous to the characterization of the hyperspace monad H in terms of functionals on open sets (Section 2). For V, the role that the open sets play for H is played by lower semicontinuous functions.

Definition 3.14 (Pushforward)**.** *Let $f : X \to Y$ be continuous and consider $\nu \in VX$. We define the pushforward of ν along f as the valuation $f_*\nu \in VY$ that assigns to an open set $U \subseteq Y$ the mass*

$$f_*\nu(U) \;=\; \nu(f^{-1}(U)).$$

Equivalently, we could define the pushforward by the requirement that

$$\langle f_*\nu, g\rangle \;=\; \langle \nu, g \circ f\rangle$$

for every $g \in \mathrm{L}X$.

To see that $f_*\nu$ is a continuous valuation, we apply Theorem 3.6, the assumptions of which we need to verify.

Proposition 3.15. *Let $f : X \to Y$ be a continuous map and let $\nu \in VX$ be a continuous valuation. Then the assignment $\langle \nu, - \circ f\rangle : \mathrm{L}Y \to [0, \infty]$ is Scott-continuous.*

Proof. Consider a directed set $\{g_\alpha\}_{\alpha \in A}$ of lower semicontinuous functions. Then

$$\begin{aligned} \left\langle f_*\nu, \sup_{\alpha\in A} g_\alpha \right\rangle &= \left\langle \nu, \left(\sup_{\alpha\in A} g_\alpha\right) \circ f \right\rangle = \left\langle \nu, \sup_{\alpha\in A} (g_\alpha \circ f) \right\rangle \\ &= \sup_{\alpha\in A} \langle \nu, g_\alpha \circ f\rangle = \sup_{\alpha\in A} \langle f_*\nu, g_\alpha\rangle, \end{aligned}$$

where the second step uses that directed suprema in $\mathrm{L}X$ are pointwise. □

The other properties required for the duality theorem are straightforward to check. The map $f_* : VX \to VY$ is well-defined. We need to verify that it is continuous. The following statements follow directly from the definition.

Lemma 3.16. *Let $f : X \to Y$ be a continuous function between topological spaces. Let $g \in \mathrm{L}Y$ and $r \in [0, \infty)$. Then*

$$f_*^{-1}(\Theta(g, r)) \;=\; \Theta(g \circ f, r).$$

Corollary 3.17. *The map $f_* : VX \to VY$ is continuous, and therefore a morphism of* Top.

Lemma 3.18. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between topological spaces. Then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Let $\nu \in VX$ and let $h : Z \to [0, \infty]$ be lower semicontinuous. Then

$$\langle (g \circ f)_* \nu, h \rangle = \langle \nu, h \circ (g \circ f) \rangle = \langle \nu, (h \circ g) \circ f \rangle = \langle f_* \nu, h \circ g \rangle = \langle g_* f_* \nu, h \rangle.$$

□

Moreover, it is clear that $(\mathrm{id}_X)* = \mathrm{id}_{VX}$. Hence we have a functor $V : \mathsf{Top} \to \mathsf{Top}$ which assigns to a space X its space of continuous valuations VX and to each continuous map $f : X \to Y$ the continuous map $f_* : VX \to VY$.

It is sometimes useful to know that V preserves subspace embeddings.

Lemma 3.19. *Let X be a topological space and $i : Y \hookrightarrow X$ a subspace inclusion. Then $i_* : VY \hookrightarrow VX$ is a subspace inclusion as well.*

Proof. We first show injectivity of i_*. If $\nu, \rho \in VY$ are such that $i_*(\nu) = i_*(\rho)$, then this means that $\nu(U \cap Y) = \rho(U \cap Y)$ for all open sets $U \subseteq X$. Since every open subset of Y is of this form, we have $\nu = \rho$, making i_* injective. Similarly, every basic open subset of VX is of the form $\theta(U \cap Y, r)$ and contains exactly those $\nu \in VY$ for which $i_*(\nu) \in \theta(U, r)$. We hence have a subspace embedding. □

Remark 3.20. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps with $f \leq g$. Then $f_* \leq g_*$. In other words, V preserves 2-cells, making it into a 2-functor.

Proof. Let $\nu \in VX$. By Lemma A.3, for all open $U \subseteq X$,

$$(f_* \nu)(U) = \nu(f^{-1}(U)) \leq \nu(g^{-1}(U)) = (g_* \nu)(U).$$

By Lemma 3.10, this means that $f_* \nu \in \mathrm{cl}(\{g_* \nu\})$. □

3.4. Monad structure

As we did for H in Section 2, we equip V with a monad structure using the duality theory developed in Section 3.1.

3.4.1. Unit

Definition 3.21. *Let X be a topological space. We define the map $\delta : X \to VX$ which assigns $x \in X$ to the valuation $\delta(x) \coloneqq \delta_x$ defined by $\langle \delta_x, g \rangle := g(x)$ for every $g \in \mathrm{L}X$.*

Lemma 3.22. *Let X be a topological space and consider $f \in \mathrm{L}X$ and $r \in [0, \infty)$. Then*

$$\delta^{-1}(\Theta(f, r)) = f^{-1}\big((r, \infty]\big).$$

Hence the map $\delta : X \to VX$ is continuous, making it a morphism of Top.

We turn to the question of naturality.

Proposition 3.23. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta} \\ VX & \xrightarrow{\ f_*\ } & VY \end{array} \tag{3.3}$$

In other words, $\delta : \mathrm{id} \Rightarrow V$ is a natural transformation.

Proof. For any $x \in X$ and $g \in \mathrm{L}Y$,

$$\langle f_*(\delta(x)), g \rangle \;=\; \langle \delta(x), g \circ f \rangle \;=\; g(f(x)) \;=\; \langle \delta(f(x)), g \rangle.$$

□

The unit map δ is not necessarily injective.

Proposition 3.24. *Let X be a topological space. Then $\delta : X \to VX$ is injective if and only if X is T_0.*

Proof. $\delta(x) = \delta(y)$ is equivalent to $x \sim y$ in the specialization preorder. □

3.4.2. Multiplication

Definition 3.25. *Let X be a topological space. We define the map $\mathcal{E} : VVX \to VX$ as the one assigning to $\xi \in VVX$ the valuation $\mathcal{E}\xi \in VX$ such that, for every $g \in \mathrm{L}X$,*

$$\langle \mathcal{E}\xi, g \rangle \;\equiv\; \langle \xi, \langle -, g \rangle \rangle.$$

Recall that the map $\langle -, g \rangle : VX \to [0, \infty]$ is indeed in $\mathrm{L}VX$ by the definition of the topology on VX. The continuity condition of the duality Theorem 3.6 is the following,

$$\left\langle \mathcal{E}\xi, \sup_{\alpha \in A} g_\alpha \right\rangle = \left\langle \xi, \left\langle -, \sup_{\alpha \in A} g_\alpha \right\rangle \right\rangle = \left\langle \xi, \sup_{\alpha \in A} \langle -, g_\alpha \rangle \right\rangle$$

$$= \sup_{\alpha \in A} \langle \xi, \langle -, g_\alpha \rangle \rangle = \sup_{\alpha \in A} \langle \mathcal{E}\xi, g_\alpha \rangle.$$

The other properties required by Theorem 3.6 are straightforward to check.

The definition directly implies the following.

Lemma 3.26. *Let X be a topological space and $g \in \mathrm{L}X$. Then*

$$\mathcal{E}^{-1}(\Theta(g,r)) \;=\; \Theta(\langle -, g\rangle, r).$$

Hence the map $\mathcal{E} : VVX \to VX$ is continuous and $\mathcal{E}$ is a morphism of Top.

We now turn to naturality.

Proposition 3.27. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then the following diagram commutes.*

$$\begin{array}{ccc} VVX & \xrightarrow{\;f_{**}\;} & VVY \\ \big\downarrow{\scriptstyle \mathcal{E}} & & \big\downarrow{\scriptstyle \mathcal{E}} \\ VX & \xrightarrow{\;f_{*}\;} & VY \end{array}$$

Proof. Let $\xi \in VVX$ and $g \in \mathrm{L}X$. Then

$$\begin{aligned} \langle \mathcal{E}(f_{**}(\xi)), g\rangle &= \langle f_{**}(\xi), \langle -, g\rangle\rangle = \langle \xi, \langle f_* -, g\rangle\rangle \\ &= \langle \xi, \langle -, g \circ f\rangle\rangle = \langle \mathcal{E}(\xi), g \circ f\rangle \\ &= \langle f_*(\mathcal{E}(\xi)), g\rangle. \end{aligned}$$ □

3.4.3. Monad axioms

Proposition 3.28. *Let X be a topological space. Then the following three diagrams commute.*

$$\begin{array}{ccc} VX & \xrightarrow{\;\delta\;} & VVX \\ & \searrow^{=} & \big\downarrow{\scriptstyle \mathcal{E}} \\ & & VX \end{array} \qquad \begin{array}{ccc} VX & \xrightarrow{\;\delta_*\;} & VVX \\ & \searrow^{=} & \big\downarrow{\scriptstyle \mathcal{E}} \\ & & VX \end{array} \qquad \begin{array}{ccc} VVVX & \xrightarrow{\;\mathcal{E}_*\;} & VVX \\ \big\downarrow{\scriptstyle \mathcal{E}} & & \big\downarrow{\scriptstyle \mathcal{E}} \\ VVX & \xrightarrow{\;\mathcal{E}\;} & VX \end{array} \tag{3.4}$$

Proof. We start with the first diagram, left unitality. For every $\nu \in VX$ and $g \in \mathrm{L}X$,

$$\langle \mathcal{E}(\delta_\nu), g\rangle \;=\; \langle \delta_\nu, \langle -, g\rangle\rangle \;=\; \langle \nu, g\rangle.$$

For right unitality, we have

$$\langle \mathcal{E}(\delta_*\nu), g\rangle \;=\; \langle \delta_*\nu, \langle -, g\rangle\rangle \;=\; \langle \nu, \langle \delta(-), g\rangle\rangle \;=\; \langle \nu, g\rangle,$$

since $\langle \delta(-), g\rangle = g(-) = g$. It remains to show associativity. Let $\xi \in VVVX$ and $g \in \mathrm{L}X$. Then

$$\begin{aligned} \langle \mathcal{E}(\mathcal{E}_*\xi), g\rangle &= \langle \mathcal{E}_*\xi, \langle -, g\rangle\rangle = \langle \xi, \langle \mathcal{E} -, g\rangle\rangle \\ &= \langle \xi, \langle -, \langle -, g\rangle\rangle\rangle = \langle \mathcal{E}\xi, \langle -, g\rangle\rangle \\ &= \langle \mathcal{E}\mathcal{E}\xi, g\rangle. \end{aligned}$$ □

We have proven the following statement.

Theorem 3.29. *The triple* $(V, \delta, \mathcal{E})$ *is a monad on* Top*.*

We will call $(V, \delta, \mathcal{E})$, or more briefly V, the *valuation monad.* By Remark 3.20, V is a strict 2-monad if we view Top as a 2-category (see Appendix A).

3.5. On the algebras

In general, algebras of probability monads have some convex structure, such that the algebra map can be interpreted as calculating the barycenter. For example, the algebras of the Radon monad on the category of compact Hausdorff spaces are the compact convex subsets of locally convex topological vector spaces [Świ74; Kei08b]. The algebras of the Kantorovich monad on the category of complete metric spaces are the closed convex subsets of Banach spaces [FP19]. If we drop the normalization of probability, as we do for V, then the algebra map $VA \to A$ can similarly be interpreted as assigning to every continuous valuation the integral of the identity function $A \to A$; in particular, such an algebra is a space where positive linear combinations of points can be evaluated to points. In contrast to the Radon and Kantorovich monads, a complete characterization of the category of algebras of V seems to be quite difficult. However, partial characterizations are possible. In work concurrent with ours, Goubault-Larrecq and Jia [GJ19] have proven that the algebras of V on the subcategory of T_0 spaces are topological cones in the sense of Keimel [Kei08a]. Topological cones generalize the properties of convex cones in topological vector spaces; in particular, addition may not be cancellative. We review Keimel's definition, and state some of the results of Goubault-Larrecq and Jia [GJ19]. For details we refer to the original papers [Kei08a; GJ19]. Just as for the monad H (see Section 2.6), the restriction to T_0 spaces does not lead to a real loss of generality, as explained below.

Definition 3.30 (The category of topological cones)**.** *A topological cone is a* T_0 *topological space* K *equipped with two operations.*

(a) An operation of addition $+ : K \times K \to K$*, jointly continuous, with a neutral element* $0 \in K$*.*

(b) An operation of scalar multiplication $\cdot : \mathbb{R}_{\geq 0} \times K \to K$*, jointly continuous (where* $\mathbb{R}_{\geq 0}$ *is equipped with the topology of lower semicontinuity).*

These operations satisfy the axioms of an $\mathbb{R}_{\geq 0}$*-semimodule with respect to the usual semiring structure of* $\mathbb{R}_{\geq 0}$*. A morphism of topological cones is a continuous map which preserves the* $\mathbb{R}_{\geq 0}$*-semimodule structure.*

Proposition 3.31 (Goubault-Larrecq and Jia [GJ19])**.** *Every V-algebra $e : VA \to A$ in the category of T_0 spaces admits a canonical topological cone structure given by*

$$a + b \;\coloneqq\; e(\delta_a + \delta_b) \qquad \text{and} \qquad r\,a \;\coloneqq\; e(r\,\delta_a),$$

where the sum and scalar multiplication on the right-hand sides are the pointwise sum and scalar multiplication of valuations.

Since VX is sober, and retracts of sober spaces are sober, every V-algebra is sober, in particular T_0. Therefore the theorem is true on the whole of Top.

Let A be a topological cone. We say that A is *cancellative* if it is cancellative as a monoid, which means that for all $a, b, c \in A$,

$$a + c \;=\; b + c \quad\Longrightarrow\quad a \;=\; b.$$

Topological lattices are the most prominent example of non-cancellative cones.

Example 3.32. Consider the lattice $W \coloneqq \{0, x, y, x \vee y\}$ with the topological cone structure where the sum is given by the join, $0w = 0$ and for and $\lambda w = w$ for all $w \in W$ and $\lambda > 0$, and the open sets are the upper sets. This is a topological cone, and it is clearly not cancellative. In fact it is a topological complete join-semilattice in the sense of Definition 2.38, and therefore an H-algebra by Theorem 2.39. A systematic way in which lattices become V-algebras is given in Section 5.3, where we show that every H-algebra is also canonically a V-algebra.

3.6. Product and marginal valuations

In applications of measure theory, especially in those to probability theory, it is crucial to form products of measures and to project measures on product spaces to their marginal measures on the factor spaces. Even the fundamental probabilistic concept of stochastic independence can be understood in these terms: A product measure exhibits independence if the product of its marginal measures equals itself. This section is concerned with the structure of products and marginals for the V-monad. We will show that V is a commutative monad (see Appendix C).

Instead of constructing products of valuations directly, it is much easier to equip the monad with the equivalent structure of a *commutative strength* (see Appendix C). This simpler approach is known to measure-theorists—even if not under this name. For example, a map corresponding to the strength has been used by Ressel in his study of products of τ-smooth Borel measures [Res77] (See our Corollary 4.20). Conceptually, the use of the strength is as old as the concept of product measures; it is, for example, implicit in Halmos' treatment of product measures [Hal50, Paragraph 35]. The content of this section is mostly a unified treatment of results due to Heckmann [Hec95].

The following definition uses the slice map $j_x : Y \to X \times Y$ from Notation 2.46.

Definition 3.33. *Let X and Y be topological spaces. Let $x \in X$ and $\nu \in VY$. We define $s(x, \nu) := (j_x)_*(\nu)$.*

As a functional maping $\mathrm{L}(X \times Y) \to [0, \infty]$, this continuous valuation is given by $g \mapsto \langle \nu, g \circ j_x \rangle$. The following statement and its proof are analogous to Proposition 2.48.

Proposition 3.34 (Proposition 11.1 in [Hec95])**.** *The assignment $s : X \times VY \to V(X \times Y)$ is continuous.*

Proof. Let $W \subseteq X \times Y$ be open, let $r \geq 0$, and consider the basic open set $\theta(W, r)$. We have to show that

$$s^{-1}(\theta(W, r)) \;=\; \{(x, \nu) \in X \times VY \mid \nu(W_x) > r\}$$

is open. We will show that any $(x, \nu) \in s^{-1}(\theta(W, r))$ is interior. For every $y \in W_x$, choose open neighborhoods $U_y \ni x$ and $V_y \ni y$ such that $U_y \times V_y \subseteq W$. We have $W_x = \bigcup_{y \in W_x} V_y$ and therefore, by Scott-continuity of ν, the assumption $\nu(W_x) > r$ implies that there are finitely many $y_1, \dots, y_n$ such that $\nu(\bigcup_{i=1}^n V_{y_i}) > r$. Consider the basic open subset

$$\left(\bigcap_{i=1}^n U_{y_i}\right) \times \theta\left(\bigcup_{i=1}^n V_{y_i}, r\right)$$

of $X \times VY$. By construction, this set contains (x, ν). Suppose $x' \in \bigcap_{i=1}^n U_{y_i}$ and $\nu' \in \theta(\bigcup_{i=1}^n V_{y_i}, r)$. Since $x' \in U_{y_i}$ for all i, we have $V_{y_i} \subseteq W_{x'}$ for all i too. Therefore

$$\nu'(W_{x'}) \;\geq\; \nu'\left(\bigcup_{i=1}^n V_{y_i}\right) \;>\; r,$$

and hence $s(x', \nu') \in \theta(W, r)$, as was to be shown. □

Just as Lemma 2.49, the following lemma allows us to simplify calculations by testing valuations only against products.

Lemma 3.35. *Let X and Y be topological spaces. A continuous valuation on $X \times Y$ is uniquely determined by the values it has on the open sets of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.*

More is true: A continuous valuation is uniquely determined by its values on any basis which is closed under intersections [Hec95, Proposition 3.2].

Proof. Given sets X, Y and subsets $A_1, \dots, A_n \subseteq X$ and $B_1, \dots, B_n \subseteq Y$, we have

$$\bigcap_{i=1}^n (A_i \times B_i) \;=\; \left(\bigcap_{i=1}^n A_i\right) \times \left(\bigcap_{i=1}^n B_i\right). \tag{3.5}$$

We will also make use of the n-ary modularity law for valuations,

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \;=\; \sum_{I\subseteq\{1,\dots,n\}} (-1)^{|I|+1}\,\nu\left(\bigcap_{j\in I} U_j\right) \tag{3.6}$$

for open sets $U_1,\dots,U_n$, which follows by induction from the binary modularity law of Definition 3.1(c) and the distributivity of intersections over unions.

The collection

$$\{U\times V \mid U\subseteq X \text{ and } V\subseteq Y \text{ are open }\}$$

is a basis of the product topology, hence every open subset of $X\times Y$ is a union of sets from that collection. Every union can be written as a directed union of finite unions. Therefore every open subset $W\subseteq X\times Y$ can be expressed as a directed union

$$W \;=\; \bigcup_{\alpha\in A} W_\alpha$$

where, for each α, the open set W_α is a finite union

$$W_\alpha \;=\; \bigcup_{i=1}^{n_\alpha} \big(U_{\alpha,i}\times V_{\alpha,i}\big)$$

where, for each α and i, the sets $U_{\alpha,i}\subseteq X$ and $V_{\alpha,i}\subseteq Y$ are open.

Let ν be a continuous valuation on $X\times Y$. By Scott continuity,

$$\nu(W) \;=\; \sup_{\alpha\in A}\nu(W_\alpha).$$

For each α, by the n-ary modularity law (3.6) and by (3.5),

$$\begin{aligned}
\nu(W_\alpha) &= \nu\left(\bigcup_{i=1}^{n_\alpha} U_{\alpha,i}\times V_{\alpha,i}\right)\\
&= \sum_{I\subseteq\{1,\dots,n_\alpha\}} (-1)^{|I|+1}\,\nu\left(\bigcap_{j\in I}\big(U_{\alpha,j}\times V_{\alpha,j}\big)\right)\\
&= \sum_{I\subseteq\{1,\dots,n_\alpha\}} \nu\left(\left(\bigcap_{j\in I} U_{\alpha,j}\right)\times\left(\bigcap_{j\in I} V_{\alpha,j}\right)\right).
\end{aligned}$$

In summary, for every open set $W\subseteq X\times Y$ we have an equation of the type

$$\nu(W) \;=\; \sup_{\alpha\in A}\sum_{I\subseteq\{1,\dots,n_\alpha\}} (-1)^{|I|+1}\,\nu\left(\left(\bigcap_{j\in I} U_{\alpha,j}\right)\times\left(\bigcap_{j\in I} V_{\alpha,j}\right)\right). \tag{3.7}$$

The claim follows because on the right-hand side, ν is evaluated only on open subsets of the desired form. □

Corollary 3.36. *A continuous valuation on the topological space Y is, as a functional on $\mathrm{L}(X \times Y)$, uniquely determined by its values on lower semicontinuous functions of the form $(x, y) \mapsto g(x)h(y)$ where $g : X \to [0, \infty]$ and $h : Y \to [0, \infty]$ are lower semicontinuous.*

Let X and Y be topological spaces. Let $x \in X$ and $\nu \in VY$. The valuation $s(x, \nu) \in V(X \times Y)$ evaluated on functions of the above form yields

$$\langle s(x, \nu), g \cdot h \rangle \;\coloneqq\; g(x) \cdot \langle \nu, h \rangle.$$

Just as we have done for H, we use this characterization to show that s is natural in both arguments. This means that for all continuous functions $f : X \to Z$ and $g : Y \to W$, the following two diagrams commute.

$$\begin{array}{ccc} X \times VY & \xrightarrow{\;s\;} & V(X \times Y) \\ \big\downarrow{\scriptstyle f \times \mathrm{id}} & & \big\downarrow{\scriptstyle (f \times \mathrm{id})_*} \\ Z \times VY & \xrightarrow{\;s\;} & V(Z \times Y) \end{array} \qquad \begin{array}{ccc} X \times VY & \xrightarrow{\;s\;} & V(X \times Y) \\ \big\downarrow{\scriptstyle \mathrm{id} \times g_*} & & \big\downarrow{\scriptstyle (\mathrm{id} \times g)_*} \\ X \times VW & \xrightarrow{\;s\;} & V(X \times W) \end{array}$$

To see why the first diagram commutes, let $k \in \mathrm{L}Z$ and $h \in \mathrm{L}Y$. Then, for each $x \in X$ and $\nu \in VY$, we have

$$\begin{aligned} \langle (f \times \mathrm{id})_* s(x, \nu), k \cdot h \rangle &= \langle s(x, \nu), (k \circ f) \cdot h \rangle \\ &= k(f(x)) \cdot \langle \nu, h \rangle \\ &= \langle s(f(x), \nu), k \cdot h \rangle. \end{aligned}$$

To see why the second diagram commutes, let $k \in \mathrm{L}X$ and $h \in \mathrm{L}W$. For each $x \in X$ and $\nu \in VY$, we have

$$\begin{aligned} \langle (\mathrm{id} \times g)_* s(x, \nu), k \cdot h \rangle &= \langle s(x, \nu), k \cdot (h \circ g) \rangle \\ &= k(x) \cdot \langle \nu, h \circ g \rangle \\ &= k(x) \cdot \langle g_* \nu, h \rangle \\ &= \langle s(x, g_* \nu), k \cdot h \rangle. \end{aligned}$$

Proposition 3.37. *The natural transformation s is a strength for the monad V.*

In other words, for all topological spaces X and Y, the following four diagrams commute. The first two involve the unitor u and the associator a of the (Cartesian) monoidal structure of Top, the others involve the structure maps of the monad.

$$\begin{array}{ccc} 1 \times VX & \xrightarrow{\;s\;} & V(1 \times X) \\ & {\scriptstyle u}\searrow^{\cong} & \big\downarrow{\scriptstyle u_*}\,{\scriptstyle \cong} \\ & & VX \end{array}$$

$$\begin{CD}
(X\times Y)\times VZ @>{s}>> V((X\times Y)\times Z)\\
@V{a}V{\cong}V @V{a_*}V{\cong}V\\
X\times(Y\times VZ) @>{\mathrm{id}\times s}>> X\times V(Y\times Z) @>{s}>> V(X\times(Y\times Z))
\end{CD}$$

$$\begin{array}{ccc}
X\times Y & \xrightarrow{\mathrm{id}\times\delta} & X\times VY\\
& \searrow^{\delta} & \downarrow{\scriptstyle s}\\
& & V(X\times Y)
\end{array}$$

$$\begin{CD}
X\times VVY @>{s}>> V(X\times VY) @>{s_*}>> VV(X\times Y)\\
@V{\mathrm{id}\times\mathcal{E}}VV @V{\mathcal{E}}VV\\
X\times VY @>{s}>> V(X\times Y)
\end{CD}$$

Proof. For the first diagram, let $\nu\in VX$ and $g\in \mathrm{L}X$. Denoting by $\bullet$ the unique point of 1, we have

$$\langle u_*(s(\bullet,\nu)),g\rangle \;=\; \langle(s(\bullet,\nu)),g\circ u\rangle \;=\; \langle\nu,g\rangle \;=\; \langle u(\bullet,\nu),g\rangle.$$

For the second diagram, let $f\in\mathrm{L}X$, $g\in\mathrm{L}Y$, and $h\in\mathrm{L}Z$. For each $x\in X$, $y\in Y$ and $\nu\in VZ$, we have

$$\begin{aligned}
\langle a_*(s((x,y),\nu)),f\cdot(g\cdot h)\rangle &= \langle(s((x,y),\nu)),(f\cdot(g\cdot h))\circ a\rangle\\
&= \langle(s((x,y),\nu)),(f\cdot g)\cdot h\rangle\\
&= (f\cdot g)(x,y)\cdot\langle\nu,h\rangle\\
&= f(x)\cdot(g(y)\cdot\langle\nu,h\rangle)\\
&= f(x)\cdot\langle s(y,\nu),g\cdot h\rangle\\
&= \langle s(x,s(y,\nu)),f\cdot(g\cdot h)\rangle,
\end{aligned}$$

which suffices by Corollary 3.36.

For the third diagram, let $g\in\mathrm{L}X$ and $h\in\mathrm{L}Y$. For each $x\in X$ and $y\in Y$, we have

$$\begin{aligned}
\langle s(x,\delta_y),g\cdot h\rangle &= g(x)\cdot\langle\delta_y,h\rangle\\
&= g(x)\cdot h(y)\\
&= (g\cdot h)(x,y)\\
&= \left\langle\delta_{(x,y)},g\cdot h\right\rangle.
\end{aligned}$$

For the last diagram, let $f\in\mathrm{L}X$ and $g\in\mathrm{L}Y$. For each $x\in X$ and $\psi\in VVY$, we have

$$\langle\mathcal{E}s_*s(x,\psi),g\cdot h\rangle \;=\; \langle s_*s(x,\psi),\langle -,g\cdot h\rangle\rangle$$

$$
\begin{aligned}
&= \langle s(x,\psi), \langle s-, g\cdot h\rangle\rangle \\
&= \langle s(x,\psi), g(-)\cdot\langle -, h\rangle\rangle \\
&= g(x)\cdot\langle \psi, \langle -, h\rangle\rangle \\
&= g(x)\cdot\langle \mathcal{E}\psi, h\rangle \\
&= \langle s(x,\mathcal{E}\psi), g\cdot h\rangle.
\end{aligned}
$$

□

We can define the costrength $t : VX \times Y \to V(X \times Y)$ by symmetry, which yields

$$\langle t(\nu, y), g\cdot h\rangle \;=\; \langle \nu, g\rangle \cdot h(y)$$

for all $\nu \in VX$, $y \in Y$, $g \in \mathrm{L}X$, and $h \in \mathrm{L}Y$. By symmetry, the costrength satisfies the properties analogous to those of Proposition 3.37.

The following proposition also defines the product valuations via the diagonal composite map of the diagram. Following Kock [Koc12, Section 5], this result can be thought of as a version of Fubini's theorem for continuous valuations.

Proposition 3.38. *The strength and costrength of V are compatible in the sense that the following diagram commutes.*

$$
\begin{array}{ccccc}
VX \times VY & \xrightarrow{\;s\;} & V(VX \times Y) & \xrightarrow{\;t_*\;} & VV(X \times Y) \\
\big\downarrow{\scriptstyle t} & & & & \big\downarrow{\scriptstyle \mathcal{E}} \\
V(X \times VY) & \xrightarrow{\;s_*\;} & VV(X \times Y) & \xrightarrow{\;\mathcal{E}\;} & V(X \times Y)
\end{array}
$$

Proof. Let $\nu \in VX$, $\rho \in VY$, $g \in \mathrm{L}X$ and $h \in \mathrm{L}Y$. Then

$$
\begin{aligned}
\langle \mathcal{E}t_* s(\nu,\rho), g\cdot h\rangle &= \langle t_* s(\nu,\rho), \langle -, g\cdot h\rangle\rangle \\
&= \langle s(\nu,\rho), \langle t-, g\cdot h\rangle\rangle \\
&= \langle s(\nu,\rho), \langle -, g\rangle \cdot h(-)\rangle \\
&= \langle \nu, g\rangle \cdot \langle \rho, h\rangle.
\end{aligned}
$$

A similar computation shows that $\langle \mathcal{E}s_* t(\nu,\rho), g\cdot h\rangle = \langle \nu, g\rangle \cdot \langle \rho, h\rangle$ as well. □

Corollary 3.39. $(V, \delta, \mathcal{E})$ *is a symmetric monoidal monad.*

As the proof above shows, the product of ν and ρ evaluates on functions of the form $g \cdot h$ as

$$g\cdot h \;\longmapsto\; \langle \nu, g\rangle \cdot \langle \rho, h\rangle.$$

On open sets, the product valuation assigns

$$U \times V \longmapsto \nu(U) \cdot \rho(V).$$

The universal property of the product makes $(V, \delta, \mathcal{E})$ an oplax, and therefore also a bilax, monoidal monad. The comultiplication $V(X \times Y) \to VX \times VY$ is the marginalization of valuations. The bilax monoidal structures of monads such as V admit a probabilistic interpretation [FP18].

4. The probability monad on Top

The first probability functor was defined by Lawvere [Law62]: It assigns to a measurable space X with σ-algebra Σ_X the space of probability measures on Σ_X endowed with the initial σ-algebra with respect to the family of evaluation maps $p \mapsto p(M)$ for $M \in \Sigma_X$. This functor on the category of measurable spaces carries a canonical monad structure which makes it into a probability monad [Gir82]. Due to the importance of topological concepts to measure theory and probability, such as weak convergence of measures, one can argue that a probability monad should be defined on a suitable category of topological spaces with respect to their Borel σ-algebras.

The subcategory of topological spaces that is traditionally considered in analytical settings is the category of Polish spaces. Giry [Gir82] also introduced a probability monad on the category of Polish spaces. Similar to the above, a Polish space X is mapped to the space of Borel probability measures on X equipped with the weak topology with respect to the integration maps $p \mapsto \int f \, dp$ for bounded continuous $f : X \to \mathbb{R}$.

A convenient subcategory for point-set topological purposes is the category of compact Hausdorff spaces. The Radon monad arising from the functor that assigns to a compact Hausdorff space the respective space of Borel probability measures equipped with the weak topology has been introduced by Swirszcz [Świ74]. A thorough treatment is due to Fedorchuk [Fed91].

Neither of these monads restricts to the other, since the full subcategories of Polish spaces and of compact Hausdorff spaces overlap only partially in Top. But they are both contained in the category of $T_{3\frac{1}{2}}$ spaces. This case has been treated by Banakh [Ban95], who studies the functor that assigns to a $T_{3\frac{1}{2}}$ space the space of inner regular and τ-smooth Borel probability measures endowed with the weak topology, as well as the respective monad structure.

In this section, we introduce the probability monad of τ-smooth (but not necessarily inner regular) Borel probability measures on all of Top. The underlying functor assigns to a topological space X the space of τ-smooth Borel probability measures on X equipped with the A-topology, which coincides with the weak topology with respect to bounded continuous functions whenever X is $T_{3\frac{1}{2}}$. Equipping this functor with a monad structure yields a generalization of all the probability monads mentioned above. Since a probability measure on a Polish space is automatically τ-smooth, our monad restricts to Giry's on Polish spaces. On compact Hausdorff spaces, the Radon probability measures are exactly the τ-smooth ones, and we recover the Radon monad. On $T_{3\frac{1}{2}}$ spaces, Banakh's monad is a proper submonad of ours.

While our generalization is of independent interest, it is also motivated by applications to theoretical computer science: the study of probabilistic nondeterminism in denotational semantics requires the treatment of probability measures on topological spaces which may not even be T_1, let alone $T_{3\frac{1}{2}}$. Such spaces arise for example as partially ordered sets equipped with an order-compatible topology [Gou13].

We finish this short review with a discussion of probability monads in metric settings. Breugel defined a probability monad on the category of compact metric spaces and 1-Lipschitz maps [Bre05]. The respective functor assigns to a compact metric space the space of its Borel probability measures endowed with the optimal transportation distance. Since the underlying space is compact, the optimal transportation distance induces the weak topology. This monad is a restriction of the probability monad of compact Hausdorff spaces of Swirszcz. However, the subcategory of topological spaces that is most convenient for geometrical purposes is the category of complete metric spaces and 1-Lipschitz maps. A probability monad on this category has been studied by Fritz and Perrone [FP19], based on a categorical construction which does not involve measure theory. Its underlying functor assigns to a complete metric space the space of Radon probability measures with finite first moment, the largest subset of the Radon probability measures metrized by the optimal transportation distance.

4.1. τ-smooth Borel measures

Definition 4.1 (τ-smooth Borel measure)**.** *Let X be a topological space. A Borel measure m on X is called τ-smooth if, for every directed net $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets of X,*

$$m\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigvee_{\lambda} m(U_\lambda).$$

A more common regularity assumption on measures is the Radon property. It is known that, in the category of Hausdorff spaces, every Radon measure is τ-smooth, and that, in the category of compact Hausdorff spaces, Radon measures and τ-smooth measures coincide [Bog00, Proposition 7.2.2]. In particular, all results of this paper hold for the case of Radon measures on Hausdorff spaces.

Every τ-smooth Borel measure restricts to a continuous valuation. We now elaborate on the converse statement.

Definition 4.2. *Let ν be a continuous valuation on a topological space X. We say that ν is extendable, or that it extends to a measure, if there exists a Borel measure m such* *that, for every open set $U \subseteq X$, we have $m(U) = \nu(U)$.*

Since a Borel measure is uniquely determined by its values on open sets, such an extension, if it exists, is unique and τ-smooth.

Proposition 4.3. *Let X be a topological space. Suppose that a continuous valuation $\nu \in VX$ is extendable to a Borel measure m on X. Then, for every lower semicontinuous function $g \in \mathrm{L}X$, we have*

$$\langle \nu, g\rangle \;=\; \int_X g\, dm.$$

To prove the statement, we use the following standard result. Note that all functions in $\mathrm{L}X$ are measurable and nonnegative, therefore their Lebesgue integral against a probability measure is either a well-defined nonnegative number or $+\infty$.

Proposition 4.4 (Corollary 414B.a in [Fre06])**.** *Let X be a topological space and μ a τ-smooth Borel measure on X. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a directed set in $\mathrm{L}X$. Define $f(x) := \sup_\lambda f_\lambda(x)$ for all $x \in X$. Then f is lower semicontinuous and*

$$\int f\, d\mu \;=\; \sup_\lambda \int f_\lambda\, d\mu.$$

Proof of Proposition 4.3. If g is simple, the two quantities agree by definition of integral of a simple function (both for measures and for valuations), using the fact that ν and m agree on open sets. If g is not simple, we use the defining supremum of $\langle \nu, g\rangle$ and Proposition 4.4 to obtain the desired equality. ☐

If X is not sober, a continuous valuation need not be extendable, as the following counterexample illustrates. It is based on the fact that the $\{0,\infty\}$-valued continuous valuations with $\nu(X) = \infty$ correspond to completely prime filters on $\mathcal{O}(X)$.

Example 4.5. Let X be the set $(0, 1)$ with the topology whose open subsets are in the form $(a, 1)$ for $a \in [0, 1]$. The Borel σ-algebra of X coincides with the Borel σ-algebra of $(0, 1)$ with its usual topology. Consider the continuous valuation $\nu : \mathcal{O}(X) \to [0,\infty]$ given by

$$\nu(U) \;:=\; \begin{cases} 0 & \text{if } U = \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that there exists a Borel measure m on X which agrees with ν on the open sets. Consider the set $(a, b]$ for $0 < a < b < 1$. We have

$$m\big((a, b]\big) \;=\; m\big((a, 1)\backslash(b, 1)\big) \;=\; m\big((a, 1)\big) - m\big((b, 1)\big) \;=\; 1 - 1 \;=\; 0.$$

The space X can be expressed as a countable disjoint union,

$$X \;=\; (0, 1) \;=\; \bigsqcup_{n=1}^{\infty}\Big(1 - \frac{1}{n}\;,\; 1 - \frac{1}{n+1}\Big].$$

Note that $m(X) = \nu(X) = 1$, while

$$\sum_{n=1}^{\infty} m\left(\left(1 - \frac{1}{n}\,,\, 1 - \frac{1}{n+1}\right]\right) \;=\; \sum_{n=1}^{\infty} 0 \;=\; 0.$$

Therefore m is not countably additive.

The interpretation of the above example is that ν represents a Dirac mass at 1, a point in the sobrification of X. If we sobrify X to $(0, 1]$ by including the point 1, then ν can be extended to δ_1.

It is known that, on a $T_{3\frac{1}{2}}$ space, every finite continuous valuation extends to a measure [AES98]. The same holds on spaces that are sober and locally compact [Alv02]. In particular, the sets of finite τ-smooth Borel measures and of finite continuous valuations are in bijection for all locally compact Hausdorff spaces and for all metric spaces. Whether extensions exist for all continuous valuations on sober spaces seems to be an open question.

4.2. The A-topology

We will construct our monad P as a submonad of the continuous valuations monad V from Section 3.

Definition 4.6. *Let X be a topological space. We define the space PX to be the set of τ-smooth Borel probability measures on X, equipped with the subspace topology inherited from VX, which is generated by sets of the form*

$$O(U, r) \coloneqq \{q : \; q(U) > r\}$$

for open $U \subseteq X$ and $r \in [0, \infty)$.

In other words, the A-topology is the weakest topology which makes the evaluation maps $p \mapsto p(U)$ lower semicontinuous. By Proposition 3.7 and Proposition 4.3, we have the following well-known result [Top70, Theorem 8.1(iii)-(iv)].

Corollary 4.7. *Let X be a topological space and let $f : X \to \mathbb{R}$ be bounded and lower semicontinuous. The map $PX \to \mathbb{R}$ given by $p \mapsto \int f\, dp$ is lower semicontinuous.*

The topology of Definition 4.6 is called the *A-topology* [Bog00, 8.10(iv)] after Pavel Alexandrov, who was the first to prove a result of the following type. Recall that a set is *functionally open* if it is the preimage of an open set under a continuous real-valued function. The functionally open sets generate the *Baire σ-algebra* which carries the *Baire measures.*

Theorem 4.8 (Alexandrov's theorem, e.g. Theorem 8.2.1 in [Bog00])**.** *Let X be a topological space and let $\mathrm{B}X$ be the set of Baire probability measures. Then the weak topology on $\mathrm{B}X$ with respect to integration against bounded continuous functions coincides with the topology generated by the sets $O(U, r)$ for functionally open U.*

For a thorough study of the A-topology in the case of a Hausdorff space, consult the monographs by Topsøe [Top70, Part II] and Bogachev [Bog00, Section 8.10(iv)]. Note that Topsøe calls the A-topology the weak topology.

Corollary 4.9. *Let X be a $T_{3\frac{1}{2}}$ space. Then the weak topology with respect to bounded continuous functions and the A-topology on PX coincide.*

Proof. In a $T_{3\frac{1}{2}}$ space, every open set is functionally open which implies that the Borel and Baire σ-algebras coincide. By the classical Alexandrov theorem (Theorem 4.8), the respective topologies coincide as well. □

In general, in order for the weak topology to be well-behaved, one either requires the space to be at least $T_{3\frac{1}{2}}$, or one restricts to Baire measures. The A-topology, instead, is well-behaved on all spaces and all Borel measures, having the properties of a pointwise topology on the space of functionals on open sets. This is why we work with the A-topology. In any case, due to Corollary 4.9, all results stated for the A-topology hold for the weak topology in the $T_{3\frac{1}{2}}$ case. This includes all metric spaces and all compact Hausdorff spaces.

4.3. Functoriality

For topological spaces X and Y, every continuous $f : X \to Y$ is Borel measurable. Therefore, for $p \in PX$, we have the pushforward measure $f_*p \in PY$ defined by $(f_*p)(B) := p(f^{-1}(B))$ for all Borel sets $B \subseteq Y$. It is easy to see that p_*f is again τ-smooth. Suppose further that $p \in PX$ restricts to $\nu \in VX$. Then we have

$$(f_*\nu)(U) = \nu\big(f^{-1}(U)\big) = p\big(f^{-1}(U)\big) = (f_*p)(U),$$

where the first equation holds by Definition 3.14. Hence f_*p restricts to $f_*\nu$. We have proven the following proposition.

Proposition 4.10. *P is a subfunctor of V.*

We can further conclude from Lemma 3.19 that if $i : Y \hookrightarrow X$ is a subspace embedding, then so is $i_* : PY \hookrightarrow PX$.

4.4. Monad structure

Here we prove that P can be extended to a submonad of $(V, \delta, \mathcal{E})$. Since P is already a subfunctor of V, this monad structure is necessarily unique: we only need to show that both the unit and the multiplication of V restrict to P.

Consider first the unit $\delta : X \to VX$. Since the Dirac valuation $\delta_x(U) = \mathbb{1}_U(x)$ for open $U \subseteq X$ obviously extends to the Dirac measure $\delta_x(B) := \mathbb{1}_B(x)$ for Borel sets $B \subseteq X$, the map δ factors through the inclusion $PX \hookrightarrow VX$. We also write $\delta : X \to PX$ by abuse of notation.

In many cases, the unit $\delta : X \to PX$ is a closed embedding.

Theorem 4.11 (Theorem 11.1 of [Top70]). *Let X be a Hausdorff space. Then the map $\delta : X \to PX$ is a closed embedding.*

The remaining issue in showing that P is a submonad of V is to prove that the multiplication $\mathcal{E} : VVX \to VX$ restricts to a map $\mathrm{E} : PPX \to PX$. To simplify the exposition, we construct $\mathrm{E} : PPX \to PX$ first and then show that E is indeed the restriction of $\mathcal{E}$.

Proposition 4.12. *Let X be a topological space and let $A \subseteq X$ be a Borel set. Then the evaluation map $\varepsilon_A : PX \to \mathbb{R}$ given by $p \mapsto p(A)$ is Borel measurable.*

Proof. If $A \subseteq X$ is open, then ε_A is lower semicontinuous by definition of the A-topology on PX, and therefore also Borel measurable.

In general, we denote by Σ the set of Borel measurable $A \subseteq X$ for which $\varepsilon_A : PX \to \mathbb{R}$ is measurable.

To see that Σ is closed under countable unions, suppose that a measurable set $A \subseteq X$ can be written as a countable union of disjoint measurable subsets,

$$B = \bigcup_{n=1}^{\infty} A_n,$$

such that $A_n \in \Sigma$ for every n. Then, for each $p \in PX$,

$$\varepsilon_A(p) \;=\; p(A) \;=\; \sum_{n=1}^{\infty} p(A_n) \;=\; \sum_{n=1}^{\infty} \varepsilon_{A_n}(p).$$

Since pointwise suprema of measurable functions are measurable, and every partial sum on the right is measurable in p, we conclude that ε_A is also measurable in p. Therefore Σ is closed under countable disjoint unions. Consider $A, B \in \Sigma$ with $A \subseteq B$. Clearly $\varepsilon_{B\setminus A} = \varepsilon_B - \varepsilon_A$ is also measurable. Therefore Σ is closed under complements. This makes Σ into a Dynkin system. Since it contains the π-system of open subsets, the π-λ theorem implies that Σ is the σ-algebra of Borel sets. □

Definition 4.13. *Let X be a topological space and $\mu \in PPX$. Let $A \subseteq X$ be Borel measurable. We define*

$$(\mathrm{E}\mu)(A) := \int_{PX} p(A)\, d\mu(p).$$

The integrand $p \mapsto p(A) \in [0, 1]$ is measurable by Proposition 4.12 and bounded. Therefore the integral exists.

Proposition 4.14. *Let X be a topological space and consider $\mu \in PPX$. Then the assignment $A \mapsto (\mathrm{E}\mu)(A)$ is a τ-smooth probability measure on X.*

Proof. The nontrivial properties to establish are σ-additivity and τ-smoothness. For the former, let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of disjoint measurable subsets of X and let A be their union. Then

$$\begin{aligned}(\mathrm{E}\mu)(A) &= \int_{PX} p(A)\, d\mu(p) = \int_{PX} \left(\sum_{n=1}^{\infty} p(A_n)\right) d\mu(p) \\ &= \sum_{n=1}^{\infty} \int_{PX} p(A_n)\, d\mu(p) = \sum_{n=1}^{\infty} \mathrm{E}\mu(A_n),\end{aligned}$$

where the third step can be thought of as an application of Fubini's theorem to $PX \times \mathbb{N}$.

Now we turn to τ-smoothness, which follows from the monotone convergence theorem for integrals of directed nets. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a directed net of open sets with union U. Then, since every measure $p \in PX$ is τ-smooth,

$$\begin{aligned}(\mathrm{E}\mu)(U) &= \int_{PX} p(U)\, d\mu(p) = \int_{PX} \sup_\lambda p(U_\lambda)\, d\mu(p) \\ &= \sup_\lambda \int_{PX} p(U_\lambda)\, d\mu(p) = \sup_\lambda\, (\mathrm{E}\mu)(U_\lambda),\end{aligned}$$

as was to be shown. □

Hence we have a well-defined map $\mathrm{E} : PPX \to PX$. The next proposition will also prove that this map is continuous.

The inclusion $\iota : PX \hookrightarrow VX$ is a subspace embedding. Therefore so is $\iota_* : VPX \hookrightarrow VVX$ by Lemma 3.19. By composing these two embeddings, we consider PPX as a subspace of VVX.

Proposition 4.15. *Let X be a topological space. Then the following diagram commutes.*

$$\begin{array}{ccccc} PPX & \xrightarrow{\iota} & VPX & \xrightarrow{\iota_*} & VVX \\ \big\downarrow{\scriptstyle \mathrm{E}} & & & & \big\downarrow{\scriptstyle \mathcal{E}} \\ PX & & \xrightarrow{\qquad\quad \iota \quad\qquad} & & VX \end{array}$$

Proof. Let $\mu \in PPX$ and let $\nu \in VVX$ be its image. Then we show that $\mathcal{E}\nu$ extends to the measure $\mathrm{E}\mu$, by showing that they evaluate to the same number on any open set $U \subseteq X$,

$$\begin{aligned}(\mathcal{E}\nu)(U) &= \langle \mathcal{E}\nu, \mathbb{1}_U\rangle = \langle \nu, \langle -, \mathbb{1}_U\rangle\rangle = \int_{VX} \langle \rho, \mathbb{1}_U\rangle\, d\mu(\rho) \\ &= \int_{PX} \varepsilon_U(p)\, d\mu(p) = \int_{PX} p(U)\, d\mu(p) = (\mathrm{E}\mu)(U).\end{aligned}$$

□

Since the unit and the multiplication of V restrict to P, the equations required for P to be a monad follow automatically from those of V. In particular, we do not need to prove the associativity of E. We have proven the following theorem.

Theorem 4.16. *The triple* (P, δ, E) *is a submonad of* $(V, \delta, \mathcal{E})$ *on* Top.

Just as V, the monad P is a strict 2-monad if we consider Top as a 2-category.

Through the submonad embedding, every algebra of V is also an algebra of P in a canonical way, resulting in a faithful functor from V-algebras to P-algebras. The question whether this forgetful functor is full or essentially surjective is related to the extension problem and is, at present, unanswered.

4.5. Product and marginal probabilities

Here, we show that P is a commutative monad as well by equipping it with a commutative strength transformation $s : X \times PY \to P(X \times Y)$. This in particular implies that we have a natural transformation $PX \times PY \to P(X \times Y)$ implementing the formation of product probability measures. This is a non-trivial statement due to the discrepancy between the Borel σ-algebra on $X \times Y$ and the product of the Borel σ-algebras on X and Y, and even if the Borel product measure exists, it is a priori unclear whether it is τ-smooth.

As for the monad multiplication, this strength $X \times PY \to P(X \times Y)$ is inherited from the strength of V. And while we only need to prove that the strength of V restricts to P, it is more convenient to first construct the map $X \times PY \to P(X \times Y)$ explicitly.

Lemma 4.17. *Let X and Y be topological spaces, and let $A \subseteq X \times Y$ be a Borel subset. Then, for each $x \in X$, the slice $A_x := \{y \in Y \mid (x, y) \in A\}$ is a Borel subset of Y.*

Proof. The map $j_x : Y \to X \times Y$ of Notation 2.46 is continuous. Hence it is Borel-measurable. In particular, preimages of Borel sets are Borel. □

Now for given $x \in X$ and $p \in PY$, define the Borel probability measure $\delta_x \otimes p$ on $X \times Y$ as follows. For every Borel set $A \subseteq X \times Y$, set

$$(\delta_x \otimes p)(A) \;:=\; p(A_x).$$

This is a τ-smooth probability measure because taking the slice preserves arbitrary unions, intersections, and complements.

By abuse of notation, we also write $s : X \times PY \to P(X \times Y)$ as we do for $s : X \times VY \to V(X \times Y)$. The following guarantees that this will not lead to confusion.

Proposition 4.18. *Let X and Y be topological spaces. Then the following diagram commutes.*

$$\begin{array}{ccc} X \times PY & \xrightarrow{\mathrm{id} \times \iota} & X \times VY \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ P(X \times Y) & \xrightarrow{\iota} & V(X \times Y) \end{array}$$

Proof. Suppose that $p \in PY$ and denote $\nu := \iota(p)$. For every open $W \subseteq X \times Y$, we have

$$(\delta_x \otimes p)(W) \;=\; p(W_x) \;=\; \nu(W_x) \;=\; s(x,\nu)(W),$$

as was to be shown. ☐

It is clear that $s : X \times PY \to P(X \times Y)$ inherits the naturality and strength properties from the strength of V. Thus we have shown the following.

Corollary 4.19. *The strength s of V restricts to a strength of the submonad P which makes P into a commutative monad.*

The commutativity of the strength of P is Fubini's theorem [Koc12].

Therefore P is also a symmetric monoidal monad (Appendix C), and the operation of forming product measures is a natural transformation $PX \times PY \to P(X \times Y)$.

Corollary 4.20 (Theorem 1 in [Res77])**.** *The product of two τ-smooth Borel measures on any two topological spaces X and Y extends to a τ-smooth Borel measure on the product space $X \times Y$.*

Moreover, since ι is a morphism of commutative monads by construction, Proposition C.5 implies that ι is also a morphism of monoidal monads.

Corollary 4.21. *The inclusion $\iota : P \to V$ is a monoidal natural transformation.*

In other words, the following diagram commutes, which implies that the product of two extendable valuations is extendable.

$$\begin{array}{ccc} PX \times PY & \xrightarrow{\nabla} & P(X \times Y) \\ \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \iota} \\ VX \times VY & \xrightarrow{\nabla} & V(X \times Y) \end{array}$$

5. The support as a morphism of monads

We now consider the support of a continuous valuation, or more specifically of a τ-smooth Borel measure. Our main result is that taking supports is a natural transformation supp : $V \Rightarrow H$ which is a morphism of commutative monads. Since $P \subseteq V$ is a commutative submonad, it follows that taking the support of a τ-smooth probability measure is an operation described by a morphism of commutative monads supp : $P \Rightarrow H$. The definition of support of a valuation is straightforward and analogous to the one of a measure, but it seems that it has not yet appeared in the literature.

Intuitively, the support is the set of points of positive mass. The commonly used notion of support is the following one, defined for a finite Borel measure m on a space X. A measurable set $A \subseteq X$ has full m-measure if and only if $m(X \setminus A) = 0$, or, equivalently, if $m(A) = m(X)$.

Definition 5.1. *Let X be a topological space and let m be a Borel measure on X. The support* $\operatorname{supp}(m)$ *of m is the intersection of all closed subsets of X that have full m-measure.*

The support is the intersection of closed sets and therefore closed. The support of τ-smooth measures is particularly well-behaved, as the following result shows.

Proposition 5.2 (Proposition 7.2.9 of [Bog00])**.** *Let X be a topological space and let m be a τ-smooth Borel measure on X. Then* $\operatorname{supp}(m)$ *has full measure.*

Proof. The open set $U = X \setminus \operatorname{supp}(m)$ is the union of the family $(U_\lambda)_{\lambda\in\Lambda}$ of all open sets of measure zero. Given any two open sets of measure zero, their union has measure zero too, hence $(U_\lambda)_{\lambda\in\Lambda}$ is a directed net. The τ-smoothness of m implies

$$m(U) = \sup_\lambda m(U_\lambda) = 0.$$

Therefore $\operatorname{supp}(m) = X \setminus U$ has full measure. □

It is clear from the proof that the support has full measure *if and only if* m is τ-smooth on the open null sets. The following standard example, due to Dieudonné, shows that the support of a measure that is not τ-smooth may not have full measure.

Example 5.3 (Dieudonné measure, Example 7.1.3 in [Bog00])**.** We consider the initial segment of the ordinal numbers $X = [0, \omega_1]$, up to and including the first uncountable ordinal ω_1. We equip this totally ordered set with the topology generated by intervals of the form $\{x : a < x\}$, $\{x : x < b\}$, or $\{x : a < x < b\}$ for some $a, b \in X$. The Dieudonné measure is the Borel measure on X defined as

$$m(B) := \begin{cases} 1 & \text{if there exists } F \subseteq B \setminus \{\omega_1\} \text{ which is closed and uncountable} \\ 0 & \text{otherwise.} \end{cases}$$

The measure m is Borel [Bog00, Examples 7.1.3 and 6.1.21]. The family $([0, x))_{x<\omega_1}$ is a directed net of open sets. We have $m([0, x)) = 0$, since $[0, x)$ is countable for every $x < \omega_1$. But $\bigcup_{x<\omega_1}[0, x)$ is uncountable. We have

$$0 = \sup_{x<\omega_1} m([0, x)) < m\left(\bigcup_{x<\omega_1} [0, x)\right) = 1.$$

Hence the Dieudonné measure is not τ-smooth. Since every closed interval of the form $[x, \omega_1]$ for $x < \omega_1$ has full measure, we have $\operatorname{supp}(m) = \{\omega_1\}$. But since $m(\{\omega_1\}) = 0$, the support does not have full measure.

Due to the above example, some authors, for example Bogachev [Bog00], require the support to have full measure by definition. In this case, some measures do not have a support. Since this work only treats τ-smooth measures (except in counterexamples), the difference of definitions will not lead to ambiguity.

Corollary 5.4. *Let X be a topological space, let m be a τ-smooth Borel measure on X, and let $U \subseteq X$ be open. Then $m(U) > 0$ if and only if $U \cap \operatorname{supp}(m) \neq \varnothing$.*

We turn to the case of valuations. In view of the discussion in Section 2.1 and Corollary 5.4, we give the following definition of the support of a continuous valuation.

Definition 5.5 (Support of a continuous valuation)**.** *Let X be a topological space and consider $\nu \in VX$. Then, for every open set $U \subseteq X$, the support of ν hits U if and only if $\nu(U) > 0$,*

$$\langle \operatorname{supp}(\nu), U \rangle \ := \ [\![\nu(U) > 0]\!].$$

It is straightforward to verify that this indeed defines a closed set $\operatorname{supp}(\nu) \in HX$ via the duality of Proposition 2.14. By definition, the support of an extendable valuation equals the support of its extension as defined in Definition 5.1.

We can alternatively define the support in the following way. Consider the sign function $\operatorname{sgn} : [0, \infty] \to \{0, 1\}$ defined as

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the support is characterized by

$$\langle \operatorname{supp}(\nu), U \rangle := \operatorname{sgn}(\langle \nu, \mathbb{1}_U \rangle). \tag{5.1}$$

Proposition 5.6. *Let X be a topological space, $\nu \in VX$ a continuous valuation, and $g \in \mathrm{L}X$. Then the following two statements are equivalent.*

(a) $\langle \nu, g \rangle > 0$

(b) There exists $x \in \operatorname{supp}(\nu)$ such that $g(x) > 0$.

In other words,

$$\operatorname{sgn}(\langle \nu, g \rangle) = \big\langle \operatorname{supp}(\nu), g^{-1}((0, \infty]) \big\rangle. \tag{5.2}$$

Proof. (a) $\implies$ (b): Let $\langle g, \nu \rangle > 0$. By the definition of the pairing, the lower integral, there exists a simple lower semicontinuous function $g' = \sum_{i=1}^{n} r_i \, \mathbb{1}_{U_i}$ dominated by g such that

$$\langle \nu, g' \rangle = \sum_i r_i \, \nu(U_i) > 0.$$

We conclude that there exists U_i with $\nu(U_i) > 0$ and $r_i > 0$ such that $g > r_i \, \mathbb{1}_{U_i}$, which implies in particular that g is strictly positive on all of U_i. Since $\nu(U_i) > 0$, the support hits U_i. These two facts prove the claim.

(b) $\implies$ (a): Choose any $r > 0$ with $g(x) > r$. Since g is lower semicontinuous, there exists an open neighborhood $U \ni x$ such that $g|_U > r$ as well. We have

$$\langle \nu, g \rangle \geq \langle \nu, g \cdot \mathbb{1}_U \rangle \geq r \cdot \nu(U) > 0.$$

□

5.1. Continuity and naturality

Proposition 5.7. *Let X be a topological space and let $U \subseteq X$ be open. Then*

$$\operatorname{supp}^{-1}(\operatorname{Hit}(U)) \;=\; \theta(U, 0).$$

Proof. We have $\nu \in \operatorname{supp}^{-1}(\operatorname{Hit}(U))$ if and only if $\operatorname{supp}(\nu) \in \operatorname{Hit}(U)$, which, by Definition 5.5, is equivalent to $\nu(U) > 0$. Therefore

$$\operatorname{supp}^{-1}(\operatorname{Hit}(U)) \;=\; \big\{\nu \in VX \mid \nu(U) > 0\big\} \;=\; \theta(U, 0). \qquad \square$$

Corollary 5.8. *The map* $\operatorname{supp} : VX \to HX$ *is continuous.*

Similar statements for different choices of topologies are known (for example Theorem 17.1 in [AB06]).

We now turn to naturality, which relies on the Scott continuity of valuations.

Proposition 5.9. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then, for every $\nu \in VX$,*

$$\operatorname{supp}(f_*\nu) = f_\sharp \operatorname{supp}(\nu).$$

In other words, the following diagram commutes.

$$\begin{array}{ccc} VX & \xrightarrow{\operatorname{supp}} & HX \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_\sharp} \\ VY & \xrightarrow{\operatorname{supp}} & HY \end{array}$$

Proof. Let $U \subseteq Y$ be open. Then

$$\begin{aligned} \langle \operatorname{supp}(f_*\nu), U\rangle &= \operatorname{sgn}(\langle f_*\nu, \mathbb{1}_U\rangle) = \operatorname{sgn}(\langle \nu, \mathbb{1}_U \circ f\rangle) = \operatorname{sgn}(\big\langle \nu, \mathbb{1}_{f^{-1}(U)}\big\rangle) \\ &= \big\langle \operatorname{supp}(\nu), f^{-1}(U)\big\rangle = \langle f_\sharp(\operatorname{supp}(\nu)), U\rangle. \end{aligned} \qquad \square$$

Therefore the support induces a natural transformation $\operatorname{supp} : V \Rightarrow H$, which restricts to a natural transformation $\operatorname{supp} : P \Rightarrow H$.

On the set of *all* Borel measures, the support as given in Definition 5.1 is not natural. We give an example using the Dieudonné measure of Example 5.3.

Example 5.10 (Dieudonné measure, continued)**.** Consider $Y = [0, \omega_1)$, the space of countable ordinals (which can be identified with ω_1). We equip this totally ordered set with the subspace topology of X from Example 5.3 and consider the Dieudonné measure m from Example 5.3, which is not τ-smooth. The measure m restricted to Y has $\operatorname{supp}(m) = \varnothing$. Consider the map $f : Y \to \{1\}$ into the singleton space. We have the following diagram.

$$\begin{array}{ccc} m & \xmapsto{\operatorname{supp}} & \varnothing \\ \big\downarrow{\scriptstyle f_*} & & \searrow{\scriptstyle f_\sharp} \\ \delta_1 & \xmapsto{\operatorname{supp}} & \{1\} \neq \varnothing \end{array}$$

The following counterexample shows that the notion of support is also not natural for signed measures, which is why we do not expect a generalization of our results to the signed case.

Example 5.11. Consider the two-point space $\{a, b\}$ with the finite signed measure $m := \delta_a - \delta_b$. The map into the singleton space $f : \{a, b\} \to 1$ yields $f_* m = 0$. The usual definition of the support of a signed measure leads to $\text{supp}(m) = \{a, b\}$, and hence $f_\sharp(\text{supp}(m)) = 1$. But we have $\text{supp}(f_* m) = \varnothing$.

5.2. The support is a morphism of monads

Proposition 5.12. *For any topological space X, the following two diagrams commute.*

$$\begin{array}{ccc} & & VX \\ & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle \text{supp}} \\ X & & \\ & \underset{\sigma}{\searrow} & \\ & & HX \end{array} \qquad \begin{array}{ccc} VVX & \xrightarrow{\mathcal{E}} & VX \\ \downarrow{\scriptstyle\text{supp}} & & \\ HVX & & \downarrow{\scriptstyle\text{supp}} \\ \downarrow{\scriptstyle\text{supp}_\sharp} & & \\ HHX & \xrightarrow{\mathcal{U}} & HX \end{array} \tag{5.3}$$

We refer to these diagrams as the *unit* and *multiplication* diagram, respectively, since they express the compatibility of the support map with the respective monad maps. The unit diagram says that the support of the Dirac valuation δ_x equals $\text{cl}(\{x\})$. The multiplication diagram says: If the valuation $\nu \in VX$ is the integral of the valuation $\xi \in VVX$, then the support of ν is the closure of the union of the supports of all the valuations in the support of ξ.

To illustrate this statement in the simplest case, suppose that X is finite and discrete and that $\xi \in VVX$ is finitely supported. Then VX can be identified with the simplex of probability vectors $(p_x)_{x \in X}$ and $\mathcal{E}\xi$ is a finite convex combination of these. The statement is then that the set of nonzero components of $\mathcal{E}\xi$ coincides with the union of the sets of nonzero components in each term contributing to the convex combination. In the discrete case this statement is straightforward to prove, however, to the best of our knowledge, it has never appeared in a published document. (We did receive an independent proof of the statement for the finite case from G. van Heerdt, J. Hsu, J. Ouaknine and A. Silva in a personal communication.) Our result can be thought of as a generalization to continuous distributions.

Proof of Proposition 5.12. Considering the unit diagram, we have, for any $x \in X$ and any open $U \subseteq X$,

$$\langle \text{supp}(\delta_x), U \rangle \;=\; [\![\delta_x(U) > 0]\!] \;=\; [\![x \in U]\!] \;=\; \langle \sigma(x), U \rangle.$$

Considering the multiplication diagram, let $\xi \in VVX$ and let $U \subseteq X$ be open. Using (5.1) and (5.2), we obtain

$$\langle \text{supp}(\mathcal{E}\xi), U \rangle \;=\; \text{sgn}(\langle \mathcal{E}\xi, \mathbb{1}_U \rangle)$$

$$
\begin{aligned}
&= \operatorname{sgn}(\langle \xi, \langle -, \mathbb{1}_U \rangle \rangle) \\
&= \langle \operatorname{supp}(\xi), \operatorname{sgn}(\langle -, \mathbb{1}_U \rangle \rangle \\
&= \big\langle \operatorname{supp}(\xi), \operatorname{supp}^{-1}(\operatorname{Hit}(U)) \big\rangle \\
&= \big\langle \operatorname{supp}_\sharp(\operatorname{supp}(\xi)), U \big\rangle \\
&= \big\langle \mathcal{U}(\operatorname{supp}_\sharp(\operatorname{supp}(\xi))), U \big\rangle. \qquad \square
\end{aligned}
$$

Corollary 5.13. *The support induces a morphism of monads* $\operatorname{supp} : (V, \delta, \mathcal{E}) \to (H, \sigma, \mathcal{U})$.

5.3. Consequences for algebras

It is well-known that a morphism of monads on the same category induces a functor between the respective categories of algebras in the opposite direction.

Proposition 5.14. *Let* (S, η_S, μ_S) *and* (T, η_T, μ_T) *be monads on the category* C *and let* $m : S \Rightarrow T$ *be a morphism of monads. Then every* T*-algebra* (A, a) *can be equipped with an* S*-algebra structure via* $(A, a) \mapsto (A, a \circ m)$*. Moreover, a* T*-algebra morphism* $f : (A, a) \to (B, b)$ *induces an* S*-algebra morphism* $f : (A, a \circ m) \to (B, b \circ m)$ *in a functorial way.*

The above combined with Corollary 5.13 yields the following.

Corollary 5.15. *Every* H*-algebra is a* V*-algebra, and therefore also a* P*-algebra, in a canonical way. Concretely, if* (A, a) *is an* H*-algebra, then* $(A, a \circ \operatorname{supp})$ *is a* V*-algebra.*

The algebras of H are the complete topological join-semilattices (Section 2.6). Hence the following statement.

Corollary 5.16. *Every topological complete join-semilattice is a* V*-algebra with structure map* $\nu \mapsto \bigvee \operatorname{supp}(\nu)$.

Every V-algebra is also a topological cone (Section 3.5). Hence the following statement.

Corollary 5.17. *Every topological complete join-semilattice* A *is a topological cone with addition given by the binary join, and scalar multiplication*

$$
(\lambda, x) \longmapsto \begin{cases} x & \text{if } \lambda > 0 \\ \bot & \text{if } \lambda = 0, \end{cases}
$$

which is jointly continuous in both arguments.

Clearly a nontrivial cone cannot be embedded into a vector space. This is analogous to Example 4.4 and Proposition 4.14 of Goubault-Larrecq and Jia [GJ19], as well as to the convex spaces of combinatorial type [Fri09].

5.4. The support of products and marginals

The support is not only a morphism of monads, but also respects the monoidal structures.

Proposition 5.18. *Let X and Y be topological spaces. The following diagram commutes.*

$$\begin{array}{ccc} X \times VY & \xrightarrow{s} & V(X \times Y) \\ {\scriptstyle \mathrm{id}\times\mathrm{supp}}\downarrow & & \downarrow{\scriptstyle \mathrm{supp}} \\ X \times HY & \xrightarrow{s} & H(X \times Y) \end{array}$$

Proof. Consider $x \in X$ and $\rho \in VY$ and let $U \subseteq X$ and $V \subseteq Y$ be open. Since the sign function is multiplicative, we have

$$\begin{aligned} \langle \mathrm{supp}(s(x,\rho)), U \times V \rangle &= \mathrm{sgn}(\langle s(x,\rho), \mathbb{1}_{U\times V} \rangle) \\ &= \mathrm{sgn}(\langle s(x,\rho), \mathbb{1}_U \cdot \mathbb{1}_V \rangle) \\ &= \mathrm{sgn}(\mathbb{1}_U(x) \cdot \langle \rho, \mathbb{1}_V \rangle) \\ &= \mathrm{sgn}(\mathbb{1}_U(x)) \cdot \mathrm{sgn}(\langle \rho, \mathbb{1}_V \rangle) \\ &= [\![x \in U]\!] \wedge \langle \mathrm{supp}(\rho), V \rangle \\ &= \langle s(x, \mathrm{supp}(\rho)), U \times V \rangle. \end{aligned}$$

This is enough by Lemma 2.49. □

Corollary 5.19. *The support map* supp : $V \to H$ *is a monoidal natural transformation.*

Proof. See Proposition C.5. □

In particular, the following diagram commutes.

$$\begin{array}{ccc} VX \times VY & \xrightarrow{\nabla} & V(X \times Y) \\ {\scriptstyle \mathrm{supp}\times\mathrm{supp}}\downarrow & & \downarrow{\scriptstyle \mathrm{supp}} \\ HX \times HY & \xrightarrow{\nabla} & H(X \times Y) \end{array}$$

In other words, the support of the product distribution is the product of the supports of the factors. For the unitors we have the trivial statement that the support of the valuation in the image of $u : 1 \to V1$ is the unique point of 1.

By the universal property of the Cartesian product, we know that the support is also an opmonoidal natural transformation. This means that the supports of the marginals are the projections of the support.

A. Top as a 2-category

Here, we recall some background material on the specialization preorder, which makes Top into a category enriched in preordered sets, and therefore into a 2-category.

Every topological space is canonically equipped with a preorder, its specialization preorder. Since a preorder is a category, this equips Top with a 2-categorical structure. This is not the higher categorical structure that arises from homotopies of maps. Rather, it is closely related to the usual 2-categorical structure on the category of locales.

Definition A.1 (Specialization preorder)**.** *Let X be a topological space. Given $x, y \in X$, we define the specialization preorder on X by setting $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$.*

Equivalently, $x \leq y$ if and only if every open neighborhood of x is also a neighborhood of y. Equivalently, $x \leq y$ if and only if $\mathrm{cl}(\{x\}) \subseteq \mathrm{cl}(\{y\})$. Equivalently, any net or filter which converges to y also converges to x. These reformulations show that the specialization preorder is indeed a preorder (it is reflexive and transitive). A space is T_0 if and only if its specialization preorder is a partial order (it is antisymmetric). A space is T_1 if and only if its specialization preorder is the discrete relation. Every preorder on any set arises as the specialization preorder of some topology, for example its Alexandrov topology.

We write $x \sim y$ if $x \leq y$ as well as $y \leq x$. Any two equivalent points $x \sim y$ have the same neighborhood filter, and any net or filter tending to one also tends to the other. Since two points of a space X are equivalent in the specialization preorder if and only if they have the same open neighborhoods, a space is T_0 if and only if $x \sim y$ implies $x = y$. The *Kolmogorov quotient* is the quotient space $X/\sim$, which is the initial T_0 space equipped with a continuous map from X, making the category of T_0 spaces into a reflective subcategory of Top.

Definition A.2. *Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. We say that there is a 2-cell $f \leq g$ if and only if, for every $x \in X$, we have $f(x) \leq g(x)$ in the specialization preorder of Y.*

Every continuous map is monotone for the specialization preorder. Hence, equipped with these 2-cells, Top is a strict 2-category.

Lemma A.3. *Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Then $f \leq g$ if and only if $f^{-1}(U) \subseteq g^{-1}(U)$ for every open $U \subseteq Y$.*

Proof. Assuming $f \leq g$, we prove the claim by showing that every $x \in f^{-1}(U)$ is also in $g^{-1}(U)$. Since $x \in U$ and $f(x) \leq g(x)$, we indeed have $g(x) \in U$.

Now suppose that $f^{-1}(U) \subseteq g^{-1}(U)$ for all open $U \subseteq Y$. Then, for every $x \in X$ and any open neighborhood $U \ni f(x)$, we have $x \in f^{-1}(U) \subseteq g^{-1}(U)$, which implies that $g(x) \in U$. We have shown that every open neighborhood of $f(x)$ is an open neighborhood of $g(x)$. □

By definition of equivalence in a 2-category, a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ is an equivalence if and only if $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. In particular, two T_0 spaces are equivalent if and only if they are homeomorphic. If X and Y are not necessarily T_0, then a map $f : X \to Y$ that induces an equivalence need not be bijective. In particular, the topological spaces X and Y are equivalent if and only if their Kolmogorov quotients $X_\sim$ and $Y_\sim$, which are T_0 by construction, are homeomorphic. For example, all codiscrete spaces are equivalent.

A simpler way to characterize equivalence is the following criterion, which is analogous to the well-known fact that a functor is an equivalence if and only if it is faithful.

Lemma A.4. *A continuous map $f : X \to Y$ is an equivalence if and only if $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is bijective, and for every point $y \in Y$ there is a point $x \in X$ with $f(x) \sim y$ in the specialization preorder.*

The second condition is necessary, since otherwise we could, for example, pick a non-sober space X and take f to be the universal map into its sobrification.

Proof. The "only if" direction is clear, so we focus on the "if" direction.

The assumption implies that $x \leq x'$ in X if and only if $f(x) \leq f(x')$ in Y. It follows that the same equivalence holds for the specialization equivalence $\sim$. For each $y \in Y$, we arbitrarily choose a point $g(y) \in X$ with $f(g(y)) \sim y$. These choices amount to a map $g : Y \to X$. Since we have $f(g(f(x))) \sim f(x)$ for all $x \in X$, cancelling f results in $g(f(x)) \sim x$. A similar argument shows that g also preserves specialization equivalence. It remains to show that g is continuous by showing that $g^{-1}(U)$ is open for every open $U \subseteq X$. Pick the unique open set $V \subseteq Y$ with $f^{-1}(V) = U$. We claim that $g^{-1}(U) = V$. Indeed, for $y \in Y$, the assumption $y \in g^{-1}(U)$, or equivalently $g(y) \in U$, means that $y \sim f(g(y)) \in V$. Conversely, if $y \in V$, choosing $x \in X$ with $y \sim f(x)$ gives $f(x) \in V$, and therefore $x \in U$, such that $g(y) \sim g(f(x)) \sim x \in U$ as well, and hence $y \in g^{-1}(U)$. Finally, since we have $g(f(x)) \sim x$ for all $x \in X$ and $f(g(y)) \sim y$ for all $y \in Y$, we conclude that $g \circ f \sim \mathrm{id}_X$ and that $f \circ g \sim \mathrm{id}_Y$. ☐

B. Topology of mapping spaces

Here we briefly discuss the issue of equipping the set of maps between topological spaces with a suitable topology.

Recall that a subset $V \subseteq U$ of a topological space is *relatively compact* in U if and only if every open cover of U admits a finite subcover of V, or, equivalently, if and only if V is way below U with respect to the the inclusion order [Gie+03, p. 50]. For this reason, we also denote relative compactness by $V \ll U$.

Definition B.1 (Core-compact spaces [HL78])**.** *A topological space X is core-compact if, for every $x \in X$ and every open neighborhood $U \ni x$, there exists an open neighborhood V such that $x \in V \subseteq U$ and $V \ll U$.*

It follows that a space is core-compact if and only if its lattice of open sets is a continuous lattice [Isb86]. If X is sober, then it is core-compact if and only if it is locally compact (see Theorem V-5.6 in [Gie+03]).

Theorem B.2 (See [Isb86; EH01] and Proposition II-4.6 in [Gie+03])**.** *Let X be a topological space. The functor $- \times X : \mathsf{Top} \to \mathsf{Top}$ has a right adjoint $(-)^X : \mathsf{Top} \to \mathsf{Top}$ if and only if X is core-compact. In this case, for any space Y, the topology on Y^X is the Isbell topology generated by the subbasis*

$$O(U, V) := \left\{ f : X \to Y : U \ll f^{-1}(V) \right\}$$

for open sets $U \subseteq X$ and $V \subseteq Y$. Under the same assumption on X, the Isbell topology on S^X and on $[0, \infty]^X$ coincides with the respective Scott topology (where on $[0, \infty]$ we take the topology of upper open sets, as in the rest of this paper).

Therefore Scott continuous functionals on $[0, \infty]^X$ and on S^X, such as the continuous valuations and the closed sets studied in the main text, are related to double dualization monads in the sense of Lucyshyn-Wright [Luc17]. This double dual construction does not rely on Cartesian closedness (since Top is not Cartesian closed) but it equips the function spaces with the topology of the exponential objects whenever these exist.

Whenever X is not core-compact, the exponential space S^X does not exist, and neither does $[0, \infty]^X$. This is well-known and may be shown by translating a theorem of Niefield [Nie82, Theorem 2.3] into our setting. (Niefield's "cartesianness" corresponds to our "exponentiability" and we take her T to be 1 so that Top/T becomes Top.) The equivalence of conditions (b), (c) and (d) in Niefield's theorem reads as follows.

Theorem B.3 (Niefield)**.** *Let X be a topological space. The following three conditions are equivalent.*

(a) X is exponentiable.

(b) The exponential object S^X exists in Top.

(c) Given any open $U \subseteq X$ and any point $x \in U$, there exists a Scott open $\mathcal{V} \subseteq O(X)$ such that $U \in \mathcal{V}$ and $\bigcap \mathcal{V}$ is a neighborhood of x.

Proposition B.4. *Condition (c) holds if and only if X is core-compact.*

Proof of B.4. Suppose that X satisfies (c). Then, for given $x \in X$ and open $U \ni x$, we have a Scott open $\mathcal{V} \subseteq \mathcal{O}(X)$ such that $U \in \mathcal{V}$ and $\bigcap \mathcal{V}$ is a neighborhood of x. The latter means that there exists an open V such that $x \in V \subseteq \bigcap \mathcal{V}$. We claim that $V \ll U$. Let

$\{U_\alpha\}_{\alpha\in A}$ be an open cover of U. Since $U \in \mathcal{V}$, Scott openness implies that there is a finite subfamily $(U_{\alpha_i})_{i=1}^n$ such that $\bigcup_{i=1}^n U_{\alpha_i}$ is already a member of $\mathcal{V}$, and therefore contains $\bigcap \mathcal{V}$. But then this finite subfamily also covers V.

Conversely, let X be core-compact. For open $U \subseteq X$ and $x \in U$, there exists an open $V \subseteq X$ such that $x \in V \ll U$. Define the set

$$\mathcal{V} \coloneqq \{U' \in \mathcal{O}(X) : V \ll U'\}.$$

We have $U \in \mathcal{V}$ and $V \subseteq \bigcap \mathcal{V}$. The set $\mathcal{V} \subseteq \mathcal{O}(X)$ is Scott open by the assumption that $\mathcal{O}(X)$ is a continuous lattice, which implies that principal upsets with respect to "$\ll$" are Scott open (Proposition II.1.6 in [Gie+03]). □

C. Commutative and symmetric monoidal monads

We recall the notion of monad and the equivalence between a commutative monad and a symmetric monoidal monad. We assume familiarity with symmetric monoidal categories, lax symmetric monoidal functors, and monoidal natural transformations. All our monoidal functors will be lax monoidal. The monoidal category of primary interest in the main text is Top with its Cartesian product structure. We start with the definition of monads and their morphisms for convenient reference.

Definition C.1 (Category of monads). *Let* C *be a category.*

(a) A functor $T : \mathsf{C} \to \mathsf{C}$ *is a monad if it is equipped with natural transformations* $\eta : \mathrm{id} \Rightarrow T$ *and* $\mu : TT \Rightarrow T$ *such that the following three diagrams commute for every* $X \in \mathsf{C}$.

$$\begin{array}{ccc} TX & \xrightarrow{\eta} & TTX \\ & \searrow^{=} & \downarrow \mu \\ & & TX \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{T\eta} & TTX \\ & \searrow^{=} & \downarrow \mu \\ & & TX \end{array} \qquad \begin{array}{ccc} TTTX & \xrightarrow{T\mu} & TTX \\ \downarrow \mu & & \downarrow \mu \\ TTX & \xrightarrow{\mu} & TX \end{array} \tag{C.1}$$

(b) If (T, μ, ν) *and* (T', μ', ν') *are monads, then a morphism of monads is a natural transformation* $\alpha : T \to T'$ *such that the following two diagrams commute for every* $X \in \mathsf{C}$.

$$\begin{array}{ccc} & & TX \\ & \nearrow^{\eta'} & \\ X & & \downarrow \alpha \\ & \searrow_{\eta} & \\ & & T'X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{\mu} & TX \\ \downarrow \alpha & & \\ T'TX & & \downarrow \alpha \\ \downarrow \alpha & & \\ T'T'X & \xrightarrow{\mu'} & T'X \end{array} \tag{C.2}$$

An introduction to monads from the probabilistic perspective is given by Perrone [Per18, Chapter 1]. We use the term *probability monad* to loosely refer to monads T where for every object $X \in \mathsf{C}$, the object TX is the object of probability measures (of a given kind) on X. Then $T(X \otimes Y)$ is the object of *joint* probability distributions on X and Y. The multiplication $\mu : TTX \to TX$ averages a probability measure on probability measures to the probability measure representing the expectation value or barycenter of the measure on measures. All three monads considered in this paper are variations on this theme.

An important structure in probability theory is the formation of product distributions. In the probability monad formalism, this is encoded as a natural transformation $\nabla : TX \otimes TY \to T(X \otimes Y)$ which makes T into a lax symmetric monoidal functor that interacts nicely with the monad structure.

Definition C.2 (Symmetric monoidal monad)**.** *Suppose that a functor $T : \mathsf{C} \to \mathsf{C}$ carries both the structure of a monad and of a symmetric monoidal functor with structure maps $\nabla : T(-) \otimes T(-) \to T(- \otimes -)$ and $u : 1 \to T1$. Then T is a symmetric monoidal monad if $u = \eta$ as morphisms $1 \to T1$ and the following two diagrams commute.*

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle \eta\otimes\eta}\swarrow & & \searrow{\scriptstyle \eta} \\ TX \times TY & \xrightarrow{\quad\nabla\quad} & T(X \times Y) \end{array}$$

$$\begin{array}{ccccc} TTX \times TTY & \xrightarrow{\nabla} & T(TX \times TY) & \xrightarrow{T\nabla} & TT(X \times Y) \\ \downarrow{\scriptstyle \mu\times\mu} & & & & \downarrow{\scriptstyle \mu} \\ TX \times TX & & \xrightarrow{\quad\nabla\quad} & & T(X \times Y) \end{array}$$

It is well-known that, given a monad T, there is a bijective correspondence between symmetric monoidal structures on T and *strengths* on T which make T into a *commutative monad* [Koc72]. We introduce the relevant definitions.

Definition C.3 (Strength)**.** *A strength on a monad T is a family of maps $X \otimes TY \to T(X \otimes Y)$, natural in X and Y, such that the following four diagrams commute for all $X, Y \in \mathsf{C}$, where the unnamed isomorphisms are the monoidal structure isomorphisms.*

$$\begin{array}{ccc} 1 \otimes TX & \xrightarrow{s} & T(1 \otimes X) \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \cong} \\ & & TX \end{array}$$

$$\begin{array}{ccccc} (X \otimes Y) \otimes TZ & & \xrightarrow{\quad s\quad} & & T((X \otimes Y) \otimes Z) \\ \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\ X \otimes (Y \otimes TZ) & \xrightarrow{\mathrm{id}\otimes s} & X \otimes T(Y \otimes Z) & \xrightarrow{s} & T(X \otimes (Y \otimes Z)) \end{array}$$

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\mathrm{id}\otimes\eta} & X \otimes TY \\
& \searrow^{\eta} & \downarrow{\scriptstyle s} \\
& & T(X \otimes Y)
\end{array}
$$

$$
\begin{array}{ccccc}
X \otimes TTY & \xrightarrow{s} & T(X \otimes TY) & \xrightarrow{Ts} & TT(X \otimes Y) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\mu} & & & & \downarrow{\scriptstyle \mu} \\
X \otimes TY & & \xrightarrow{\qquad s \qquad} & & T(X \otimes Y)
\end{array}
$$

Similarly, a *costrength* is a natural transformation with components $t : TX \otimes Y \to T(X \otimes Y)$ satisfying the analogous equations. Since C is symmetric monoidal, every strength induces a costrength and vice versa. A *strong monad* is a monad equipped with a strength.

Definition C.4 (Commutative monad)**.** *Let T be a strong monad with strength s and induced costrength t. Then T is a commutative monad if the following diagram commutes for all $X, Y \in \mathsf{C}$.*

$$
\begin{array}{ccccc}
TX \otimes TY & \xrightarrow{s} & T(TX \otimes Y) & \xrightarrow{Tt} & TT(X \otimes Y) \\
\downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle \mu} \\
T(X \otimes TY) & \xrightarrow{Ts} & TT(X \otimes Y) & \xrightarrow{\mu} & T(X \otimes Y)
\end{array}
\tag{C.3}
$$

The bijection between a symmetric monoidal structure on a monad T and a commutative structure on T can be obtained by explicit construction of each piece of structure in terms of the other. In one direction, we start with $\nabla : TX \otimes TY \to T(X \otimes Y)$ and obtain a strength as the composite

$$
X \otimes TY \xrightarrow{\eta\otimes\mathrm{id}} TX \otimes TY \xrightarrow{\nabla} T(X \otimes Y).
$$

In the other direction, we start with $s : X \otimes TY \to T(X \otimes Y)$ and obtain a symmetric monoidal structure as the diagonal of (C.3).

The following observation is fairly elementary, but seems to be hard to find in the literature. It says that the correspondence between commutative strong monads and symmetric monoidal monads is functorial.

Proposition C.5. *If S and T are commutative monads, then the following two properties of a morphism of monads $\alpha : S \Rightarrow T$ are equivalent.*

(a) The morphism α is a monoidal natural transformation, that is the following two diagrams commute for all $X, Y \in \mathsf{C}$.

$$
\begin{array}{ccc}
& & S1 \\
& \nearrow^{\eta} & \downarrow{\scriptstyle \alpha} \\
1 & & \\
& \searrow_{\eta} & \\
& & T1
\end{array}
\qquad
\begin{array}{ccc}
SX \otimes SY & \xrightarrow{\nabla} & S(X \otimes Y) \\
\downarrow{\scriptstyle \alpha\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\
TX \otimes TX & \xrightarrow{\nabla} & T(X \otimes Y)
\end{array}
\tag{C.4}
$$

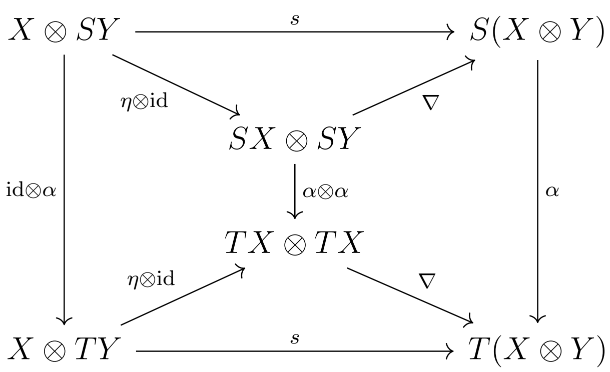
.

(b) The morphism α preserves the strengths in the sense that the following diagram commutes for all $X, Y \in \mathsf{C}$.

$$\begin{array}{ccc} X \otimes SY & \xrightarrow{\;s\;} & S(X \otimes Y) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\alpha} & & \Big\downarrow{\scriptstyle \alpha} \\ X \otimes TY & \xrightarrow{\;s'\;} & T(X \otimes Y) \end{array} \tag{C.5}$$

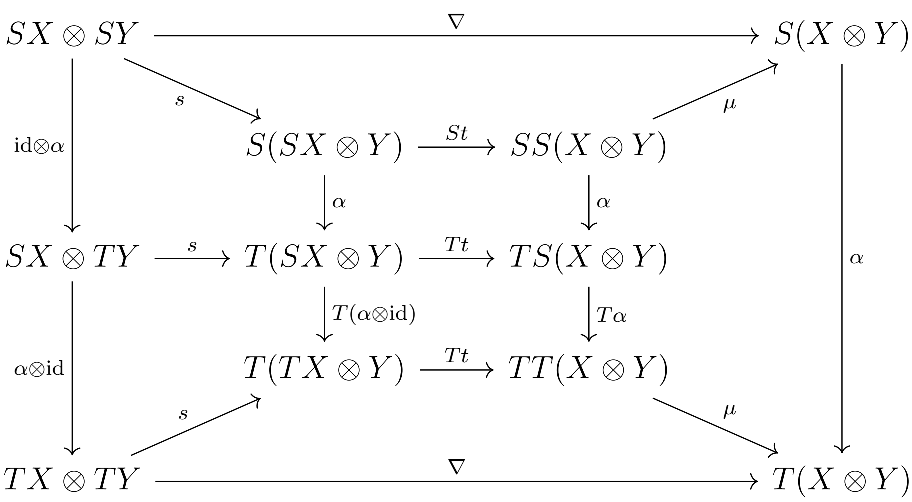
The proof proceeds by construction of the lax monoidal structure from the strength and vice versa. Both directions use the assumption that α preserves the monad structure. The more tedious direction from (b) to (a) also uses the naturality of s and α. Since the proof is hard to locate in the literature, we give it here.

Proof. We start by (a)⇒(b). We can decompose the diagram (C.5) as follows.



The triangles at the top and at the bottom commute; in fact they are the standard way of obtaining a strength from a monoidal structure. The trapezium on the left commutes because α is a morphism of monads and therefore preserves the units. The trapezium on the right is the second diagram of (C.4).

We continue by (b)⇒(a). The first diagram of (C.4) commutes since α is a morphism of monads and therefore preserves the units. The second diagram of (C.4) can be decomposed as follows.



The upper and lower trapezia commute; in fact they are the standard way of obtaining a monoidal structure from a strength. The upper trapezium on the left is the diagram (C.5), which commutes by hypothesis. The lower trapezium on the left commutes by naturality of s. The upper central square commutes by naturality of α. The lower central square is the image under T of a naturality square for t. The trapezium on the right commutes since α is a morphism of monads and therefore preserves the multiplication. $\square$

In probability theory, one is also interested in the formation of *marginals*, which is implemented by an *opmonoidal* structure $T(X \otimes Y) \to TX \otimes TY$ [FP18]. In the setting of this paper, where $\mathsf{C} = \mathsf{Top}$, the monoidal structure of our category is the Cartesian product structure. Hence the projections $X \times Y \to X$ and $X \times Y \to Y$ induce maps $T(X \times Y) \to TX$ and $T(X \times Y) \to TY$ by functoriality. These induce the opmonoidal structure describing the formation of marginal distributions from joint distributions. The well-behaved interaction of this opmonoidal structure with the monoidal structure is immediately implied [FP18, Proposition 3.4].

References

[AB06] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis*. Springer, 2006 (cit. on p. 54).

[ACT10] A. Akhvlediani, M.M. Clementino, and W. Tholen. “On the categorical meaning of Hausdorff and Gromov distances 1”. In: *Topology and its Applications* 157.8 (2010). arXiv:0901.0618, pp. 1275–1295 (cit. on p. 18).

[AES98] M. Alvarez-Manilla, A. Edalat, and N. Saheb-Djahromi. “An extension result for continuous valuations”. In: *Electronic Notes in Theoretical Computer Science* 13 (1998) (cit. on p. 46).

[AJK04] Mauricio Alvarez-Manilla, Achim Jung, and Klaus Keimel. “The probabilistic powerdomain for stably compact spaces”. In: *Theoretical Computer Science* 328.3 (2004), pp. 221–244 (cit. on pp. 4, 26, 28).

[Alv02] M. Alvarez-Manilla. “Extension of valuations on locally compact sober spaces”. In: *Topology and its Applications* 124.2 (2002), pp. 397–443 (cit. on p. 46).

[Ban95] Taras Banakh. “The topology of spaces of probability measures 1”. In: *Matematychni Studii* 5.1-2 (1995). Russian, pp. 65–87 (cit. on p. 43).

[Bog00] V.I. Bogachev. *Measure Theory*. Springer, 2000 (cit. on pp. 44, 46, 47, 52).

[Bre05] Franck van Breugel. “The Metric Monad for Probabilistic Nondeterminism”. Available at http://www.cse.yorku.ca. 2005 (cit. on p. 44).

[CT97] Maria Manuel Clementino and Walter Tholen. “A characterization of the Vietoris topology”. In: *Topology Proceedings* 22 (1997), pp. 71–95 (cit. on pp. 6, 7).

[EH01] Martín Escardó and Reinhold Heckmann. “Topologies on spaces of continuous functions”. In: *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York).* Vol. 26. 2. 2001, pp. 545–564 (cit. on p. 60).

[EM65] S. Eilenberg and J. Moore. “Adjoint functors and triples”. In: *Illinois Journal of Mathematics* 9 (1965), pp. 381–389 (cit. on p. 3).

[Fed91] V.V. Fedorchuk. “Probability measures in topology”. In: *Russian Mathematical Surveys* 46.1 (1991), pp. 45–93 (cit. on p. 43).

[FP18] Tobias Fritz and Paolo Perrone. “Bimonoidal Structure of Probability Monads”. In: *Proceedings of MFPS 34, ENTCS* (2018). arXiv:1804.03527 (cit. on pp. 43, 65).

[FP19] Tobias Fritz and Paolo Perrone. “A probability monad as the colimit of spaces of finite samples”. In: *Theory and Applications of Categories* 34.7 (2019), pp. 170–220 (cit. on pp. 36, 44).

[Fre06] D.H. Fremlin. *Measure theory Vol. 4.* Topological measure spaces. Part I, II, Corrected second printing of the 2003 original. Torres Fremlin, Colchester, 2006, Part I: 528 pp., Part II: 439+19 pp. (errata) (cit. on p. 45).

[Fri09] Tobias Fritz. *Convex Spaces 1: Definition and Examples.* arXiv:0903.5522. 2009 (cit. on p. 56).

[Gie+03] G. Gierz et al. *Continuous Lattices and Domains.* Cambridge University Press, 2003 (cit. on pp. 59–61).

[Gir82] Michèle Giry. “A categorical approach to probability theory”. In: *Lecture Notes in Mathematics 915: Categorical Aspects of Topology and Analysis.* Springer, 1982, pp. 68–85 (cit. on pp. 3, 43).

[GJ19] Jean Goubault-Larrecq and Xiaodong Jia. “Algebras of the extended probabilistic powerdomain monad”. In: *Electronic Notes in Theoretical Computer Science* 345 (2019), pp. 37–61 (cit. on pp. 26, 36, 37, 56).

[Gou13] Jean Goubault-Larrecq. *Non-Hausdorff topology and domain theory.* Cambridge University Press, 2013 (cit. on p. 44).

[Hal50] Paul R. Halmos. *Measure Theory.* Van Nostrand, 1950 (cit. on p. 37).

[Hau14] Felix Hausdorff. *Grundzüge der Mengenlehre.* German. Veit und Co., 1914 (cit. on p. 5).

[Hec95] R. Heckmann. "Spaces of valuations". Working paper version. Available at http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.45.5845. 1995 (cit. on pp. 26, 37, 38).

[Hec96] R. Heckmann. "Spaces of valuations". In: *Papers on General Topology and Applications: Eleventh Summer Conference at the University of Southern Maine*. Ed. by S. Andima et al. New York Academy of Sciences, 1996, pp. 174–200 (cit. on pp. 26, 31).

[HL78] Karl H. Hofmann and Jimmie D. Lawson. "The spectral theory of distributive continuous lattices". In: *Transactions of the American Mathematical Society* 246 (1978), pp. 285–310 (cit. on p. 60).

[Hof79] Rudolf-E. Hoffmann. "Essentially complete T_0-spaces". In: *Manuscripta Mathematica* 27.4 (1979), pp. 401–432 (cit. on pp. 8, 17, 18, 20).

[Isb86] John Isbell. "General function spaces, products and continuous lattices". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 100.2 (1986), pp. 193–205 (cit. on p. 60).

[Jac11] Bart Jacobs. "Probabilities, distribution monads, and convex categories". In: *Theoretical Computer Science* 412.28 (2011), pp. 3323–3336 (cit. on p. 3).

[Joh82] Peter Johnstone. *Stone Spaces*. Cambridge University Press, 1982 (cit. on pp. 9, 19).

[JP89] C. Jones and J.D. Plotkin. "A Probabilistic Powerdomain of Evaluations". In: *Proceedings of the Fourth Annual Symposium of Logics in Computer Science* (1989) (cit. on pp. 26, 31).

[Jun04] A. Jung. "Stably compact spaces and the probabilistic powerspace construction". In: *Electronic Notes in Theoretical Computer Science 87: Domain-theoretic Methods in Probabilistic Processes*. Ed. by J. Desharnais and P. Panangaden. 2004, pp. 5–20 (cit. on p. 27).

[Kei08a] K. Keimel. "Topological cones: Functional analysis in a T_0-setting". In: *Semigroup Forum* 77 (2008), pp. 109–142 (cit. on p. 36).

[Kei08b] Klaus Keimel. "The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras". In: *Topology and Applications* 156.2 (2008), pp. 227–239 (cit. on pp. 5, 36).

[Kir93] O. Kirch. "Bereiche und Bewertungen". German. MA thesis. Technische Hochschule Darmstadt, 1993 (cit. on pp. 26–28).

[KL97] Max Kelly and Steve Lack. "On property-like structures". In: *Theory and Applications of Categories* 3.9 (1997) (cit. on p. 18).

[Koc12] Anders Kock. "Commutative monads as a theory of distributions". In: *Theory and Applications of Categories* 26.4 (2012), pp. 97–131 (cit. on pp. 42, 51).

[Koc72] Anders Kock. “Strong functors and monoidal monads”. In: *Archiv der Mathematik* 23 (1972), pp. 113–120 (cit. on p. 62).

[Koc95] Anders Kock. “Monads for which structures are adjoint to units”. In: *Journal of Pure and Applied Algebra* 104 (1995), pp. 41–59 (cit. on p. 18).

[Law62] F.W. Lawvere. “The category of probabilistic mappings with applications to stochastic processes, statistics, and pattern recognition”. Available at nlab. 1962 (cit. on p. 43).

[Luc17] Rory B.B. Lucyshyn-Wright. “Functional distribution monads in functional-analytic contexts”. In: *Advances in Mathematics* 322 (2017), pp. 806–860 (cit. on pp. 4, 8, 60).

[Man03] Ernie Manes. “Monads of Sets”. In: *Handbook of Algebra Vol. 3*. Ed. by M. Hazewinkel. Elsevier, 2003, pp. 67–153 (cit. on p. 3).

[Mic51] Ernest Michael. “Topologies on spaces of subsets”. In: *Transactions of the American Mathematical Society* 71 (1951), pp. 152–182 (cit. on p. 5).

[Nie82] S.B. Niefield. “Cartesianness: Topological spaces, uniform spaces, and affine schemes”. In: *Journal of Pure and Applied Algebra* 23 (1982), pp. 146–167 (cit. on p. 60).

[Per18] Paolo Perrone. “Categorical Probability and Stochastic Dominance in Metric Spaces”. Available at http://paoloperrone.org/phdthesis.pdf. PhD thesis. University of Leipzig, 2018 (cit. on pp. 3, 61).

[Res77] Paul Ressel. “Some continuity and measurability results on spaces of measures”. In: *Mathematica Scandinavica* 40.1 (1977) (cit. on pp. 37, 51).

[RS70] M. Rothschild and J.E. Stiglitz. “Increasing risk 1: A definition”. In: *Journal of Economic Theory* 2 (1970), pp. 225–243 (cit. on p. 31).

[Sch93] Andrea Schalk. “Algebras for generalized power constructions”. Available at www.cs.man.ac.uk/ schalk/publ/diss.ps.gz. PhD thesis. University of Darmstadt, 1993 (cit. on pp. 4, 6, 8, 17–20).

[Smy83] M.B. Smyth. “Power domains and predicate transformers: A topological view”. In: *Automata, Languages and Programming. ICALP 1983. Lecture Notes in Computer Science Vol. 154*. Ed. by J. Diaz. Springer, 1983, pp. 662–675 (cit. on p. 6).

[Świ74] T. Świrszcz. “Monadic functors and convexity”. In: *Bulletin de l’Académie Polonaise des Sciences: Série des sciences mathematiques, astronomique et physique* 22.1 (1974), pp. 39–42 (cit. on pp. 5, 36, 43).

[Top70] Flemming Topsøe. *Topology and Measure*. Springer, 1970 (cit. on pp. 46–48).

[Vie22] Leopold Vietoris. “Bereiche zweiter Ordnung”. In: *Monatshefte für Mathematik und Physik* 32 (1922). German, pp. 258–280 (cit. on pp. 5, 7).

[Zöb76] Volker Zöberlein. “Doctrines on 2-categories”. In: *Mathematische Zeitschrift* 148 (1976), pp. 267–279 (cit. on p. 18).