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by

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We study the quantum separability problem by using general symmetric informationally complete measurements and present a separability criterion for arbitrary dimensional bipartite systems. We show by detailed examples that our criterion is more powerful than the existing ones in entanglement detection.

I. INTRODUCTION

Quantum entanglement is one of the most fundamental resources in quantum information processing [1, 2, 3]. Operational and efficient criteria for the detection of entanglement is of great significance. It has been discussed that the problem of determining whether or not a given state is entangled is NP-hard [4, 5, 6]. There have been numerous criteria to distinguish quantum entangled states from the separable ones, such as positive partial transposition(PPT) criterion [7, 8, 9], realignment criterion [10, 11, 12, 13, 14, 15, 16], covariance matrix criterion [17], correlation matrix criterion [18, 19], and generalized form of the correlation matrix criterion [20].

While numerous mathematical tools have been employed in entanglement detection of quantum states, experimental implementation of entanglement detection for unknown quantum states has fewer results [21, 22, 23, 24]. In [25], the authors connected the separability problem with the concept of mutually unbiased bases (MUBs) [26] for two-qubit, multipartite and continuous-variable quantum systems. These entanglement criteria are shown to be powerful and can be implemented experimentally. After that, the authors in [27, 28] generalized such idea and provided an entanglement criterion based on mutually unbiased measurements (MUMs) [29]. Moreover, it has been shown that the criterion based on MUMs is more effective than the criterion based on MUBs.

Besides mutually unbiased bases, another intriguing topic in quantum information theory is the symmetric informationally complete positive operator-valued measures (SIC-POVMs). Most of the literature on SIC-POVMs focus on rank-1 SIC-POVMs that all the positive operator-valued measure (POVM) elements are proportional to rank-1 projectors. Nevertheless, the existence of SIC-POVMs in arbitrary dimension is still an open problem [38]. In [30], the author introduced the general symmetric informationally complete measurements (general SIC-POVMs) in which the elements need not to be rank one, and showed that general SIC-POVMs exist in all finite dimensions. Furthermore, Gour and Kalev [31] constructed all general SIC measurements from the generalized Gell-Mann matrices. In [32, 33], the authors presented separability criteria for both d-dimensional bipartite and multipartite systems based on general SIC-POVMs. Very recently, Bae *et.al.*[34] studied entanglement detection via quantum 2-designs, which includes SIC-POVMs as a special example. In [35] the authors investigated the entanglement properties of multipartite systems with tight informationally complete measurements including SIC-POVMs. In addition, the authors in [36] considered a nonlinear entanglement criterion based on SIC-POVMs. In [37] the authors used the general SIC-POVMs to derive the entanglement criterion and demonstrated the superiority of the criterion by various examples.

In this paper, we study the quantum separability problem by using general SIC-POVMs and present a separability criterion for arbitrary high dimensional bipartite systems of a d_A-dimensional subsystem and a d_B-dimensional subsystem. The paper is organized as follows. In Sec. II, we recall some basic notions of SIC-POVMs and general SIC-POVMs. Section III provides an entanglement criterion based on the general SIC-POVMs and some remarks. In Sec. IV, we compare the criterion with the ones in [32] and [37] via detailed examples, and show that our criterion is more efficient than the existing ones. We conclude the paper in Sec. V.

II. SIC-POVMS AND GENERAL SIC-POVMS

We first review some basic knowledge of symmetric informationally complete measurements and general symmetric informationally complete measurements. A POVM $\{P_j\}$ with d^2 rank-1 operators acting on $\mathbb{C}^d$ is symmetric informationally complete, if

$$P_j = \frac{1}{d}|\phi_j\rangle\langle\phi_j|, \tag{1}$$

$$\sum_{j=1}^{d^2} P_j = \mathbb{I}, \tag{2}$$

where $j = 1, 2, \cdots, d^2$, $\mathbb{I}$ is the identity operator, and the vectors $|\phi_j\rangle$ satisfy $|\langle\phi_j|\phi_k\rangle|^2 = 1/(d+1)$, $j \neq k$. The existence of SIC-POVMs in arbitrary dimension d is still an open problem. Only analytical solutions have been found in dimensions $d = 2-24, 28, 30, 31, 35, 37, 39, 43, 48, 124$ and numerical solutions have been found up to dimension 151 [38].

The concept and constructions of general SIC measurements were introduced in Refs. [30, 31]. A set of d^2 positive semidefinite operators $\{P_\alpha\}_{\alpha=1}^{d^2}$ on $\mathbb{C}^d$ is said to be a general SIC measurements if

$$\sum_{\alpha=1}^{d^2} P_\alpha = \mathbb{I}, \tag{3}$$

$$\mathrm{Tr}(P_\alpha^2) = a, \tag{4}$$

$$\mathrm{Tr}(P_\alpha P_\beta) = \frac{1-da}{d(d^2-1)}, \tag{5}$$

where $\alpha, \beta \in \{1, 2, \cdots, d^2\}$, $\alpha \neq \beta$, the parameter a satisfies $\frac{1}{d^3} < a \leqslant \frac{1}{d^2}$, and $a = \frac{1}{d^2}$ if and only if all P_α are rank one, which gives rise to a SIC-POVM. It can be shown that $\mathrm{Tr}(P_\alpha) = \frac{1}{d}$ for all α. Contrasting to SIC-POVM, the general SIC-POVM can be explicitly constructed [31]. Let $\{F_\alpha\}_{\alpha=1}^{d^2-1}$ be a set of d^2-1 Hermitian, traceless operators acting on $\mathbb{C}^d$, satisfying $\mathrm{Tr}(F_\alpha F_\beta) = \delta_{\alpha,\beta}$. Set $F = \sum_{\alpha=1}^{d^2-1} F_\alpha$. The d^2 operators

$$P_\alpha = \frac{1}{d^2}\mathbb{I} + t[F - d(d+1)F_\alpha], \alpha = 1, 2, \cdots, d^2-1, \tag{6}$$

$$P_{d^2} = \frac{1}{d^2}\mathbb{I} + t(d+1)F \tag{7}$$

form a general SIC measurement. Here t should be chosen such that $P_\alpha \geqslant 0$ and the parameter a is given by

$$a = \frac{1}{d^3} + t^2(d-1)(d+1)^3. \tag{8}$$

III. ENTANGLEMENT DETECTION VIA GENERAL SIC-POVMS

Entanglement detection via SIC-POVMs had been discussed in [37]. However, the method subjects to the existence of SIC-POVMs, which is an open problem. Unlike the SIC-POVMs, the general symmetric informationally complete measurements do exist for arbitrary dimension d.

Consider a quantum state ρ and a general SIC-POVM $\mathcal{M}_s = \{P_\alpha\}_{\alpha=1}^{d^2}$. We have the probability $p_\alpha = \langle P_\alpha\rangle = \mathrm{Tr}(P_\alpha\rho)$ of outcome α. Conversely, the quantum state ρ can be reconstructed from these probabilities:

$$\rho = \frac{d(d^2-1)}{ad^3-1}\sum_{\alpha=1}^{d^2} p_\alpha P_\alpha - \frac{d(1-ad)}{ad^3-1}\mathbb{I}. \tag{9}$$

Denote $(e| = (p_1\ p_2\ \cdots\ p_{d^2})$ and $|e) = (p_1\ p_2\ \cdots\ p_{d^2})^T$. Calculation shows that

$$\sum_{\alpha=1}^{d^2} p_\alpha^2 = \frac{(ad^3-1)\mathrm{Tr}(\rho^2) + d(1-ad)}{d(d^2-1)} \leqslant \frac{ad^2+1}{d(d+1)}, \tag{10}$$

where the upper bound is saturated iff ρ is pure.

Now consider a $d_A \times d_B$ bipartite state ρ, and two general SIC-POVMs: $\{P_\alpha^A\}_{\alpha=1}^{d_A^2}$ with parameter a_A and $\{P_\alpha^B\}_{\alpha=1}^{d_B^2}$ with parameter a_B for the two subsystems, respectively. The linear correlations between P^A and P^B read

$$[\mathcal{P}]_{ij} = \langle P_i^A \otimes P_j^B \rangle. \quad (11)$$

Denote $\mathcal{P}$ the matrix with entries given by $[\mathcal{P}]_{ij}$.

Theorem 1 If a bipartite state ρ is separable, then

$$\|\mathcal{P}\|_{\mathbf{tr}} \leqslant \sqrt{\frac{a_A d_A^2+1}{d_A(d_A+1)}}\sqrt{\frac{a_B d_B^2+1}{d_B(d_B+1)}}, \quad (12)$$

where $\|\mathcal{P}\|_{\mathbf{tr}} = \mathrm{Tr}(\sqrt{\mathcal{P}\mathcal{P}^\dagger})$.

Proof. We consider a pure separable state $\rho = \rho_A \otimes \rho_B$ at first. We have

$$\mathcal{P} = \begin{pmatrix} p_{A,1} \\ \vdots \\ p_{A,d_A^2} \end{pmatrix} \begin{pmatrix} p_{B,1} & \cdots & p_{B,d_B^2} \end{pmatrix} \equiv |e_A)(e_B|, \quad (13)$$

where $p_{A,i} = \mathrm{Tr}(P_i^A \rho)$ for $i = 1, 2, \cdots, d_A^2$ and $p_{B,j} = \mathrm{Tr}(P_j^B \rho)$ for $j = 1, 2, \cdots, d_B^2$. Then

$$\begin{aligned} \|\mathcal{P}\|_{\mathbf{tr}} &= (e_A \mid e_A)^{\frac{1}{2}} (e_B \mid e_B)^{\frac{1}{2}} \\ &\leqslant \sqrt{\frac{a_A d_A^2+1}{d_A(d_A+1)}}\sqrt{\frac{a_B d_B^2+1}{d_B(d_B+1)}}. \end{aligned} \quad (14)$$

By employing the convexity property of the trace norm, we have

$$\|\mathcal{P}\|_{\mathbf{tr}} \leqslant \sqrt{\frac{a_A d_A^2+1}{d_A(d_A+1)}}\sqrt{\frac{a_B d_B^2+1}{d_B(d_B+1)}} \quad (15)$$

for separable states. □

Remark 1. If one takes $a = \frac{1}{d^2}$, the criterion of Theorem 1 reduces to the criterion based on SIC-POVM [37], i.e., if a bipartite state ρ is separable, then $\|\mathcal{P}\|_{\mathbf{tr}} \leqslant \sqrt{\frac{2}{d_A(d_A+1)}}\sqrt{\frac{2}{d_B(d_B+1)}}$.

Remark 2. If $d_A = d_B$ and $a_A = a_B$, we have $\|\mathcal{P}\|_{\mathbf{tr}} \leqslant \frac{ad^2+1}{d(d+1)}$. Furthermore, for a product state, one gets

$$J_a(\rho) \leqslant \|\mathcal{P}_s\|_{\mathbf{tr}}, \quad (16)$$

where $J_a(\rho) = \sum_{j=1}^{d^2} \mathrm{Tr}(P_j \otimes Q_j \rho)$ [32].

IV. EXAMPLES

Let us consider some examples to illustrate the effectiveness and superiority of our criterion compared with the previously known criterion in [32] and the recently criterion in [37].

Let $\{P_\alpha\}_{\alpha=1}^{d^2}$ be a set of general SIC-POVM on $\mathbb{C}^d$ with the parameter a. Let $\bar{P}_\alpha$ denote the conjugation of P_α. Then $\{\bar{P}_\alpha\}_{\alpha=1}^{d^2}$ is another set of general SIC-POVM with the same parameter a. We consider the case of $d = 3$.

It can be shown that for any non-zero $t \in [-0.012, 0.012]$, the following nine matrices

$$P_\alpha = \frac{1}{9}\mathbb{I} + t(G_9 - 12G_\alpha), \text{ for } \alpha = 1, 2, \cdots, 8, \quad (17)$$

$$P_9 = \frac{1}{9}\mathbb{I} + 4tG_9 \quad (18)$$

form a general SIC-POVM, where

$$G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_5 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$G_7 = \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad G_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$G_9 = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} & \frac{1-i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & \frac{1+i}{\sqrt{2}} & -\sqrt{\frac{2}{3}} \end{pmatrix}.$$

We can use the two general SIC-POVMs $\{G_\alpha\}_{\alpha=1}^9$ and $\{\bar{G}_\alpha\}_{\alpha=1}^9$ to recognize entanglement.

Example 1. Consider the isotropic states that are locally unitarily equivalent to a maximally entangled state mixed with white noise:

$$\rho_{\text{iso}} = q \mid \Phi^+\rangle\langle\Phi^+| + (1-q)\frac{\mathbb{I}}{d^2}, \quad 0 \leqslant q \leqslant 1, \quad (19)$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$. For $d=3$, by directly calculating the correlation entries $[\mathcal{P}]_{ij} = \langle G_i \otimes \bar{G}_j\rangle$, $i,j = 1,\cdots,9$, we have $\|\mathcal{P}\|_{\mathbf{tr}} - \frac{4a+1}{6} = 96t^2(4q-1) > 0$ for $\frac{1}{4} < q \leqslant 1$. Thus our criterion can detect the entanglement of the state ρ_{iso} for $\frac{1}{4} < q \leqslant 1$.

Example 2. Consider the Werner states [39]

$$W_d \equiv \frac{1}{d^3-d}((d-f)\mathbb{I}_{d^2} + (df-1)V), \tag{20}$$

where $-1 \leqslant f \leqslant 1$, $V = \sum_{i,j=0}^{d-1}|ij\rangle\langle ji|$. W_d is entangled if and only if $-1 \leqslant f < 0$. For $d=3$, by direct calculation we have $\|\mathcal{P}\|_{\mathbf{tr}} - \frac{9a+1}{12} = 48t^2(\sqrt{(3f-1)^2} - 2) > 0$ for $-1 \leq f < -\frac{1}{3}$. Thus our criterion recognizes the entanglement for $-1 \leqslant f < -\frac{1}{3}$. From the criterion in [32], one has $J_a(W_3) - \frac{9a+1}{12} = \sum_{j=1}^{d^2}\mathrm{Tr}(G_j \otimes \bar{G}_j W_3) - \frac{9a+1}{12} = 36(f-3)t^2 < 0$, since $-1 \leqslant f \leqslant 1$. Hence, our criterion is more efficient than the criterion in [32].

Example 3. Consider the 3×3 bound entangled state ρ^x [40],

$$\rho^x = \begin{pmatrix}
\frac{x}{8x+1} & 0 & 0 & 0 & \frac{x}{8x+1} & 0 & 0 & 0 & \frac{x}{8x+1} \\
0 & \frac{x}{8x+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{x}{8x+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{x}{8x+1} & 0 & 0 & 0 & 0 & 0 \\
\frac{x}{8x+1} & 0 & 0 & 0 & \frac{x}{8x+1} & 0 & 0 & 0 & \frac{x}{8x+1} \\
0 & 0 & 0 & 0 & 0 & \frac{x}{8x+1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{x+1}{2(8x+1)} & 0 & \frac{\sqrt{1-x^2}}{2(8x+1)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{8x+1} & 0 \\
\frac{x}{8x+1} & 0 & 0 & 0 & \frac{x}{8x+1} & 0 & \frac{\sqrt{1-x^2}}{2(8x+1)} & 0 & \frac{x+1}{2(8x+1)}
\end{pmatrix}, \tag{21}$$

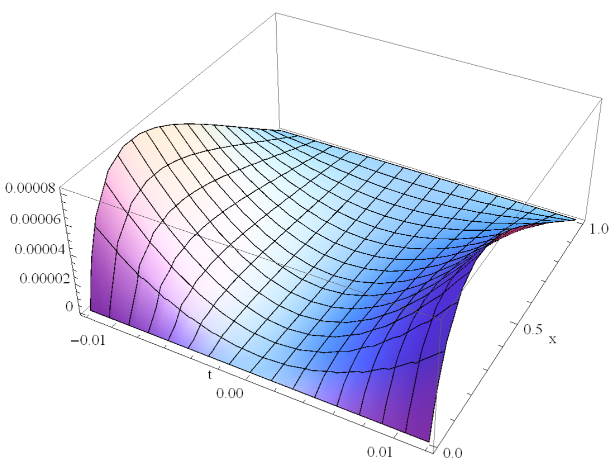

FIG. 1. The value of $|\mathcal{P}\|_{\mathbf{tr}} - \frac{9a+1}{12}$ for the state ρ^x, where $x \in (0,1)$ and $t \in [-0.012, 0.012]$.

where $0 < x < 1$.

By straightforward computation, we have that $\|\mathcal{P}\|_{\mathbf{tr}} > \frac{9a+1}{12}$ for $0 < x < 1$. Thus our criterion can detect the entanglement for the whole family of 3×3 bound entangled states. In Fig. 1 we plot the value of $|\mathcal{P}\|_{\mathbf{tr}} - \frac{9a+1}{12}$ as a function of x and t.

Now we add white noise to ρ^x, and consider

$$\rho(x,q) = q\rho^x + \frac{(1-q)}{9}\mathbb{I}, \quad 0 \leqslant q \leqslant 1. \tag{22}$$

Using the same general SIC-POVMs, we have

$$\begin{aligned} J_a(\rho(x,q)) - \frac{9a+1}{12} &= \sum_{j=1}^{d^2}\mathrm{Tr}(G_j \otimes \bar{G}_j\rho(x,q)) - \frac{9a+1}{12} \\ &= 24t^2(-4 + \frac{q+35qx}{1+8x}). \end{aligned} \tag{23}$$

From Fig. 2, one can easily find that $J_a(\rho(x,q)) - \frac{9a+1}{12} < 0$ for all permissable x, q. Thus our criterion is shown to be more efficient in detecting entanglement of $\rho(x,q)$ than the criterion of Ref. [32].

Moreover, our criterion can successfully detect entanglement for some states that cannot be detected by the criterion in [37]. Let us consider the following three

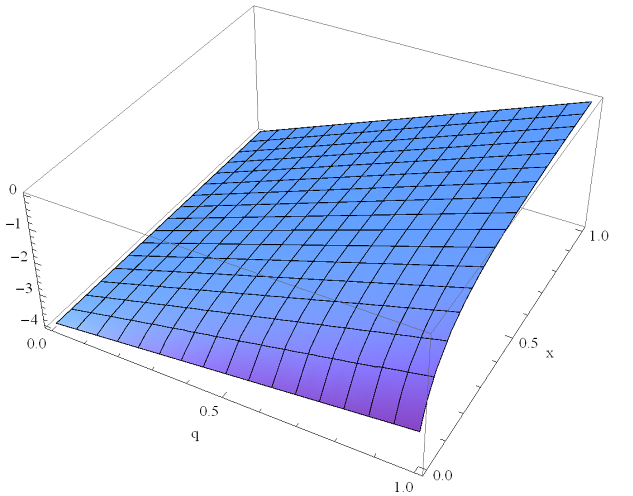

FIG. 2. The value of $-4+\frac{q+35qx}{1+8x}$ for $x\in(0,1)$ and $q\in[0,1]$.

states, $\rho(0.25,0.994)$, $\rho(0.45,0.995)$ and $\rho(0.57,0.996)$, whose entanglement cannot be identified by the criterion in [37]. Denote the correlation entries by $[\mathcal{P}^{\alpha}]_{ij}=\langle G_i\otimes\bar{G}_j\rangle$, $i,j=1,\cdots,9$, $\alpha=1,2,3$, for three states $\rho(0.25,0.994)$, $\rho(0.45,0.995)$ and $\rho(0.57,0.996)$, respectively. We have $\|\mathcal{P}^1\|_{\mathbf{tr}}>\frac{9a+1}{12}$, $\|\mathcal{P}^2\|_{\mathbf{tr}}>\frac{9a+1}{12}$ and $\|\mathcal{P}^3\|_{\mathbf{tr}}>\frac{9a+1}{12}$ in the respective fixed parameter interval, see FIG. 3. Thus our criterion can successfully detect the entanglement of these states by suitably choosing the general SIC measurements, namely, the parameter t. Therefore, in this case our criterion is more efficient than the criterion in [37].

V. CONCLUSION

We have presented an entanglement detection criterion constructed from the general SIC measurements. Interestingly, this construction includes the criterion constructed by the SIC measurements as a special case that the parameter a of general SIC measurements is equal to $1/d^2$. The criterion has been shown to be more efficient in detecting entanglement of some quantum states than the existing criteria. Moreover, our separability criterion is experimentally feasible.

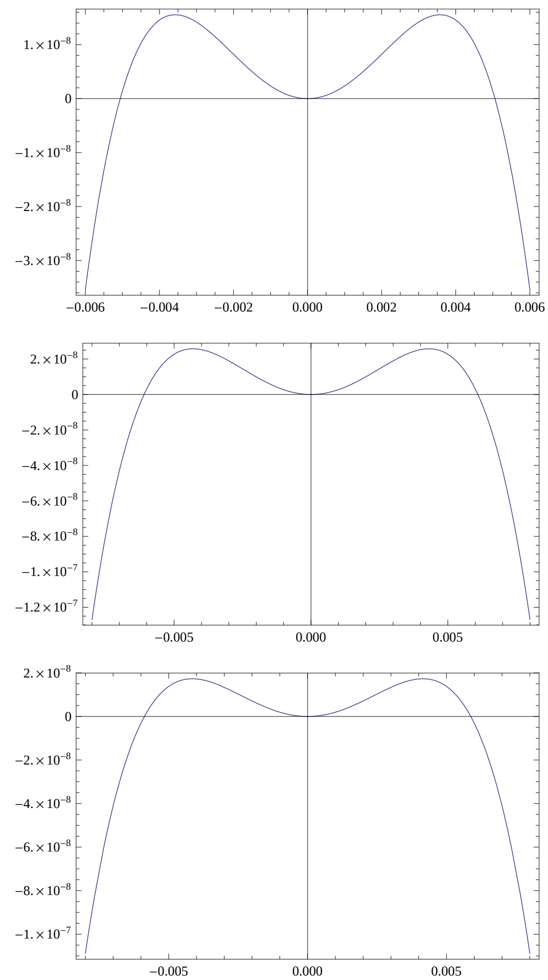

FIG. 3. The value $\|\mathcal{P}\|_{\mathbf{tr}}-\frac{9a+1}{12}$ for the states $\rho(0.25,0.994)$, $\rho(0.45,0.995)$ and $\rho(0.57,0.996)$ (form top to bottom).

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