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entanglement beyond qubits

by

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# Polygamy relations of multipartite entanglement beyond qubits

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We investigate the polygamy relations related to the concurrence of assistance for any multipartite pure states. General polygamy inequalities given by the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence of assistance is first presented for multipartite pure states in arbitrary-dimensional quantum systems. We further show that the general polygamy inequalities can even be improved to be tighter inequalities under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement is provided in a multipartite system. Moreover, the $\beta$th ($0 \leq \beta \leq 1$) power of polygamy inequalities are obtained for the entanglement of assistance as a by-product, which are shown to be tighter than the existing ones. A detailed example is presented.

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## INTRODUCTION

Quantum entanglement [1–8] has been extensively studied due to its importance in quantum communication and quantum information processing in recent years. The study of quantum entanglement from various viewpoints has been a very active area and has led to many impressive results. Monogamy of entanglement is one of the nonintuitive phenomena of quantum physics that distinguish quantum from classical physics. Different from the classical world, it is not possible to prepare three qubits in a way that any two qubits are maximally entangled. Qualitatively, monogamy of entanglement measures the shareability of entanglement in a composite quantum system. Moreover, the monogamy property has emerged as the ingredient in the security analysis of quantum key distribution [9].

The monogamy relation was first quantified by Coffman, Kundu, and Wootters [10] for three qubits, which satisfies $\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}$. The CKW inequality shows the mutually exclusive nature of multipartite quantum entanglement in a quantitative way: more entanglement shared between two qubits ($A$ and $B$) necessarily implies less entanglement between the other two qubits ($A$ and $C$). CKW inequality was generalized for multiqubit systems [11] and also studied intensively in more general settings [12, 13]. However, the CKW inequality fails for higher-dimensional quantum systems. It is also not generally true for three-qubit systems with other entanglement measures like entanglement of formation [14]. Monogamy of multiqubit entanglement and some higher-dimensional quantum systems were later characterized in terms of various entanglement measures [15–17].

Whereas the monogamy of entanglement shows the restricted sharability of multipartite quantum entanglement, the distribution of entanglement in multipartite quantum systems was shown to have a dually monogamous property. Using concurrence of assistance [18] as the measure of distributed entanglement, the polygamy of entanglement provides a lower bound for distribution of bipartite entanglement in a multipartite system [19]. Polygamy of entanglement is characterized as a polygamy inequality, $E_{a\,A|BC} \leq E_{a\,AB} + E_{a\,AC}$ for a tripartite quantum state $\rho_{ABC}$, where $E_{a\,A|BC}$ is the assisted entanglement [20] between $A$ and $BC$. Polygamy of entanglement was generalized to multiqubit systems [19] and arbitrary dimensional multipartite states [19, 21–23].

The study of quantum entanglement in higher-dimensional quantum systems is of importance in quantum information processing. Monogamy and polygamy of entanglement can restrict the possible correlations between the authorized users and the eavesdroppers, which tightens security bounds in quantum cryptography. And optimized monogamy and polygamy relations give rise to finer characterizations of the entanglement distributions. Furthermore, to optimize the efficiency of entanglement usage as a resource in quantum cryptography, higher-dimensional quantum systems rather than qubits are preferred in some physical systems for stronger security in quantum key distribution [24].

In this paper, we provide a tighter polygamy inequalities for arbitrary dimensional quantum systems. General polygamy inequalities given by the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence of assistance are first presented for multipartite pure states in arbitrary-dimensional quantum systems. We further show that the general polygamy inequalities can even be improved to be tighter inequalities under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement is provided for multipartite systems. Moreover, the $\beta$th ($0 \leq \beta \leq 1$) power of polygamy inequalities are obtained for the entanglement of assistance, which are shown to be tighter than the existing ones.

We first recall monogamy and polygamy inequalities related to concurrence and concurrence of assistance. Let $\mathbb{H}_X$ denote a discrete finite-dimensional complex vector space associated with a quantum subsystem $X$. For a

bipartite pure state $|\psi\rangle_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$, the concurrence is given by [25–27], $C(|\psi\rangle_{AB}) = \sqrt{2\left[1 - \text{Tr}(\rho_A^2)\right]}$, where $\rho_A$ is the reduced density matrix obtained by tracing over the subsystem $B$, $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state $\rho_{AB}$ is defined by the convex roof extension, $C(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$, where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$, $\sum_i p_i = 1$ and $|\psi_i\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance is defined by [28, 29],

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. For pure states $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$, one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For an $N$-qubit state $\rho_{AB_1\cdots B_{N-1}} \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$, viewed as a bipartite state under the partition $A$ and $B_1, B_2, \cdots, B_{N-1}$, satisfies the Coffman-Kundu-Wootters inequality [11, 30],

$$C^2(\rho_{A|B_1,B_2\cdots,B_{N-1}}) \geq \sum_{i=1}^{N-1} C^2(\rho_{AB_i}), \tag{1}$$

where $\rho_{AB_i} = \text{Tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{AB_1\cdots B_{N-1}})$. Further improved monogamy relations are presented in [15] and [16].

The dual inequality in terms of the concurrence of assistance for $N$-qubit states have the form [19],

$$C_a^2(\rho_{A|B_1,B_2\cdots,B_{N-1}}) \leq \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}). \tag{2}$$

For a bipartite arbitrary dimensional pure state $|\phi\rangle_{AB} = \sum_{i=1}^{d_1} \sum_{k=1}^{d_2} a_{ik}|ik\rangle_{AB}$ in $C^{d_1} \otimes C^{d_2}$, the concurrence is given by [31]

$$C^2(|\phi\rangle_{AB}) = 2(1 - \text{Tr}(\rho_A^2)) = 4\sum_{i<j}^{d_1}\sum_{k<l}^{d_2} |a_{ik}a_{jl} - a_{il}a_{jk}|^2. \tag{3}$$

And for a mixed state $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB}\langle\phi_i|$, from (3) its concurrence of assistance satisfies $C_a(\rho_{AB}) = \max_{\{p_i,|\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle) \leq \sum_{m=1}^{D_1}\sum_{n=1}^{D_2}(\max\sum_i p_i|\langle\phi_i|(L_A^m \otimes L_B^n)|\phi_i^*\rangle|) = \sum_{m=1}^{D_1}\sum_{n=1}^{D_2} C_a((\rho_{AB})_{mn}) := \tau_a(\rho_{AB})$ [32], where $D_1 = d_1(d_1-1)/2$, $D_2 = d_2(d_2-1)/2$, $L_A^m = P_A^m(-|i\rangle_A\langle j| + |j\rangle_A\langle i|)P_A^m$, $L_B^n = P_B^n(-|k\rangle_B\langle l| + |l\rangle_B\langle k|)P_B^n$, and $P_A^m = |i\rangle_A\langle i| + |j\rangle_A\langle j|$, $P_B^n = |k\rangle_B\langle k| + |l\rangle_B\langle l|$ are the projections onto the subspaces spanned by the local bases $\{|i\rangle_A, |j\rangle_A\}$ and $\{|k\rangle_B, |l\rangle_B\}$, respectively. A

general polygamy inequality for any multipartite pure state $|\phi\rangle_{A_1\cdots A_n} \in C^{d_1} \otimes \cdots \otimes C^{d_n}$ was established in [32],

$$\tau_a^2(|\phi\rangle_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^{n} \tau_a^2(\rho_{A_1A_i}), \tag{4}$$

where $\rho_{A_1A_k}$ is the reduced density matrix of $|\phi\rangle_{A_1|A_2\cdots A_n}$ associated with the subsystems $A_1A_k$, $k = 2, \cdots, n$.

## WEIGHTED POLYGAMY RELATION FOR CONCURRENCE OF ASSISTANCE

Polygamy of entanglement states that if a multipartite state is maximally entangled with respect to a given kind of multipartite entanglement, then it must be pure [33]. This observation implies that all maximally entangled states are necessarily uncorrelated with any other systems. One can even propose this condition as another requisite for a good multipartite entanglement quantifier. Furthermore, it is also important to note that this polygamy holds for all kinds of entanglement, that is, whenever a system reaches a maximum amount of entanglement under any partitions, it becomes "free" of its environment.

Therefore, for states that do not reach the maximum amount of entanglement of assistance under any partition, the polygamy inequality of entanglement provides a lower bound for the distribution of bipartite entanglement in a multipartite system. Meanwhile, the bipartite sharability of entanglement in a multipartite system gives an upper bound of the entanglement. Tighter polygamy inequalities give rise to finer characterization of the entanglement distributions, which are tightly related to the security of quantum cryptographic protocols based on entanglement [9] (it limits the amount of correlations that an eavesdropper can have with the honest parties). In the following, we give a class of polygamy inequalities that are tighter than existing ones. First, we give the definition of Hamming weight.

For any non-negative integer $j$ and its binary expansion

$$j = \sum_{i=0}^{n-1} j_i 2^i,$$

with $\log_2 j \leq n$ and $j_i \in \{0, 1\}$, $i = 0, 1, \cdots, n-1$, we can always define a unique binary vector $\vec{j}$ associated with $j$,

$$\vec{j} = (j_0, j_1, \cdots, j_{n-1}). \tag{5}$$

For the binary vector $\vec{j}$ defined in (5), the Hamming weight $w_H(\vec{j})$ is defined by the number of $1's$ in $\{j_0, j_1, \cdots, j_{n-1}\}$ [1].

[**Lemma 1**]. For any real numbers $x$ and $t$, $0 \leq t \leq 1$, $0 \leq x \leq 1$, we have $(1 + t)^x \leq 1 + (2^x - 1)t^x$.

[Proof]. Let $f(x,y) = (1+y)^x - y^x$ with $0 \leq x \leq 1$, $y \geq 1$. Then $\frac{\partial f}{\partial y} = x[(1+y)^{x-1} - y^{x-1}] \leq 0$. Therefore, $f(x,y)$ is an decreasing function of $y$, i.e., $f(x,y) \leq f(x,1) = 2^x - 1$. Set $y = \frac{1}{t}$, $0 < t \leq 1$, we obtain $(1+t)^x \leq 1 + (2^x - 1)t^x$. When $t = 0$, the inequality is trivial. $\square$

The following theorem provides states that a class of polygamy inequalities satisfied by the $\alpha$-power of $\tau_a$. For convenience, we denote $\tau_a(\rho_{AB_i}) = \tau_{a\,AB_i}$ the concurrence of assistance $\rho_{AB_i}$ and $\tau_a(\rho_{A|B_0B_1\cdots B_{N-1}}) = \tau_{a\,A|B_0B_1\cdots B_{N-1}}$.

[**Theorem 1**]. For any multiparty pure state $\rho_{AB_0\cdots B_{N-1}}$, we have

$$\tau_{a\,A|B_0B_1\cdots B_{N-1}}^\alpha \leq \sum_{j=0}^{N-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})} \tau_{a\,AB_j}^\alpha \qquad (6)$$

for $0 \leq \alpha \leq 2$, where $\vec{j} = (j_0, j_1, \cdots, j_{N-1})$ is the vector from the binary representation of $j$ and $w_H(\vec{j})$ is the Hamming weight of $\vec{j}$.

[Proof]. Without loss of generality, we can always have

$$\tau_{a\,AB_j} \geq \tau_{a\,AB_{j+1}} \geq 0, \qquad (7)$$

by relabeling the subsystems. From (4), it is sufficient to show that

$$\left(\sum_{j=0}^{N-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \leq \sum_{j=0}^{N-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})} \tau_{a\,AB_j}^\alpha. \qquad (8)$$

We first prove the inequality (8) for the case that $N$ is a power of 2, $N = 2^n$, by mathematical induction. For $n = 1$, by using Lemma 1 we have

$$\tau_{a\,A|B_0B_1}^\alpha \leq \tau_{a\,AB_0}^\alpha + (2^{\frac{\alpha}{2}} - 1)\tau_{a\,AB_1}^\alpha,$$

which is just the inequality (8) for $N = 2$.

Now let us assume that the inequality (8) is true for $N = 2^{n-1}$ with $n \geq 2$, and consider the case that $N = 2^n$. For an $(N+1)$-partite quantum state $\rho_{AB_0\cdots B_{N-1}}$ and its bipartite reduced density matrices $\rho_{AB_j}$, $j = 0, 1, \cdots, N-1$, we have

$$\left(\sum_{j=0}^{N-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}}$$
$$= \left(\sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2 + \sum_{j=2^{n-1}}^{2^n-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}}$$
$$= \left(\sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \left(1 + \frac{\sum_{j=2^{n-1}}^{2^n-1} \tau_{a\,AB_j}^2}{\sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2}\right)^{\frac{\alpha}{2}}. \quad (9)$$

Due to (7) we have

$$\sum_{j=2^{n-1}}^{2^n-1} \tau_{a\,AB_j}^2 \leq \sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2. \qquad (10)$$

By using Lemma 1 we get

$$\left(\sum_{j=0}^{N-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \leq \left(\sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}}$$
$$+ (2^{\frac{\alpha}{2}} - 1)\left(\sum_{j=2^{n-1}}^{2^n-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}}. \quad (11)$$

Here, the induction hypothesis assures that

$$\left(\sum_{j=0}^{2^{n-1}-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \leq \sum_{j=0}^{2^{n-1}-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})} \tau_{a\,AB_j}^\alpha. \quad (12)$$

From above relations we obtain

$$\left(\sum_{j=2^{n-1}}^{2^n-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \leq \sum_{j=2^{n-1}}^{2^n-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})-1} \tau_{a\,AB_j}^\alpha. (13)$$

Taking into account (11), (12) and (13) we have

$$\left(\sum_{j=0}^{2^n-1} \tau_{a\,AB_j}^2\right)^{\frac{\alpha}{2}} \leq \sum_{j=0}^{2^n-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})} \tau_{a\,AB_j}^\alpha, \quad (14)$$

which proves the inequality (8) for $N = 2^n$.

Now for an arbitrary positive integer $N$, consider an $(N+1)$-partite state $\rho_{AB_0\cdots B_{N-1}}$. We can always assume that $0 \leq N \leq 2^n$ for some $n$. Consider a $(2^n+1)$-partite quantum state

$$\rho'_{AB_0\cdots B_{2^n-1}} = \rho_{AB_0\cdots B_{N-1}} \otimes \delta_{B_N\cdots B_{2^n-1}}, \qquad (15)$$

which is a product of $\rho_{AB_0\cdots B_{N-1}}$ and an arbitrary $(2^n - N)$-partite quantum state $\delta_{B_N\cdots B_{2^n-1}}$.

Because $\rho'_{AB_0\cdots B_{2^n-1}}$ is a $(2^n+1)$-partite state, inequality (14) leads to

$$\tau_a^\alpha(\rho'_{A|B_0B_1\cdots B_{2^n-1}}) \leq \sum_{j=0}^{2^n-1} (2^{\frac{\alpha}{2}} - 1)^{w_H(\vec{j})} \tau_a^\alpha(\sigma_{AB_j}), (16)$$

where $\sigma_{AB_j}$ is the bipartite reduced density matrix of $\rho'_{AB_0\cdots B_{2^n-1}}$ for $j = 0, 1, \cdots, 2^n-1$. Since $\rho'_{AB_0\cdots B_{2^n-1}}$ is a separable state with respect to the bipartition between $AB_0\cdots B_{N-1}$ and $B_N\cdots B_{2^n-1}$, one has

$$\tau_a(\rho'_{A|B_0B_1\cdots B_{2^n-1}}) = \tau_a(\rho_{A|B_0B_1\cdots B_{N-1}}), \qquad (17)$$

and

$$\tau_a(\sigma_{AB_j}) = 0, \qquad (18)$$

for $j = N, \cdots, 2^n-1$. Moreover, for $j = 0, 1, \cdots, N-1$ one has

$$\sigma_{AB_j} = \rho_{AB_j}. \qquad (19)$$

From (16 - 19), we have

$$
\begin{aligned}
&\tau_a^\alpha(\rho_{A|B_0B_1\cdots B_{N-1}})\\
&= \tau_a^\alpha(\rho'_{A|B_0B_1\cdots B_{2^n-1}})\\
&\leq \sum_{j=0}^{2^n-1}(2^{\frac{\alpha}{2}}-1)^{w_H(\vec{j})}\tau_a^\alpha(\sigma_{AB_j})\\
&= \sum_{j=0}^{N-1}(2^{\frac{\alpha}{2}}-1)^{w_H(\vec{j})}\tau_a^\alpha(\rho_{AB_j}).
\end{aligned}
\tag{20}
$$

This completes the proof. $\square$

We have obtained the general polygamy inequality of the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence of assistance for arbitrary-dimensional quantum systems. In fact, (4) is a special case of (6) for $\alpha = 2$. Besides, based on the improved polygamy relations, we get a new upper bound for bipartite entanglement in multipartite systems for $0 \leq \alpha < 2$, which is better than (4). To illustrate the advantage of (6), we give an example as follows.

Let us consider the three-qubit state $\rho = |\psi\rangle\langle\psi|$ in the generalized Schmidt decomposition form, where $|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, $\lambda_i \geq 0$, $i = 0,1,2,3,4$ and $\sum_{i=0}^{4}\lambda_i^2 = 1$. We have $\tau_{a\,A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $\tau_{a\,AB} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_4^2}$, and $\tau_{a\,AC} = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2}$. Take $\lambda_0 = \lambda_1 = \frac{1}{2}$, $\lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$, one has $\tau_{a\,A|BC} = \frac{\sqrt{2}}{2}$, $\tau_{a\,AB} = \tau_{a\,AC} = \frac{\sqrt{3}}{3}$, and the marginal quantum relations is $\tau_{a\,AB}^2 + \tau_{a\,AC}^2 - \tau_{a\,A|BC}^2 \approx 0.167$ for $\alpha = 2$. For $\alpha = 1$, the marginal quantum relations from (6) is $\tau_{a\,AB} + (\sqrt{2}-1)\tau_{a\,AC} - \tau_{a\,A|BC} \approx 0.109$, which is smaller than the one for $\alpha = 2$.

Since $0 \leq (2^{\frac{\alpha}{2}}-1)^{w_H(\vec{j})} \leq 1$ for any $0 \leq \alpha \leq 2$, we have

$$
\begin{aligned}
\tau_a^\alpha{}_{A|B_0B_1\cdots B_{N-1}} &\leq \sum_{j=0}^{N-1}(2^{\frac{\alpha}{2}}-1)^{w_H(\vec{j})}\tau_a^\alpha{}_{AB_j}\\
&\leq \sum_{j=0}^{N-1}\tau_a^\alpha{}_{AB_j},
\end{aligned}
\tag{21}
$$

for any multipartite quantum state $\rho_{A|B_0B_1\cdots B_{N-1}}$. Thus, we have the following corollary.

[**Corollary 1**]. For any multiparty pure state $\rho_{AB_0\cdots B_{N-1}}$, we have

$$
\tau_a{}^\alpha_{A|B_0B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1}\tau_a{}^\alpha_{AB_j}
$$

for $0 \leq \alpha \leq 2$.

The class of weighted polygamy inequalities in Theorem 1 can be further tightened under some condition on bipartite quantum relations.

[**Theorem 2**]. For any multipartite pure state $\rho_{AB_0\cdots B_{N-1}}$, if

$$
\tau_a^2{}_{AB_i} \geq \sum_{j=i+1}^{N-1}\tau_a^2{}_{AB_j}
\tag{22}
$$

for $i = 0,1,\cdots N-2$, we have

$$
\tau_a^\alpha{}_{A|B_0B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1}(2^{\frac{\alpha}{2}}-1)^j \tau_a^\alpha{}_{AB_j}
\tag{23}
$$

for $0 \leq \alpha \leq 2$.

[Proof]. From Lemma 1, we have

$$
\begin{aligned}
&\tau_a^\alpha{}_{A|B_0B_1\cdots B_{N-1}}\\
&\leq \tau_a^\alpha{}_{AB_0} + (2^{\frac{\alpha}{2}}-1)\left(\sum_{j=1}^{N-1}\tau_a^2{}_{AB_j}\right)^{\frac{\alpha}{2}}\\
&\leq \tau_a^\alpha{}_{AB_0} + (2^{\frac{\alpha}{2}}-1)\tau_a^\alpha{}_{AB_1} + (2^{\frac{\alpha}{2}}-1)^2\left(\sum_{j=2}^{N-1}\tau_a^2{}_{AB_j}\right)^{\frac{\alpha}{2}}\\
&\leq \cdots\\
&\leq \tau_a^\alpha{}_{AB_0} + (2^{\frac{\alpha}{2}}-1)\tau_a^\alpha{}_{AB_1} + \cdots + (2^{\frac{\alpha}{2}}-1)^{N-1}\tau_a^\alpha{}_{AB_{N-1}}.
\end{aligned}
$$

$\square$

In Theorem 2, the condition (22) are always satisfied by some states. Let us consider a four-qubit state $\rho = |W\rangle_{ABCD}\langle W|$, where $|W\rangle_{ABCD} = a|1000\rangle + b|0100\rangle + c|0010\rangle + d|0001\rangle$, and $a^2 + b^2 + c^2 + d^2 = 1$. We have $\tau_a(\rho_{A|BCD}) = 2a\sqrt{1-a^2}$, $\tau_a(\rho_{AB}) = 2ab$, $\tau_a(\rho_{AC}) = 2ac$, $\tau_a(\rho_{AD}) = 2ad$. The condition (22) is satisfied as long as $b^2 \geq c^2 + d^2$. For example, we set $b = \frac{1}{\sqrt{2}}$, $a = c = d = \frac{1}{\sqrt{6}}$. Then the state $\rho = |W\rangle_{ABCD}\langle W|$ satisfies the condition (22). On the other hand, if $b^2 \leq c^2 + d^2$, e.g., $c = \frac{1}{\sqrt{2}}$ and $a = b = d = \frac{1}{\sqrt{6}}$, then $\rho$ does not satisfy the condition (22).

[**Remark 1**]. For any non-negative integer $j$ and the corresponding binary vector $\vec{j}$ in Eq. (5), the Hamming weight $w_H(\vec{j})$ is upper bounded by $\log_2 j$. Thus, we have $w_H(\vec{j}) \leq \log_2 j \leq j$, which implies that $\tau_a^\alpha{}_{A|B_0B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1}(2^{\frac{\alpha}{2}}-1)^j\tau_a^\alpha{}_{AB_j} \leq \sum_{j=0}^{N-1}(2^{\frac{\alpha}{2}}-1)^{w_H(\vec{j})}\tau_a^\alpha{}_{AB_j}$, for $0 \leq \alpha \leq 2$. In other words, inequality (23) in Theorem 2 is tighter than the inequality (6) in Theorem 1 for states satisfying the conditions $\tau_a^2{}_{AB_i} \geq \sum_{j=i+1}^{N-1}\tau_a^2{}_{AB_j}$, $i = 0,\cdots,N-2$.

## POLYGAMY RELATIONS FOR ENTANGLEMENT OF ASSISTANCE

Now we study the polygamy relations for entanglement of assistance. For polygamy inequality beyond qubits, it was shown that the von Neumann entropy

can be used to establish a polygamy inequality of tripartite quantum systems [34]. For any arbitrary dimensional tripartite pure state $|\psi\rangle_{ABC}$, one has $E(|\psi\rangle_{A|BC}) \leq E_a(\rho_{AB}) + E_a(\rho_{AC})$, where $E(|\psi\rangle_{A|BC}) = S(\rho_A)$ is the entropy of entanglement between $A$ and $BC$ in terms of the von Neumann entropy $S(\rho) = -\text{Tr}\rho \ln \rho$, and $E_a(\rho_{AB}) = \max \sum_i p_i E(|\psi_i\rangle_{AB})$, with the maximization taking over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. Later, a general polygamy inequality for any multipartite state $\rho_{A_1|A_2\cdots A_n}$ was established [35],

$$E_a(\rho_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^{n} E_a(\rho_{A_1 A_i}). \qquad (24)$$

Recently, another class of multipartite polygamy inequalities in terms of the $\beta$th power of entanglement of assistance (EOA) has been introduced [23]. For any multipartite state $\rho_{A|B_0 B_1 \cdots B_{N-1}}$ and $0 \leq \beta \leq 1$,

$$E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1} \beta^{w_H(\vec{j})} E_{a\ AB_j}^{\beta}, \qquad (25)$$

if $E_{a AB_i} \geq E_{a AB_{i+1}}$ for $i = 0, 1, \cdots, N-2$; and

$$E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1} \beta^j E_{a\ AB_j}^{\beta},$$

if $E_{a AB_i} \geq \sum_{j=i+1}^{N-1} E_{a A|B_j}$ for $i = 0, 1, \cdots, N-2$. With a similar consideration to $\tau_{AB_0 \cdots B_{N-1}}$, we have the following result for EOA.

[**Theorem 3**]. For any multipartite state $\rho_{AB_0 \cdots B_{N-1}}$, we have

$$E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1} (2^\beta - 1)^{w_H(\vec{j})} E_{a\ AB_j}^{\beta} \qquad (26)$$

for $0 \leq \beta \leq 1$.

To illustrate the tightness of the inequality (26) compared with the inequality (25) in [23], we consider the three-qubit state $\rho_{ABC} = |W\rangle_{ABC}\langle W|$, where $|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. We have $E_a(\rho_{A|BC}) = S(\rho_A) = \log_2 3 - \frac{2}{3}$ and $E_a(\rho_{AB}) = E_a(\rho_{AC}) = \frac{2}{3}$. Thus the marginal EOA from inequality (25) is $E_a^\beta(\rho_{AB}) + \beta E_a^\beta(\rho_{AC}) - E_a(\rho_{A|BC}) = (1+\beta)(\frac{2}{3})^\beta + \frac{2}{3} - \log_2 3$. The marginal EOA from inequality (26) is $E_a^\beta(\rho_{AB}) + (2^\beta - 1)E_a^\beta(\rho_{AC}) - E_a(\rho_{A|BC}) = 2^\beta(\frac{2}{3})^\beta + \frac{2}{3} - \log_2 3$. Fig. 1 shows that our inequality gives a smaller upper bound than (25) in [23], namely, our marginal EOA is smaller than inequallity (25) in [23] for $0 < \beta < 1$.
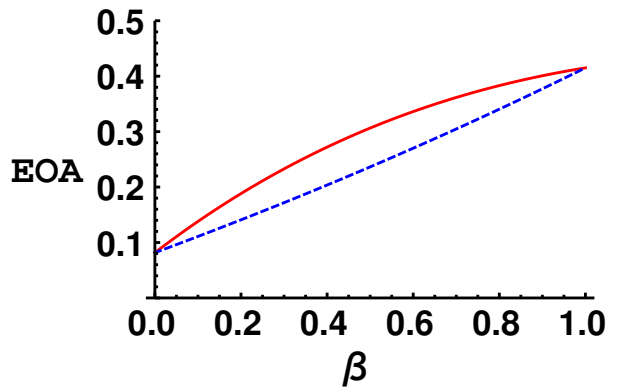


FIG. 1: The blue dashed line represents the marginal EOA from inequality (26) for three-qubit $W$ state, the red thick line represents the marginal EOA from inequality (25) in [23].

Since $0 \leq (2^\beta - 1)^{w_H(\vec{j})} \leq 1$ for any $0 \leq \beta \leq 1$, we have

$$\begin{aligned} E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} &\leq \sum_{j=0}^{N-1} (2^\beta - 1)^{w_H(\vec{j})} E_{a\ AB_j}^{\beta} \\ &\leq \sum_{j=0}^{N-1} E_{a\ AB_j}^{\beta}, \end{aligned}$$

for any multipartite quantum state $\rho_{A|B_0 B_1 \cdots B_{N-1}}$. Thus, we have the following corollary.

[**Corollary 2**]. For any multipartite pure state $\rho_{AB_0 \cdots B_{N-1}}$, we have

$$E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1} E_{a\ AB_j}^{\beta}$$

for $0 \leq \beta \leq 1$.

With a similar consideration to Theorem 2, we can tighten the class of weighted polygamy inequalities in Theorem 3 under certain conditions on bipartite quantum correlations.

[**Theorem 4**]. For any multipartite state $\rho_{AB_0 \cdots B_{N-1}}$, we have

$$E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1} (2^\beta - 1)^j E_{a\ AB_j}^{\beta}, \qquad (27)$$

conditioned that

$$E_{a\ AB_i}^2 \geq \sum_{j=i+1}^{N-1} E_{a\ AB_j}^2,$$

for $i = 0, 1, \cdots N-2$, $0 \leq \beta \leq 1$.

[Remark 2]. For any non-negative integer $j$, since $w_H(\vec{j}) \leq \log_2 j \leq j$, one has $E_{a\ A|B_0 B_1 \cdots B_{N-1}}^{\beta} \leq \sum_{j=0}^{N-1}(2^\beta - 1)^j E_{a\ AB_j}^{\beta} \leq \sum_{j=0}^{N-1}(2^\beta - 1)^{w_H(\vec{j})} E_{a\ AB_j}^{\beta}$ for $0 \leq \beta \leq 1$. Therefore, inequality (27) in Theorem 4

is tighter than the inequality (26) in Theorem 3 for states satisfying the conditions $E_{a\,AB_i}^2 \geq \sum_{j=i+1}^{N-1} E_{a\,AB_j}^2$, $i = 0, \cdots, N-2$.

In particular, (27) reduces to (24) in [35] for $\beta = 1$. For $0 < \beta < 1$, (27) is a tighter polygamy inequality compared with (24). Since $w_H(\vec{j}) \leq j$, (27) in Theorem 4 is in general tighter than the (26) in Theorem 3. From the example shown in Fig. 1, one can see that (26) is generally tighter than the result in [23]. Hence our weighted polygamy relations give finer characterizations of the entanglement distributions among the subsystems, and help better security analysis of quantum key distribution [9] in quantum information processing.

## CONCLUSION

Entanglement monogamy and polygamy are fundamental properties of multipartite entanglement. We have investigated the polygamy relations related to the concurrence of assistance. General polygamy inequalities given by the $\alpha$th $(0 \leq \alpha \leq 2)$ power of concurrence of assistance have been presented for multipartite states in arbitrary-dimensional quantum systems. We have further shown that the general polygamy inequalities can even be improved to be tighter ones under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement has been provided for multipartite systems. Moreover, the $\beta$th $(0 \leq \beta \leq 1)$ power of polygamy inequalities have been obtained for the entanglement of assistance as a byproduct, which are shown to be tighter than the existing ones.

The higher-dimensional quantum systems are the key resources in various quantum information and communication processing tasks. For instance, the qudit $(d > 2)$ systems are preferred in some quantum key distributions, where the use of qudits increases the coding density and provides stronger security compared to qubits [36]. Our results apply to general polygamy relations of multipartite entanglement in arbitrary higher-dimensional quantum systems. Moreover, our polygamy inequalities provide tighter constraints and finer characterizations of the entanglement distributions among the multipartite systems. These results may highlight future works on the study of multipartite quantum entanglement.

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