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by

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In the quantitative theory of quantum coherence, the amount of coherence is defined as the distance between the given state to to the closest incoherent state. The set of incoherent states is conventionally defined as any state with a diagonal density matrix. One of the objections to this formulation is that the incoherent states are intrinsically basis-dependent, which makes the amount of coherence also a basis-dependent quantity. Basis-independent measures have been recently been proposed where the incoherent state is taken as the maximally mixed state. We show that this is the only possible choice of reference incoherent state, without modifying the original definition of coherence. We find a relation between the two formulations by defining a contribution to the coherence due to the basis choice. The hierarchical relationship between quantum coherence and the various quantum correlations is explored in detail. Finally, we illustrate some operational uses of the basis-independent quantum coherence in quantum information theory tasks.

I. INTRODUCTION

One of the fundamental properties that distinguishes quantum mechanics from classical physics is coherence, arising from the superposition principle of quantum states [?]. One of the features of quantum coherence is that it can arise even in single party systems, and underlies all the other types of quantum correlations. Quantum coherence therefore can be viewed to be the prime ingredient required for various quantum technologies including quantum computing [?], quantum simulation [?], quantum metrology [? ?] and quantum cryptography [?]. Although the concept of coherence has been present since the dawn of quantum mechanics and studied extensively in many fields such as quantum optics [? ? ?], quantum biology [? ? ? ?] and quantum systems out of equilibrium [?], in all these works coherence was only studied as a physical phenomenon. Recently, a rigorous method to quantify coherence was recently proposed in Ref. [?] with its main focus on bringing in quantum coherence within the ambit of resource theories.

Based on the framework introduced in Ref. [?], several quantifiers such as distance-based coherence measures [? ? ? ? ? ? ? ?], distillable coherence [? ?], coherence cost [?] and robustness of coherence [?] were introduced. Most of these quantifiers, however, are basis-dependent i.e., the value of coherence in a physical system depends on the basis in which the coherence is measured. For example, the coherence of a quantum state $|+\rangle$ as measured using the relative entropy of coherence is +1 in the σ_z-basis, but vanishes in the σ_x-basis. Thus the estimation of coherence does not have an universal value, due to its dependence on the choice of the measurement basis. This is in contrast to the measurement of other quantum correlations like entanglement and discord which are basis-independent. This also makes measurements of coherence ambiguous in the sense that the amount of coherent in a given quantum state changes according to what basis is chosen.

The basis-dependent aspect of quantum coherence has been a point of concern since the introduction of the theory, and several alternatives were suggested to remove this ambiguity. The approach proposed in Ref. [?] was to optimize the quantum coherence over all possible local basis. This method makes the quantum coherence in quantum correlated systems basis-independent, but classically correlated systems always have zero coherence. It was shown in this work that the coherence defined in this way is equivalent to the amount of quantum correlations, which is conventionally quantified using quantum discord. This has raised questions of whether such a definition truly captures the conventional notion of quantum coherence. An alternative method was introduced in Ref. [?], where the the set of incoherent states is replaced by the maximally mixed state. This method works equally

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well for correlated and uncorrelated systems and it follows that the maximal amount of basis-dependent coherence that can be extracted is when the maximally mixed state is the closest incoherent state [**?**].

In this paper, we mathematically establish that to achieve a basis-independent theory of quantum coherence by modifying the set of incoherent states, the only possible choice is to make the reference state a maximally mixed state. The hierarchical relationship between quantum coherence and the various quantum correlations is explored, and we show that a basis-independent formulation allows for a coherence to fit within this heirarchy in a consistent fashion. Then we illustrate some operational uses of the basis-independent quantum coherence. The manuscript is organized as follows: In Sec. II, we mathematically show how to construct a basis independent measure of quantum coherence using the maximally mixed states and also explore the set of the relevant free operations. The various basis-independent formulations of measuring quantum coherence is discussed in Sec. III to point out the conceptual edge of our method. We present three applications of the basis-independent quantum coherence in Section IV. Finally in Sec. V the results are summarized.

II. BASIS-INDEPENDENT QUANTUM COHERENCE

In this section we define a basis-independent measure of quantum coherence using relative entropy. The basis-dependent quantum coherence based on relative entropy was introduced in Ref. [**?**] which measures the distance of a quantum state to the closest state in the set $\mathcal{I}$, which is the set of incoherent (diagonal) states. This set of incoherent states is basis-dependent since a basis transformation on a diagonal state can convert it to a coherent state with off-diagonal terms. Since the basis dependence comes about through the incoherent state, it is desirable that there is a diagonal state which remains incoherent under basis transformations.

We find this set of states rigorously through the following propositions. Let X and Z denote two projective measurements with measurement operators $X_m = |X_m\rangle\langle X_m|$ and $Z_m = |Z_m\rangle\langle Z_m|$, respectively. Here $|X_m\rangle$ and $|Z_m\rangle$ with $m = 1, ..., d$ are the eigenvectors of the observables X and Z which form a complete orthogonal basis in a d-dimensional Hilbert space. The quantum coherence in the X, Z reference basis is $C_r^{X,Z} = S(\rho_d^{X,Z}) - S(\rho)$, where $\rho_d^{X,Z}$ is the diagonal part of the density matrix ρ in the corresponding basis. From the quantity $C_r^X(\rho) + C_r^Z(\rho)$ we have:

Proposition 1. *Given two projective measurements X and Z, the following inequality holds for any state ρ,*

$$C_r^X(\rho) + C_r^Z(\rho) \geqslant -\log_2 c - 2S(\rho) \tag{1}$$

where $c = \max_{j,k} |\langle X_j|Z_k\rangle|^2$.
The proof is given in Appendix A.

Proposition 1 states that there is a trade-off between the coherence of an arbitrary quantum state ρ in different reference bases. This implies that for any two non-commuting measurements, if the non-negative lower bound $-\log_2 c - 2S(\rho)$ is not zero, and the coherence is zero in one measurement basis, then it is strictly positive in the other measurement basis. Using this proposition we can identify the set of incoherent states for a basis-independent measure of quantum coherence based on the relative entropy as explained in the next proposition.

Proposition 2. *For orthonormal bases X and Z there is a quantum state $\rho \neq I/d$ such that $C_r^Z(\rho) = C_r^X(\rho) = 0$ iff there exists a permutation π of the set $\{1, ..., d\}$ such that*

$$\Delta^Z\left(\sum_{i=1}^{k} |X_{\pi(i)}\rangle\langle X_{\pi(i)}|\right) = \sum_{i=1}^{k} |X_{\pi(i)}\rangle\langle X_{\pi(i)}| \tag{2}$$

for some $k < d$.
The proof is given in Appendix B.

Proposition 2 gives the criteria for the vanishing trade-off in (1). For the maximal trade-off we need the concept of mutually unbiased bases (MUB) [**? ?**]. Two bases are unbiased if a basis state in one basis is overlaps equally with all basis states in the other basis. A set of K bases are called mutually unbiased if any pair of them are unbiased. In a d-dimensional Hilbert space at least two MUB exists [**? ?**]. It is obvious that choosing MUB as the projective measurements in Proposition 1 maximizes the trade-off lower bound. In Appendix B we show that for any state $\rho \neq I/d$, the choice of MUB as X and Z violates the criterion in Proposition 2, which means by changing suitable basis a state may always display some nonzero coherence, unless it is the maximally mixed state.

From the above we establish that the maximally mixed state I/d is the *only basis-independent incoherent* state. Therefore, to formulate a basis-independent measure of quantum coherence by modifying the set of incoherent states, the only choice is to take I/d as the reference incoherent state. The relative entropy distance between an arbitrary density matrix ρ and I/d is

$$C(\rho) = S(\rho \| I_d/d) = \log_2 d - S(\rho). \tag{3}$$

Since the reference incoherent state is always the maximally mixed state, the measured coherence is basis-independent in nature and no optimization is required.

To qualify as a resource for applications in quantum information theory, the basis-independent quantum coherence should satisfy the properties listed in Ref. [**?**]. Any resource theory has three major ingredients *viz* the resource, the free states and the free operations. We therefore identify basis independent quantum coherence as the resource and the maximally mixed state as the free state [**?**]. In the current work we introduce the free operations which maps the maximally mixed state on to itself. The physically feasible incoherent operations (PFIO's) which form a physically relevant subset of the unital completely positive trace preserving operations are the relevant incoherent operations and are defined below:

Proposition 3. *A completely positive trace preserving map $\mathcal{E}$ is a PFIO if and only if it can be expressed in the following form:*

$$\mathcal{E}(\rho_A) = \frac{1}{d}\sum_{y,j} K_j^{(y)} \rho_A K_j^{(y)\dagger} \tag{4}$$

where $\forall y$, $K_j^{(y)} = U_j^{(y)} P_j^{(y)}$, $U_j^{(y)} = \sum_x e^{i\theta_{xy}} |\pi_j^{(y)}(x)\rangle\langle x|$, $\pi_j^{(y)}$ are permutations, and $P_j^{(y)}$ form a complete set of incoherent projectors which are mutually orthogonal.

The axiomatic framework of quantum coherence defined in Ref. [**?**] requires that the measure of quantum coherence obeys the following postulates namely: (a) $C(\rho) \geqslant 0$ and $C(\rho) = 0$ if and only if ρ is an incoherent state; (b) Monotonicity under incoherent operation; (c) Monotonicity under selective measurements on average; (d) Non-increasing under mixing of quantum states. To verify these properties we use the maximally mixed state as the incoherent state and the PFIO's as the incoherent operations.

The condition (a) follows from the fact that the relative entropy is always non-negative and equals to zero only when the two quantum states are identical. The second condition (b) can be verified by noting that $C(\rho)$ is monotonic under noisy operations [**?**], which can be decomposed into two steps as adding an ancillary subsystem and use a unitary operation on the system plus ancilla and finally removing the ancilla. Since the PFIO's are contained in noisy operations, $C(\rho)$ is monotonic. The criterion (c) follows as a property of relative entropy. To verify condition (d) we use the fact that relative entropy is jointly concave [**?**] and we have

$$\begin{aligned} C\left(\sum_n p_n \rho_n\right) &\leq S\left(\sum_n p_n \rho_n \| \sum_n p_n \frac{I}{d}\right) \\ &\leq \sum_n p_n S\left(\rho_n \| \frac{I}{d}\right) = \sum_n p_n C(\rho_n) \end{aligned} \tag{5}$$

Thus we verify that the relative entropy based measure of coherence with the maximally mixed state as the incoherent state is a valid coherence measure.

The basis-independent quantum coherence introduced in our work can be related to the basis-dependent coherence [**?**] by introducing the following quantity

$$\delta C^{(b)}(\rho) = S(I/d) - S(\rho_d) \equiv \log_2 d - S(\rho_d), \tag{6}$$

which measures the coherence in a diagonal state which is not maximally mixed. Eqn. (6) can be expressed as an exact difference between the basis-independent and basis-dependent coherence

$$\delta C^{(b)}(\rho) = C(\rho) - C^{(b)}(\rho). \tag{7}$$

From the entire discussion above we can see that

$$C(\rho) \geqslant C^{(b)}(\rho) \tag{8}$$

and the equality holds only when the closest incoherent state in the basis-dependent coherence becomes a maximally mixed state. For the coherence quantifier $\delta C^{(b)}$, the free states are the maximally mixed state I/d and the PFIO's are the free operations.

III. RELATION BETWEEN QUANTUM COHERENCE AND QUANTUM CORRELATIONS

So far we have argued that for a basis-independent theory of quantum coherence from the point of view that this gives an unambiguous definition that depends only upon the quantum state. There is another point of view which favors a basis-independent formulation, in terms of the relationship between other quantum information quantities. As mentioned previously, quantities such as the quantum discord and quantum entanglement measures are formulated to be basis-independent quantities. The set of quantum correlated states, entangled states, steerable states, and non-local Bell violating states are successively more specialized states following a hierarchical structure [**?**]. Since quantum coherence is a fundamental property of quantum mechanical states, it is reasonable to expect that it has the structure as shown in Fig. 1, where quantum coherence is the common element to all quantum correlations. Furthermore, it has been shown in Ref. [**?**] that the amount of entanglement is bounded by the quantum coherence in the system. Since quantum coherence can also be present in a localized way [**?**], this is suggestive that a consistent theory of coherence follows the hierarchical structure as shown in Fig. 1.

We can explicitly show via a counterexample that basis-independent coherence does not fit this heirachical structure. For example, consider a two qubit system, and let the set of incoherent states be any diagonal state in the Bell basis

$$\begin{aligned} \rho_d = & p_{00}|\Psi^+\rangle\langle\Psi^+| + p_{01}|\Phi^+\rangle\langle\Phi^+| + p_{01}|\Psi^-\rangle\langle\Psi^-| \\ & + p_{11}|\Phi^-\rangle\langle\Phi^-|, \end{aligned} \tag{9}$$

where $|\Psi^\pm\rangle, |\Phi^\pm\rangle$ are the Bell states and p_{ij} are probabilities summing to 1. In this case the coherence of any Bell state is zero, since it is an instance of ρ_d. A Bell state however has all the types of quantum correlations (i.e. quantum discord and all subsets) shown in Fig. 1, and is therefore incompatible with such a hierarchy.

For a basis-dependent theory of quantum coherence, the hierarchical structure of Fig. 1 holds. The proof is trivial. For the case that $\rho_d = I/d$, the set of all states that have non-zero coherence are all states except for $\rho = I/d$ itself. Such a set of states includes all quantum correlated states, and other states, including classically correlated states, or any product state. This has a larger set of states than quantum correlated states, hence has the structure of Fig. 1.

The basis-independent approach that we describe above is more consistent with established approaches for

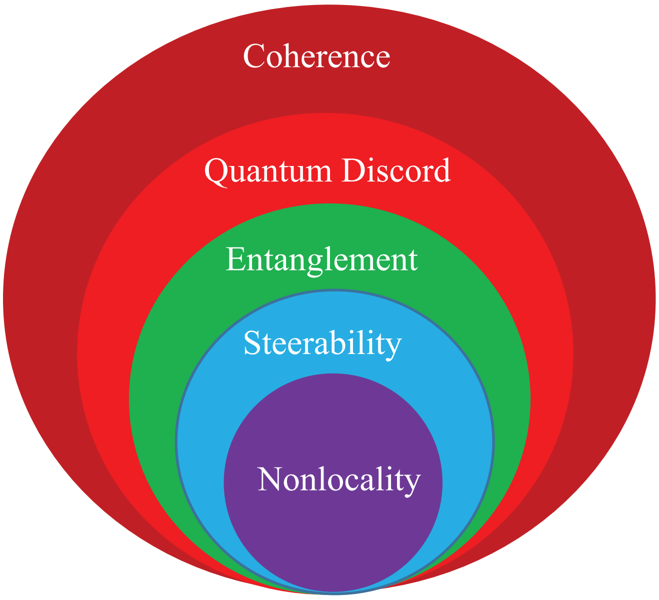

FIG. 1. The hierarchy among quantum coherence and the various quantum correlations are shown in the figure. In a basis-independent theory of quantum coherence, the most general form of quantumness is coherence and includes all the different forms of correlations such as discord, entanglement, steering and nonlocality as subsets. Quantum dissonance corresponds to the set of all states with quantum discord, removing all entangled states.

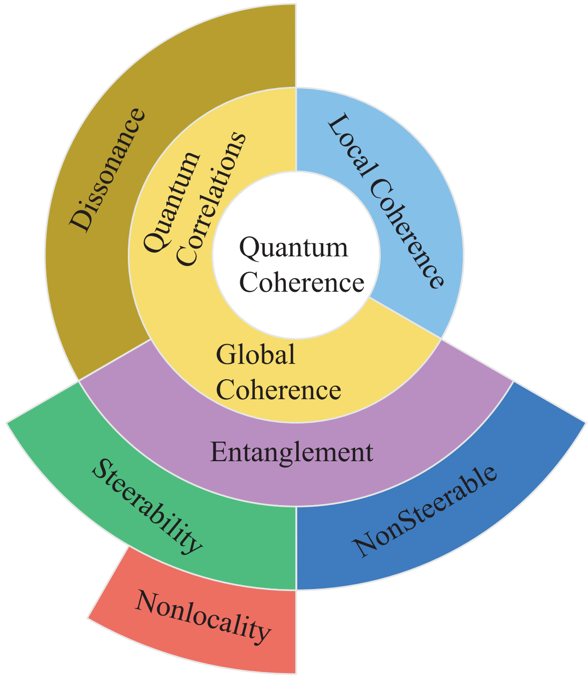

FIG. 2. The hierarchy dependencies between various quantum correlations. Quantum coherence is at the centre of quantum correlations and can be divided into the local coherence and the global coherence due to the correlations. The quantum correlations are further classified into the non-local correlations (entanglement) and local correlations (dissonance). Entanglement is again again classified into steerable and non-steerable correlations and non-locality is a more specific type of steerable quantum correlation.

other types of quantum correlations. For example, entanglement is defined as the minimal distance between the state in question and a separable state. The set of separable states is left invariant under local unitary transformations, but clearly non-local unitary transformations will change the nature of the state since this may introduce entanglement into the separable state. Thus when finding the closest separable state, some types of unitary transformations are excluded since they do not preserve the quantity in question. Similarly when we measure the coherence, we define it as the distance to the closest incoherent state. Even local unitary transformations might change an incoherent state into a coherent state. To avoid this we need to define a basis independent measure of coherence. Such a basis independent measure would have to find the distance to the closest state which does not become coherent under local unitary operations. The only state which is does not gain coherence on subjection to a local unitary operation is the maximally mixed state. Hence any basis independent measure of coherence should be the distance to the maximally mixed state.

The quantum coherence and the various correlations are shown in Fig. 2, representing the hierarchical relationship between them. Here the base of the hierarchy is represented by a circle at the center of the chart while the rest of dependent quantities radiate outwards from the center. At the core of all the quantum correlations lies quantum coherence and hence it is represented by a central circle. The quantum coherence can originate due to correlations between the qubits or because of the superposition between the levels of the qubit which are in general referred to as global coherence and local coherence respectively [**?**]. The global coherence is a manifestation of the quantum correlations in the system and hence is equal to the quantum discord. Quantum discord is composed of the quantum dissonance which arises due to local quantum correlations and quantum entanglement due to the non-local quantum correlations in the system. The non-local correlations can in general be classified into the steerable and the non-steerable correlations. Finally, we represent the well known fact that non-locality is a special case of steerability.

IV. APPLICATIONS

We now show several examples of utilizing basis-independent quantum coherence and its connection to other quantum information theoretic quantities. In particular, we show that the basis-independent coherence is equal to the rate of purification of a state, the amount of

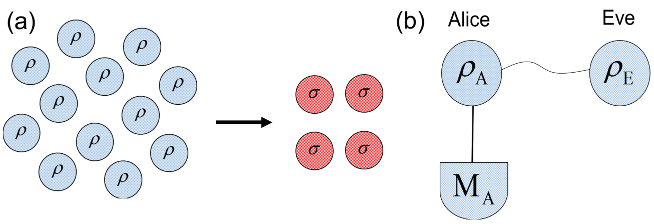

FIG. 3. (a) State transformation: Multiple copies of ρ are transformed into multiple copies of the pure state σ by using PFIOs. The transformation rate can be estimated using the amount of basis-independent quantum coherence in the system. (b) Quantum randomness: In a bipartite state ρ_{AE}, where Alice has qubit A and Eve has qubit E, Alice would like to extract the randomness of the system unknown to Eve. To this end, Alice selects a measurement, whose basis may depend on the quantum state ρ_A and performs measurement on her share.

randomness, and is related to average work extracted in quantum thermodynamical systems.

IV.1. Connection to conversion rate

Let us consider the transformation of n copies of state ρ to m_n copies of the state σ, $\rho^{\otimes n} \to \sigma^{\otimes m_n}$ as shown in Fig. 3(a). In general an exact transformation is usually difficult. Hence we consider the transformation of the form $\rho^{\otimes n} \to \sigma_n \approx \sigma^{\otimes m_n}$. The inaccuracy between the states σ_n and $\sigma^{\otimes m_n}$ vanishes in the limit of large n and hence satisfies the relation $\|\sigma_n - \sigma^{\otimes m_n}\| \to 0$ for $n \to \infty$. The ratio m_n/n is the transformation ratio $R(\rho \to \sigma)$ and we prove that this is equal to the basis-independent quantum coherence.

Through the law of large numbers, we know that there exists a subset of eigenvalues T of the system $\rho^{\otimes n}$ such that for $p_i \in T$

$$\sum_{p_i \in T} p_i \geq 1 - \epsilon; \quad 2^{-n(S+\delta)} < p_i < 2^{-n(S-\delta)} \tag{10}$$

$\forall \epsilon, \delta > 0$. These eigenvalues carry most of the information about the state and they are uniform. Based on whether the condition $p_i \in T$ is satisfied or not we can classify the state $\rho^{\otimes n}$ into two states ρ_{typ} and ρ_{atyp} as given below:

$$\rho_{\text{typ}} = \frac{1}{c} \sum_{p_i \in T} p_i |i\rangle\langle i|; \quad \rho_{\text{atyp}} = \frac{1}{1-c} \sum_{p_i \notin T} p_i |i\rangle\langle i| \tag{11}$$

Here $|i\rangle$ are the eigenvectors corresponding to p_i and $c = \sum_{p_i \in T} p_i$ is the normalization constant. We can see that $\rho^{\otimes n}$ is a mixture of ρ_{typ} and ρ_{atyp}

$$\rho^{\otimes n} = c\rho_{\text{typ}} + (1-c)\rho_{\text{atyp}} \tag{12}$$

It can be immediately seen from the relation $\|\rho_{\text{typ}} - \rho^{\otimes n}\| \leq 2\epsilon$ that ρ_{typ} is closer to $\rho^{\otimes n}$. So we can use ρ_{typ} instead of $\rho^{\otimes n}$ and convert ρ_{typ} into approximately $n(\log_2 d - S(\rho))$ of pure states. The eigenvalues of ρ_{typ} are

$$\lambda_i = \frac{p_i}{c} \geq \frac{2^{-n(S+\delta)}}{c}. \tag{13}$$

The state ρ_{typ} majorizes over a state ρ_{out} with D eigenvalues of $1/D$ and $d^n - D$ eigenvalues of 0 where $D = c/2^{-n(S+\delta)}$. It is well know that ρ_{typ} can be transformed to ρ_{out} using a noisy operation [?]. Since PFIO's are noisy operations we can effect this transformation using them. The state ρ_{out} is a tensor product of $\log_2 D$ qubits in the maximally mixed state and $n \log_2 d - \log_2 D \geq n(\log_2 d - S(\rho) - \delta) - 1$ qubits in the pure state. We can discard the maximally mixed states and retain the pure states. The rate of transformation is

$$R(\rho \to \sigma) = \frac{n(\log_2 d - S(\rho) - \delta) - 1}{n}. \tag{14}$$

In the asymptotic limit, when n is very large and δ can be chosen to be arbitrarily small, the transformation rate is

$$R(\rho \to \sigma) = \log_2 d - S(\rho) = C(\rho). \tag{15}$$

Hence we find that the asymptotic transformation rate of state ρ into another state is equal to the basis-independent measure of quantum coherence. This shows that the basis-independent quantum coherence is operationally equal to the rate of transformation from the state ρ to a pure state σ.

IV.2. Quantum Randomness and basis-independent quantum coherence

Let us consider a bipartite quantum system ρ_{AE} shared between two observers Alice and Eve as shown in Fig. 3 (b). When Alice performs a measurement on her system it can be in any basis F_i. Hence her outcome will follow a distribution $\{p_i\}$ with $p_i = \text{Tr}(\rho_A F_i)$. The randomness is given by the Shannon entropy $H(\{p_i\})$.

We can consider the shared state ρ_{AE} to be a pure state $|\psi\rangle_{AE}$. If it is not a pure state we can still extend Eve's part to create a purified version of ρ_{AE}. This does not decrease Eve's information on Alice's outcome. Now if the subsystem ρ_A held by Alice is a pure state $|\phi\rangle_A$ then the best measurement performed by Alice will give her the maximal entropy. This is because Eve is completely disentangled with Alice and hence has no knowledge about the outcome. The randomness in this situation is $\log_2 d$ which is exactly equal to the basis-independent quantum coherence.

When Alice has a mixed state ρ_A, we can consider the spectral decomposition $\rho_A = \sum_i \lambda_i |i\rangle\langle i|$, where $\{|i\rangle\}$ is assumed to be the eigenbasis without any loss of generality. The basis-independent measure of quantum coherence corresponding to the state ρ_A is

$$C(\rho_A) = \log_2 d + \sum_i \lambda_i \log \lambda_i. \tag{16}$$

The total amount of randomness in the system is the entropy of a maximally mixed state which has the eigenvalues $\lambda_i = 1/d \ \forall i$ and the expression reads:

$$R_t|_{max} = -\underbrace{\left(\frac{1}{d} + \cdots + \frac{1}{d}\right)}_{d-\text{terms}} \log_2\left(\frac{1}{d}\right) = \log_2 d \qquad (17)$$

From the property of spectral decomposition we know that the inherent classical randomness of the system ρ_A is

$$R_c = -\lambda_i \log_2 \lambda_i. \qquad (18)$$

In general the total randomness R_t of the system is related to the classical randomness R_c and quantum randomness R_Q through the inequality

$$R_Q \geqslant R_t - R_c. \qquad (19)$$

Since the maximum total randomness in the system is $\log_2 d$, the quantum randomness is

$$R_Q = R_t|_{max} - R_c = \log_2 d + \sum_i \lambda_i \log_2 \lambda_i = C(\rho_A). \qquad (20)$$

Hence we find that the quantum randomness in the system is exactly the basis-independent quantum coherence in the system.

IV.3. Applications in Quantum thermodynamics

The application of information theoretic properties to standard thermodynamics helps to include quantum effects of small ensemble sizes leading to the emerging field of quantum thermodynamics. The unconventional competition between the thermal fluctuations and quantum fluctuations leads to the emergence of unique properties in the study of quantum thermodynamics. Since quantum coherence is a manifestation of the quantumness of a system, it is expected to play a major role in quantum thermodynamic processes. The amount of work extractable from a given state ρ in contact with a thermal reservoir at a fixed temperature [**?**] is

$$\langle W\rangle = k_BT(\log_2 d - S(\rho)) \equiv k_BTC(\rho), \qquad (21)$$

where $C(\rho)$ is the basis-independent quantum coherence of the system and k_B and T are the Boltzmann constant and temperature of the system.

The projection mechanism in quantum thermodynamics is the quantum analogy of data erasure in information processing. Of particular interest is the projection

$$\rho \to \eta^H = \sum_k \mathrm{Tr}(\rho \Pi_\mathrm{k}^\mathrm{H})\Pi_\mathrm{k}^\mathrm{H}, \qquad (22)$$

which is a projection onto the energy eigenstates $\{\Pi_k^H\}$ of the system Hamiltonian $H = \sum_k E_k \Pi_k^H$. The conversion of quantum coherence into work can be achieved through the cyclical process consisting of the following steps: (i) The qubit is isolated from the environment and unitarily evolved to the thermal state at temperature T. (ii) A quasi-static evolution of the Hamiltonian is carried out by bringing the qubit in contact with a thermal bath at temperature T. Throughout this evolution the qubit state is maintained as the thermal state of the instantaneous Hamiltonian. The state η^H denotes the state of the qubit at the end of this step. (iii) The Hamiltonian is quickly changed back to the initial state. It has been shown in Ref. [**?**] that the average work extracted from the qubit systems during the above steps is

$$\langle W\rangle = k_BT(S(\eta^H) - S(\rho)). \qquad (23)$$

In terms of the basis-independent measure of quantum coherence (3) the work extracted from the qubit given in Eqn. (23) can be written as

$$\langle W\rangle = k_BT(C(\rho) - C(\eta^H)). \qquad (24)$$

Hence we find that the average work extracted from the qubit is the difference between the coherence in the state ρ and the coherence of the state which is projected in the energy eigenbasis.

In a generic setting, when the ρ state is projected in an arbitrary basis $\{\Pi_k^{\mathcal{P}}\}$ to state $\eta^{\mathcal{P}}$ the optimal extractable work [**?**] from the qubit is

$$\begin{aligned}\langle W\rangle &= k_BT[S(\eta^{\mathcal{P}}) - S(\rho)] - \mathrm{Tr}[\mathrm{H}(\eta^{\mathcal{P}} - \rho)] \\ &= k_BT[C(\rho) - C(\eta^{\mathcal{P}})] - \Delta U \end{aligned} \qquad (25)$$

where U is the internal energy of the system. A comparison with the first law of thermodynamics $W = Q + \Delta U$ shows that the net heat transferred to the qubit system can be quantified by the loss of the basis-independent coherence in the system. This seems quite natural since any increase in temperature of transfer of heat into a quantum system, should decrease the quantumness due to decoherence. This decrease in quantumness is quantified by the loss of basis-independent coherence. Hence we observe that the basis-independent quantum coherence captures the amount of quantumness of a system.

V. CONCLUSIONS

In this paper we have analyzed a basis-independent formulation of quantum coherence based on the relative entropy. We have shown that in order to formulate a basis-independent measure of quantum coherence by modifying the set of incoherent states, the only choice is to take I/d as the reference incoherent state. The free operations corresponding to a basis independent measure was defined. Using the incoherent state and the free operations we prove that this measure obeys all postulates needed to be a quantum coherence measure. To relate the basis independent and basis-dependent coherence, we propose a coherence measure which estimates the amount of coherence that is locked due to the basis choice. The hierarchical relationship of the basis-independent quantum coherence along with the various quantum correlations was explored. We find that the basis-independent formulation is compatible with the hierarchical structure with other quantum correlations, where it is viewed as the fundamental quantum quantity underlying quantum correlations and superposition.

The usefulness of basis-independent quantum coherence is shown by considering several applications in quantum information tasks. First we showed that the rate of conversion of a quantum state into a pure state in the asymptotic limit is equal to amount of the basis-independent quantum coherence in the system. We further showed that the quantum randomness generated in a system by measurement is equal to the basis-independent coherence. Since quantum coherence is a manifestation of the quantumness of a system we naturally expect it to play a role in quantum thermodynamics. We find that the loss of basis-independent quantum coherence is equal to the amount of heat supplied to the system. This is natural because the heat supplied to a system can cause a loss of quantumness of a system due to decoherence. Since any loss of quantumness can be estimated by the coherence, we find the heat supplied to be equal to the basis-independent quantum coherence.

Although we have used the relative entropy of coherence based on the von Neumann entropy we may also consider other entropic measures such as the generalized Renyi entropy, Tsallis entropy, or Jensen-Shannon divergence. For instance one may also define a basis-independent coherence by using $D(\rho, I/d)$ where we can choose any distance measure which obeys the axioms of a coherence measure. It is straightforward to show that the trace distance satisfies all the required conditions for a good measure of coherence. Investigations of basis-independent coherence using other such measures is beyond the scope of this paper and will be left as future work.

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Appendix A: Appendix

1. Appendix A: Proof of Proposition 1

Here we prove that given two projective measurements X and Z, the following holds for any state ρ

$$C_r^X(\rho) + C_r^Z(\rho) \geq \log_2 \frac{1}{c} - 2S(\rho). \tag{A1}$$

In the X-reference basis, ρ can be expressed as $\rho = \sum_{i,j=1}^d \alpha_{ij} |X_i\rangle\langle X_j|$ where $\alpha_{ij} = \langle X_i|\rho|X_j\rangle$. Then the diagonal part of the density matrix ρ in the X-reference basis is given by $\rho_{ii}^X = \alpha_{ii}$. Therefore, $C_r^R(\rho) = S(\rho_d^Z) - S(\rho) = -\sum_{i=1}^d \alpha_{ii} \log \alpha_{ii} - S(\rho) = H(X) - S(\rho)$, where $H(X) = -\sum_i^d \alpha_{ii} \log \alpha_{ii}$ is the Shannon entropy associated to the measurement X on state ρ. Similarly we have $C_r^Z(\rho) = H(Z) - S(\rho)$. Since the uncertainty $H(X) + H(Z)$ satisfies $H(X) + H(Z) \geqslant \log_2 \frac{1}{c}$ [**?**], this proves Proposition 1.

2. Appendix B: Proof of Proposition 2

Here we prove Proposition 2 in the main text. Sufficiency of the condition is obvious. Turning to necessity, suppose that $C_r^Z(\rho) = C_r^X(\rho) = 0$ for some $\rho \neq \frac{1}{d}I$. This implies that $\rho = \Delta^Z(\rho) = \Delta^X(\rho)$, and so

$$\rho = \sum_\lambda t_\lambda P_\lambda = \sum_\lambda t_\lambda \Delta^Z(P_\lambda),$$

where the t_λ are distinct non-negative eigenvalues and each P_λ is a projector diagonal in the X basis; i.e. for all λ, $P_\lambda = \sum_{i=1}^{k_\lambda} |X_{\pi(i)}\rangle\langle X_{\pi(i)}|$ with $k_\lambda < d$. Let $t_{\max}$ be the largest eigenvalue of ρ and $P_{\max}$ the associated eigenspace projector. Then applying $P_{\max}$ to both sides of the previous equation and dividing by $t_{\max}$ yields

$$P_{\max} = \Delta^Z(P_{\max})P_{\max} + \sum_{\lambda \neq max} c_\lambda \Delta^Z(P_\lambda)P_{\max},$$

where $c_\lambda = t_\lambda / t_{\max} < 1 \ \forall \ \lambda \neq \max$. Taking a trace of both sides one obtains

$$\begin{aligned}\mathrm{Tr}[\mathrm{P_{max}}] &= \mathrm{Tr}[\Delta^{\mathrm{Z}}(\mathrm{P_{max}})\mathrm{P_{max}}] + \sum_{\lambda \neq \mathrm{max}} \mathrm{c}_\lambda \mathrm{Tr}[\Delta^{\mathrm{Z}}(\mathrm{P}_\lambda)\mathrm{P_{max}}] \\ &\leq \mathrm{Tr}[\Delta^{\mathrm{Z}}(\mathrm{P_{max}})\mathrm{P_{max}}] + \sum_{\lambda \neq \mathrm{max}} \mathrm{Tr}[\Delta^{\mathrm{Z}}(\mathrm{P}_\lambda)\mathrm{P_{max}}] \\ &= \mathrm{Tr}[\mathrm{P_{max}}],\end{aligned}$$

where the last equality follows from the fact that $P_{\max} + \sum_{\lambda \neq \max} P_\lambda = I$ and $\Delta^Z(I) = I$. Since $c_\lambda < 1$ the previous equation can hold iff the second term equals zero, namely $\mathrm{Tr}[\mathrm{P_{max}}] = \mathrm{Tr}[\Delta^{\mathrm{Z}}(\mathrm{P_{max}})\mathrm{P_{max}}]$, which means that $\Delta^Z(P_{\max}) = P_{\max}$. This completes the proof because $k_{\max} < d$ implies that $P_{\max} \neq I$.

As an example, let us consider a special case when X and Z are mutually unbiased (MUBs) [**? ?**]. The

MUBs measurements are complementary to each other in the sense that any pair of bases are maximally unbiased, MUBs measurements are crucial in many quantum information tasks [**?** **?**]. Suppose that $C_r^X(\rho) = C_r^Z(\rho) = 0$ for a quantum state $\rho \neq \frac{1}{d}I$. Let $P_{\max} = \sum_{i=1}^{k} |X_{\pi(i)}\rangle\langle X_{\pi(i)}|$, $k < d$, be the eigenspace projector with respect to the largest eigenvalue of ρ. Then $\Delta^Z(P_{\max}) = P_{\max}$ holds according to the proving process of Proposition 2. However, it can be easily seen that $\Delta^Z(P_{\max}) = \frac{k}{d}I$ by use of the mutually unbiaseness of X and Z, which leads to a contradiction. Therefore we claim that $C_r^X(\rho) + C_r^Z(\rho) > 0$ for all $\rho \neq \frac{1}{d}I$. As a consequence for any state that is not complete mix state, if we change the basis, i.e., from X to Z, then its coherence can be nonzero.