

**Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig**

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by

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Preprint no.: 58

2019





# Monogamy relations of all quantum correlation measures for multipartite quantum systems

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## Abstract

The monogamy relations of quantum correlation restrict the sharability of quantum correlations in multipartite quantum states. We show that all measures of quantum correlations satisfy some kind of monogamy relations for arbitrary multipartite quantum states. Moreover, by introducing residual quantum correlations, we present tighter monogamy inequalities that are better than all the existing ones. In particular, for multi-qubit pure states, we also establish new monogamous relations based on the concurrence and concurrence of assistance under the partition of the first two qubits and the remaining ones.

Sharing of quantum correlations among many parties is an important quantum phenomena, which plays significant roles in quantum information processing ranging from quantum communication protocols [1–4] to cooperative events in quantum systems [5, 6]. It is therefore important to conceptualize and quantify quantum correlations. Any such measure of quantum correlation is expected to satisfy a monotonic (precisely, non-increasing) under an intuitively satisfactory set of local quantum operations [7, 8].

For a quantum state shared by more than two parties, one may expect that all the measures of quantum correlation would additionally follow a monogamy property [9–18], which restricts the sharability of quantum correlations among many parties. The monogamous nature of quantum correlations plays a key role in the security of quantum cryptography [19]. Monogamy relations are not always satisfied by a correlation measure, for example, the entanglement of formation [10] which quantifies the amount of entanglement required for preparation of a given bipartite quantum state. However, although the concurrence [20] and entanglement of formation do not satisfy the monogamy inequality,  $\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}$  ( $\mathcal{E}_{A|BC}$  stands for the entanglement between  $A$  and  $BC$ ), it has been shown that the  $\alpha$ th ( $\alpha \geq 2$ ) power of concurrence and the  $\alpha$ th ( $\alpha \geq \sqrt{2}$ ) power of entanglement of formation for  $N$ -qubit states do satisfy such monogamy relations [21]. In [22] a tighter monogamy relation for  $\alpha$ th ( $\alpha \geq 2$ ) power of concurrence has been presented. It has been shown that the information-theoretic quantum correlation measure, quantum discord [23], can violate the monogamy relations [24–27], but a monotonically increasing function of the quantum discord could satisfy the monogamy relation for three-qubit pure states [28].

In this paper, we first show that all quantum correlation measures satisfy some kind of monogamy relations for arbitrary multipartite quantum states. Then we introduce the residual quantum correlations, and present tighter monogamy inequalities that are better than all the existing ones. For multi-qubit pure states, we establish new monogamous relations based on the concurrence and concurrence of assistance under the partition of the first two qubits and the rest ones.

Let  $\mathcal{Q}$  be an arbitrary quantum correlation measure of bipartite systems. The quantum correlation measure  $\mathcal{Q}$  is said to be monogamous for an  $N$ -partite quantum state  $\rho_{AB_1B_2\cdots B_{N-1}}$ , if it satisfies the following inequality [29],

$$\mathcal{Q}(\rho_{AB_1}) + \mathcal{Q}(\rho_{AB_2}) + \cdots + \mathcal{Q}(\rho_{AB_{N-1}}) \leq \mathcal{Q}(\rho_{A|B_1B_2\cdots B_{N-1}}), \quad (1)$$

where  $\rho_{AB_i}$ ,  $i = 1, \dots, N - 1$ , are the reduced density matrices,  $\mathcal{Q}(\rho_{A|B_1B_2\dots B_{N-1}})$  denotes the quantum correlation  $\mathcal{Q}$  of the state  $\rho_{AB_1B_2\dots B_{N-1}}$  under bipartite partition  $A|B_1B_2\dots B_{N-1}$ . For simplicity, we denote  $\mathcal{Q}(\rho_{AB_i})$  by  $\mathcal{Q}_{AB_i}$ , and  $\mathcal{Q}(\rho_{A|B_1B_2\dots B_{N-1}})$  by  $\mathcal{Q}_{A|B_1B_2\dots B_{N-1}}$ . One can define the  $\mathcal{Q}$ -monogamy score for the  $N$ -partite state  $\rho_{AB_1B_2\dots B_{N-1}}$ ,

$$\delta_{\mathcal{Q}} = \mathcal{Q}_{A|B_1B_2\dots B_{N-1}} - \sum_{i=1}^{N-1} \mathcal{Q}_{AB_i}. \quad (2)$$

Non-negativity of  $\delta_{\mathcal{Q}}$  for all quantum states implies the monogamy of  $\mathcal{Q}$ . For instance, the square of the concurrence has been shown to be monogamous [9, 12] for all multi-qubit states. However, there are other measures like entanglement of formation, quantum discord, and quantum work deficit which are known to be nonmonogamous for pure three-qubit states [24, 25].

Given any quantum correlation measure that is non-monogamic for a multipartite quantum state, it is always possible to find a monotonically increasing function of the measure which is monogamous for the same state [30]. It has been proved that for arbitrary dimensional tripartite states, there exists  $\beta_{\min}(\mathcal{Q}) \in \mathbb{R}$  such that for any  $\gamma \geq \beta_{\min}(\mathcal{Q})$ , the quantum correlation measure  $\mathcal{Q}$  satisfies the following monogamy relation [30]

$$\mathcal{Q}_{A|BC}^{\gamma} \geq \mathcal{Q}_{AB}^{\gamma} + \mathcal{Q}_{AC}^{\gamma}. \quad (3)$$

In the following, we denote  $\beta = \beta_{\min}(\mathcal{Q})$  the minimal value such that  $\mathcal{Q}$  satisfies the above inequality. Generalizing the conclusion (3) to the  $N$  partite case, we have the following result.

**[Theorem 1].** For any  $d \otimes d_1 \otimes \dots \otimes d_{N-1}$  state  $\rho_{AB_1B_2\dots B_{N-1}}$ , we have

$$\mathcal{Q}_{A|B_1B_2\dots B_{N-1}}^{\alpha} \geq \sum_{i=1}^{N-1} \mathcal{Q}_{AB_i}^{\alpha}, \quad (4)$$

for  $\alpha \geq \beta$ ,  $N \geq 3$ .

**[Proof].** The Eq. (4) reduces to Eq. (3) for  $N = 3$ . Suppose the Theorem 1 holds for  $N - 2$ . Then, if we consider the state  $\rho_{AB_2\dots B_{N-1}}$ , we have

$$\mathcal{Q}_{A|B_2\dots B_{N-1}}^{\alpha} \geq \sum_{i=2}^{N-1} \mathcal{Q}_{AB_i}^{\alpha},$$

for any  $\alpha \geq \beta$ .

Applying Eq. (3) for the tripartite partition  $A|B_1|B_2 \cdots B_{N-1}$ , we have

$$\begin{aligned} \mathcal{Q}_{A|B_1 B_2 \cdots B_{N-1}}^\alpha &\geq \mathcal{Q}_{AB_1}^\alpha + \mathcal{Q}_{A|B_2 \cdots B_{N-1}}^\alpha \\ &\geq \mathcal{Q}_{AB_1}^\alpha + \sum_{i=2}^{N-1} \mathcal{Q}_{AB_i}^\alpha \\ &= \sum_{i=2}^{N-1} \mathcal{Q}_{AB_i}^\alpha, \end{aligned}$$

for any  $\alpha \geq \beta$ . ■

Theorem 1 gives a general result for arbitrary measure of quantum correlations. However, such relations can be further improved by tightening the lower bound of the inequality (4). Similar to the three tangle of concurrence, for tripartite quantum states  $\rho \in H_A \otimes H_B \otimes H_C$ , we define the residual quantum correlation as a function of  $\alpha$ ,

$$\mathcal{Q}_{A|B|C}^\alpha(\alpha) = \mathcal{Q}_{A|BC}^\alpha - \mathcal{Q}_{AB}^\alpha - \mathcal{Q}_{AC}^\alpha, \quad \alpha \geq \beta. \quad (5)$$

In the following, we denote  $\mathcal{Q}_{A|B|C}^\alpha = \mathcal{Q}_{A|B|C}^\alpha(\alpha)$  for convenience. Now consider a  $d \otimes d \otimes d \otimes d$  state  $\rho_{AB_1 B_2 B_3}$ . Define  $\mathcal{Q}_{A|B'_1|B'_2}^\alpha = \max\{\mathcal{Q}_{A|B_1|B_2}^\alpha, \mathcal{Q}_{A|B_1|B_3}^\alpha, \mathcal{Q}_{A|B_2|B_3}^\alpha\}$ , where  $B'_1$  and  $B'_2$  stand for two of  $B_1, B_2$  and  $B_3$  such that  $\mathcal{Q}_{A|B'_1|B'_2}^\alpha = \max\{\mathcal{Q}_{A|B_1|B_2}^\alpha, \mathcal{Q}_{A|B_1|B_3}^\alpha, \mathcal{Q}_{A|B_2|B_3}^\alpha\}$ .

**[Theorem 2].** For any  $d \otimes d_1 \otimes d_2 \otimes d_3$  state  $\rho_{AB_1 B_2 B_3}$ , we have

$$\mathcal{Q}_{A|B_1 B_2 B_3}^\alpha \geq \sum_{i=1}^3 \mathcal{Q}_{AB_i}^\alpha + \mathcal{Q}_{A|B'_1|B'_2}^\alpha, \quad (6)$$

for  $\alpha \geq \beta$ .

**[Proof].** By definition we have

$$\begin{aligned} \sum_{i=1}^3 \mathcal{Q}_{AB_i}^\alpha + \mathcal{Q}_{A|B'_1|B'_2}^\alpha &= \mathcal{Q}_{AB'_3}^\alpha + \mathcal{Q}_{A|B'_1 B'_2}^\alpha \\ &\leq \mathcal{Q}_{A|B_1 B_2 B_3}^\alpha, \end{aligned}$$

where  $B'_3$  is the complementary of  $B'_1 B'_2$  in the subsystem  $B_1 B_2 B_3$ , the equality is due to the definition of the residual quantum correlation. From (4), we get the inequality. ■

Since the last term  $\mathcal{Q}_{A|B'_1|B'_2}^\alpha$  in (6) is semi-positive, the inequality (6) is always tighter than (4) for such states  $\rho_{AB_1 B_2 B_3}$ . Let us consider the following example based on the quantum correlation measure concurrence. First, we give the definition of concurrence. For a bipartite pure state  $|\psi\rangle_{AB} \in H_A \otimes H_B$ , the concurrence is  $C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}$ ,

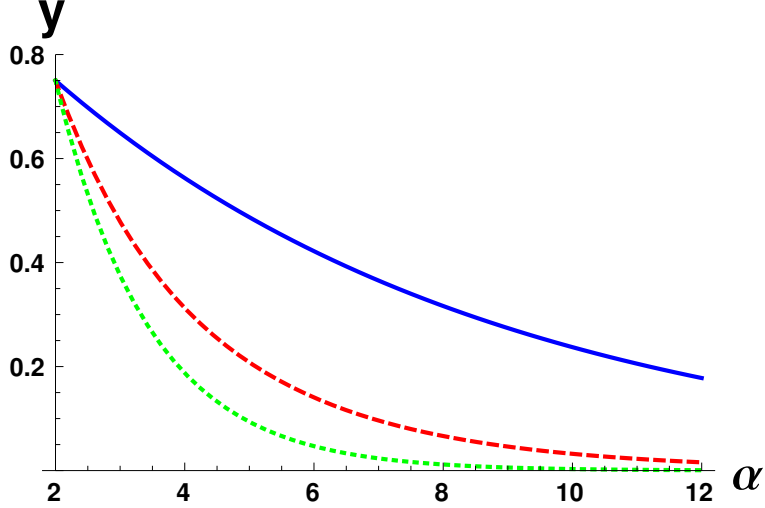


FIG. 1: Solid (blue) line  $y_1$  is the  $\alpha$ th power of concurrence under bipartition  $A|B_1B_2B_3$ ; Dashed (red) line  $y_3$  for the lower bound in (6); Dotted (green) line  $y_2$  for the result (7) in [21].

where  $\rho_A$  is the reduced density matrix obtained by tracing over the subsystem  $B$ ,  $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ . The concurrence for a bipartite mixed state  $\rho_{AB}$  is defined by the convex roof extension,  $C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$ , where the minimum is taken over all possible decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , with  $p_i \geq 0$  and  $\sum_i p_i = 1$  and  $|\psi_i\rangle \in H_A \otimes H_B$ . In [21], the authors show that

$$C^\alpha(\rho_{A|B_1B_2\dots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \dots + C^\alpha(\rho_{AB_{N-1}}), \quad (7)$$

for an  $N$ -qubit state  $\rho_{AB_1\dots B_{N-1}}$ .

*Example 1.* For the concurrence of the  $W$  state,

$$|W\rangle_{A|B_1B_2B_3} = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle), \quad (8)$$

we have  $\beta = 2$ ,  $C_{AB_i} = \frac{1}{2}$ ,  $i = 1, 2, 3$ , and  $C_{A|B_1B_2} = C_{A|B_1B_3} = C_{A|B_2B_3} = \frac{\sqrt{2}}{2}$ . Therefore  $C_{A|B_1|B_2}^\alpha = C_{A|B_1|B_3}^\alpha = C_{A|B_2|B_3}^\alpha = (\frac{\sqrt{2}}{2})^\alpha - 2(\frac{1}{2})^\alpha$ . Set  $y_1 = C_{A|B_1B_2B_3}^\alpha = (\frac{\sqrt{3}}{2})^\alpha$ ,  $y_2 = \sum_{i=1}^3 C_{AB_i}^\alpha = 3(\frac{1}{2})^\alpha$ ,  $y_3 = \sum_{i=1}^3 C_{AB_i}^\alpha + C_{A|B_1|B_2}^\alpha = (\frac{\sqrt{2}}{2})^\alpha + (\frac{1}{2})^\alpha$ , one can see that our result is better than (7) in [21], see Fig. 1.

Generalizing the conclusion in Theorem 2 to  $N$  partite case, we have the following result.

**[Theorem 3].** For any  $d \otimes d_1 \otimes \dots \otimes d_{N-1}$  state  $\rho_{A|B_1B_2\dots B_{N-1}}$ , we have

$$\mathcal{Q}_{A|B_1B_2\dots B_{N-1}}^\alpha \geq \sum_{i=1}^{N-1} \mathcal{Q}_{AB_i}^\alpha + \sum_{k=2}^{N-2} \mathcal{Q}_{A|B'_1|B'_2|\dots|B'_k}^\alpha, \quad (9)$$

for  $\alpha \geq \beta$ , where  $\mathcal{Q}_{A|B'_1|B'_2|\dots|B'_k}^\alpha = \max_{1 \leq l \leq k+1} \{ \mathcal{Q}_{A|B_1|\dots|\hat{B}_l|\dots|B_{k+1}}^\alpha \}$  (where  $\hat{B}_l$  stands for  $B_l$  being omitted in the sub-indices),  $\mathcal{Q}_{A|B_1|B_2|\dots|B_{k+1}}^\alpha = \mathcal{Q}_{A|B_1 B_2 \dots B_{k+1}}^\alpha - \sum_{i=1}^{k+1} \mathcal{Q}_{AB_i}^\alpha - \sum_{i=2}^k \mathcal{Q}_{A|B'_1|B'_2|\dots|B'_i}^\alpha$ ,  $2 \leq k \leq N-2$ ,  $1 \leq l \leq k+1$ ,  $N \geq 4$ .

[Proof]. We prove the theorem by induction. For  $N=4$  it reduces to Theorem 2. Suppose the Theorem 2 holds for  $N=n$ , i.e.,

$$\mathcal{Q}_{A|B_1 B_2 \dots B_{n-1}}^\alpha \geq \sum_{i=1}^{n-1} \mathcal{Q}_{AB_i}^\alpha + \mathcal{Q}_{A|B'_1|B'_2}^\alpha + \dots + \mathcal{Q}_{A|B'_1|B'_2|\dots|B'_{n-2}}^\alpha. \quad (10)$$

Then for  $N=n+1$ , we have

$$\begin{aligned} & \sum_{i=1}^n \mathcal{Q}_{AB_i}^\alpha + \mathcal{Q}_{A|B'_1|B'_2}^\alpha + \dots + \mathcal{Q}_{A|B'_1|B'_2|\dots|B'_{n-1}}^\alpha \\ & \leq \mathcal{Q}_{A|B'_1 B'_2 \dots B'_{n-1}}^\alpha + \mathcal{Q}_{AB'_n}^\alpha \\ & \leq \mathcal{Q}_{A|B_1 B_2 \dots B_n}^\alpha, \end{aligned}$$

where  $B'_n$  is the complementary of  $B'_1 B'_2, \dots, B'_{n-1}$  in the subsystem  $B_1 B_2, \dots, B_n$ . The first inequality is due to (10). By (4) we get the last inequality.  $\blacksquare$

In Theorems 1 and 2 we have taken into account the maximum value among  $\mathcal{Q}_{A|B_1|\dots|\hat{B}_l|\dots|B_k}^\alpha$ . If instead of the maximum value, one just considers the mean value of  $\mathcal{Q}_{A|B_1|\dots|\hat{B}_l|\dots|B_k}^\alpha$ , one may have the following corollary.

[Corollary 1]. For any  $d \otimes d_1 \otimes \dots \otimes d_{N-1}$  state  $\rho_{A|B_1 B_2 \dots B_{N-1}}$ , we have

$$\mathcal{Q}_{A|B_1 B_2 \dots B_{N-1}}^\alpha \geq \sum_{i=1}^{N-1} \mathcal{Q}_{AB_i}^\alpha + \sum_{k=3}^{N-1} \left( \frac{1}{k} \sum_{l=1}^k \mathcal{Q}_{A|B_1|\dots|\hat{B}_l|\dots|B_k}^\alpha \right), \quad (11)$$

for all  $\alpha \geq \beta$ ,  $N \geq 4$ , where

$$\mathcal{Q}_{A|B_1|B_2|\dots|B_j}^\alpha = \mathcal{Q}_{A|B_1 B_2 \dots B_j}^\alpha - \sum_{i=1}^j \mathcal{Q}_{AB_i}^\alpha - \sum_{k=3}^j \left( \frac{1}{k} \sum_{l=1}^k \mathcal{Q}_{A|B_1|\dots|\hat{B}_l|\dots|B_k}^\alpha \right), \quad (12)$$

$3 \leq j \leq N-1$ ,  $3 \leq k \leq N-1$  and  $1 \leq l \leq k$ .

*Example 2.* Let us consider the concurrence of the four-qubit pure state,

$$|\psi\rangle_{ABCD} = \frac{1}{\sqrt{3}}(|0000\rangle + |0010\rangle + |1011\rangle). \quad (13)$$

We have  $\rho_{ACD} = \text{Tr}_B(|\psi\rangle_{ABCD}\langle\psi|) = \frac{1}{3}(|000\rangle + |010\rangle + |111\rangle)(\langle 000| + \langle 010| + \langle 111|)$ ,  $\rho_{BCD} = \text{Tr}_A(|\psi\rangle_{ABCD}\langle\psi|) = \frac{1}{3}(|000\rangle\langle 000| + |000\rangle\langle 010| + |010\rangle\langle 000| + |010\rangle\langle 010| + |011\rangle\langle 011|)$ ,  $C_{AB} = C_{AC} = 0$ ,  $C_{AD} = \frac{2}{3}$ ,  $C_{BC} = C_{BD} = 0$ ,  $C_{A|BC} = 0$ ,  $C_{A|BD} = \frac{2}{3}$ ,  $C_{A|CD} = \frac{2\sqrt{2}}{3}$ . Therefore,



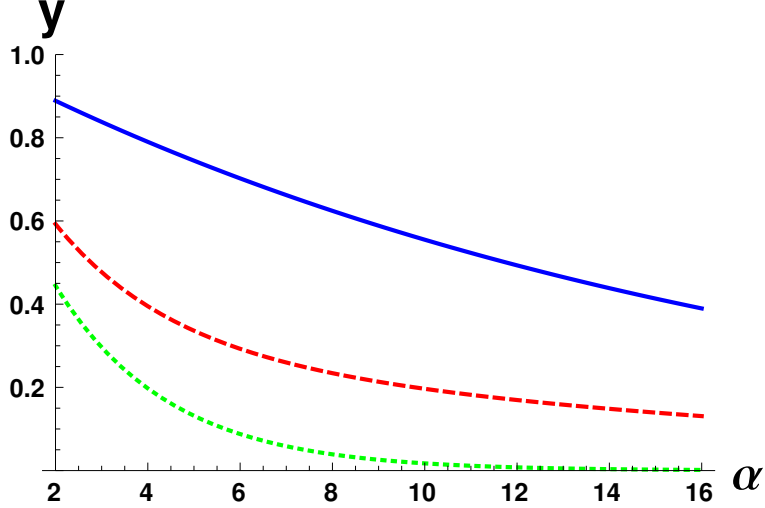


FIG. 2: Solid (blue) line  $y_1$  for the  $\alpha$ th power of concurrence under bipartition  $A|B_1B_2B_3$ ; Dashed (red) line  $y_3$  for the lower bound in (11); Dotted (green) line  $y_2$  for the result in [21].

$C_{A|B|C} = C_{A|B|D} = 0$ ,  $C_{A|C|D}^{\alpha} = (\frac{2\sqrt{2}}{3})^{\alpha} - (\frac{2}{3})^{\alpha}$ . Set  $y_1 = C_{A|BCD}^{\alpha} = (\frac{2\sqrt{2}}{3})^{\alpha}$ ,  $y_2 = C_{AB}^{\alpha} + C_{AC}^{\alpha} + C_{AD}^{\alpha} = (\frac{2}{3})^{\alpha}$ ,  $y_3 = C_{AB}^{\alpha} + C_{AC}^{\alpha} + C_{AD}^{\alpha} + \frac{1}{3} (C_{A|B|C}^{\alpha} + C_{A|B|D}^{\alpha} + C_{A|C|D}^{\alpha}) = (\frac{2}{3})^{\alpha+1} + \frac{1}{3}(\frac{2\sqrt{2}}{3})^{\alpha}$ , one can see that our result is better than that in [21], see Fig. 2.

Next, we adopt an approach used in [22] to improve the further above results on monogamy relations for multipartite quantum correlation measures. First, we give a Lemma.

**[Lemma].** For any  $d_1 \otimes d_2 \otimes d_3$  mixed state  $\rho \in H_A \otimes H_B \otimes H_C$ , if  $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$ , we have

$$\mathcal{Q}_{A|BC}^{\alpha} \geq \mathcal{Q}_{AB}^{\alpha} + \frac{\alpha}{\beta} \mathcal{Q}_{AC}^{\alpha}, \quad (14)$$

for all  $\alpha \geq \beta$ .

**[Proof].** For arbitrary  $d_1 \otimes d_2 \otimes d_3$  tripartite state  $\rho_{ABC}$ . If  $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$ , we have

$$\begin{aligned} \mathcal{Q}_{A|BC}^{\alpha} &= (\mathcal{Q}_{AB}^{\beta} + \mathcal{Q}_{AC}^{\beta})^{\frac{\alpha}{\beta}} = \mathcal{Q}_{AB}^{\alpha} \left( 1 + \frac{\mathcal{Q}_{AC}^{\beta}}{\mathcal{Q}_{AB}^{\beta}} \right)^{\frac{\alpha}{\beta}} \\ &\geq \mathcal{Q}_{AB}^{\alpha} \left[ 1 + \frac{\alpha}{\beta} \left( \frac{\mathcal{Q}_{AC}^{\beta}}{\mathcal{Q}_{AB}^{\beta}} \right)^{\frac{\alpha}{\beta}} \right] = \mathcal{Q}_{AB}^{\alpha} + \frac{\alpha}{\beta} \mathcal{Q}_{AC}^{\alpha}, \end{aligned}$$

where the first equality is due to (3), the inequality is due to the inequality  $(1+t)^x \geq 1+xt \geq 1+xt^x$  for  $x \geq 1$ ,  $0 \leq t \leq 1$ .  $\blacksquare$

In the above Lemma, without loss of generality, we have assumed that  $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$ , as the subsystems  $A$  and  $B$  are equivalent. Moreover, in the proof of the Lemma we have

assumed  $\mathcal{Q}_{AB} > 0$ . If  $\mathcal{Q}_{AB} = 0$  and  $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$ , then  $\mathcal{Q}_{AB} = \mathcal{Q}_{AC} = 0$ . The lower bound is trivially zero. Generalizing the Lemma to multipartite quantum systems, we have the following Theorem.

**[Theorem 4].** For any  $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$  state  $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ , if  $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1}\cdots B_{N-1}}$  for  $i = 1, 2, \dots, m$ , and  $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$  for  $j = m+1, \dots, N-2$ ,  $\forall 1 \leq m \leq N-3$ ,  $N \geq 4$ , we have

$$\begin{aligned} \mathcal{Q}_{A|B_1 B_2 \cdots B_{N-1}}^\alpha &\geq \mathcal{Q}_{AB_1}^\alpha + \frac{\alpha}{\beta} \mathcal{Q}_{AB_2}^\alpha + \cdots + \left(\frac{\alpha}{\beta}\right)^{m-1} \mathcal{Q}_{AB_m}^\alpha \\ &\quad + \left(\frac{\alpha}{\beta}\right)^{m+1} (\mathcal{Q}_{AB_{m+1}}^\alpha + \cdots + \mathcal{Q}_{AB_{N-2}}^\alpha) + \left(\frac{\alpha}{\beta}\right)^m \mathcal{Q}_{AB_{N-1}}^\alpha, \end{aligned} \quad (15)$$

for all  $\alpha \geq \beta$ .

**[Proof].** By using the Lemma repeatedly, one gets

$$\begin{aligned} \mathcal{Q}_{A|B_1 B_2 \cdots B_{N-1}}^\alpha &\geq \mathcal{Q}_{AB_1}^\alpha + \frac{\alpha}{\beta} \mathcal{Q}_{A|B_2 \cdots B_{N-1}}^\alpha \\ &\geq \mathcal{Q}_{AB_1}^\alpha + \frac{\alpha}{\beta} \mathcal{Q}_{AB_2}^\alpha + \left(\frac{\alpha}{\beta}\right)^2 \mathcal{Q}_{A|B_3 \cdots B_{N-1}}^\alpha \\ &\geq \cdots \geq \mathcal{Q}_{AB_1}^\alpha + \frac{\alpha}{\beta} \mathcal{Q}_{AB_2}^\alpha + \cdots \\ &\quad + \left(\frac{\alpha}{\beta}\right)^{m-1} \mathcal{Q}_{AB_m}^\alpha + \left(\frac{\alpha}{\beta}\right)^m \mathcal{Q}_{A|B_{m+1} \cdots B_{N-1}}^\alpha. \end{aligned} \quad (16)$$

As  $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$  for  $j = m+1, \dots, N-2$ , by (16) we get

$$\begin{aligned} \mathcal{Q}_{A|B_{m+1} \cdots B_{N-1}}^\alpha &\geq \frac{\alpha}{\beta} \mathcal{Q}_{AB_{m+1}}^\alpha + \mathcal{Q}_{A|B_{m+2} \cdots B_{N-1}}^\alpha \\ &\geq \frac{\alpha}{\beta} (\mathcal{Q}_{AB_{m+1}}^\alpha + \cdots + \mathcal{Q}_{AB_{N-2}}^\alpha) + \mathcal{Q}_{AB_{N-1}}^\alpha. \end{aligned} \quad (17)$$

Combining (16) and (17), we have Theorem 4. ■

Similar to the Theorem 3, (15) can be improved by adding a term for residual quantum correlation. By a similar derivation to Theorem 3, we have

**[Theorem 5].** For any  $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$  state  $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ , if  $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1}\cdots B_{N-1}}$  for  $i = 1, 2, \dots, m$ , and  $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$  for  $j = m+1, \dots, N-2$ ,

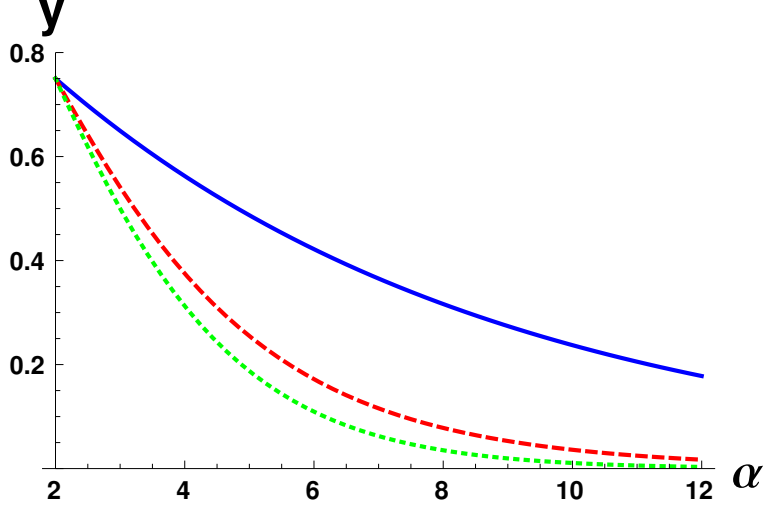


FIG. 3: Solid (blue) line  $y_1$  is the  $\alpha$ th power of concurrence under bipartition  $A|B_1B_2B_3$ . The dashed (red) line  $y_2$  is the lower bound in (18) and the dotted (green)  $y_3$  line for (15).

$\forall 1 \leq m \leq N - 3, N \geq 4$ , we have

$$\begin{aligned}
\mathcal{Q}_{A|B_1B_2\cdots B_{N-1}}^\alpha &\geq \mathcal{Q}_{AB_1}^\alpha + \frac{\alpha}{\beta} \mathcal{Q}_{AB_2}^\alpha + \cdots + \left(\frac{\alpha}{\beta}\right)^{m-1} \mathcal{Q}_{AB_m}^\alpha \\
&+ \left(\frac{\alpha}{\beta}\right)^{m+1} (\mathcal{Q}_{AB_{m+1}}^\alpha + \cdots + \mathcal{Q}_{AB_{N-2}}^\alpha) + \left(\frac{\alpha}{\beta}\right)^m \mathcal{Q}_{AB_{N-1}}^\alpha \\
&+ \sum_{k=2}^{N-2} \hat{\mathcal{Q}}_{A|B'_1|B'_2|\cdots|B'_k}^\alpha \\
&= \sum_{i=1}^{N-1} \hat{\mathcal{Q}}_{AB_i}^\alpha + \sum_{k=2}^{N-2} \hat{\mathcal{Q}}_{A|B'_1|B'_2|\cdots|B'_k}^\alpha,
\end{aligned} \tag{18}$$

for all  $\alpha \geq \beta$ , where for simplicity, we have denoted  $\hat{\mathcal{Q}}_{AB_1}^\alpha = \mathcal{Q}_{AB_1}^\alpha$ ,  $\hat{\mathcal{Q}}_{AB_2}^\alpha = \frac{\alpha}{\beta} \mathcal{Q}_{AB_2}^\alpha$ ,  $\cdots$ ,  $\hat{\mathcal{Q}}_{AB_m}^\alpha = \left(\frac{\alpha}{\beta}\right)^{m-1} \mathcal{Q}_{AB_m}^\alpha$ ,  $\hat{\mathcal{Q}}_{AB_{m+1}}^\alpha = \left(\frac{\alpha}{\beta}\right)^{m+1} \mathcal{Q}_{AB_{m+1}}^\alpha$ ,  $\cdots$ ,  $\hat{\mathcal{Q}}_{AB_{N-2}}^\alpha = \left(\frac{\alpha}{\beta}\right)^{m+1} \mathcal{Q}_{AB_{N-2}}^\alpha$ ,  $\hat{\mathcal{Q}}_{AB_{N-1}}^\alpha = \left(\frac{\alpha}{\beta}\right)^m \mathcal{Q}_{AB_{N-1}}^\alpha$ . The residual quantum correlation term  $\hat{\mathcal{Q}}_{A|B'_1|B'_2|\cdots|B'_{k-1}}^\alpha = \max_{1 \leq l \leq k} \{\hat{\mathcal{Q}}_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\}$ ,  $\hat{\mathcal{Q}}_{A|B_1|B_2|\cdots|B_k}^\alpha = \mathcal{Q}_{A|B_1B_2\cdots B_k}^\alpha - \sum_{i=1}^k \hat{\mathcal{Q}}_{AB_i}^\alpha - \sum_{i=2}^{k-1} \hat{\mathcal{Q}}_{A|B'_1|B'_2|\cdots|B'_i}^\alpha$ ,  $2 \leq k \leq N - 2, 1 \leq l \leq k$ .

As an example, let us consider again the the concurrence of the state (8). We have  $\hat{C}_{A|B_1|B_2}^\alpha = \hat{C}_{A|B_1|B_3}^\alpha = \hat{C}_{A|B_2|B_3}^\alpha = \left(\frac{\sqrt{2}}{2}\right)^\alpha - \left(1 + \frac{\alpha}{2}\right)\left(\frac{1}{2}\right)^\alpha$ . Set  $y_1 = C_{A|B_1B_2B_3}^\alpha = \left(\frac{\sqrt{3}}{2}\right)^\alpha$ ,  $y_2 = \sum_{i=1}^3 \hat{C}_{AB_i}^\alpha + \hat{C}_{A|B_1|B_2}^\alpha = \left(\frac{\sqrt{2}}{2}\right)^\alpha + \frac{\alpha}{2}\left(\frac{1}{2}\right)^\alpha$ ,  $y_3 = \sum_{i=1}^3 \hat{C}_{AB_i}^\alpha = (\alpha + 1)\left(\frac{1}{2}\right)^\alpha$ . We see in Fig. 3 that the bound (15) is improved.

In the following, we consider the multi-qubit states,  $d = d_1 = \cdots = d_{N-1} = 2$ . For this

case, it has been shown in [32] that

$$C_{A|B_1B_2\dots B_{N-1}}^2 \leq C_{aAB_1}^2 + C_{aAB_2}^2 + \dots + C_{aAB_{N-1}}^2, \quad (19)$$

where the concurrence of assistance  $C_a$  is defined by  $C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$ , with the maximum taking over all possible decompositions of  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ , and  $C_{aAB} = C_a(\rho_{AB})$ .

The residual quantum correlation for the concurrence can be also used to improve other kinds of monogamous relations based on concurrence and concurrence of assistance [31]. For  $N$ -qubit systems  $ABC_1\dots C_{N-2}$ , the monogamy relations satisfied by the concurrence of  $N$ -qubit pure states under the partition  $AB$  and  $C_1\dots C_{N-2}$  have been first time established in [33]. In following we give an improved one.

**[Theorem 6].** For any  $2 \otimes 2 \otimes \dots \otimes 2$  pure state  $|\psi\rangle_{ABC_1\dots C_{N-2}}$ , we have

$$C_{AB|C_1\dots C_{N-2}}^2 \geq \max \left\{ \begin{array}{l} \sum_{i=0}^{N-2} (C_{AC_i}^2 - C_{aBC_i}^2) + \sum_{k=1}^{N-3} C_{A|C'_0|C'_1|\dots|C'_k}^2, \\ \sum_{i=0}^{N-2} (C_{BC_i}^2 - C_{aAC_i}^2) + \sum_{k=1}^{N-3} C_{B|C'_0|C'_1|\dots|C'_k}^2, \end{array} \right. \quad (20)$$

where  $\rho_{AC_0} = \rho_{AB}$ ,  $\rho_{BC_0} = \rho_{BA}$ ,  $C_{A|C'_0|C'_1|\dots|C'_k}^2 = \max_{0 \leq l \leq k+1} \{C_{A|C_0|\dots|\hat{C}_l|\dots|C_{k+1}}^2\}$ ,  $C_{A|C_0|C_1|\dots|C_k}^2 = C_{A|C_0C_1\dots C_k}^2 - \sum_{i=0}^k C_{AC_i}^2 - \sum_{i=2}^{k-1} C_{A|C'_0|C'_1|\dots|C'_i}^2$ ,  $1 \leq k \leq N-3$ ,  $0 \leq l \leq k+1$ .

**[Proof].** For  $2 \otimes 2 \otimes \dots \otimes 2$  state  $|\psi\rangle_{ABC_1\dots C_{N-2}}$ , one has

$$\begin{aligned} C_{AB|C_1\dots C_{N-2}}^2 &= 2T(\rho_{AB}) \\ &\geq 2T(\rho_A) - 2T(\rho_B) \\ &= C_{A|BC_1\dots C_{N-2}}^2 - C_{B|AC_1\dots C_{N-2}}^2 \\ &\geq \sum_{i=0}^{N-2} C_{AC_i}^2 + \sum_{k=1}^{N-3} C_{A|C'_0|C'_1|\dots|C'_k}^2 - C_{B|AC_1\dots C_{N-2}}^2 \\ &\geq \sum_{i=0}^{N-2} C_{AC_i}^2 + \sum_{k=1}^{N-3} C_{A|C'_0|C'_1|\dots|C'_k}^2 - \sum_{i=0}^{N-2} C_{aBC_i}^2, \end{aligned}$$

where  $T(\rho) = 1 - \text{Tr}(\rho^2)$ , the first inequality is due to a property of the linear entropy. Using the Theorem 3, one can get the second inequality. The last inequality is obtained from (19).

■

The second summation term in (20) improves the result in [33]. Consider the concurrence of the state (13),  $|\psi\rangle_{AB_1B_2B_3} = |\psi\rangle_{ABCD}$ . From Theorem 6 we have  $C_{AB|CD} \geq \frac{8}{9}$ , which is better than the result  $C_{AB|CD} \geq \frac{4}{9}$  from [33].

Now we generalize our results to the concurrence  $C_{ABC_1|C_2\cdots C_{N-2}}$  under partition  $ABC_1$  and  $C_2\cdots C_{N-2}$  ( $N \geq 6$ ) for pure state  $|\psi\rangle_{ABC_1\cdots C_{N-2}}$ . Similar to Theorem 6, we can obtain the following corollary:

**[Corollary 2].** For any  $N$ -qubit pure state  $|\psi\rangle_{ABC_1\cdots C_{N-2}}$ , we have

$$C_{ABC_1|C_2\cdots C_{N-2}}^2 \geq \max\{J_A, J_B\} - J_{C_1}, \quad (21)$$

where  $J_A = \sum_{i=0}^{N-2} (C_{AC_i}^2 - C_{aBC_i}^2) + \sum_{k=1}^{N-3} C_{A|C'_0|C'_1|\cdots|C'_k}^2$ ,  $J_B = \sum_{i=0}^{N-2} (C_{BC_i}^2 - C_{aAC_i}^2) + \sum_{k=1}^{N-3} C_{B|C'_0|C'_1|\cdots|C'_k}^2$ ,  $J_{C_1} = C_{aC_1A}^2 + C_{aC_1B}^2 + \sum_{i=2}^{N-2} C_{aC_1C_i}^2$ .

**[Proof].** For any  $N$ -qubit pure state  $|\psi\rangle_{ABC_1\cdots C_{N-2}}$ , we have

$$\begin{aligned} C_{ABC_1|C_2\cdots C_{N-2}}^2 &= 2T(\rho_{ABC_1}) \\ &\geq 2T(\rho_{AB}) - 2T(\rho_{C_1}) \\ &= C_{AB|C_1\cdots C_{N-2}}^2 - C_{C_1|ABC_2\cdots C_{N-2}}^2, \end{aligned}$$

where the inequality is due to the property of the linear entropy  $T(\rho_{ABC_1}) \geq T(\rho_{AB}) - T(\rho_{C_1})$ . Combining (19) and (20), we obtain (21).  $\blacksquare$

We have presented general monogamy relations for any quantum correlation measures and multipartite quantum states. Similar to the three tangle of concurrence, we defined the  $\alpha$ th ( $\alpha \geq \beta$ ) power of the residual quantum correlation. Based on this, we have established tighter monogamy inequalities for arbitrary quantum correlation measures. For qubit systems, the bound for concurrence, given by concurrence of assistance, has been also improved. Finally, we have presented a different kind of monogamy relations satisfied by the concurrence of  $N$ -qubit pure states under partition  $AB$  and  $C_1\cdots C_{N-2}$ , as well as under partition  $ABC_1$  and  $C_2\cdots C_{N-2}$ , which is also shown to be better than the existing ones. The residual quantum correlation we introduced may also contribute to improve other relations satisfied by the measures of quantum correlations.

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