

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

Unextendible Maximally Entangled  
Bases in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$

by

*Gui-Jun Zhang, Yuan-Hong Tao, Yi-Fan Han,  
Xin-Lei Yong, and Shao-Ming Fei*

Preprint no.: 6

2019





# Unextendible Maximally Entangled Bases in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$

Gui-Jun Zhang<sup>1</sup>, Yuan-Hong Tao<sup>1</sup> \*, Yi-Fan Han<sup>1</sup>, Xin-Lei Yong<sup>1</sup>, Shao-Ming Fei<sup>2,3</sup>

<sup>1</sup> Department of Mathematics College of Sciences, Yanbian University, Yanji 133002, China

<sup>2</sup> School of Mathematics Sciences, Capital Normal University, Beijing 100048, China

<sup>3</sup> Max-Planck-Institute for Mathematics in the Science, 04103 Leipzig, Germany

\* Correspondence to taoyuanhong12@126.com

## Abstract

The construction of unextendible maximally entangled bases is tightly related to quantum information processing like local state discrimination. We put forward two constructions of UMEBs in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  ( $p \leq q$ ) based on the constructions of UMEBs in  $\mathbb{C}^d \otimes \mathbb{C}^d$  and in  $\mathbb{C}^p \otimes \mathbb{C}^q$ , which generalize the results in [Phys. Rev. A. 94, 052302 (2016)] by two approaches. Two different 48-member UMEBs in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  have been constructed in detail.

## 1 Introduction

It is well known that the quantum states are divided into two parts: separable states and entanglement states. Quantum entanglement, as a potential resource, is widely applied into many quantum information process, such as quantum computation [1], quantum teleportation [2], quantum cryptography [3] as well as nonlocality [4]. Nonlocality is a very useful concept in quantum mechanics [4, 5, 6] and is tightly related to entanglement. However, it is proved that the unextendible product bases (UPBs) reveal some nonlocality without entanglement [7, 8]. The UPB is a set of incomplete orthogonal product states in bipartite quantum system  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  consisting of fewer than  $dd'$  vectors which have no additional product states orthogonal to each element of the set [9].

In 2009, S. Bravyi and J. A. Smolin [10] first proposed the notion of unextendible maximally entangled basis (UMEB): a set of incomplete orthogonal maximally entangled states in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  consisting of fewer than  $dd'$  vectors which have no additional maximally entangled vectors that are orthogonal to all of them. These incomplete bases have some special properties. In bipartite space  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ , one can get a state on the UMEB's complementary subspace, whose entanglement of assistance (EoA) is strictly smaller than  $\log dd'$ , the asymptotic EoA [10]. As for in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ,

one can also get a state on the complementary subspace of UMEB, corresponding to a quantum channel, which would not be unital. Besides, it cannot be convex mixtures of unitary operators too [11]. In addition, for a given mixed state, its Schmidt number is hard to calculate. If we can get a  $n$ -member UMEB  $\{|\phi_i\rangle\}$  in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ , the Schmidt number of the following state

$$\rho^\perp = \frac{1}{dd' - n} (I - \sum_{i=1}^n |\phi_i\rangle\langle\phi_i|),$$

is smaller than  $d$  [12]. Therefore, different UMEBs can be used to construct different mixed entangled states with limited Schmidt number, even state with different Schmidt number.

In [11], B. Chen and S. M. Fei provided a way to construct UMEBs in some special cases of bipartite system. Then H. Nan et al. [13], M. S. Li et al. [14], Y. L. Wang et al. [15, 16], Y. Guo [17], G. J. Zhang et al. [18] developed some new results of UMEB in bipartite system. Later, Y.J. Zhang et al. [19] and Y. Guo et al. [20] generalized the notion of UMEB from bipartite systems to multipartite quantum systems. In [20], Y. Guo showed that if there exists an  $N$ -member UMEB  $\{|\psi_j\rangle\}$  in  $\mathbb{C}^d \otimes \mathbb{C}^d$ , then there exists a  $qd^2 - q(d^2 - N)$ -member UMEB in  $\mathbb{C}^{qd} \otimes \mathbb{C}^{qd}$  for any  $q \in \mathbb{N}$ . Y. Guo et al. [21] also proposed the definition of entangled bases with fixed Schmidt number.

In this paper, we study UMEBs in bipartite system  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  ( $p \leq q$ ). A systematic way of constructing UMEBs in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  from that in  $\mathbb{C}^d \otimes \mathbb{C}^d$  is presented firstly, and a construction of 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  is given as an example. Furthermore, an explicit method to construct UMEBs containing  $pqd^2 - d(pq - N)$  maximally entangled vectors in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  from a  $N$ -member UMEB in  $\mathbb{C}^p \otimes \mathbb{C}^q$  is presented. Moreover, another construction of 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  is obtained, thus generalizing the results in Yu Guo[phys.Rev.A.94,052302(2016)] by two approaches.

## 2 Preliminaries

Throughout the paper, we denote  $[d]' = \{0, 1, \dots, d-1\}$  and  $[d]^* = \{1, 2, \dots, d\}$ .

A pure state  $|\psi\rangle$  is said to be a maximally entangled state in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  ( $d \leq d'$ ) if and only if for an arbitrary given orthonormal basis  $\{|i\rangle\}$  of  $\mathbb{C}^d$ , there exists an orthonormal basis  $\{|i'\rangle\}$  of  $\mathbb{C}^{d'}$  such that  $|\psi\rangle$  can be written as  $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i'\rangle$  [6].

A set of pure states  $\{|\phi_i\rangle\}_{i=0}^{n-1} \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$  with the following conditions is called an unex-

tendible maximally entangled bases(UMEB) [10]:

(i)  $|\phi_i\rangle, i \in [n]'$  are all maximally entangled states,

(ii)  $\langle\phi_i|\phi_j\rangle = \delta_{ij}, i, j \in [n]'$ ,

(iii)  $n < dd'$ , and if a pure state  $|\psi\rangle$  meets that  $\langle\phi_i|\psi\rangle = 0, i \in [n]'$ , then  $|\psi\rangle$  can not be maximally entangled.

Let  $\mathcal{M}_{d' \times d}$  be the Hilbert space of all  $d' \times d$  complex matrices equipped with the inner product defined by  $\langle A|B\rangle = Tr(A^\dagger B)$  for any  $A, B \in \mathcal{M}_{d' \times d}$ . If  $\{A_i\}_{i=0}^{dd'-1}$  constitutes a Hilbert-Schmidt basis of  $\mathcal{M}_{d' \times d}$ , where  $\langle A_i|A_j\rangle = d\delta_{ij}$ , then there is a one-to-one correspondence between an orthogonal basis in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$   $\{|\phi_i\rangle\}$  and  $\{A_i\}$  as follows [20, 21]:

$$|\phi_i\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{\ell'=0}^{d'-1} a_{\ell'k}^{(i)} |k\rangle|\ell'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \Leftrightarrow A_i = [a_{\ell'k}^{(i)}] \in \mathcal{M}_{d' \times d},$$

$$Sr(|\phi_i\rangle) = rank(A_i), \quad \langle\phi_i|\phi_j\rangle = \frac{1}{d} Tr(A_i^\dagger A_j), \quad (1)$$

where  $Sr(|\phi_i\rangle)$  denotes the Schmidt number of  $|\phi_i\rangle$ . Obviously,  $|\phi_i\rangle$  is a maximally entangled pure state in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  iff  $A_i$  is a  $d' \times d$  singular-value-1 matrix (a matrix whose singular values all equal to 1). Specially,  $A_i$  is a unitary matrix when  $d = d'$ .

For simplicity we adopt the following definitions [17]. We call a Hilbert-Schmidt basis  $\Omega = \{A_i\}_{i=0}^{d^2-1}$  in  $\mathcal{M}_{d \times d}$  a unitary Hilbert-Schmidt basis (UB) of  $\mathcal{M}_{d \times d}$  if  $A_i$ s are unitary matrices, and a Hilbert-Schmidt basis  $\Omega = \{A_i\}_{i=0}^{dd'-1}$  in  $\mathcal{M}_{d' \times d}$  a singular-value-1 Hilbert-Schmidt basis (SV1B) of  $\mathcal{M}_{d' \times d}$  if  $A_i$ s are singular-value-1 matrices. A set of  $d \times d$  unitary matrices  $\Omega = \{A_i\}_{i=0}^{n-1}$  ( $n < d^2$ ) is called an unextendible unitary Hilbert-Schmidt basis (UUB) of  $\mathcal{M}_{d \times d}$  if (i)  $Tr(A_i^\dagger A_j) = d\delta_{ij}$ ; (ii) if  $Tr(A_i^\dagger X) = 0, i \in [n]'$ , then X is not unitary.

[Definition] A set of  $d \times d'$  ( $d < d'$ ) singular-value-1 matrices  $\Omega = \{A_i\}_{i=0}^{dd'-1}$  ( $n < dd'$ ) is called an unextendible singular-value-1 Hilbert-Schmidt basis (USV1B) of  $\mathcal{M}_{d' \times d}$  if (a)  $Tr(A_i^\dagger A_j) = d\delta_{ij}$ ; (b) if  $Tr(A_i^\dagger X) = 0, i \in [n]'$ , then X is not a singular-value-1 matrix.

According to the Eq.(1), it is obvious that  $\Omega = \{A_i\}_{i=0}^{d^2-1}$  is a UB iff  $\{|\phi_i\rangle\}$  is a maximally entangled basis (MEB) of  $\mathbb{C}^d \otimes \mathbb{C}^d$  while  $\Omega = \{A_i\}_{i=0}^{dd'-1}$  is a SV1B iff  $\{|\phi_i\rangle\}$  is a MEB of  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ . And  $\Omega = \{A_i\}_{i=0}^{n-1}$  ( $n < d^2$ ) is a UUB iff  $\{|\phi_i\rangle\}$  is a UMEB of  $\mathbb{C}^d \otimes \mathbb{C}^d$  while  $\Omega = \{A_i\}_{i=0}^{n-1}$  ( $n < dd'$ ) is a USV1B iff  $\{|\phi_i\rangle\}$  is a UMEB of  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  [17].

### 3 UMEBs in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ ( $p \leq q$ ) from UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$

**Theorem 1.** If there is an  $N$ -member UMEB  $\{|\psi_j\rangle\}$  in  $\mathbb{C}^d \otimes \mathbb{C}^d$ , then there exists a  $pqd^2 - p(d^2 - N)$ -member UMEB in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  ( $p \leq q$ ).

**Proof.** Let  $\{W_j = [w_{i'i}^j]_{j=0}^{N-1}\}$  be a UUB of  $\mathcal{M}_{d \times d}$  corresponding to  $\{|\psi_j\rangle\}$ ,

$$|\psi_j\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{i'=0}^{d-1} w_{i'i}^j |i\rangle \otimes |i'\rangle, \quad j \in [N]'$$

Denote

$$U_{nm} = \sum_{a=0}^{d-1} e^{\frac{2\pi n a \sqrt{-1}}{d}} |a \oplus_d m\rangle \langle a|,$$

$$V_{kl} = \sum_{a=0}^{p-1} e^{\frac{2\pi k a \sqrt{-1}}{p}} |a \oplus_q l\rangle \langle a|,$$

where  $m, n \in [d]'$ ;  $l \in [q]'$ ;  $k \in [p]'$ ;  $j \in [N]'$ , and

$$B_{k0}^j = V_{k0} \otimes W_j, \quad k \in [p]'; \quad j \in [N]'$$

$$B_{kl}^{nm} = V_{kl} \otimes U_{nm}, \quad k \in [p]'; l \in [q-1]^*; m, n \in [d]'$$

Set  $C_1 = \{B_{k0}^j\}$  and  $C_2 = \{B_{kl}^{nm}\}$ . Then  $C_1 \cup C_2$  is exactly a USV1B in  $\mathcal{M}_{qd \times pd}$ .

Firstly, all  $B_{k0}^j$  and  $B_{kl}^{nm}$  are  $qd \times pd$  singular-value-1 matrices, which satisfy the conditions in the Definition:

(a)  $Tr[(B_{k0}^j)^\dagger B_{k'0}^{j'}] = pd\delta_{kk'}\delta_{jj'}$ ,  $Tr[(B_{kl}^{nm})^\dagger B_{k'l'}^{n'm'}] = pd\delta_{kk'}\delta_{ll'}\delta_{nn'}\delta_{mm'}$  and  $Tr[(B_{kl}^{nm})^\dagger B_{k'0}^{j'}] = 0$ , where  $j, j' \in [N]'$ ;  $k, k' \in [p]'$ ;  $l, l' \in [q-1]^*$ ;  $n, n', m, m' \in [d]'$ .

(b) Denote  $S$  the matrix space of  $(I_p, O_{p \times (q-p)})^t \otimes R$ , where  $t$  stands for matrix transpose,  $I_p$  is the  $p \times p$  identity matrix,  $O_{p \times (q-p)}$  is the  $p \times (q-p)$  zero matrix and  $R \in \mathcal{M}_{d \times d}$ . Obviously the dimension of  $S^\perp$  is  $p(q-1)d^2$ . Thus  $C_2$  is an SV1B of  $S^\perp$  with  $p(q-1)d^2$  elements.

Assume that  $D$  is a singular-value-1 matrix in  $\mathcal{M}_{qd \times pd}$ , which is orthogonal to all matrices in  $C_1 \cup C_2$ . Since  $C_2$  is a SV1B of  $S^\perp$ , then  $D \in S$ . No loss of generality, set

$$D = \begin{pmatrix} A \\ O_1 \end{pmatrix}, \quad A = \text{diag}(A_1, A_2, \dots, A_p), \quad O_1 = O_{(q-p)d \times pd},$$

where  $A_h$  ( $h \in [p]^*$ ) are all  $d \times d$  matrices. Note that  $D$  is orthogonal to each  $B_{k0}^j$  in  $C_1$ , i.e.,

$$Tr(D^\dagger B_{k0}^j) = 0, \quad k \in [p]'; \quad j \in [N]'$$

Then

$$\text{Tr} \left[ \left( \begin{array}{cc} A^\dagger & O_1^\dagger \end{array} \right)_{pd \times qd} \cdot \left( \begin{array}{c} G \\ O_1 \end{array} \right)_{qd \times pd} \right] = 0,$$

where  $G = \text{diag}(\omega_p^{0k}W_j, \omega_p^{1k}W_j, \dots, \omega_p^{(p-1)k}W_j)$ , i.e.,

$$\omega_p^{0k} \text{Tr}(A_1^\dagger W_j) + \omega_p^{-1k} \text{Tr}(A_2^\dagger W_j) + \dots + \omega_p^{(1-p)k} \text{Tr}(A_p^\dagger W_j) = 0.$$

Hence,

$$HX_j = 0,$$

where

$$H = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_p^{p-1} & \omega_p^{p-2} & \dots & \omega_p^1 \\ 1 & \omega_p^{p-2} & \omega_p^{p-4} & \dots & \omega_p^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_p^1 & \omega_p^2 & \dots & \omega_p^{p-1} \end{pmatrix}, \quad X_j = \begin{pmatrix} \text{Tr}(A_1^\dagger W_j) \\ \text{Tr}(A_2^\dagger W_j) \\ \text{Tr}(A_3^\dagger W_j) \\ \vdots \\ \text{Tr}(A_p^\dagger W_j) \end{pmatrix}.$$

Obviously,  $X_j = O$  for  $j \in [N]'$  since  $\det H \neq 0$ . That is to say,  $\text{Tr}(A_h^\dagger W_0) = \text{Tr}(A_h^\dagger W_1) = \dots = \text{Tr}(A_h^\dagger W_{N-1}) = 0$ ,  $h \in [p]^*$ . As every  $A_n$  is orthogonal to each  $W_j$ , whereas  $\{W_j\}$  is a UUB in  $\mathcal{M}_{d \times d}$ , none of  $A_h$  is unitary. Moreover, all the singular values of  $A_h$ s are also the singular values of  $D$ . Therefore,  $D$  is not a singular-value-1 matrix, which contradicts to the assumption. Thus,  $C_1 \cup C_2$  is a USV1B in  $\mathcal{M}_{qd \times pd}$ .  $\square$

**Example 1.** A 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  from a 6-member UMEB in  $\mathbb{C}^3 \otimes \mathbb{C}^3$ .

A 6-member UMEB in  $\mathbb{C}^3 \otimes \mathbb{C}^3$  from Ref.[10] is as follows:

$$W_j = I - (1 - e^{\sqrt{-1}\theta})|\psi_j\rangle\langle\psi_j|, j = [6]',$$

where

$$\begin{aligned} |\psi_{0,1}\rangle &= \frac{1}{\sqrt{1+\phi^2}}(|0\rangle \pm \phi|1\rangle), \\ |\psi_{2,3}\rangle &= \frac{1}{\sqrt{1+\phi^2}}(|1\rangle \pm \phi|2\rangle), \\ |\psi_{4,5}\rangle &= \frac{1}{\sqrt{1+\phi^2}}(|2\rangle \pm \phi|0\rangle), \end{aligned}$$

with  $\phi = (1 + \sqrt{5})/2$ .

Then, denote

$$U_{nm} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}^n, \quad m, n \in [3]',$$

$$V_{kl} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^l \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k, \quad k \in [2]'; \quad l \in [3]',$$

where  $\omega_3 = e^{\frac{2\pi\sqrt{-1}}{3}}$ .

Let

$$\begin{aligned} B_{01}^{nm} &= \begin{pmatrix} 0 & 0 \\ U_{nm} & 0 \\ 0 & U_{nm} \end{pmatrix}, & B_{11}^{nm} &= \begin{pmatrix} 0 & 0 \\ U_{nm} & 0 \\ 0 & -U_{nm} \end{pmatrix}, \\ B_{02}^{nm} &= \begin{pmatrix} 0 & U_{nm} \\ 0 & 0 \\ U_{nm} & 0 \end{pmatrix}, & B_{12}^{nm} &= \begin{pmatrix} 0 & U_{nm} \\ 0 & 0 \\ -U_{nm} & 0 \end{pmatrix}, \\ B_{00}^j &= \begin{pmatrix} W_j & 0 \\ 0 & W_j \\ 0 & 0 \end{pmatrix}, & B_{10}^j &= \begin{pmatrix} W_j & 0 \\ 0 & -W_j \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where  $n, m \in [3]'; j \in [6]'$ . Set  $C_1 = \{B_{k0}^j\}$ ,  $C_2 = \{B_{kl}^{nm}\}$ , for  $k \in [2]'; j \in [6]'; l \in [2]^*$ ;  $n, m \in [3]'$ . According to Theorem 1, we have that  $C_1 \cup C_2$  is a 48-number UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$ .

*Remark 1.* Theorem 1 in Ref.[15] is a special case of the above Theorem 1 for  $p = q$ .

#### 4 UMEBs in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ ( $p \leq q$ ) from UMEBs in $\mathbb{C}^p \otimes \mathbb{C}^q$

Next, we will present a general approach to construct UMEBs in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  from UMEBs in  $\mathbb{C}^p \otimes \mathbb{C}^q$ .

**Theorem 2.** If there is an  $N$ -member UMEB  $\{|\psi_j\rangle\}$  in  $\mathbb{C}^p \otimes \mathbb{C}^q$ , then there exists a  $pqd^2 - d(pq - N)$ -member UMEB in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  ( $p \leq q$ ).

**Proof.** Let  $\{W_j = [w_{i'i}^j]_{j=0}^{N-1}\}$  be a USV1B of  $\mathcal{M}_{q \times p}$  corresponding to  $\{|\psi_j\rangle\}$ , then

$$|\psi_j\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{i'=0}^{d'-1} w_{i'i}^j |i\rangle \otimes |i'\rangle, \quad j \in [N]'$$

Denote

$$\begin{aligned} U_{nm} &= \sum_{a=0}^{d-1} e^{\frac{2\pi na\sqrt{-1}}{d}} |a \oplus_d m\rangle \langle a|, \\ V_{kl} &= \sum_{a=0}^{p-1} e^{\frac{2\pi ka\sqrt{-1}}{p}} |a \oplus_q l\rangle \langle a|, \end{aligned}$$

where  $m, n \in [d]'$ ;  $l \in [q]'$ ;  $k \in [p]'$ ;  $j \in [N]'$ . Let

$$B_{n0}^j = U_{n0} \otimes W_j, \quad n \in [d]'; \quad j \in [N]',$$

$$B_{nm}^{kl} = U_{nm} \otimes V_{kl}, \quad m \in [d-1]^*; n \in [d]'; l \in [q]'; k \in [p]',$$

and

$$C_1 = \{B_{n0}^j\}, \quad C_2 = \{B_{nm}^{kl}\}.$$

then,  $C_1 \cup C_2$  is exactly a USV1B in  $\mathcal{M}_{qd \times pd}$ .

Firstly, all  $B_{n0}^j$  and  $B_{nm}^{kl}$  are  $qd \times pd$  singular-value-1 matrices, satisfying the conditions in the Definition:

(a)  $Tr[(B_{n0}^j)^\dagger B_{n'0}^{j'}] = qd\delta_{nn'}\delta_{jj'}$ ,  $Tr[(B_{nm}^{kl})^\dagger B_{n'm'}^{k'l'}] = qd\delta_{nn'}\delta_{mm'}\delta_{kk'}\delta_{ll'}$  and  $Tr[(B_{nm}^{kl})^\dagger B_{n'0}^j] = 0$ , where  $j, j' \in [N]'$ ;  $k, k' \in [q]'$ ;  $l, l' \in [p]'$ ;  $n, n' \in [d]'$ ;  $m, m' \in [d-1]^*$ .

(b) Denote  $S$  the matrix space of  $I_d \otimes R$ , where  $R \in \mathcal{M}_{q \times p}$ . Obviously, the dimension of  $S^\perp$  is  $pq(d-1)d$ .

Setting  $C_1 = \{B_{n0}^j\}$  and  $C_2 = \{B_{nm}^{kl}\}$ , we have that  $C_2$  with  $pq(d-1)d$  elements is an SV1B of  $S^\perp$ , and  $C_1 \cup C_2$  is just a USV1B in  $\mathcal{M}_{qd \times pd}$ .

Assume that  $D$  is a singular-value-1 matrix in  $\mathcal{M}_{qd \times pd}$ , which is orthogonal to all matrices in  $C_1 \cup C_2$ . Since  $C_2$  is a SV1B of  $S^\perp$ , then  $D \in S$ . No loss of generality, set

$$D = \text{diag}(A_1, A_2, \dots, A_d)_{qd \times pd},$$

where  $A_h (h \in [d]^*)$  are all  $q \times p$  matrices. Similar to the proof of Theorem 1, we can prove that none of  $A_h$  is singular-value-1 matrices. Moreover, all the singular values of all  $A_h$ s are also the singular values of  $D$ , namely,  $D$  is not a singular-value-1 matrix, which contradicts to the assumption. Thus,  $C_1 \cup C_2$  is a USV1B in  $\mathcal{M}_{qd \times pd}$ .  $\square$

**Example 2.** A 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  from a 4-member UMEB in  $\mathbb{C}^2 \otimes \mathbb{C}^3$ .

A 4-member UMEB in  $\mathbb{C}^2 \otimes \mathbb{C}^3$  is as follows:

$$W_{0,1} = |0'\rangle\langle 0| \pm |1'\rangle\langle 1|,$$

$$W_{2,3} = |0'\rangle\langle 1| \pm |1'\rangle\langle 0|.$$

Denote

$$U_{nm} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}^n, \quad m, n \in [3]',$$

$$V_{kl} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^l \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k, \quad k \in [3]'; \quad l \in [2]',$$

where  $\omega_3 = e^{\frac{2\pi\sqrt{-1}}{3}}$ . Let

$$\begin{aligned} B_{01}^{kl} &= \begin{pmatrix} 0 & 0 & V_{kl} \\ V_{kl} & 0 & 0 \\ 0 & V_{kl} & 0 \end{pmatrix}, B_{11}^{kl} = \begin{pmatrix} 0 & 0 & \omega_3^2 V_{kl} \\ V_{kl} & 0 & 0 \\ 0 & \omega_3 V_{kl} & 0 \end{pmatrix}, B_{21}^{kl} = \begin{pmatrix} 0 & 0 & \omega_3 V_{kl} \\ V_{kl} & 0 & 0 \\ 0 & \omega_3^2 V_{kl} & 0 \end{pmatrix}, \\ B_{02}^{kl} &= \begin{pmatrix} 0 & V_{kl} & 0 \\ 0 & 0 & V_{kl} \\ V_{kl} & 0 & 0 \end{pmatrix}, B_{12}^{kl} = \begin{pmatrix} 0 & \omega_3 V_{kl} & 0 \\ 0 & 0 & \omega_3^2 V_{kl} \\ V_{kl} & 0 & 0 \end{pmatrix}, B_{22}^{kl} = \begin{pmatrix} 0 & \omega_3^2 V_{kl} & 0 \\ 0 & 0 & \omega_3 V_{kl} \\ V_{kl} & 0 & 0 \end{pmatrix}, \\ B_{00}^j &= \begin{pmatrix} W_j & 0 & 0 \\ 0 & W_j & 0 \\ 0 & 0 & W_j \end{pmatrix}, B_{10}^j = \begin{pmatrix} W_j & 0 & 0 \\ 0 & \omega_3 W_j & 0 \\ 0 & 0 & \omega_3^2 W_j \end{pmatrix}, B_{20}^j = \begin{pmatrix} W_j & 0 & 0 \\ 0 & \omega_3^2 W_j & 0 \\ 0 & 0 & \omega_3 W_j \end{pmatrix}, \end{aligned}$$

where  $k \in [3]'; l \in [2]'; j \in [4]'$ . Set  $C_1 = \{B_{n0}^j\}$ ,  $C_2 = \{B_{nm}^{kl}\}$ , for  $k \in [3]'; l \in [2]'; j \in [4]'; n \in [3]'; m \in [2]^*$ . Then according to Theorem 2,  $C_1 \cup C_2$  is a 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$ .

**Remark 2.** The Constructions of UMEB in Theorem 1 and Theorem 2 are different, which can be easily seen from the Examples 1 and 2. We can give a state with Schmidt number 4 in the subspace of the UMEB in Example 1. While what we can get in the subspace of the UMEB in Example 2 are the states with Schmidt number no more than 3. In fact, according to Theorem 2, one can construct a UMEB in  $\mathbb{C}^4 \otimes \mathbb{C}^6$  from the UMEB in  $\mathbb{C}^2 \otimes \mathbb{C}^3$ , while one can not do this way from the Theorem 1. Here Theorem 1 in [15] is a also special case of the above Theorem 2 for  $p = q$ .

**Remark 3.** By using Theorem 2 in [18], we can give a  $p(q-r)$ -member UMEB in  $\mathbb{C}^p \otimes \mathbb{C}^q$ . According to Theorem 2 in this paper, we can obtain a  $pd(qd-r)$ -member UMEB in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ , in whose subspace we can get some states with Schmidt number  $dr$ . We can also get a  $pd(qd-r)$ -member UMEB directly by Theorem 2 in [18], nevertheless, in the associated subspace, one can only attain the states with Schmidt number no greater than  $r$ . Therefore, they are different constructions. Actually, there are many  $N$ -number UMEBs in  $\mathbb{C}^p \otimes \mathbb{C}^q$ , where  $p \nmid N$ . In this case, it doesn't hold that  $pd|(pqd^2 - d(pq - N))$ . Namely, we can not even get a UMEB with the same number of members by Theorem 2 in [18].

## 5 Conclusion

We have provided an explicit way of constructing a  $pqd^2 - p(d^2 - N)$ -member UMEB in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  from an  $N$ -member UMEB in  $\mathbb{C}^d \otimes \mathbb{C}^d$ , and constructed a 48-number UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$  as a detailed example. We have also established a method to construct a  $pqd^2 - d(pq - N)$ -member UMEB in  $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$  from an  $N$ -member UMEB in  $\mathbb{C}^p \otimes \mathbb{C}^q$ , and presented another 48-member UMEB in  $\mathbb{C}^6 \otimes \mathbb{C}^9$ . These results may highlight the further investigations on the construction of unextendible bases and the theory of quantum entanglement.

## References

- [1] A. Barenco, A. K. Ekert. Dense Coding Based on Quantum Entanglement[J]. Journal of Modern Optics, 1995, 42(6): 1253-1259
- [2] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki. Quantum entanglement[J]. Reviews of Modern Physics, 2009, 81: 865-942
- [3] W. Tittel, J. Brendel, H. Zbinden, N. Gisin. Quantum cryptography using entangled photons in energy-time bell states[J]. Physical Review Letters, 2000, 84(20): 4737
- [4] S. B. Zheng. Quantum nonlocality for a three-particle nonmaximally entangled state without inequalities[J]. Physical Review A, 2002, 66(1): 90-95
- [5] W. J. Guo, D. H. Fan, L. F. Wei. Experimentally testing Bells theorem based on Hardys nonlocal ladder proofs[J]. Science China Physics, Mechanics Astronomy. 2015, 58: 024201.
- [6] H. X. Meng, H. X. Cao, W. H. Wang, L. Chen, Y. J. Fan. Continuity of the robustness of contextuality and the contextuality cost of empirical models[J]. Science China Physics, Mechanics & Astronomy. 2016, 59(4): 640303
- [7] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, B. M. Terhal. Unextendible product bases, uncompletable product bases and bound entanglement[J]. Commun. Math. Phys. 2013, 238, 379
- [8] S. D. Rinaldis. Distinguishability of complete and unextendible product bases[J]. Phys. Rev. A. 2004, 70: 022309

- [9] C. H. Bennett et al. Unextendible product bases and bound entanglement[J]. Phys. Rev. Lett. 1999, 82: 5385
- [10] S. Bravyi, J. A. Smolin. Unextendible maximally entangled bases[J]. Physical Review A, 2011, 84: 042306
- [11] B. Chen, S. M. Fei. Unextendible maximally entangled bases and mutually unbiased bases[J]. Physical Review A, 2013, 88: 034301
- [12] Y. Guo, S. J. Wu. Unextendible entangled bases with fixed Schmidt number[J]. Physical Review A, 2014, 48(24): 245301
- [13] H. Nan, Y. H. Tao, L. S. Li, J. Zhang. Unextendible Maximally Entangled Bases and Mutually Unbiased Bases in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  [J]. International Journal of Theoretical Physics, 2015, 54: 927
- [14] M. S. Li, Y. L. Wang, S. M. Fei, Z. J. Zheng. Unextendible maximally entangled bases in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  [J]. Physical Review A, 2014, 89: 062313
- [15] Y.L. Wang, M. S. Li, S.M. Fei. Unextendible maximally entangled bases in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  [J]. Physical Review A, 2014, 90: 034301
- [16] Y. L. Wang, M. S. Li, S. M. Fei. Connecting the UMEB in  $\mathbb{C}^d \otimes \mathbb{C}^d$  with partial Hadamard matrices[J]. Quantum Information Processing, 2017, 16(3): 84
- [17] Y. Guo. Constructing the unextendible maximally entangled basis from the maximally entangled basis[J]. Physical Review A, 2016, 94: 052302
- [18] G. J. Zhang, Y. H. Tao, Y. F. Han, X. L. Yong, S. M. Fei. Constructions of Unextendible Maximally Entangled Bases in  $\mathbb{C}^d \otimes \mathbb{C}^{d'}$  [J]. Scientific Reports, 2018, 8(1): 319
- [19] Y. J. Zhang, H. Zhao, N. Jing, S. M. Fei. Multipartite unextendible entangled basis[J]. International Journal of Theoretical Physics, 2017, 56(11): 3425-3430
- [20] Y. Guo, Y. P. Jia, X. L. Li. Multipartite unextendible entangled basis[J]. Quantum Inf Process. 2015, 14: 3553
- [21] Y. Guo, S. P. Du, X. L. Li, S. J. Wu. Entangled bases with fixed Schmidt number[J]. Journal of Physics A: Mathematical and Theoretical, 2015, 48: 245301

## Acknowledgements

The work is supported by the NSFC under number 11675113, 11761073 and NSF of Beijing under No. KZ201810028042.

## Acknowledgements

G.-J.Z and Y.-H.T. wrote the main manuscript text. All of the authors reviewed the manuscript.