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by

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Entanglement polygamy, like entanglement monogamy, is a fundamental property of multipartite quantum states. We investigate the polygamy relations related to the concurrence C and the entanglement of formation E for general n-qubit states. We extend the results in [Phys. Rev. A 90, 024304 (2014)] from the parameter region $\alpha \leq 0$ to $\alpha \leq \alpha_0$, where $0 < \alpha_0 \leq 2$ for C, and $0 < \alpha_0 \leq \sqrt{2}$ for E.

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I. INTRODUCTION

Quantum entanglement [1–5] lies at the heart of quantum information processing and quantum computation [6]. The quantification of quantum entanglement has drawn much attention in the last decade. An fundamental difference between quantum entanglement and classical correlations is that a quantum system entangled with one of other systems limits its entanglement with the remaining systems. The monogamy and polygamy relations give rise to the structures of entanglement distribution in multipartite systems. They are also essential features allowing for security in quantum key distribution [7].

For a tripartite system A, B and C, the monogamy of an entanglement measure ε implies that [8], the entanglement between A and BC satisfies

$$\varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}. \tag{1}$$

Such monogamy relations are not always satisfied by any entanglement measures. Dually the polygamy inequality in literature is expressed as [9]:

$$\varepsilon_{A|BC} \leq \varepsilon_{AB} + \varepsilon_{AC}. \tag{2}$$

It has been shown that the squared concurrence C^2 [10, 11] and the squared entanglement of formation E^2 [12, 13] do satisfy such monogamy relations (1). In Ref. [14] it has been shown that general monogamy inequalities are satisfied by the α ($\alpha \geq 2$)th power of concurrence C^α and the α ($\alpha \geq \sqrt{2}$)th power of entanglement of formation E^α for $n-$qubit mixed states. If $C(\rho_{AB_i}) \neq 0$, $i = 1, ..., n-1$, C^α satisfies (2) for $\alpha \leq 0$. In Ref. [15] tighter monogamy inequalities for concurrence, entanglement of formation have been given.

Ref. [16] shown that the αth power of the square of convex-roof extended negativity (SCREN) provides a class of monogamy inequalities of multiqubit entanglement in a tight way for $\alpha \geq 1$, and further shown that the αth power of SCREN also provides a class of tight polygamy inequalities for $0 \leq \alpha \leq 1$. By using the αth power of entanglement of assistance for $0 \leq \alpha \leq 1$, and the Hamming weight of the binary vector related with the distribution of subsystems, Ref. [18] established a class of weighted polygamy inequalities of multiparty entanglement in arbitrary dimensional quantum systems.

However, the polygamy properties of the αth ($0 < \alpha < 2$) power of concurrence and the αth ($0 < \alpha < \sqrt{2}$) power of entanglement of formation are still unknown. In this paper, we study the polygamy inequalities of C^α for $\alpha \in (0, 2)$ and E^α for $\alpha \in (0, \sqrt{2})$.

II. POLYGAMY RELATIONS FOR CONCURRENCE

For a bipartite pure state $|\psi\rangle_{AB}$, the concurrence is given by [19–21],

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - Tr(\rho_A^2)]},$$

where ρ_A is reduced density matrix obtained by tracing over the subsystem B, $\rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence is extended to mixed states $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum_i p_i = 1$, by the convex roof construction,

$$C(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where the minimum takes over all possible pure state decompositions of ρ_{AB}.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by [22]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

for all possible ensemble realizations of $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$. When $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ is a pure state, then one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For $n-$qubit quantum states, the concurrence satisfies [14]

$$C^\alpha_{A|B_1B_2...B_{n-1}} \geq C^\alpha_{AB_1} + ... + C^\alpha_{AB_{n-1}}, \tag{3}$$

for $\alpha \geq 2$, where $C_{A|B_1B_2...B_{n-1}}$ is the concurrence of ρ under bipartite partition $A|B_1B_2...B_{n-1}$, and C_{AB_i}, $i = 1, 2..., n-1$, is the concurrence of the mixed states $\rho_{AB_i} = Tr_{B_1B_2...B_{i-1}B_{i+1}...B_{n-1}}(\rho)$. For $C_{AB_i} \neq 0$, $i = 1, ..., n-1$, the concurrence satisfies

$$C^{\alpha}_{A|B_1...B_{n-1}} < C^{\alpha}_{AB_1} + ... + C^{\alpha}_{AB_{n-1}}, \quad (4)$$

for $\alpha \leq 0$. Further, in Ref. [15] monogamy inequalities tighter than (3) are derived for the αth ($\alpha \geq 2$) power of concurrence.

Dual to the CKW inequality, the polygamy monogamy relation based on the concurrence of assistance for the $n-$qubit pure states $|\varphi\rangle_{A|B_1...B_{n-1}}$ was proved in [17]:

$$C^2(|\varphi\rangle_{A|B_1...B_{n-1}}) \leq \sum_{i=1}^{n-1} C^2_a(\rho_{AB_i}). \quad (5)$$

Theorem 1 *For any $2 \otimes 2 \otimes 2^{n-2}$ tripartite mixed state ρ_{ABC}, if $C(\rho_{AB})C(\rho_{AC}) \neq 0$, there exists a real number $\alpha_0 \in (0, 2]$, for any $\alpha \in [0, \alpha_0]$, we have*

$$C^{\alpha}(\rho_{A|BC}) \leq C^{\alpha}(\rho_{AB}) + C^{\alpha}(\rho_{AC}). \quad (6)$$

[Proof] For arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC}, if $C(\rho_{AB})C(\rho_{AC}) \neq 0$, denote $f(\alpha) = C^{\alpha}(\rho_{AB}) + C^{\alpha}(\rho_{AC})$. Obviously, $f(0) = 2$ and $f(2) = C^2_{AB} + C^2_{AC} \leq C^2(\rho_{A|BC}) \leq 1$. Since the continuity of the $f(\alpha)$, there exists a real number $\alpha_0 \in (0, 2]$ such that $f(\alpha_0) = 1$. Together with the monotonicity of $f(\alpha)$, we have $f(\alpha) \geq 1$ for $\alpha \in [0, \alpha_0]$, i.e $C^{\alpha}_{AB} + C^{\alpha}_{AC} \geq 1 \geq C^{\alpha}(\rho_{A|BC})$ for $\alpha \in [0, \alpha_0]$. $\Box$

Theorem 1 shows the polygamy of inequality (2) for arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC} in case of $C_{AB}C_{AC} \neq 0$. Specifically, for $\alpha \in (\alpha_0, 2]$, from the proof of Theorem 1, we have $C^{\alpha}_{AB} + C^{\alpha}_{AC} \leq 1$. If $C_{AB}C_{AC} = 0$, obviously we have $C^{\alpha}(\rho_{A|BC}) \geq \max\{C^{\alpha}_{AB}, C^{\alpha}_{AC}\}$ for any $\alpha \in [0, +\infty)$.

Example 1. Let us consider the three-qubit state, $\rho_{ABC} = \frac{1-t}{8} I_8 + t|\psi\rangle_{ABC}\langle\psi|$ with $t \geq 0.783612$, where $|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, and I_8 is the 8×8 identity matrix. We have $C(|\psi\rangle_{A|BC}) = \frac{2\sqrt{2}}{3}$ and $C(\rho_{AB}) = C(\rho_{AC}) = \frac{2t}{3} - \sqrt{\frac{3-2t-t^2}{3}}$. Therefore,

$$f(\alpha) = 2\left(\frac{2t}{3} - \sqrt{\frac{3-2t-t^2}{3}}\right)^{\alpha}.$$

We have $f\left([\log_2(\frac{3}{2t-\sqrt{9-6t-3t^2}})]^{-1}\right) = 1$, i.e., $\alpha_0 = [\log_2(\frac{3}{2t-\sqrt{9-6t-3t^2}})]^{-1}$, see Fig. 1. It is clear that $C^{\alpha}(\rho_{A|BC}) \leq C^{\alpha}(\rho_{AB}) + C^{\alpha}(\rho_{AC})$ for $\alpha \in [0, \alpha_0]$. In particular, take $t = 1$. Then $\alpha_0 \approx 1.70951$, see Fig. 2.

Generalizing the result of Theorem 1 , we have the following theorem for multipartite qubit systems.

Theorem 2 *For any n-qubit quantum state ρ, if there are at least two substates $\rho_{AB_{i_1}}$ and $\rho_{AB_{i_2}}$ such that $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$, $i_1 \neq i_2$ and $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\alpha_0 \in (0, 2]$ such that*

$$C^{\alpha}(\rho_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} C^{\alpha}(\rho_{AB_i}), \quad (7)$$

for $0 \leq \alpha \leq \alpha_0$.

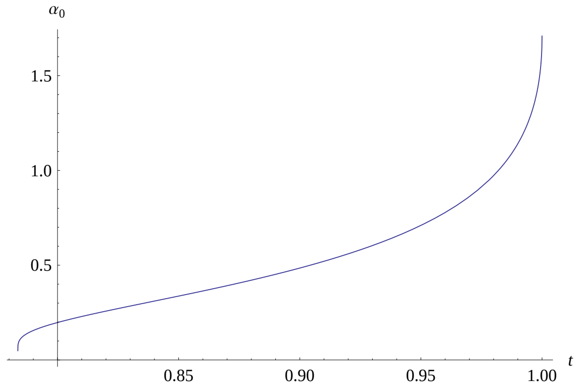

FIG. 1: α_0 with $0.783612 \leq t \leq 1$.

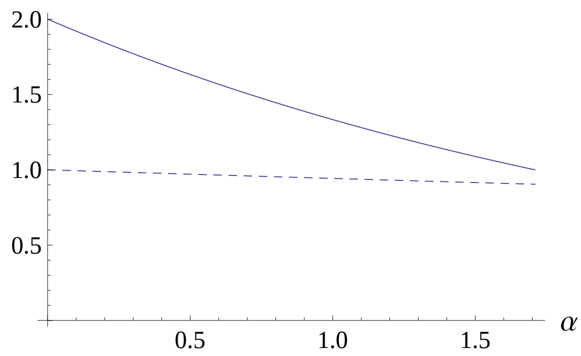

FIG. 2: The solid line is $C^{\alpha}(\rho_{AB}) + C^{\alpha}(\rho_{AC})$ and the dashed line is $C^{\alpha}(\rho_{A|BC})$ for $\alpha \in [0, 1.70951]$ with $t = 1$.

[Proof] For convenience, we denote $f(\alpha) = \sum_{i=1}^{n-1} C^{\alpha}(\rho_{AB_i})$ with $\alpha \geq 0$. For any $2 \otimes 2 \otimes 2 \otimes ... \otimes 2$ quantum states $\rho_{AB_1...B_{n-1}}$, $f(2) = \sum_{i=1}^{n-1} C^2(\rho_{AB_i}) \leq C^2(\rho_{A|B_1...B_{n-1}}) \leq 1$. Since $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$, we have $f(0) \geq 2$. Taking into account that $f(\alpha)$ is continuous, we have that there must be a real number $\alpha_0 \in (0, 2]$ such that $f(\alpha_0) = 1$. As $f(\alpha)$ is monotonically decreasing, we have $f(\alpha) \geq 1$ for $\alpha \in [0, \alpha_0]$. $\Box$

From Theorem 2, inequalities (3) and (4), we have the following result for $n-$qubit quantum states $\rho_{AB_1...B_{n-1}}$:

(1) If there is only one substate $\rho_{AB_{i_0}}$, $i_0 \in \{1, ..., n-1\}$, is entangled, then $C^{\alpha}(\rho_{AB_1...B_{n-1}}) \geq C^{\alpha}(\rho_{AB_{i_0}})$ for any $\alpha \geq 0$; and $C^{\alpha}(\rho_{AB_1...B_{n-1}}) \leq C^{\alpha}(\rho_{AB_{i_0}})$ for any $\alpha \leq 0$;

(2) If there are at least two entangled substates, then there must be $\alpha_0 \in (0, 2]$, such that $C^{\alpha}(\rho_{AB_1...B_{n-1}}) \geq \sum_{i=1}^{n-1} C^{\alpha}(\rho_{AB_i})$ for any $\alpha \geq 2$; and $C^{\alpha}(\rho_{AB_1...B_{n-1}}) \leq \sum_{i=1}^{n-1} C^{\alpha}(\rho_{AB_i})$ for any $0 \leq \alpha \leq \alpha_0$.

(3) If all the substates ρ_{AB_i}, $i = 1, ..., n-1$, are entangled, then there must be $\alpha_0 \in (0, 2]$, such that $C^\alpha(\rho_{AB_1...B_{n-1}}) \geq \sum_{i=1}^{n-1} C^\alpha(\rho_{AB_i})$ for any $\alpha \geq 2$; and $C^\alpha(\rho_{AB_1...B_{n-1}}) \leq \sum_{i=1}^{n-1} C^\alpha(\rho_{AB_i})$ for any $\alpha \leq \alpha_0$.

Example 2: We consider the 4-qubit generalized W-class state, $|\psi\rangle = a|0000\rangle + b_1|1000\rangle + b_2|0100\rangle + b_3|0010\rangle + b_4|0001\rangle$, with $a = b_2 = \frac{1}{\sqrt{10}}$, $b_1 = \frac{1}{\sqrt{15}}$, $b_3 = \sqrt{\frac{2}{15}}$, $b_4 = \sqrt{\frac{3}{5}}$. One has $C(\rho_{AB_i}) = 2|b_1||b_{i+1}|$, $i = 1, 2, 3$, and $C(|\psi\rangle) = 2|b_1|\sqrt{1-|b_1|^2} = \frac{2\sqrt{14}}{15}$. Denote $f(\alpha) = \sum_{i=1}^{4} C^\alpha(\rho_{AB_i}) = (\frac{\sqrt{6}}{15})^\alpha + (\frac{2\sqrt{2}}{15})^\alpha + (\frac{2}{5})^\alpha$. We have $f(0.783586) = 1$, i.e., $C^\alpha(|\psi\rangle) \leq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + C^\alpha(\rho_{AB_3})$ for $\alpha \in [0, 0.783586]$, see Fig. 3.

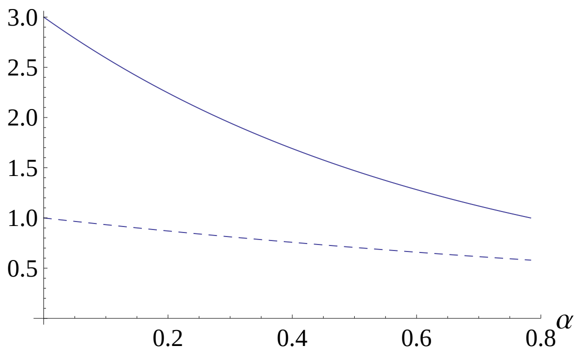

FIG. 3: The solid line is $\sum_{i=1}^{3} C^\alpha(\rho_{AB_i})$ and the dashed line is $C^\alpha(|\psi\rangle)$ for $\alpha \in [0, 0.783586]$.

From Theorem 2 and that $C(\rho_{AB}) \leq C_a(\rho_{AB})$ for any quantum states, we have the following corollaries:

Corollary 1 *For any n-qubit quantum state ρ, if there are at least two substates such that $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$ for $i_1 \neq i_2$ and $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\beta_0 \in (0, 2]$,*

$$C^\beta(\rho_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} C_a^\beta(\rho_{AB_i}), \tag{8}$$

where $0 \leq \beta \leq \beta_0$ and β_0 is a real number which satisfies $\sum_{i=1}^{n-1} C^{\beta_0}(\rho_{AB_i}) = 1$.

Denote $g(\alpha) = \sum_{i=1}^{n-1} C^\alpha(\rho_{AB_i}) - C^\alpha(\rho_{A|B_1...B_{n-1}})$. For $n-$qubit states $\rho_{AB_1...B_{n-1}}$ with at least two entangled pairs of qubits, from Theorem 2, we have $g(\alpha_0) \geq 0$ and $g(2) \leq 0$. There must be a real number α_1 such that $g(\alpha_1) = 0$. Similar to Theorem 1 and 2, we have the following corollary.

Corollary 2 *For any n-qubit quantum state $\rho_{A|B_1...B_{n-1}}$, if $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$ for $i_1 \neq i_2$ and $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\alpha_1 \in [\alpha_0, 2]$, such that*

$$C^{\alpha_1}(\rho_{A|B_1B_2...B_{n-1}}) = \sum_{i=1}^{n-1} C^{\alpha_1}(\rho_{AB_i}), \tag{9}$$

where α_0 is a real number which satisfies $\sum_{i=1}^{n-1} C^{\alpha_0}(\rho_{AB_i}) = 1$ for $0 < \alpha_0 \leq 2$.

III. POLYGAMY INEQUALITIES FOR EOF

The entanglement of formation (EoF) [23, 24] is a well-defined and important measure of quantum entanglement for bipartite systems. Let H_A and H_B be m- and n-dimensional ($m \leq n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle \in H_A \otimes H_B$ is defined by $E(|\psi\rangle) = S(\rho_A)$, where $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = -Tr(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$

with the infimum taking over all possible decompositions of ρ_{AB} in a mixture of pure states $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. $E_a(\rho_{AB})$ is the entanglement of assistance (EOA) of ρ_{AB} defined as

$$E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$

maximizing over all possible ensemble realizations of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$.

It has been shown that the entanglement of formation does not satisfy monogamy inequality such as $E_{AB} + E_{AC} \leq E_{A|BC}$ [13]. In [14] the authors showed that

$$E^\alpha(\rho_{A|B_1B_2...B_{n-1}}) \geq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i}) \tag{10}$$

for $\alpha \geq \sqrt{2}$.

A general polygamy inequality of multipartite quantum entanglement was established as

$$E_a(\rho_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} E_a(\rho_{AB_i}) \tag{11}$$

for any multipartite quantum state $\rho_{AB_1...B_{n-1}}$ of arbitrary dimension [25]. For any multipartite quantum state $\rho_{AB_1B_2...B_{n-1}}$, one has for any $\beta \in [0, 1]$ [18]:

$$[E_a(\rho_{A|B_1B_2...B_{n-1}})]^\beta \leq \sum_{i=1}^{n-1} \beta^i [E_a(\rho_{AB_i})]^\beta, \tag{12}$$

conditioned that

$$E_a(\rho_{AB_i}) \leq \sum_{j=i+1}^{n-1} E_a(\rho_{AB_j}).$$

In fact, by using applying the approach for Theorems 1 and 2, we can prove the following results generally for EoF:

Theorem 3 *For any n-qubit quantum state $\rho_{AB_1B_2...B_{n-1}}$, if there are at least two substates such that $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$ for $i_1 \neq i_2$ and $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\alpha_0 \in (0, \sqrt{2}]$, such that*

$$E^\alpha(\rho_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i}), \qquad (13)$$

where $0 \leq \alpha \leq \alpha_0$.

[Proof] For convenience, we denote $f(\alpha) = \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i})$ with $\alpha \geq 0$. Then $f(\sqrt{2}) = \sum_{i=1}^{n-1} E^{\sqrt{2}}(\rho_{AB_i}) \leq E^{\sqrt{2}}(\rho_{A|B_1...B_{n-1}}) \leq 1$. Since $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$, we have $E(\rho_{AB_{i_1}})E(\rho_{AB_{i_2}}) \neq 0$, i.e., $f(0) \geq 2$. As $f(\alpha)$ is continuous, there must be a real number $\alpha_0 \in (0, \sqrt{2}]$ so that $f(\alpha_0) = 1$. Since $f(\alpha)$ is monotonically decreasing, we have $f(\alpha) \geq 1$ for $\alpha \in [0, \alpha_0]$. $\square$

From Theorem 3, inequalities (10) and (11), we have the following results for n-qubit quantum states $\rho_{AB_1...B_{n-1}}$:

(1) If only there is only one substate $\rho_{AB_{i_0}}$, $i_0 \in \{1, ..., n-1\}$, is entangled, then $E^\alpha(\rho_{A|B_1...B_{n-1}}) \geq E^\alpha(\rho_{AB_{i_0}})$ for any $\alpha \geq 0$; and $E^\alpha(\rho_{A|B_1...B_{n-1}}) \leq E^\alpha(\rho_{AB_{i_0}})$ for any $\alpha < 0$.

(2) If at least two of the substates ρ_{AB_i}, $i = 1, ..., n-1$, are entangled, then there must be $\alpha_0 \in (0, \sqrt{2}]$ so that $E^\alpha(\rho_{A|B_1...B_{n-1}}) \geq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i})$ for any $\alpha \geq \sqrt{2}$; and $E^\alpha(\rho_{A|B_1...B_{n-1}}) \leq \sum_{i=1}^{n-1} E^\alpha(\rho_{AB_i})$ for any $0 \leq \alpha \leq \alpha_0$.

Example 3. Consider the pure state in Example 1, $|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. We have $E_{A|BC} = 0.918296$, $E_{AB} = E_{AC} = 0.550048$. Let $f(\alpha) = E^\alpha_{AB} + E^\alpha_{AC}$. Then $f(1.15959) = 1$. It is easily verified that $E^\alpha(|\psi\rangle_{A|BC}) \leq E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC})$ for $\alpha \leq 1.15959$, see Fig. 3.

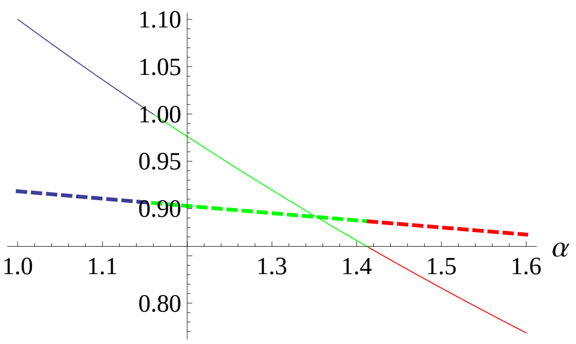

FIG. 4: The solid line is $E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC})$ and the dashed line is $E^\alpha(|\psi\rangle)$. The blue, green and red lines are for $0 \leq \alpha \leq 1.15959$, $\alpha \in (1.15959, \sqrt{2})$ and $\alpha \geq \sqrt{2}$, respectively.

From Theorem 3 and that $E(\rho_{AB}) \leq E_a(\rho_{AB})$ for any quantum states, we have the follow result:

Corollary 3 *For any n-qubit quantum state $\rho_{AB_1B_2...B_{n-1}}$, if there are at least two substates such that $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$ for $i_1 \neq i_2$, $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\beta_0 \in (0, \sqrt{2}]$, so that*

$$E^\beta(\rho_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} E_a^\beta(\rho_{AB_i}), \qquad (14)$$

for $0 \leq \beta \leq \beta_0$, where β_0 is a real number which satisfies $\sum_{i=1}^{n-1} E^{\beta_0}(\rho_{AB_i}) = 1$.

Since $E(|\varphi\rangle_{AB_1...B_{n-1}}) = E_a(|\varphi\rangle_{AB_1...B_{n-1}})$ for any pure state $|\varphi\rangle_{AB_1...B_{n-1}}$, for pure states (14) becomes

$$E_a^\beta(|\varphi\rangle_{A|B_1B_2...B_{n-1}}) \leq \sum_{i=1}^{n-1} E_a^\beta(\rho_{AB_i}), \qquad (15)$$

for $0 \leq \beta \leq \beta_0$, where β_0 is a real number which satisfies $\sum_{i=1}^{n-1} E^{\beta_0}(\rho_{AB_i}) = 1$. If $\beta_0 > 1$, we have

$$\begin{aligned} E_a^\beta(|\varphi\rangle_{A|B_1B_2...B_{n-1}}) &\leq \sum_{i=1}^{n-1} E_a^\beta(\rho_{AB_i}) \\ &\leq \sum_{i=1}^{n-1} \beta^i E_a^\beta(\rho_{AB_i}), \end{aligned}$$

for $\beta \in [1, \beta_0]$. (15) also gives the polygamy inequalities when $1 < \beta \leq 1.15959$, see the example 3, while (12) fails in this case.

Similarly, for any n-qubit quantum state $\rho_{AB_1B_2...B_{n-1}}$, if there are at least two substates such that $C(\rho_{AB_{i_1}})C(\rho_{AB_{i_2}}) \neq 0$ for $i_1 \neq i_2$, $i_1, i_2 \in \{1, ..., n-1\}$, there must be a real number $\alpha_1 \in [\alpha_0, \sqrt{2}]$, so that

$$E^{\alpha_1}(\rho_{A|B_1B_2...B_{n-1}}) = \sum_{i=1}^{n-1} E^{\alpha_1}(\rho_{AB_i}). \qquad (16)$$

For Example 3, Fig. 4 shows that the solid line and dashed line have only one intersection at $\alpha_1 = 1.35244$. The relations between $E^\alpha(|\psi\rangle)$ and $E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC})$ fall into two classes: $E^\alpha(|\psi\rangle_{A|BC}) \leq E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC})$ for $0 \leq \alpha \leq 1.35244$, and $E^\alpha(|\psi\rangle) \geq E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC})$ for $\alpha \geq 1.35244$.

IV. CONCLUSION

Like entanglement monogamy, entanglement polygamy is a fundamental property of multipartite quantum states. It characterizes the entanglement distribution in multipartite quantum systems. We have investigated the polygamy relations related to the concurrence C and the entanglement of formation E for general n-qubit states. We have extended the results (4) and (11) in Ref. [14] from $\alpha \leq 0$ to $\alpha \leq \alpha_0$, where $0 < \alpha_0 \leq 2$ for C, and $0 < \alpha_0 \leq \sqrt{2}$ for E. When $\alpha_0 > 2$ ($\alpha_0 > \sqrt{2}$), the polygamy relation of concurrence $C\ (E)$ can not be obtained. It remains an open question if for this case, like Example 3, there is only one intersection α_1.

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