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by

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Originated from the superposition principle in quantum mechanics, coherence has been extensively studied as a kind important resource in quantum information processing. We investigate the distinguishability of coherence-breaking channels with the help of quantum entanglement. By explicitly computing the minimal error probability of channel discrimination, it is shown that entanglement can enhance the capacity of channel distinguishability.

Introduction Quantum coherence is a fundamental aspect of quantum physics, which encapsulates the defining features of the theory from the superposition principle to quantum correlations [1]. Quantum coherence is also an essential ingredient in quantum information processing [2]. It constitutes a powerful resource for quantum metrology [3, 4] and entanglement creation [5, 6], and plays a central role in the emergent fields such as nanoscale thermodynamics [7–10] and quantum biology [11, 12]. The quantification and implication of quantum coherence have been extensively studied recently [13–15]. A related important problem is the transformation of quantum coherence under quantum channels. It is of particular significance to study the class of coherence-breaking channelstrace preserving completely positive maps for which the output state is always incoherence [21]. More precisely, a quantum channel Φ is called coherence breaking if $\Phi(\rho)$ is always incoherent for any density matrix ρ. And one of the key problems in quantum information-processing task is the channel discrimination. In [18, 19] the optimal discrimination of quantum operations has been investigated. It has been shown that the distinguishability of entanglement-breaking channels can be enhanced. Comparing coherence with entanglement, one may naturally ask whether the entanglement can enhance the distinguishability of the coherence-breaking channel. In this paper, we study the relation between entanglement and coherence-breaking channels, and show that entanglement can indeed enhance the distinguishability of coherence-breaking channels.

Preliminaries We first recapitulate some concepts required in presenting our main results. Let $\mathcal{H}$ denote a discrete finite-dimensional complex vector space associated with a quantum system. A completely positive and trace-preserving (CPTP) map Φ on a state ρ can be expressed as $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$, where K_n are Kraus operators on $\mathcal{H}$ satisfying $\sum_n K_n^\dagger K_n = I$ [20]. A qubit coherence-breaking channel is a CPTP map for which the Kraus operators K_n can be

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only one of the following three types [22]:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

$$E_1 = \begin{pmatrix} 0 & 0 \\ 0 & e^{i\xi} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ e^{i\xi} & 0 \end{pmatrix}, \tag{2}$$

or

$$E_1 = \begin{pmatrix} 0 & 0 \\ -\sin\phi & e^{i\xi}\cos\phi \end{pmatrix}, \quad E_2 = \begin{pmatrix} \cos\phi & e^{i\xi}\sin\phi \\ 0 & 0 \end{pmatrix}. \tag{3}$$

The problem of optimal discrimination of two quantum channels Φ_1 and Φ_2, given with a prior probability p_1 and $p_2 = 1 - p_1$, respectively, can be reformulated into the problem of finding out a state ρ such that the error probability in the discrimination of the output states $\Phi_1(\rho)$ and $\Phi_2(\rho)$ is minimal [18]. Without taking into account of entanglement, the minimal error probability is given by

$$p'_E = \frac{1}{2}(1 - \max_{\rho\in\mathcal{H}} \|p_1\Phi_1(\rho) - p_2\Phi_2(\rho)\|_1), \tag{4}$$

where $\|\cdot\|_1$ denotes the trace norm. However, if the system is entangled with another system $\mathcal{K}$, the minimal error probability is changed to be

$$p_E = \frac{1}{2}(1 - \max_{\rho\in\mathcal{H}\otimes\mathcal{K}} \|p_1(\Phi_1\otimes\mathcal{I})(\rho) - p_2(\Phi_2\otimes\mathcal{I})(\rho)\|_1). \tag{5}$$

Here the maximum in (4) and (5) are both achieved by pure states.

Denote $|A\rangle\rangle = \sum_{mn}\langle n|A|m\rangle|n\rangle\otimes|m\rangle = A\otimes I|I\rangle\rangle = I\otimes A^t|I\rangle\rangle$ [18]. p_E can be rewritten as,

$$p_E = \frac{1}{2}(1 - \max_{\mathrm{Tr}[\zeta^\dagger\zeta]} \|I\otimes\zeta^\dagger \triangle I\otimes\zeta^*\|_1), \tag{6}$$

where $\triangle = p_1\sum_n |E_n^{(1)}\rangle\rangle\langle\langle K_n^{(1)}| - p_2\sum_m |K_m^{(2)}\rangle\rangle\langle\langle K_m^{(2)}|$. p_E can also be written in the form,

$$p_E = \frac{1}{2}(1 - \max_{P\geqslant 0,\ \mathrm{Tr}[P^2]=1} \|I\otimes P \triangle I\otimes P\|_1), \tag{7}$$

where for qubit channels, P can be written as $P = \begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ with $x, y \geqslant 0$, $xy \geqslant |z|^2$ and $x^2 + y^2 + 2|z|^2 = 1$. Particularly, one has $x + y = 1$ and $|z| = \sqrt{xy}$ if $\mathrm{rank}(P) = 1$. The rank of P that achieves the maximum gives directly information about the usefulness of entanglement. The entanglement is not needed for optimal discrimination if and only if the maximum in (6) can be achieved by a rank-1 operator P.

Coherence-breaking channels of the same type We first consider qubit coherence-breaking channels of the the same types given in (1), (2) and (3). Obviously the discrimination for two channels of the form either (1) or (2) is trivial. We only need to study the problem for two coherence-breaking channels that both are of the form (3). Let Φ_i, $i = 1, 2$, be two coherence-breaking channels with, respectively, $E_1^{(i)} = \begin{pmatrix} 0 & 0 \\ -\sin\phi_i & e^{i\xi_i}\cos\phi_i \end{pmatrix}$, $E_2^{(i)} = \begin{pmatrix} \cos\phi_i & e^{i\xi_i}\sin\phi_i \\ 0 & 0 \end{pmatrix}$ for $i = 1, 2$. Then,

$$\triangle = p_1(|E_1^{(1)}\rangle\rangle\langle\langle E_1^{(1)}| + |E_2^{(1)}\rangle\rangle\langle\langle E_2^{(1)}|) - p_2(|E_1^{(2)}\rangle\rangle\langle\langle E_1^{(2)}| + |E_2^{(2)}\rangle\rangle\langle\langle E_2^{(2)}|) = \begin{pmatrix} \triangle_{11} & \triangle_{12} & 0 & 0 \\ \triangle_{12}^* & \triangle_{22} & 0 & 0 \\ 0 & 0 & \triangle_{22} & -\triangle_{12} \\ 0 & 0 & -\triangle_{12}^* & \triangle_{11} \end{pmatrix}, \quad (8)$$

where $\triangle_{11} = p_1\cos^2\phi_1 - p_2\cos^2\phi_2$, $\triangle_{12} = p_1 e^{-i\xi_1}\sin\phi_1\cos\phi_1 - p_2 e^{-i\xi_2}\sin\phi_2\cos\phi_2$ and $\triangle_{22} = p_1\sin^2\phi_1 - p_2\sin^2\phi_2$. The singular values of $\triangle$ are:

$$s_0(\triangle) = s_1(\triangle) = \frac{1}{2}\left| p_1 - p_2 + \sqrt{p_1^2 + p_2^2 - 2p_1p_2\cos 2\phi_1\cos 2\phi_2 - p_1p_2(e^{i\xi_1 - i\xi_2} + e^{i\xi_2 - i\xi_1})\sin 2\phi_1\sin 2\phi_2} \right|,$$

$$s_2(\triangle) = s_3(\triangle) = \frac{1}{2}\left| p_1 - p_2 - \sqrt{p_1^2 + p_2^2 - 2p_1p_2\cos 2\phi_1\cos 2\phi_2 - p_1p_2(e^{i\xi_1 - i\xi_2} + e^{i\xi_2 - i\xi_1})\sin 2\phi_1\sin 2\phi_2} \right|.$$

Thus,

$$\sum_{i=0}^{3} s_i(\triangle) = 2\max\left\{ |p_1 - p_2|, \sqrt{p_1^2 + p_2^2 - 2p_1p_2\cos 2\phi_1\cos 2\phi_2 - p_1p_2(e^{i\xi_1 - i\xi_2} + e^{i\xi_2 - i\xi_1})\sin 2\phi_1\sin 2\phi_2} \right\}. \quad (9)$$

Note that for a matrix A, $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A} = \max_U |\mathrm{Tr}[UA]| = \sum_i s_i(A)$, where U is a unitary operator and $\{s_i(A)\}$ denote the singular values of A. For the strategy without entangled input, we get $p_E' = \frac{1}{2}(1 - \max_{P'} \|I \otimes P' \triangle I \otimes P'\|)$ with $P' = \begin{pmatrix} x & \sqrt{x(1-x)}e^{i\phi} \\ \sqrt{x(1-x)}e^{-i\phi} & 1-x \end{pmatrix}$, $0 \leqslant x \leqslant 1$ and $0 \leqslant \phi \leqslant 2\pi$. The singular values of $\triangle' = I \otimes P' \triangle I \otimes P'$ are given by

$$s_0(\triangle') = \left| (p_1 - p_2) - ((p_1\cos^2\phi_1 - p_2\cos^2\phi_2)x + (p_1\sin^2\phi_1 - p_2\sin^2\phi_2)(1-x) + (re^{i\phi} + r^* e^{-i\phi})\sqrt{x(1-x)}) \right|,$$

$$s_1(\triangle') = \left| (p_1\cos_1^{\phi} - p_2\cos^2\phi_2)x + (p_1\sin^2\phi_1 - p_2\sin^2\phi_2)(1-x) + (re^{i\phi} + r^* e^{-i\phi})\sqrt{x(1-x)} \right|,$$

and $s_2(\triangle') = s_3(\triangle') = 0$ with $r = p_1 e^{i\xi_1}\sin\phi_1\cos\phi_1 - p_2\sin\phi_2\cos\phi_2$. Therefore,

$$\sum_{i=0}^{3} s_i(\triangle') = \max\left\{ |p_1 - p_2|, |p_1\cos 2\phi_1 - p_2\cos 2\phi_2|, \sqrt{(re^{i\phi} + r^* e^{-i\phi})^2 + (p_1\cos 2\phi_1 - p_2\cos 2\phi_2)^2|} \right\}. \quad (10)$$

Although the necessary and sufficient condition for $\sum_{i=0}^{3} s_i(\triangle) > \sum_{i=0}^{3} s_i(\triangle')$ can be difficult to calculate explicitly in general, we can find that $p_E = \frac{1}{2}(1 - \frac{1}{2}\sum_{i=0}^{3} s_i(\triangle)) < p_E' = \frac{1}{2}(1 - \frac{1}{2}\sum_{i=0}^{3} s_i(\triangle'))$ for $\sqrt{(re^{i\phi} + r^* e^{-i\phi})^2 + (p_1\cos 2\phi_1 - p_2\cos 2\phi_2)^2} < 2|p_1 - p_2|$ and $0 < p_1 < \frac{1}{4}$ or $\frac{3}{4} < p_1 < 1$. Hence it is clear that the entanglement can indeed enhance the distinguishability of the coherence-breaking channels (3).

Example. Consider the case that the prior probability $p_1 = \frac{7}{8}$ and $p_2 = \frac{1}{8}$, and Φ_1 and Φ_2 satisfy conditions $\sin 2\phi_1 = \frac{7}{8}$, $\cos 2\phi_1 = \frac{\sqrt{63}}{8}$, $\sin 2\phi_2 = \frac{7}{8}$ and $\cos 2\phi_2 = \frac{\sqrt{15}}{8}$. We have, $\sum_{i=0}^{3} s_i(\triangle') = \frac{3}{4}$ and $\sum_{i=0}^{3} s_i(\triangle) = \frac{3}{2}$. Then, $p_E = \frac{1}{8} < p_E' = \frac{5}{16}$.

Coherence-breaking channels of different types We study now coherence-breaking channels with different forms. We first assume Φ_1 and Φ_2 are defined by (1) and (2), respectively. In this case, it can be found that

$$\triangle = p_1(|E_1^{(1)}\rangle\rangle\langle\langle E_1^{(1)}| + |E_2^{(1)}\rangle\rangle\langle\langle E_2^{(1)}|) - p_2(|E_1^{(2)}\rangle\rangle\langle\langle E_1^{(2)}| + |E_2^{(2)}\rangle\rangle\langle\langle E_2^{(2)}|) = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & -p_2 & 0 \\ 0 & 0 & 0 & -p_2 \end{pmatrix}. \tag{11}$$

The singular values of $\triangle$ are: $s_0(\triangle) = s_1(\triangle) = p_1,\ s_2(\triangle) = s_3(\triangle) = p_2$. Thus, $p_E = \frac{1}{2}(1 - \frac{1}{2}\sum_{i=0}^{3} s_i(\triangle)) = 0$. That is to say, the entanglement can help to distinguish these two channels perfectly.

For the strategy without entanglement, we get $p'_E = \frac{1}{2}(1 - \max_{P'} \|I \otimes P' \triangle I \otimes P'\|)$, with P' defined as above. The singular values of $\triangle' = I \otimes P' \triangle I \otimes P'$ are given by: $s_0(\triangle') = p_1$, $s_1(\triangle') = p_2$ and $s_2(\triangle') = s_3(\triangle') = 0$. Hence we have $p'_E = \frac{1}{2}(1 - \frac{1}{2}\sum_{i=0}^{3} s_i(\triangle')) = \frac{1}{4}$, which shows that the entanglement indeed improves the channel distinguishability.

Now consider channels Φ_1 and Φ_2 defined by (1) and (3), respectively. We have

$$\begin{aligned}\triangle &= p_1(|E_1^{(1)}\rangle\rangle\langle\langle E_1^{(1)}| + |E_2^{(1)}\rangle\rangle\langle\langle E_2^{(1)}|) - p_2(|E_1^{(2)}\rangle\rangle\langle\langle E_1^{(2)}| + |E_2^{(2)}\rangle\rangle\langle\langle E_2^{(2)}|) \\ &= \begin{pmatrix} p_1 - p_2\cos^2\phi_2 & -p_2 e^{-i\xi}\sin\phi_2\cos\phi_2 & 0 & 0 \\ -p_2 e^{i\xi}\sin\phi_2\cos\phi_2 & p_1 - p_2\sin^2\phi_2 & 0 & 0 \\ 0 & 0 & -p_2\sin^2\phi_2 & p_2 e^{-i\xi}\sin\phi_2\cos\phi_2 \\ 0 & 0 & p_2 e^{i\xi}\sin\phi_2\cos\phi_2 & -p_2\cos^2\phi_2 \end{pmatrix}.\end{aligned} \tag{12}$$

The singular values of $\triangle$ are: $s_0(\triangle) = p_1$, $s_1(\triangle) = p_2$, $s_2(\triangle) = |p_1 - p_2|$ and $s_3(\triangle) = 0$. Thus, $\sum_{i=0}^{3} s_i(\triangle) = 1 + |p_1 - p_2|$. While for the case without entanglement, the singular values of $\triangle' = I \otimes P' \triangle I \otimes P'$ are given by $s_0(\triangle') = |(p_1 - p_2\cos^2\phi_2)x + (p_1 - p_2\sin^2\phi_2)(1-x) - (e^{i\xi+i\phi} + e^{-i\xi-i\phi})p_2\sin\phi_2\cos\phi_2\sqrt{x(1-x)}|$, $s_1(\triangle') = |(e^{i\xi+i\phi} + e^{-i\xi-i\phi})p_2\sin\phi_2\cos\phi_2\sqrt{x(1-x)} - p_2 x\sin^2\phi_2 - p_2(1-x)\cos^2\phi_2|$ and $s_2(\triangle') = s_3(\triangle') = 0$. Thus

$$\begin{aligned}\sum_{i=0}^{3} s_i(\triangle') &= \max\left\{2\,|p_1 - p_2|\,,\ \left|p_2\frac{\cos(\xi+\phi)\sin 2\phi_2|\cos(\xi+\phi)\sin 2\phi_2| \pm \cos 2\phi_2|\cos 2\phi_2|}{\sqrt{\cos^2(\xi+\phi)\sin^2 2\phi_2 + \cos^2 2\phi_2}} - 2p_1 + p_2\right|\right\} \\ &= \max\left\{2|p_1 - p_2|,\ \left|p_2\sqrt{\cos^2(\xi+\phi)\sin^2 2\phi_2 + \cos^2 2\phi_2} - 2p_1 + p_2\right|\right\} \\ &= 2|p_1 - p_2| < \sum_{i=0}^{3} s_i(\Delta).\end{aligned}$$

Namely, the entanglement enhances again the distinguishability of the coherence-breaking channels.

One may also consider channels Φ_1 and Φ_2 given by (2) and (3) respectively, and obtain similar conclusions: entanglement can always enhance the ability of discrimination of two channels of this kind.

Conclusion and discussion We have investigated the discrimination of two coherence-breaking channels based on error probabilities. It has been shown that entanglement can improve to discriminate the coherence-breaking channels. However, for other channels, this is not always true. Let us consider two Pauli channels $\Phi_i(\rho) = \sum_{\alpha=0}^{3} q_i^{\alpha}\sigma_\alpha\rho\sigma_\alpha$ with a prior probability p_1 and $p_2 = 1 - p_1$, where $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} = \{I, \sigma_x, \sigma_y, \sigma_z\}$ and $\sum_{\alpha=0}^{3} q_i^{\alpha} = 1$. In [18], it has been shown that entanglement strictly improves the discrimination if and only if $\prod_{i=0}^{3} r_\alpha < 0$, with $r_\alpha = p_1 q_1^\alpha - p_2 q_2^\alpha$. Assume Φ_1 and Φ_2 are both coherence-breaking Pauli channels. We have $q_i^{(0)} + q_i^{(1)} - q_2^{(2)} - q_3^{(3)} = q_i^{(0)} - q_i^{(1)} + q_2^{(2)} - q_3^{(3)} = 0$, $i = 1, 2$. Thus, $r_0 = r_3 = p_1 q_1 - p_2 q_2$ and $r_1 = r_2 = \frac{1}{2}(p_1 - p_2) - (p_1 q_1 - p_2 q_2)$. Hence, $\prod_{i=0}^{3} r_\alpha \geqslant 0$, which implies that the entanglement cannot improve the channel discrimination in this case.

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