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by

Jing-Yun Zhao, Hui Zhao, Naihuan Jing, and Shao-Ming Fei

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Detection of Genuine Multipartite Entanglement in Multipartite Systems

Jing Yun Zhao,[1] Hui Zhao,[1] Naihuan Jing,[2,3] and Shao-Ming Fei[4,5]

[1] *College of Applied Sciences, Beijing University of Technology, Beijing 100124, China*

[2] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA*

[3] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

[4] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

[5] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

We investigate genuine multipartite entanglement in general multipartite systems. Based on the norms of the correlation tensors of a multipartite state under various partitions, we present an analytical sufficient criterion for detecting the genuine four-partite entanglement. We show that our criterion can detect genuine entanglement by detailed example. The results are generalized to arbitrary multipartite systems.

Keywords: genuine multipartite entanglement, correlation tensor

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1. Introduction

Quantum entanglement is one of the most fascinating features in quantum physics, with numerous applications in quantum information processing, secure communication and channel protocols [1,2,3]. In particular, the genuine multipartite entanglement appears to have more significant advantages than the bipartite ones in these quantum tasks [4].

The notion of genuine multipartite entanglement (GME) was introduced in [5]. Let H_i^d, $i = 1, 2, ..., n$, denote d-dimensional Hilbert spaces. An n-partite state $\rho \in H_1^d \otimes ... \otimes H_n^d$ can be expressed as $\rho = \sum p_\alpha \left|\psi_\alpha\right\rangle \left\langle\psi_\alpha\right|$, where $0 < p_\alpha \leq 1$, $\sum p_\alpha = 1$, $\left|\psi_\alpha\right\rangle \in H_1^d \otimes ... \otimes H_n^d$ are normalized pure states. ρ is said to be fully separable if it can be written as $\rho = \sum_i q_i \ \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^n$, where q_i is a probability distribution and ρ_i^j are density matrices with respect to the subsystem H_j. On the other hand, ρ is called genuine n-partite entangled if $|\psi_\alpha\rangle$ are not separable under any bipartite partitions.

The genuine multipartite entangled states exist in physical systems like the ground state of the XY model [6]. However, it is extremely difficult to identify the GME for general mixed multipartite states. The GME concurrence and its lower bound were studied in [7-9]. Some sufficient or necessary conditions of GEM were presented in [10-12]. As for detection of GME, the common criterion is the entanglement witnesses [13-16]. Using correlation tensors, the authors in [17] have provided a general framework to detect different classes of GME for quantum systems of arbitrary dimensions. In [18] the genuine multipartite entanglement has been investigated in terms of the norms of the correlation tensors and multipartite concurrence. The relations between the norms of the correlation vectors and the detection of GME in tripartite quantum systems have been established in [19].

In this paper, we analyze the relationship between the norms of the correlation tensors and various bipartitions of multipartite quantum systems, and present sufficient conditions of GME for four partite and multipartite quantum systems.

We generalize some inequalities of the norms of the correlation tensors for four-partite states and give a criterion to detect GME of four-partite quantum systems in Section 2. In Section 3, we generalize these concepts and conclusions to multipartite quantum systems. Comments and conclusions are given in Section 4.

2. Detection of GME for Four-partite Quantum States

We first consider the GME for four-partite qudit states $\rho \in H_1^d \otimes ... \otimes H_4^d$. Let λ_i, $i = 1, \cdots, d^2-1$, denote the mutually orthogonal generators of the special unitary Lie algebra $\mathfrak{su}(d)$ under a fixed bilinear form [20], and I the $d \times d$ identity matrix. Then ρ can be expanded in terms of λ_is,

$$\rho = \frac{1}{d^4} I\otimes I\otimes I\otimes I + \frac{1}{2d^3}\sum_{f=1}^{4}\sum_{i_1=1}^{d^2-1} t_{i_1}^{(f)}\lambda_{i_1}^{(f)}\otimes I\otimes I\otimes I+\cdots$$
$$+\frac{1}{16}\sum_{i_1,i_2,i_3,i_4=1}^{d^2-1} t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)}\lambda_{i_1}^{(1)}\otimes\lambda_{i_2}^{(2)}\otimes\lambda_{i_3}^{(3)}\otimes\lambda_{i_4}^{(4)}, \quad (1)$$

where $\lambda_{i_1}^{(f)}$((f) represents the position of λ_{i_1} in the tensor product) stand for the operators with λ_{i_1} on H_f and I on the rest spaces, $t_{i_1}^{(f)} = tr(\rho\,\lambda_{i_1}^{(f)}\otimes I\otimes I\otimes I)$, $\cdots$, $t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)} = tr(\rho\,\lambda_{i_1}^{(1)}\otimes\lambda_{i_2}^{(2)}\otimes\lambda_{i_3}^{(3)}\otimes\lambda_{i_4}^{(4)})$.

Let $T^{(f)},\cdots,T^{(1,2,3,4)}$ denote vectors with entries $t_{i_1}^{(f)},\cdots,t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)}(i_1,i_2,i_3,i_4=1,\cdots,d^2-1; f=1,2,3,4)$, respectively. From $T^{(f)},\cdots,T^{(1,2,3,4)}$ we further define the following matrices under different partitions.

We denote $T_{f|ghl}$ the $(d^2-1)\times(d^2-1)^3$ matrices with entries $t_{i_f,(d^2-1)^2(i_g-1)+(d^2-1)(i_h-1)+i_l} = t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)}$, $T_{fg|hl}$ the $(d^2-1)^2\times(d^2-1)^2$ matrices with entries $t_{(d^2-1)(i_f-1)+i_g,(d^2-1)(i_h-1)+i_l} = t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)}$ and $T_{fgh|l}$ the $(d^2-1)^3\times(d^2-1)$ matrices with entries $t_{(d^2-1)^2(i_f-1)+(d^2-1)(i_g-1)+i_h,i_l} = t_{i_1,i_2,i_3,i_4}^{(1,2,3,4)}$, where $f\neq g\neq h\neq l=1,2,3,4; i_f,i_g,i_h,i_l=1,\cdots,d^2-1$.

Let $T^{(\underline{f},g)}$ and $T^{(f,\underline{g})}$ be $(d^2-1)\times(d^2-1)$ matrices with entries $t_{i_1,i_2}=t_{i_1,i_2}^{(f,g)}$ and $t_{i_2,i_1}=t_{i_1,i_2}^{(f,g)}$, respectively. We denote $T^{(\underline{f},g,h)}$, $T^{(f,\underline{g},h)}$ and $T^{(f,g,\underline{h})}$ the $(d^2-1)\times(d^2-1)^2$ matrices with entries given by $t_{i_1,(d^2-1)(i_2-1)+i_3}=t_{i_1,i_2,i_3}^{(f,g,h)}$, $t_{i_2,(d^2-1)(i_1-1)+i_3}=t_{i_1,i_2,i_3}^{(f,g,h)}$ and $t_{i_3,(d^2-1)(i_1-1)+i_2}=t_{i_1,i_2,i_3}^{(f,g,h)}$, respectively. We denote $T^{(\underline{f,g},h)}$, $T^{(\underline{f},g,\underline{h})}$ and $T^{(f,\underline{g,h})}$ the $(d^2-1)^2\times(d^2-1)$ matrices with entries given by $t_{(d^2-1)(i_1-1)+i_2,i_3}=t_{i_1,i_2,i_3}^{(f,g,h)}$, $t_{(d^2-1)(i_1-1)+i_3,i_2}=t_{i_1,i_2,i_3}^{(f,g,h)}$ and $t_{(d^2-1)(i_2-1)+i_3,i_1}=t_{i_1,i_2,i_3}^{(f,g,h)}$, respectively.

In the following we denote $\parallel M\parallel=\sqrt{\sum_{i,j}M_{ij}^2}=\sqrt{\sum_i\sigma_i^2}$ the Frobenius norm of a matrix M, and $\parallel M\parallel_k=\sum_i^k\sigma_i$ the kth Ky Fan norm of matrix M, where σ_i, $i=1,\cdots,min(m,n)$, are the singular values of the matrix M arranged in descending order.

For any pure state $\rho\in H_1^d\otimes H_2^d\otimes H_3^d$, $\rho=\frac{1}{d^3}I\otimes I\otimes I+\frac{1}{2d^2}(\sum_{i_1}^{d^2-1}t_{i_1}^{(1)}\lambda_{i_1}^{(1)}\otimes I\otimes I+\sum_{i_2}^{d^2-1}t_{i_2}^{(2)}I\otimes\lambda_{i_2}^{(2)}\otimes I+\sum_{i_3}^{d^2-1}t_{i_3}^{(3)}I\otimes I\otimes\lambda_{i_3}^{(3)})+\frac{1}{4d}(\sum_{i_1,i_2}^{d^2-1}t_{i_1,i_2}^{(1,2)}\lambda_{i_1}^{(1)}\otimes\lambda_{i_2}^{(2)}\otimes I+\sum_{i_2,i_3}^{d^2-1}t_{i_2,i_3}^{(2,3)}I\otimes\lambda_{i_2}^{(2)}\otimes\lambda_{i_3}^{(3)}+\sum_{i_1,i_3}^{d^2-1}t_{i_1,i_3}^{(1,3)}\lambda_{i_1}^{(1)}\otimes I\otimes\lambda_{i_3}^{(3)})+\frac{1}{8}\sum_{i_1,i_2,i_3}^{d^2-1}t_{i_1,i_2,i_3}^{(1,2,3)}\lambda_{i_1}^{(1)}\otimes\lambda_{i_2}^{(2)}\otimes\lambda_{i_3}^{(3)}$, we have $tr(\rho^2)=\frac{1}{d^3}+\frac{1}{2d^2}[\sum(t_{i_1}^{(1)})^2+\sum(t_{i_2}^{(2)})^2+\sum(t_{i_3}^{(3)})^2]+\frac{1}{4d}[\sum(t_{i_1,i_2}^{(1,2)})^2+\sum(t_{i_1,i_3}^{(1,3)})^2+\sum(t_{i_2,i_3}^{(2,3)})^2]+\frac{1}{8}\sum(t_{i_1,i_2,i_3}^{(1,2,3)})^2=1$.

Therefore

$$\begin{aligned}\sum(t^{(1,2,3)}_{i_1,i_2,i_3})^2 &= \frac{8(d^3-1)}{d^3}-\{\frac{4}{d^2}[\sum(t^{(1)}_{i_1})^2+\sum(t^{(2)}_{i_2})^2+\sum(t^{(3)}_{i_3})^2]+\\ &\frac{2}{d}[\sum(t^{(1,2)}_{i_1,i_2})^2+\sum(t^{(1,3)}_{i_1,i_3})^2+\sum(t^{(2,3)}_{i_2,i_3})^2]\}\\ &\leq \frac{8(d^3-1)}{d^3}.\end{aligned}$$

Thus, $\parallel T^{(1,2,3)} \parallel = \sqrt{\sum(t^{(1,2,3)}_{i_1,i_2,i_3})^2} \leq \frac{2}{d}\sqrt{\frac{2(d^3-1)}{d}}$. Concerning the relations between the correlation tensors and the separability under various partitions, we have the following results:

Lemma 1. *Let $\rho \in H_1^d \otimes H_2^d \otimes H_3^d \otimes H_4^d$ be a pure state. If ρ is fully separable, then for any $k = 1, \cdots, d^2-1$,*

$$\parallel T_{1|2|3|4} \parallel_k = \frac{4(d-1)^2}{d^2}. \tag{2}$$

Proof. Since ρ is fully separable, $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4$, where $\rho_1, \rho_2, \rho_3, \rho_4$ are the reduced density matrices of ρ. By the inequality for 1-body correlation tensors, $\parallel T^{(f)} \parallel \leq \sqrt{\frac{2(d-1)}{d}}$ [17], $f = 1, 2, 3, 4$, with the equality holding iff the state is pure, we have

$$\begin{aligned}\parallel T_{1|2|3|4} \parallel_k &= \parallel T^{(1)}(T^{(2)} \otimes T^{(3)} \otimes T^{(4)})^t \parallel_k = \parallel T^{(1)} \parallel \cdot \parallel (T^{(2)} \otimes T^{(3)} \otimes T^{(4)})^t \parallel_k\\ &= \parallel T^{(1)} \parallel \cdot \parallel (T^{(2)} \otimes T^{(3)} \otimes T^{(4)})^t \parallel = \parallel T^{(1)} \parallel \cdot \parallel T^{(2)} \otimes T^{(3)} \otimes T^{(4)} \parallel\\ &= \parallel T^{(1)} \parallel \cdot \parallel T^{(2)} \parallel \cdot \parallel T^{(3)} \parallel \cdot \parallel T^{(4)} \parallel = \frac{4(d-1)^2}{d^2},\end{aligned} \tag{3}$$

which proves the Theorem. □

Lemma 2. *Let $\rho \in H_1^d \otimes H_2^d \otimes H_3^d \otimes H_4^d$ be a pure state such that ρ is separable under at least one bipartition. Then for any $k = 1, \cdots, d^2-1$, and $f \neq g \neq h \neq l \in \{1, 2, 3, 4\}$, we have*

(i) if ρ is separable under bipartition $f|ghl$, then

$$\parallel T_{f|ghl} \parallel_k \leq \frac{4(d-1)\sqrt{d^2+d+1}}{d^2}; \tag{4}$$

(ii) if ρ is entangled under bipartition $f|ghl$, then

$$\parallel T_{f|ghl} \parallel_k \leq \frac{4\sqrt{k}(d^2-1)}{d^2}. \tag{5}$$

Proof. (i) If ρ is separable under bipartition $f|ghl$, $\rho = \rho_f \otimes \rho_{ghl}$, it follows from $\parallel T^{(f,g,h)} \parallel \leq \frac{2}{d}\sqrt{\frac{2(d^3-1)}{d}}$ that

$$\begin{aligned} \parallel T_{f|ghl} \parallel_k \quad & = \parallel T^{(f)} \cdot (T^{(g,h,l)})^t \parallel_k = \parallel T^{(f)} \parallel \cdot \parallel (T^{(g,h,l)})^t \parallel_k \\ & = \parallel T^{(f)} \parallel \cdot \parallel (T^{(g,h,l)})^t \parallel = \parallel T^{(f)} \parallel \cdot \parallel T^{(g,h,l)} \parallel \\ & \leq \frac{4(d-1)\sqrt{d^2+d+1}}{d^2}. \end{aligned} \tag{6}$$

(ii) ρ is entangled under bipartition $f|ghl$, without loss of generality, say, under the bipartition $1|234$. If ρ is separable under some bipartition of one subsystem vs the rest three subsystems, we have

$$\parallel T_{f|ghl} \parallel_k \leq \frac{4(d-1)\sqrt{d^2+d+1}}{d^2}. \tag{7}$$

If ρ is separable under some bipartition of two subsystems vs the rest two subsystems, from the inequality of 2-body correlation tensors $\parallel T^{(f,g)} \parallel \leq \sqrt{\frac{4(d^2-1)}{d^2}}$ [17], we have

$$\begin{aligned} \parallel T_{f|ghl} \parallel_k \quad & = \parallel T^{(\underline{f},g)} \otimes (T^{(h,l)})^t \parallel_k = \parallel T^{(\underline{f},g)} \parallel_k \cdot \parallel (T^{(h,l)})^t \parallel_k \\ & \leq \sqrt{k} \parallel T^{(\underline{f},g)} \parallel \cdot \parallel T^{(h,l)} \parallel \leq \frac{4\sqrt{k}(d^2-1)}{d^2}, \end{aligned} \tag{8}$$

where we have used the inequality $\parallel M \parallel_k \leq k \parallel M \parallel$ for any matrix M. If ρ is separable under some bipartition of three subsystems vs the rest one subsystem, we have

$$\begin{aligned} \parallel T_{f|ghl} \parallel_k \quad & = \parallel T^{(\underline{f},g,h)} \otimes (T^{(l)})^t \parallel_k = \parallel T^{(\underline{f},g,h)} \parallel_k \cdot \parallel (T^{(l)})^t \parallel_k \\ & \leq \sqrt{k} \parallel T^{(\underline{f},g,h)} \parallel \cdot \parallel T^{(l)} \parallel \leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}. \end{aligned} \tag{9}$$

Hence, if ρ is entangled under bipartition $1|234$, we have $\parallel T_{f|ghl} \parallel_k \leq max\{\frac{4(d-1)\sqrt{d^2+d+1}}{d^2}, \frac{4\sqrt{k}(d^2-1)}{d^2}, \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}\} = \frac{4\sqrt{k}(d^2-1)}{d^2}$. Similar discussion applies to other bipartitions $2|134$, $3|124$ and $4|123$. It indicates that these norms have the same upper bound. Hence, $\parallel T_{f|ghl} \parallel_k \leq \frac{4\sqrt{k}(d^2-1)}{d^2}$, if ρ is entangled under bipartition $f|ghl$. $\square$

We may analyze the bipartition $fgh|l$ by using similar methods above and obtain the following Lemma.

Lemma 3. *Let $\rho \in H_1^d \otimes H_2^d \otimes H_3^d \otimes H_4^d$ be a pure state such that ρ is separable under at least one bipartition. Then for any $k = 1, \cdots, d^2-1$, and $f \neq g \neq h \neq l \in \{1,2,3,4\}$, we have*
(i) if ρ is separable under bipartition $fgh|l$, then

$$\| T_{fgh|l} \|_k \leq \frac{4(d-1)\sqrt{d^2+d+1}}{d^2}; \tag{10}$$

(ii) if ρ is entangled under bipartition $fgh|l$, then

$$\| T_{fgh|l} \|_k \leq \frac{4\sqrt{k}(d^2-1)}{d^2}. \tag{11}$$

Now we consider the relations between the correlation tensors and the separability under the bipartition $fg|hl$.

Lemma 4. *Let $\rho \in H_1^d \otimes H_2^d \otimes H_3^d \otimes H_4^d$ be a pure state such that ρ is separable under at least one bipartition. Then for any $k = 1, \cdots, d^2-1$, and $f \neq g \neq h \neq l \in \{1,2,3,4\}$, we have*
(i) if ρ is separable under bipartition $fg|hl$, then

$$\| T_{fg|hl} \|_k \leq \frac{4(d^2-1)}{d^2}; \tag{12}$$

(ii) if ρ is entangled under bipartition $fg|hl$, then

$$\| T_{fg|hl} \|_k \leq \frac{4k(d^2-1)}{d^2}. \tag{13}$$

Proof. (i) If ρ is separable under bipartition $fg|hl$, $\rho = \rho_{fg} \otimes \rho_{hl}$, then

$$\begin{aligned} \| T_{fg|hl} \|_k &= \| T^{(f,g)} \cdot (T^{(h,l)})^t \|_k = \| T^{(f,g)} \| \cdot \| (T^{(h,l)})^t \|_k = \| T^{(f,g)} \| \cdot \| (T^{(h,l)})^t \| \\ &= \| T^{(f,g)} \| \cdot \| T^{(h,l)} \| \leq \frac{4(d^2-1)}{d^2}, \end{aligned} \tag{14}$$

by using the inequality for 2-body correlation tensors.

(ii) ρ is entangled under bipartition $fg|hl$, say, $12|34$. If ρ is separable under some bipartition of one subsystem vs the rest three subsystems, we have

$$\begin{aligned} \| T_{fg|hl} \|_k &= \| T^{(f)} \otimes T^{(\underline{g},h,l)} \|_k = \| T^{(f)} \| \cdot \| T^{(\underline{g},h,l)} \|_k \\ &\leq \sqrt{k} \| T^{(f)} \| \cdot \| T^{(\underline{g},h,l)} \| \leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}. \end{aligned} \tag{15}$$

If ρ is separable under some bipartition of two subsystems vs the rest two subsystems, we have

$$\parallel T_{fg|hl} \parallel_k \leq \frac{4(d^2-1)}{d^2}. \tag{16}$$

If ρ is separable under some bipartition of three subsystems vs the rest one subsystem, we have

$$\begin{aligned} \parallel T_{fg|hl} \parallel_k &= \parallel T^{(\underline{f,g},h)} \otimes (T^{(l)})^t \parallel_k = \parallel T^{(\underline{f,g},h)} \parallel_k \cdot \parallel (T^{(l)})^t \parallel_k \\ &\leq \sqrt{k} \parallel T^{(\underline{f,g},h)} \parallel \cdot \parallel T^{(l)} \parallel \leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}. \end{aligned} \tag{17}$$

Hence, if ρ is entangled under bipartition 12|34, we have $\parallel T_{fg|hl} \parallel_k \leq \max\{\frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}, \frac{4(d^2-1)}{d^2}\} = \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$, $k \geq 2$. If $k=1$, $\parallel T_{fg|hl} \parallel_1 \leq \frac{4(d^2-1)}{d^2}$.

Similarly, if ρ is entangled under bipartition 13|24, 14|23 23|14, 24|13 and 34|12, we have the upper bound of the norm as follows. Let *i vs j* denote that ρ is separable under some bipartition of i subsystem vs the rest j subsystems.

	1 *vs* 3	2 *vs* 2	3 *vs* 1
13|24	$\parallel T_{13\|24} \parallel_k$ $= \parallel T^{(f)} \otimes T^{(g,\underline{h},l)} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$	$\parallel T_{13\|24} \parallel_k$ $= \parallel T^{(\underline{f},g)} \otimes T^{(\underline{h},l)} \parallel_k$ $\leq \frac{4k(d^2-1)}{d^2}$	$\parallel T_{13\|24} \parallel_k$ $= \parallel T^{(\underline{f},g,\underline{h})} \otimes (T^{(l)})^t \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$
14|23	$\parallel T_{14\|23} \parallel_k$ $= \parallel T^{(f)} \otimes T^{(g,h,\underline{l})} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$	$\parallel T_{14\|23} \parallel_k$ $= \parallel T^{(\underline{f},g)} \otimes T^{(h,\underline{l})} \parallel_k$ $\leq \frac{4k(d^2-1)}{d^2}$	$\parallel T_{14\|23} \parallel_k$ $= \parallel T^{(\underline{f},g,h)} \otimes T^{(l)} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$
23|14	$\parallel T_{23\|14} \parallel_k$ $= \parallel T^{(f)^t} \otimes T^{(\underline{g,h},l)} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$	$\parallel T_{23\|14} \parallel_k$ $= \parallel T^{(f,\underline{g})} \otimes T^{(\underline{h},l)} \parallel_k$ $\leq \frac{4k(d^2-1)}{d^2}$	$\parallel T_{23\|14} \parallel_k$ $= \parallel T^{(f,\underline{g,h})} \otimes (T^{(l)})^t \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$
24|13	$\parallel T_{24\|13} \parallel_k$ $= \parallel T^{(f)^t} \otimes T^{(\underline{g},h,\underline{l})} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$	$\parallel T_{24\|13} \parallel_k$ $= \parallel T^{(f,\underline{g})} \otimes T^{(h,\underline{l})} \parallel_k$ $\leq \frac{4k(d^2-1)}{d^2}$	$\parallel T_{24\|13} \parallel_k$ $= \parallel T^{(f,\underline{g},h)} \otimes T^{(l)} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$
34|12	$\parallel T_{34\|12} \parallel_k$ $= \parallel (T^{(f)})^t \otimes T^{(g,\underline{h,l})} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$	$\parallel T_{34\|12} \parallel_k$ $= \parallel (T^{f,g})^t \otimes T^{(h,l)} \parallel_k$ $\leq \frac{4(d^2-1)}{d^2}$	$\parallel T_{34\|12} \parallel_k$ $= \parallel T^{(f,g,\underline{h})} \otimes T^{(l)} \parallel_k$ $\leq \frac{4(d-1)\sqrt{k(d^2+d+1)}}{d^2}$

Altogether we have $\parallel T_{fg|hl} \parallel_k \leq \frac{4k(d^2-1)}{d^2}$ if ρ is entangled under bipartition $fg|hl$. $\square$

Next we present a sufficient condition to detect GME for four-partite systems. By the Lemma 2 we have that $\| T_{f|ghl} \|_k \leq \frac{4(d-1)\sqrt{d^2+d+1}}{d^2}$ if ρ is separable, and $\| T_{f|ghl} \|_k \leq \frac{4\sqrt{k}(d^2-1)}{d^2}$ if ρ is entangled. However, $\| T_{fg|hl} \|_k \leq \frac{4k(d^2-1)}{d^2}$ is a rather weak condition. We define the average matricization norm, $M_k = \frac{1}{4}(\| T_{1|234} \|_k + \| T_{2|134} \|_k + \| T_{3|124} \|_k + \| T_{4|123} \|_k)$.

Theorem 1. *If ρ is a four-qudit state, and*

$$M_k(\rho) > \frac{(d-1)[\sqrt{d^2+d+1} + 3(d+1)\sqrt{k}]}{d^2} \tag{18}$$

for any $k \in \{1, 2, 3, \cdots, d^2-1\}$, then ρ is genuine multipartite entangled.

Remark 1: Compared with the Theorem 3 in [17] for four-qubit states, our result detects GME for any general four-qudit states.

Example: Consider the four-qubit state $\rho = \frac{1-x}{16} + x|\varphi\rangle\langle\varphi|$, where $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ is the Greenberger-Horne-Zeilinger(GHZ) state. By Theorem 3 in [17], the inequality is $\| M_{22}(T_{i_1 i_2 i_3}) \|_k > 2\sqrt{k}(1 \leq k \leq 3)$, ρ contains genuine four-qubit entangled when $0.692820 < x \leq 1$. By our result in Theorem 1, with $d = 2$ and $k = 3$, we have that ρ contains genuine four-qubit entangled for $0.911710 < x \leq 1$, see Fig. 1.

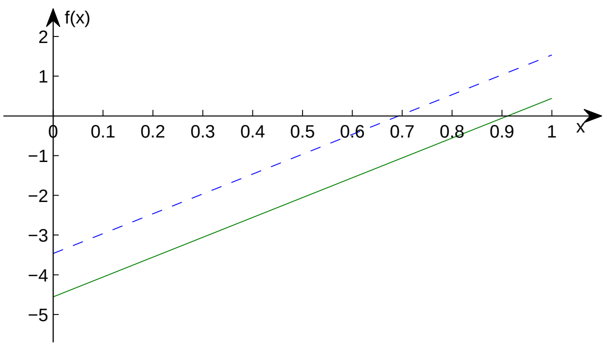

Fig. 1: Function $f(x)$ denotes the difference between the left and right hand side of the inequality (18) and the inequality of the Theorem 3 of [17]. From the Theorem 3 of [17], one has $f(x) = 5x - 3.464103$(dashed line) which contains the GME for $0.692820 < x \leq 1$. Our Theorem gives rise to $f(x) = 5x - 4.558552$(solid line), which contains GME for $0.911710 < x \leq 1$.

3. Detection of GME for Multipartite Quantum States

In this section, we study the GME for multipartite qudit states. Any n-partite density matrix $\rho \in H_1^d \otimes H_2^d \otimes \cdots \otimes H_n^d$ can be expressed as

$$\begin{aligned}\rho \quad &= \frac{1}{d^n} I \otimes \cdots \otimes I + \frac{1}{2d^{n-1}} \sum_{j_1=1}^{n} \sum_{i_1=1}^{d^2-1} t_{i_1}^{(j_1)} \lambda_{i_1}^{(j_1)} \otimes I \otimes \cdots \otimes I + \cdots \\ &+ \frac{1}{2^n} \sum_{i_1,\cdots,i_n=1}^{d^2-1} t_{i_1,\cdots,i_n}^{(1,\cdots,n)} \lambda_{i_1}^{(1)} \otimes \lambda_{i_2}^{(2)} \otimes \cdots \otimes \lambda_{i_n}^{(n)}, \end{aligned} \tag{19}$$

where (j_1) represents the position of λ_{i_1} in the tensor product, $t_{i_1}^{(j_1)} = tr(\rho \lambda_{i_1}^{(j_1)} \otimes I \otimes \cdots \otimes I), \cdots, t_{i_1,\cdots,i_n}^{(1,\cdots,n)} = tr(\rho \lambda_{i_1}^{(1)} \otimes \lambda_{i_2}^{(2)} \otimes \cdots \otimes \lambda_{i_n}^{(n)})$, and $T^{(j_1)}, \cdots, T^{(1,\cdots,n)}$ are the vectors (tensors) with elements $t_{i_1}^{(j_1)}, \cdots, t_{i_1,\cdots,i_n}^{(1,\cdots,n)}$, respectively.

For a pure state ρ, one has

$$tr(\rho^2) \quad = \frac{1}{d^n} + \frac{1}{2d^{n-1}} \sum_{j_1}^{n} \sum_{i_1}^{d^2-1} (t_{i_1}^{(j_1)})^2 + \cdots + \frac{1}{2^n} \sum_{i_1,\cdots,i_n}^{d^2-1} (t_{i_1,\cdots,i_n}^{(1,\cdots,n)})^2 = 1. \tag{20}$$

Hence

$$\sum_{i_1,\cdots,i_n}^{d^2-1} (t_{i_1,\cdots,i_n}^{(1,\cdots,n)})^2 \quad = 2^n - \frac{2^n}{d^n} - \cdots - \frac{2^n}{2d^{n-1}} \sum_{j_1}^{n} \sum_{i_1}^{d^2-1} (t_{i_1}^{(j_1)})^2 \le \frac{2^n(d^n-1)}{d^n}, \tag{21}$$

which implies that

$$\| T^{(1,2,\cdots,n)} \| = \sqrt{\sum_{i_1,\cdots,i_n}^{d^2-1} (t_{i_1,\cdots,i_n}^{(1,\cdots,n)})^2} \le \sqrt{\frac{2^n(d^n-1)}{d^n}}. \tag{22}$$

We now consider multipartite systems and their T matrices.

Theorem 2. *Let $\rho \in H_1^d \otimes \cdots \otimes H_n^d$ be a pure state. If ρ is fully separable, then for any $k = 1, \cdots, d^2-1$,*

$$\| T_{1|\cdots|n} \|_k = \sqrt{\frac{2^n(d-1)^n}{d^n}}. \tag{23}$$

Proof. According to the Proposition 1 of Ref. [21], i.e., if ρ is fully separable then $t^{(1,\cdots,n)}_{i_1,\cdots,i_n} = t^{(1)}_{i_1}\cdots t^{(n)}_{i_n}$, using the bound $\| T^{(j_1)} \| \leq \sqrt{\frac{2(d-1)}{d}}$, $j_1 = 1,\cdots,n$, we have

$$\begin{aligned}\| T_{1|\cdots|n} \|_k &= \| T^{(1)} \cdot (T^{(2)} \otimes \cdots \otimes T^{(n)})^t \|_k = \| T^{(1)} \| \cdot \| (T^{(2)} \otimes \cdots \otimes T^{(n)})^t \|_k \\ &= \| T^{(1)} \| \cdot \| T^{(2)} \otimes \cdots \otimes T^{(n)} \|_k = \| T^{(1)} \| \cdot \| T^{(2)} \otimes \cdots \otimes T^{(n)} \| \\ &= \| T^{(1)} \| \cdot \| T^{(2)} \| \cdots \| T^{(n)} \| = \sqrt{\frac{2^n(d-1)^n}{d^n}}. \end{aligned} \tag{24}$$

Hence, if ρ is fully separable, then $\| T_{1|\cdots|n} \|_k = \sqrt{\frac{2^n(d-1)^n}{d^n}}$. □

Let A_1 be subsets of the set $\{H_1, H_2, \cdots, H_n\}$ and A_2 the complement of A_1, n_{A_1} and n_{A_2} be the number of spaces contained in A_1 and A_2, respectively. For the bipartition $A_1|A_2 = j_1\cdots j_{n_{A_1}}|j_{n_{A_1+1}}\cdots j_n$, $j_1 \neq j_2 \neq \cdots \neq j_n \in \{1,2,\cdots,n\}$, let $T_{A_1|A_2}$ be a matrix with entries $t_{a,b} = t^{(1,\cdots,n)}_{i_1,\cdots,i_n}$, where $a = (d^2-1)^{n_{A_1}-1}(i_{j_1}-1)+\cdots+i_{j_{n_{A_1}}}$, $b = (d^2-1)^{n_{A_2}-1}(i_{j_{n_{A_1}}+1}-1)+\cdots+i_{j_n}$, $i_{j_1}, i_{j_2}, \ldots, i_{j_n} = 1,2,\ldots,d^2-1$.

Theorem 3. *Let $\rho \in H^d_1 \otimes \cdots \otimes H^d_n$ be a pure state. If ρ is separable under bipartition $A_1|A_2$, then for any $k = 1,\cdots,d^2-1$,*

$$\| T_{A_1|A_2} \|_k \leq \sqrt{\frac{2^n(d^{n_{A_1}}-1)(d^{n_{A_2}}-1)}{d^n}}. \tag{25}$$

Proof. If ρ is separable under bipartition $A_1|A_2$, then $\rho_{A_1} \otimes \rho_{A_2}$. Using the inequality (22), we get

$$\begin{aligned}\| T_{A_1|A_2} \|_k &= \| T^{(A_1)} \cdot (T^{(A_2)})^t \|_k = \| T^{(A_1)} \| \cdot \| (T^{(A_2)})^t \|_k \\ &= \| T^{(A_1)} \| \cdot \| (T^{(A_2)})^t \| = \| T^{(A_1)} \| \cdot \| (T^{(A_2)}) \| \\ &\leq \sqrt{\frac{2^n(d^{n_{A_1}}-1)(d^{n_{A_2}}-1)}{d^n}}. \end{aligned} \tag{26}$$

□

Theorem 4. *Let $\rho \in H^d_1 \otimes \cdots \otimes H^d_n$ be a pure state such that ρ is separable under at least one bipartition. For any $k = 1,\cdots,d^2-1$ and $j_1 \neq j_2 \neq \cdots \neq j_n \in \{1,2,\cdots,n\}$, we have*

(i) if ρ is entangled under a certain bipartition $j_1|j_2\cdots j_n$, then

$\| T_{j_1|j_2\cdots j_n} \|_k \leq \sqrt{\frac{2^n k(d^{[\frac{n}{2}]}-1)(d^{n-[\frac{n}{2}]}-1)}{d^n}}$ ([] *denotes integer function*), *when* n *is odd;*
$\| T_{j_1|j_2\cdots j_n} \|_k \leq \sqrt{\frac{2^n k(d^{\frac{n}{2}}-1)^2}{d^n}}$, *when* n *is even;*
(ii) *if* ρ *is entangled under a certain bipartition* $j_1\cdots j_{n-1}|j_n$, *then*
$\| T_{j_1\cdots j_{n-1}|j_n} \|_k \leq \sqrt{\frac{2^n k(d^{[\frac{n}{2}]}-1)(d^{n-[\frac{n}{2}]}-1)}{d^n}}$, *when* n *is odd;*
$\| T_{j_1\cdots j_{n-1}|j_n} \|_k \leq \sqrt{\frac{2^n k(d^{\frac{n}{2}}-1)^2}{d^n}}$, *when* n *is even.*

Proof. (i) If ρ is entangled under bipartition $j_1|j_2\cdots j_n$, then there is at least one bipartition $j'_1\cdots j'_p|j'_{p+1}\cdots j'_n$ $(p=1,2\cdots,n-1)$ such that ρ is separable. Let $j'_1\cdots j'_p|j'_{p+1}\cdots j'_n = A_1|A_2$, then $n_{A_1}=p$.

① $j_1=1$. If $p=1$ and $j'_1\neq 1$, we have

$$\begin{aligned}\| T_{j_1|j_2\cdots j_n} \|_k &= \| T^{(j'_1)}\cdot (T^{(j'_2,\cdots,j'_n)})^t \|_k = \| T^{(j'_1)} \| \cdot \| (T^{(j'_2,\cdots,j'_n)})^t \|_k \\ &= \| T^{(j'_1)} \| \cdot \| T^{(j'_2,\cdots,j'_n)} \| \leq \sqrt{\frac{2^n(d-1)(d^{n-1}-1)}{d^n}}. \end{aligned} \tag{27}$$

If $p=2,3,\cdots,n-1$, we get

$$\begin{aligned}\| T_{j_1|j_2\cdots j_n} \|_k &= \| T^{(\underline{j'_1},\cdots,j'_p)} \otimes (T^{(j'_{p+1},\cdots,j'_n)})^t \|_k = \| T^{(\underline{j'_1},\cdots,j'_p)} \|_k \cdot \| (T^{(j'_{p+1},\cdots,j'_n)})^t \|_k \\ &\leq \sqrt{k} \| T^{(\underline{j'_1},\cdots,j'_p)} \| \cdot \| T^{(j'_{p+1},\cdots,j'_n)} \| \leq \sqrt{\frac{2^n k(d^p-1)(d^{n-p}-1)}{d^n}}. \end{aligned} \tag{28}$$

② $j_1=2,\cdots,n-1$. For any p we have

$$\begin{aligned}\| T_{j_1|j_2\cdots j_n} \|_k &= \| T^{(j'_1,\cdots,\underline{j'_{j_1}},\cdots,j'_p)} \otimes (T^{(j'_{p+1},\cdots,j'_n)})^t \|_k = \| T^{(j'_1,\cdots,\underline{j'_{j_1}},\cdots,j'_p)} \|_k \cdot \| (T^{(j'_{p+1},\cdots,j'_n)})^t \|_k \\ &\leq \sqrt{k} \| T^{(j'_1,\cdots,\underline{j'_{j_1}},\cdots,j'_p)} \| \cdot \| T^{(j'_{p+1},\cdots,j'_n)} \| \leq \sqrt{\frac{2^n k(d^p-1)(d^{n-p}-1)}{d^n}}. \end{aligned} \tag{29}$$

③ $j_1=n$. If $p=1,\cdots,n-2$, we have

$$\begin{aligned}\| T_{j_1|j_2\cdots j_n} \|_k &= \| (T^{(j'_1,\cdots,j'_p)})^t \otimes T^{(j'_{p+1},\cdots,\underline{j'_n})} \|_k = \| (T^{(j'_1,\cdots,j'_p)})^t \|_k \cdot \| T^{(j'_{p+1},\cdots,\underline{j'_n})} \|_k \\ &\leq \sqrt{k} \| T^{(j'_1,\cdots,j'_p)} \| \cdot \| T^{(j'_{p+1},\cdots,\underline{j'_n})} \| \leq \sqrt{\frac{2^n k(d^p-1)(d^{n-p}-1)}{d^n}}. \end{aligned} \tag{30}$$

If $p=n-1$, we get

$$\begin{aligned}\| T_{j_1|j_2\cdots j_n} \|_k &= \| (T^{(j'_1,\cdots,j'_{n-1})})^t \otimes T^{(j'_n)} \|_k = \| (T^{(j'_1,\cdots,j'_{n-1})})^t \| \cdot \| T^{(j'_n)} \|_k \\ &= \| T^{(j'_1,\cdots,j'_{n-1})} \| \cdot \| T^{(j'_n)} \| \leq \sqrt{\frac{2^n(d-1)(d^{n-1}-1)}{d^n}}. \end{aligned} \tag{31}$$

Now consider $\max\{\sqrt{\frac{2^nk(d^p-1)(d^{n-p}-1)}{d^n}},\sqrt{\frac{2^n(d-1)(d^{n-1}-1)}{d^n}}\}$ $p=1,\cdots,n-1$. Let $y=(d^h-1)(d^{n-h}-1)$ $(h\in R^+)$ be a continuous function. Then the maximal value is $y_{max}=(d^{\frac{n}{2}}-1)^2$. If n is odd, $\parallel T_{A_1|A_2}\parallel_k\leq\sqrt{\frac{2^nk(d^{[\frac{n}{2}]}-1)(d^{n-[\frac{n}{2}]}-1)}{d^n}}$. If n is even, $\parallel T_{A_1|A_2}\parallel_k\leq\sqrt{\frac{2^nk(d^{\frac{n}{2}}-1)^2}{d^n}}$.

(ii) If ρ is entangled under bipartition $j_1\cdots j_{n-1}|j_n$, then there is at least one bipartition $j'_1\cdots j'_p|j'_{p+1}\cdots j'_n$ $p=1,2\cdots,n-1$, such that ρ is separable. Similarly, let $j'_1\cdots j'_p|j'_{p+1}\cdots j'_n=A_1|A_2$, then $n_{A_1}=p$. The proof can be done in three cases.

① $j_n=1$. If $p=1$, we have

$$\begin{aligned}\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k &= \parallel (T^{(j'_1)})^t\otimes T^{(j'_2,\cdots,j'_n)}\parallel_k=\parallel (T^{(j'_1)})^t\parallel_k\cdot\parallel T^{(j'_2,\cdots,j'_n)}\parallel_k\\ &= \parallel T^{(j'_1)}\parallel\cdot\parallel T^{(j'_2,\cdots,j'_n)}\parallel\leq\sqrt{\frac{2^n(d-1)(d^{n-1}-1)}{d^n}}.\end{aligned}\tag{32}$$

If $p=2,\cdots,n-1$, we get

$$\begin{aligned}\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k &= \parallel T^{(j'_1,\underline{j'_2,\cdots,j'_p})}\otimes T^{(j'_{p+1},\cdots,j'_n)}\parallel_k=\parallel T^{(j'_1,\underline{j'_2,\cdots,j'_p})}\parallel_k\cdot\parallel T^{(j'_{p+1},\cdots,j'_n)}\parallel_k\\ &\leq \sqrt{k}\parallel T^{(j'_1,\underline{j'_2,\cdots,j'_p})}\parallel\cdot\parallel T^{(j'_{p+1},\cdots,j'_n)}\parallel\leq\sqrt{\frac{2^nk(d^p-1)(d^{n-p}-1)}{d^n}}.\end{aligned}\tag{33}$$

② $j_n=2,\cdots,n-1$. For any p we have

$$\begin{aligned}\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k &= \parallel T^{(\underline{j'_1,\cdots},j'_{j_n},\underline{\cdots,j'_p})}\otimes T^{(j'_{p+1},\cdots,j'_n)}\parallel_k=\parallel T^{(\underline{j'_1,\cdots},j'_{j_n},\underline{\cdots,j'_p})}\parallel_k\cdot\parallel T^{(j'_{p+1},\cdots,j'_n)}\parallel_k\\ &\leq \sqrt{k}\parallel T^{(\underline{j'_1,\cdots},j'_{j_n},\underline{\cdots,j'_p})}\parallel\cdot\parallel T^{(j'_{p+1},\cdots,j'_n)}\parallel\leq\sqrt{\frac{2^nk(d^p-1)(d^{n-p}-1)}{d^n}}.\end{aligned}\tag{34}$$

③ $j_n=n$. If $p=1,\cdots,n-2$, we get

$$\begin{aligned}\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k &= \parallel T^{(j'_1,\cdots,j'_p)}\otimes T^{(\underline{j'_{p+1},\cdots,j'_{n-1}},j'_n)}\parallel_k=\parallel T^{(j'_1,\cdots,j'_p)}\parallel_k\cdot\parallel T^{(\underline{j'_{p+1},\cdots,j'_{n-1}},j'_n)}\parallel_k\\ &\leq \sqrt{k}\parallel T^{(j'_1,\cdots,j'_p)}\parallel\cdot\parallel T^{(\underline{j'_{p+1},\cdots,j'_{n-1}},j'_n)}\parallel\leq\sqrt{\frac{2^nk(d^p-1)(d^{n-p}-1)}{d^n}}.\end{aligned}\tag{35}$$

If $p=n-1$ and $j'_n\neq n$, we have

$$\begin{aligned}\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k &= \parallel T^{(j'_1,\cdots,j'_{n-1})}\cdot(T^{(j'_n)})^t\parallel_k=\parallel T^{(j'_1,\cdots,j'_{n-1})}\parallel\cdot\parallel (T^{(j'_n)})^t\parallel_k\\ &= \parallel T^{(j'_1,\cdots,j'_{n-1})}\parallel\cdot\parallel T^{(j'_n)}\parallel\leq\sqrt{\frac{2^n(d-1)(d^{n-1}-1)}{d^n}}.\end{aligned}\tag{36}$$

If n is odd, $\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k\leq\sqrt{\frac{2^nk(d^{[\frac{n}{2}]}-1)(d^{n-[\frac{n}{2}]}-1)}{d^n}}$. If n is even, $\parallel T_{j_1\cdots j_{n-1}|j_n}\parallel_k\leq\sqrt{\frac{2^nk(d^{\frac{n}{2}}-1)^2}{d^n}}$.

□

4. Conclusion

We have studied genuine multipartite entanglement in four-partite and multipartite qudit quantum systems, and derived the relationship between the norms of the correlation tensors and the specific matrix T. Based on these relations we have presented a criterion to detect GME in four-partite quantum systems. These results are generalized to multipartite systems. Our main results concern with special inequalities that bound the various norms of the correlation tensors, upon which our criterion is presented to detect GME in multipartite systems. These results can help distinguishing genuine multipartite entangled states. Genuine multipartite entanglement plays significant roles in many quantum information processing. Our approach and results may highlight further researches on the theory of genuine multipartite entanglement.

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