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by


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# Characterizing Bell nonlocality and steering in generalized probabilistic theories

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**Abstract** Bell nonlocality and EPR steering are both important quantum features and quantum resources with wide applications in many quantum information processing tasks. We investigate the Bell nonlocality and steering in the framework of generalized probabilistic theories. We formulate the characterizations of Bell nonlocality and steering in this more general framework and present new insight into the essence of Bell nonlocality and steering.

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**Key Words**: Bell nonlocality; steering; generalized probabilistic theories; ordered linear space

## 1. Introduction

The past few decades witnessed the great achievements and progress of quantum information theory. Fruitful theoretical results have been derived, and many of them are widely applied to quantum information processing tasks [1]-[2]. Bell nonlocality, EPR steering and quantum entanglement are three typical quantum correlations, which form a hierarchy of relations, i.e., Bell nonlocality implies steerability, while steerability implies entanglement.

Bell nonlocality of bipartite states is demonstrated by some local quantum measurements whose statistics can not be explained by a local hidden variable (LHV) model [3], which has applications in quantum protocols for decreasing communication complexity [4], and secure quantum communication [5]-[6]. For recent study on Bell nonlocality, we refer to refs. [7]-[12].

The concept of EPR steering can be traced back to Schrödinger's paper [13], but

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was formally raised in [14]. Analogously, steerability of bipartite states is demonstrated if the obtained ensembles can not be explained by a local hidden state (LHS) model [14]. It is tightly related to many quantum information tasks, such as quantum key distribution and randomness generation [15]-[17]. The relationships among EPR steering, joint measurability, entropic uncertainty relation etc.[18]-[22], steering criteria using different approaches [23]-[26] and the quantification of EPR steering [27]-[31] have been investigated in the past few years. Recently, some characterization of Bell nonlocality and EPR steering are given in the quantum mechanics formalism [32].

The study of the theories of cones and partially ordered linear spaces provides a framework which was initially constructed to generalize quantum probability rules [33]-[35], and has been further refined by many authors [36]-[39], which was now called generalized probabilistic theories (GPTs) or convex probabilistic theories. The framework of GPTs has attracted much attention in the last decade and many quantum information measures and quantum information processing tasks such as fidelity, entropy, no-cloning, no-broadcasting, teleportation, state discrimination, steering, data hiding, etc., have been studied in the more general framework of GPTs [40]-[55]. The advantage of GPTs lies in the fact that it provides a unified framework which incorporate classical and quantum theory, as well as other theories such as Popescu-Rohrlich boxes [57], and it gives us more insight to the foundations of quantum theory. Moreover, it helps us better understand the computational power of both classical and quantum information theory [58]-[59].

In this paper, we give some characterizations of Bell nonlocality and steering in the broad context of generalized probabilistic theories, which also generalize the results in [32].

## 2. The Formalism of GPTs

In this section, we recall the formalism of GPTs (see [43, 46, 48, 50] for more details). For the sake of conciseness, we omit the motivations of the following definitions. Also, we only consider finite-dimensional systems, implying that all vector spaces mentioned below are finite dimensional, which makes it possible to identify a vector space $V$ with its double dual $V^{**}$.

An *ordered linear space* (OLS) is a real vector space $V$ equipped with a partial order satisfying that $x \leq y \Rightarrow x+z \leq y+z$ and $x \leq y \Rightarrow tx \leq ty$ for all $x, y, z \in V$ and $t \geq 0$. A convex set $K$ in a real vector space $V$ is called a *convex cone* if $x \in K \Rightarrow tx \in K$ for all $t \geq 0$. If $K \cap \{-K\} = \{0\}$, the cone is said to be *pointed.* Any above-mentioned partial order is determined by the pointed convex cone $V_+ = \{x \in V | x \geq 0\}$ which is called the *positive cone*, and conversely, any pointed convex cone induces a partial order on $V$, where $x \leq y$ iff $y - x \in V_+$.

A pointed convex cone is called *regular* if it is also generating (i.e., $V = V_+ - V_+$) and closed. In the following, by "ordered linear space", we mean one whose positive cone is regular.

Let $V$ and $W$ be two ordered linear spaces. A linear map $\phi : V \to W$ is called *positive* if it is order-preserving, or equivalently, if $\phi(x) \in B_+$ for all $x \in V_+$. Specifically, a linear functional $f \in V^*$ is positive if $f(x) \geq 0$ for all $x \in V_+$. If a positive linear isomorphism $\phi : V \to W$ has a positive inverse, then it is called an *order isomorphism*. In other words, $\phi$ is an order isomorphism if it is a bijective map satisfying that $\phi(x) \geq 0$ with $\phi(x) \in W$ iff $x \geq 0$ with $x \in V$. Denote by $\mathcal{L}(V, W)$ the set of all linear mappings from $V$ to $W$, and $\mathcal{L}_+(V, W)$ the set of all positive linear mappings from $V$ to $W$. It is easy to see that $\mathcal{L}_+(V, W)$ is a pointed, closed, generating convex cone in $\mathcal{L}(V, W)$. In particular, when $W = \mathbb{R}$, we use $V_+^*$ to denote $\mathcal{L}_+(V, \mathbb{R})$, which is usually called the *dual cone* of $V$.

An *order unit* in an ordered linear space $V$ is vector $\iota \in V_+$ satisfying that for each $x \in V_+$, there exists some $t \geq 0$ such that $x \leq t\iota$. An order unit *on* $V$ is an order unit in $V^*$ (since $V$ is finite-dimensional), that is, a *strictly* positive functional $\iota \in V^*$ which means that $\iota(\omega) > 0$ for $\omega > 0$ (i.e., $\iota$ is in the interior of $V_+^*$).

**Definition 2.1** An *abstract state space* is a pair $(A, \iota_A)$, where $A$ is an ordered linear space and $\iota_A$ is a distinguished order unit on $A$. A *normalized state* is a positive element $\omega$ in $A$ such that $\iota_A(\omega) = 1$. The set of all normalized states is a compact convex subset of $A_+$, which we denote by $\Omega_A$.

**Definition 2.2** An *effect* on an abstract state space $(A, \iota_A)$ is a positive functional $e \in A^*$ such that $e \leq \iota_A$, or equivalently, an effect $e$ is an element in the order interval $[0, \iota_A]$. In other words, $e \in A^*$ is an effect if $0 \leq e(\omega) \leq 1$ holds for each normalized state $\omega \in \Omega_A$.

Denote the sets of all affine functionals on $A$ and all the effects by $\mathcal{A}(A)$ and $\mathcal{E}(A)$, respectively. Then it follows that $\mathcal{E}(A)$ is a compact convex subset of $\mathcal{A}(A)$. The extreme effect is called a *pure effect*, which we denote by $e_p$. The zero effect 0 and the unit effect $\iota$ are two trivial pure effects. It is easy to see that the effect $\iota - e$ is pure iff the effect $e$ is pure. Now denote by $\mathcal{E}(A)_p$ the set of all pure effects. Then by the Krein-Milman theorem, $\mathcal{E}(A) \neq \emptyset$, and $\mathcal{E}(A) = \overline{co}(\mathcal{E}(A)_p)$. Moreover, in finite-dimensional case, any effect $e \in \mathcal{E}(A)$ is the convex combination of finite pure effects: $e = \sum_{i=1}^n p_i e_i$, where $p_i \geq 0, \sum_{i=1}^n p_i = 1$ and $e_i \in \mathcal{E}(A)_p$ (see e.g., Theorem 5.6 in [60]).

Note that there is a natural embedding of $A$ in $\mathcal{A}(A)^*$ (the dual space of $\mathcal{A}(A)$), given by $x \longmapsto \overline{x}$, where $\overline{x}(f) = f(x)$ for all $f \in \mathcal{A}(A)$. Hence, we can identify $x$ with $\overline{x}$, and regard $x(f)$ and $f(x)$ as the same.

**Definition 2.3** A *measurement* is a set of effects $\{e_k\}_{k \in \mathscr{K}}$ on an abstract state space $(A, \iota_A)$ satisfying $\sum_{k \in \mathscr{K}} e_k = \iota_A$, or equivalently, $\sum_{k \in \mathscr{K}} e_k(\omega) = 1$ for all normalized s-tates $\omega \in \Omega_A$, which is denoted by $M = \{e_k\}_{k \in \mathscr{K}}$, where $\mathscr{K}$ is the outcome set. Note that $e_j(\omega)$ is the probability of getting $j$th outcome (say $a_j$, for instance) by a measurement $M$ in a state $\omega$.

Now we recall the formalism of composite systems. Let $A$ and $B$ be abstract state spaces, with order units $\iota_A \in A^*$, $\iota_B \in B^*$ and normalized state spaces $\Omega_A$ and $\Omega_B$. We denote by $A \otimes_{max} B$ the set of all bilinear functionals on $A^* \times B^*$, ordered by the cone of forms nonnegative on $a \otimes b$ or positive elements (i.e., $a \in A^*_+, b \in B^*_+$), with order unit $\iota_A \otimes \iota_B$. Denote by $A \otimes_{min} B$ the set ordered by the cone generated by the product states $\alpha \otimes \beta$, where $\alpha \in A_+$ and $\beta \in B_+$.

States in $A \otimes_{max} B$ satisfy the so-called no-signaling condition, meaning that the marginal states of $A$ and $B$, which act as $\omega^A(a) = \omega(a, \iota_B)$ and $\omega^B(b) = \omega(\iota_A, b)$, respectively, are both well-defined, not depending on which measurement was conducted on the other party. Conversely, if we assign a joint probability to measurement outcomes associated with $A$ and $B$ satisfying this no-signaling condition, then it extends to a positive bilinear form on $A^* \times B^*$, and hence, to an element of $A \otimes_{max} B$ [38, 39]. Hence the no-signaling condition is important in formulating the maximal tensor product of two systems. The composite system $AB$ can thus be formulated as a "tensor product" of $A$ and $B$, denoted by $A \otimes B$, which is a convex cone lying between the minimal and maximal ones, i.e., $A \otimes_{min} B \subseteq A \otimes B \subseteq A \otimes_{max} B$.

A bipartite state on a composite system $AB$ is represented by a positive bilinear functional $\omega : A^* \times B^* \to \mathbb{R}$, which induces a positive map $\hat{\omega} : A^* \to B = B^{**}$ defined by $\hat{\omega}(a)(b) = \omega(a, b)$. It follows that $\hat{\omega}(\iota_A) = \omega^B$, the $B$ marginal of $\omega$. The adjoint map $\hat{\omega}^* : B^* \to A^{**} = A$ acts as $\hat{\omega}^*(b)(a) = \hat{\omega}(a)(b) = \omega(a, b)$, and thus $\hat{\omega}^*(\iota_B) = \omega^A$. Conversely, any positive linear map $T$ satisfying $T(\iota_A) = \omega^B$ (where $\omega^B$ is a normalized state) defines a normalized bipartite state in $A \otimes_{max} B$.

## 3. A characterization of Bell nonlocality in GPTs

Consider the scenario that Alice performs $m_A$ different measurements with $o_A$ outcomes and Bob performs $m_B$ different measurements with $o_B$ outcomes, which are denoted by $M^x = \{e^{a|x}\}_{a=1}^{o_A}$ and $N^y = \{e^{b|y}\}_{b=1}^{o_B}$, respectively, where $x \in \{1, 2, \cdots, m_A\}$ and $y \in \{1, 2, \cdots, m_B\}$ are the indices of measurements, while $a$ and $b$ are the indices of outcomes. These measurements yield *measurement assemblages* of $A$ and $B$: $\mathcal{M}_A := \{M^x\}_{x=1}^{m_A}$ and $\mathcal{N}_B := \{N^y\}_{y=1}^{m_B}$, respectively, which induce a *measurement assemblage* for the composite system $AB$:

$$\mathcal{M}_A \otimes \mathcal{N}_B := \{M^x \otimes N^y : x = 1, 2, \cdots, m_A, y = 1, 2, \cdots, m_B\},$$

where $M^x \otimes N^y = \{e^{a|x} \otimes e^{b|y}\}_{a=1,2,\cdots,o_A,b=1,2,\cdots,o_B}$.

Suppose that Alice and Bob share a joint state $\omega^{AB}$. Alice performs measurement $M^x$ on system $A$ and obtains outcome $a$, and in the mean time Bob performs measurement $N^y$ on system $B$ and obtains outcome $b$. The outcomes can be described by the probability distribution

$$p(ab|xy) = (e^{a|x} \otimes e^{b|y})(\omega^{AB}).$$

Now the task is to check whether $\omega^{AB}$ is entangled by means of $p(ab|xy)$.

We provide the following definition of Bell nonlocality in the framework of GPTs.

**Definition 3.1** A normalized state in system $AB$, $\omega^{AB} \in \Omega_{AB}$, is said to be Bell local for a given measurement scenario $\mathcal{M}_A \otimes \mathcal{N}_B$, if there exists a probability distribution $\{\pi_\lambda\}_{\lambda=1}^d$, for each $(\lambda, x)$ and $(\lambda, y)$, there exist probability distributions $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}$ and $\{P_B(b|y,\lambda)\}_{b=1}^{o_B}$, such that the following holds

$$(e^{a|x} \otimes e^{b|y})(\omega^{AB}) = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) P_B(b|y,\lambda) \tag{1}$$

for all $a, b, x, y$.

**Remark 3.1** Summing over $b$ on both sides of Eq.(1) yields

$$(e^{a|x} \otimes \iota_B)(\omega^{AB}) = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda), \ \forall a, x. \tag{2}$$

Similarly, by summing over $a$ on both sides of Eq.(1), we obtain

$$(\iota_A \otimes e^{b|y})(\omega^{AB}) = \sum_{\lambda=1}^{d} \pi_\lambda P_B(b|y,\lambda), \ \forall b, y. \tag{3}$$

This implies that Alice's measurement results with $\mathcal{M}_A$ are independent of Bob's, and vice versa.

**Remark 3.2** We claim that a separable normalized state is Bell local in any GPTs. In fact, suppose that

$$\omega^{AB} = \sum_{\lambda=1}^{d} \pi_\lambda \omega_\lambda^A \otimes \omega_\lambda^B$$

is a separable state. Then for each $\mathcal{M}_A \otimes \mathcal{N}_B$, we have

$$[e^{a|x} \otimes e^{b|y}](\omega^{AB}) = \sum_{i=1}^{d} \pi_\lambda P_A(a|x,\lambda) P_B(b|y,\lambda), \ \forall x, y, a, b,$$

where $P_A(a|x,\lambda) = e^{a|x}(\omega_\lambda^A)$, $P_B(b|y,\lambda) = e^{b|y}(\omega_\lambda^B)$. It follows from Definition 3.1 that $\omega^{AB}$ is Bell local.

**Remark 3.3** In [54] the authors derived the conditions for the compatibility of channels in general probabilistic theories. They formulated the Bell nonlocality of channels as an incompatibility test and treated the Bell nonlocality of measurements as a consequence, since a measurement is defined as a map from a state space to a simplex (a special case of a channel). In our definition, a measurement refers to a set of effects which is a direct generalization of a POVM in quantum theory. Hence the Definition 3.1 is different from the one in [54].

Now, consider the set $S_A$ of all possible maps from $S_m = \{1, 2, \cdots, m_A\}$ into $S_0 = \{1, 2, \cdots, o_A\}$. Then $S_A$ has $N_A := o_A^{m_A}$ elements, and can be written as

$$S_A = \{J_1, J_2, \cdots, J_{N_A}\}.$$

Each element $J$ of $S_A$ assigns an outcome value $a$ for each measurement $M^x$, that is, $J(x) = a$. Denote by $p_A(k, \lambda)$ the probability of $J_k$ being used when Alice receives classical information indicated by $\lambda$. Thus, $\{p_A(k, \lambda)\}_{k=1}^{N_A}$ is a probability distribution depending on both the numbers of measurements $m_A$ and outcomes $o_A$. Denote by $P_A(a|x, \lambda)$ the probability of obtaining outcome $a$ when Alice uses measurement $M^x$ and receives classical information indicated by $\lambda$.

Similarly, let $S_B$ be the set of all possible maps from $T_m = \{1, 2, \cdots, m_B\}$ into $T_o = \{1, 2, \cdots, o_B\}$. Then $S_B$ has $N_B := o_B^{m_B}$ elements, and we write it as

$$S_B = \{K_1, K_2, \cdots, K_{N_B}\}.$$

Denote by $p_B(j, \lambda)$ the probability of $K_j$ being used when Alice receives a classical message $\lambda$ and $P_B(b|y, \lambda)$ the probability of obtaining outcome $b$ when Bob uses $M^x$ and receives a classical message $\lambda$.

We are now ready to give a characterization of Bell nonlocality in the framework of GPTs.

**Theorem 3.1** A state $\omega^{AB} \in \Omega_{AB}$ is Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$ if and only if there exists a probability distribution $\{q_{kj}, 1 \leq k \leq N_A, 1 \leq j \leq N_B\}$, such that

$$(e^{a|x} \otimes e^{b|y})(\omega^{AB}) = \sum_{k=1}^{N_A} \sum_{j=1}^{N_B} q_{kj} \delta_{a,J_k(x)} \delta_{b,K_j(y)}. \tag{4}$$

**Proof.** *Necessity.* By Definition 2.1, $\omega^{AB}$ satisfies Eq. (1). Based on the notations specified above, and taking Eq. (2) into account, we get

$$P_A(a|x, \lambda) = \sum_{k=1}^{N_A} p_A(k, \lambda) \delta_{a,J_k(x)}, \tag{5}$$

where $\sum_{k=1}^{N_A} p_A(k, \lambda) = 1$ for all $\lambda$.

Similarly, noting that Eq. (3) holds, we obtain

$$P_B(b|y, \lambda) = \sum_{j=1}^{N_B} p_B(j, \lambda) \delta_{b,K_j(y)}, \tag{6}$$

where $\sum_{k=1}^{N_B} p_B(j, \lambda) = 1$ for all $\lambda$.

Combining Eqs. (1), (5) and (6), we obtain

$$(e^{a|x} \otimes e^{b|y})(\omega^{AB}) = \sum_{k=1}^{N_A} \sum_{j=1}^{N_B} q_{kj} \delta_{a,J_k(x)} \delta_{b,K_j(y)},$$

where $q_{kj} = \sum_{\lambda=1}^{d} \pi_\lambda P_A(k,\lambda) P_B(j,\lambda) \geq 0$ for all $k, j$ satisfying that $\sum_{k=1}^{N_A} \sum_{j=1}^{N_B} q_{k,j} = 1$.

*Sufficiency.* Suppose that Eq. (4) holds. Let

$$\{\pi_\lambda\}_{\lambda=1}^{N_A N_B} := \{q_{k,j}\}_{1\leq k\leq N_A, 1\leq j\leq N_B},\ P_A(a|x,\lambda) = \delta_{a,J_k(x)} \text{ and } P_B(b|y,\lambda) = \delta_{b,K_j(y)}.$$

Then Eq. (1) holds for $\{\pi_\lambda\}$, $\{P_A(a|x,\lambda)\}$ and $\{P_B(b|y,\lambda)\}$. □

## 4. Characterizations of steering in GPTs

Consider the scenario in which Alice and Bob share a joint state $\omega^{AB}$. Alice performs a chosen measurement $M^x$ on system $A$, and obtains outcome $a$. Then Bob will obtain a set of sub-normalized state $\hat{\omega}^{AB}(e^{a|x})$ depending on Alice's measurement. The task is to steer Bob's state using Alice's measurement on her system. The mathematical definition of steering in the framework of GPTs can be given as follows.

**Definition 4.1** Let $\omega^{AB} \in \Omega_{AB}$ be a normalized state in system $AB$ and $\mathcal{M}_A := \{\{e^{a|x}\}_{a=1}^{o_A} : x = 1, 2, \cdots, m_A\}$ be a measurement assemblage of $A$. A state $\omega^{AB}$ is said to be unsteerable from $A$ to $B$ with $\mathcal{M}_A$, if there exists a probability distribution $\{\pi_\lambda\}_{\lambda=1}^{d}$, a set of normalized states $\{\omega_\lambda^B\}_{\lambda=1}^{d}$ of system $B$, and a probability distribution $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}$, such that

$$\hat{\omega}^{AB}(e^{a|x}) = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\omega_\lambda^B,\ \forall x, a, \tag{7}$$

**Remark 4.1** Ensemble steering in the framework of GPTs has been also studied in [50], and further discussed in [51]. However, the formulation of steering in [52] subjects some problems, since the sub-normalized state held by Bob after the measurement is not given in a proper manner. Here, we reformulate the sub-normalized state by using $\hat{\omega}$. The authors in [54] also formulated steering by channels, which is a generalization of steering by measurements. Similarly, Definition 4.1 is also different from the one in [54] due to different implications of the measurement.

We give a characterization of steering in any GPTs as follows.

**Theorem 4.1** A state $\omega^{AB} \in \Omega_{AB}$ is unsteerable from $A$ to $B$ with $\mathcal{M}_A$ if and only if there exists a probability distribution $\{\pi_\lambda\}$, a set of normalized states $\{\omega_\lambda^B\}_{\lambda=1}^{d}$ of system $B$ and $d\,m_A$ probability distributions $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}(1 \leq x \leq m_A, 1 \leq \lambda \leq d)$, such that for each measurement $\{e^b\}_{b=1}^{o_B}$ of $B$,

$$(e^{a|x} \otimes e^b)(\omega^{AB}) = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) e^b(\omega_\lambda^B),\ \forall x, a. \tag{8}$$

**Proof.** *Necessity.* By Definition 4.1, there exists a probability distribution $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a set of normalized states $\{\omega_\lambda^B\}_{\lambda=1}^{d}$ of system $B$ such that Eq. (8) holds for all $a, x$.

For each measurement $\{e^b\}_{b=1}^{N_B}$ of $B$, Eq. (8) shows that for all $x, a, b$, we have

$$\begin{aligned}(e^{a|x} \otimes e^b)(\omega^{AB}) &= e^b(\hat{\omega}^{AB}(e^{a|x})) \\ &= e^b(\sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\omega_\lambda^B) \\ &= \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) e^b(\omega_\lambda^B). \end{aligned} \tag{9}$$

*Sufficiency.* Suppose that Eq. (8) holds for every measurement $\{e^b\}_{b=1}^{o_B}$ of $B$. For every pure effect $e_p$ on system $B$, $\{e_b\}_{b=1}^{2}$ is a measurement on $B$, where $e_1 = e_p$ and $e_2 = \iota_B - e_p$ (note that $e_1$ and $e_2$ are both pure effects). Then for each measurement $M^x = \{e^{a|x}\}_{a=1}^{o_A}$ and for every pure effect $e_p$ on system $B$,

$$\begin{aligned}e_p(\hat{\omega}^{AB}(e^{a|x})) &= (e^{a|x} \otimes e_p)(\omega^{AB}) \\ &= \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) e_p(\omega_\lambda^B) \\ &= e_p(\sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\omega_\lambda^B)\end{aligned}$$

for all $(x, a)$, which implies that

$$e(\hat{\omega}^{AB}(e^{a|x})) = e(\sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\omega_\lambda^B) \tag{10}$$

for all effects $e$ and $(x, a)$. Therefore, we obtain

$$\hat{\omega}^{AB}(e^{a|x}) = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\omega_\lambda^B, \ \forall x, a.$$

This shows that $\omega^{AB}$ is unsteerable from $A$ to $B$ with $\mathcal{M}_A$. $\square$

Now we give another characterization of steering in the framework of GPTs using similar ideas in deriving Theorem 3.1.

**Theorem 4.2** A state $\omega^{AB} \in \Omega_{AB}$ is unsteerable from $A$ to $B$ with $\mathcal{M}_A$ if and only if there exists a set of sub-normalized states $\{\xi_k\}_{k=1}^{N_A}$ of system $B$ such that

$$\hat{\omega}^{AB}(e^{a|x}) = \sum_{k=1}^{N_A} \delta_{a,J_k(x)}\xi_k, \ \forall x, a. \tag{11}$$

**Proof.** *Necessity.* By Definition 4.1, there exists a probability distribution $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a set of normalized states $\{\omega_\lambda^B\}_{\lambda=1}^{d}$of system $B$, such that Eq. (7) holds. By the same arguments in Theorem 3.1 with the same notations, we obtain Eq. (5), where $\sum_{k=1}^{N_A} p_A(k,\lambda) = 1$ for all $\lambda$.

Inserting Eq. (5) into Eq. (7) yields that

$$\hat{\omega}^{AB}(e^{a|x}) = \sum_{k=1}^{N_A} \delta_{a,J_k(x)} \sum_{\lambda=1}^{d} \pi_\lambda p(k,\lambda)\omega_\lambda^B, \ \forall x, a. \tag{12}$$

Taking $\xi_k = \sum_{\lambda=1}^{d} \pi_\lambda p(k,\lambda)\omega_\lambda^B$, the conclusion follows immediately.

*Sufficiency.* Suppose that there exists $\{\xi_k\}_{k=1}^{N_A}$ such that Eq. (11) holds. Then

$$\iota_B(\hat{\omega}^{AB}(e^{a|x})) = \sum_{k=1}^{N_A} \delta_{a,J_k(x)} \iota_B(\xi_k), \ \forall a, x. \tag{13}$$

Since $\sum_{a=1}^{o_A} \delta_{a,J_k(x)} = 1$ for all $x \in S_m$ and $k \in \{1, 2, \cdots, N_A\}$, summing over $a$ on both sides of Eq. (13) yields $\sum_{k=1}^{N_A} \iota_B(\xi_k) = 1$. Taking $\pi_k = \iota_B(\xi_k)$, $\sigma_k = \frac{1}{\pi_k}\xi_k$ and $P_A(a|x,k) = \delta_{a,J_k(x)}$, we can easily check that $\{\pi_k\}_{k=1}^{N_A}$ is a probability distribution, $\{\sigma_k\}_{k=1}^{N_A}$ is a set of normalized states, and $\{P_A(a|x,k)\}_{a=1}^{o_A}$ is a probability distribution. Thus, it follows from Eq. (11) that

$$\hat{\omega}^{AB}(e^{a|x}) = \sum_{k=1}^{N_A} \pi_k P_A(a|x,k)\sigma_k, \ \forall a, x. \tag{14}$$

From Definition 4.1 $\omega^{AB}$ is unsteerable from $A$ to $B$ with $\mathcal{M}_A$. □

## 5. Conclusions

The usual Bell nonlocality and EPR steering are two kinds of important quantum correlations which received much attention during the past few years. Their characterizations have been studied extensively by many authors. In this paper, we have investigated the Bell nonlocality and steering in the framework of generalized probabilistic theories. The characterizations of Bell nonlocality and steering in the context of generalized probabilistic theories have been derived. The results may shed new light on the informational features of both Bell nonlocality and steering.


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