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**Computing zero-dimensional tropical  
varieties via projections**

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# COMPUTING ZERO-DIMENSIONAL TROPICAL VARIETIES VIA PROJECTIONS

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ABSTRACT. We present an algorithm for computing zero-dimensional tropical varieties using projections. Our main tools are fast unimodular transforms of lexicographical Gröbner bases. We prove that our algorithm requires only a polynomial number of arithmetic operations if given a Gröbner basis, and we demonstrate that our implementation compares favourably to other existing implementations. Applying it to the computation of general positive-dimensional tropical varieties, we argue that the complexity for calculating tropical links is dominated by the complexity of the Gröbner walk.

## 1. INTRODUCTION

Tropical varieties are piecewise linear structures which arise from polynomial equations. They appear naturally in many areas of mathematics and beyond, such as geometry [Mik05], combinatorics [AK06; FS05], and optimisation [ABGJ18], as well as phylogenetics [SS04; LMY18], celestial mechanics [HM06; HJ11], and auction theory [BK19; TY15]. Wherever they emerge, tropical varieties often provide a fresh insight into existing computational problems, which is why efficient algorithms and optimised implementations are of great importance.

Computing tropical varieties from polynomial ideals is a fundamentally important yet algorithmically challenging task, requiring sophisticated techniques from computational algebra and convex geometry. Currently, GFAN [Jen17] and SINGULAR [DGPS19] are the only two programs capable of computing general tropical varieties. Both programs rely on a traversal of the Gröbner complex as initially suggested by Bogart, Jensen, Speyer, Sturmfels, and Thomas [BJSST07], and for both programs the initial bottleneck had been the computation of so-called tropical links. Experiments suggest that this bottleneck was resolved with the recent development of new algorithms [Cha13; HR18]. However the new approaches still rely on computations that are known to be very hard, [Cha13] on elimination and [HR18] on root approximation to an unknown precision.

In this paper, we study the computation of zero-dimensional tropical varieties, which is the key computational ingredient in [HR18], but using projections, which

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is the key conceptual idea in [Cha13]. We create a new algorithm for computing zero-dimensional tropical varieties that only requires a polynomial amount of field operations if we start with a Gröbner basis, and whose timings compare favourably with other implementations even if we do not. In particular, we argue that in the computation of general tropical varieties, the calculation of so-called tropical links becomes computationally insignificant compared to the Gröbner walk required to traverse the tropical variety.

Note that projections are a well-studied approach in polynomial systems solving, see [Stu02; DE05] for an overview on various techniques. Our approach can be regarded as a non-Archimedean analogue of that strategy, since tropical varieties can be regarded as zeroth-order approximation of the solutions in the topology induced by the valuation.

Our paper is organised as follows: In Section 3, we introduce a special class of unimodular transformations and study how they act on generic lexicographical Gröbner bases. In Section 4, we explain our main algorithm for reconstructing zero-dimensional tropical varieties from their projections, while Section 5 touches upon some technical details of the implementation. In Section 6, we compare the performance of our algorithm against the root approximation approach, while Section 7 analyses the complexity of our algorithm.

Implementations of all our algorithms can be found in the SINGULAR library `tropicalProjection.lib`. Together with the data for the timings, it is available at <https://software.mis.mpg.de>, and will also be made publicly available as part of the official SINGULAR distribution.

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## 2. BACKGROUND

For the sake of notation, we briefly recall some basic notions of tropical algebraic geometry and computational algebra that are of immediate relevance to us. In tropical geometry, our notation closely follows that of [MS15].

**Convention 2.1** For the remainder of the article, let  $K$  be a field with non-trivial valuation  $\nu: K^* \rightarrow \mathbb{R}$  and fix a multivariate polynomial ring  $K[\mathbf{x}] := K[x_1, \dots, x_n]$  as well as a multivariate Laurent polynomial ring  $K[\mathbf{x}^\pm] := K[x_1^\pm, \dots, x_n^\pm]$ .

Moreover, given a Laurent polynomial ideal  $I \subseteq K[\mathbf{x}^\pm]$ , we call a finite subset  $G \subseteq I$  a *Gröbner basis* with respect to a monomial ordering  $\prec$  on  $K[\mathbf{x}]$  if  $G$  consists

of polynomials and forms a Gröbner basis of the polynomial ideal  $I \cap K[\mathbf{x}]$  with respect to  $\prec$  in the conventional sense, see for example [GP02, §1.6]. All our Gröbner bases are reduced.

Finally, a *lexicographical Gröbner basis* will be a Gröbner basis with respect to the lexicographical ordering  $\prec_{\text{lex}}$  with  $x_n \prec_{\text{lex}} \cdots \prec_{\text{lex}} x_1$ .

For the purposes of this article, the following definition of tropical varieties in terms of coordinate-wise valuations of points in solution sets suffices.

**Definition 2.2** (Tropical variety) Let  $I \subseteq K[\mathbf{x}^\pm]$  be a Laurent polynomial ideal. The *tropical variety*  $\text{Trop}(I) \subseteq \mathbb{R}^n$  is given by

$$\text{Trop}(I) := \text{cl}\left(\{(\widehat{\nu}^{\text{al}}(p_1), \dots, \widehat{\nu}^{\text{al}}(p_n)) \in \mathbb{R}^n \mid (p_1, \dots, p_n) \in V_{\widehat{K}^{\text{al}}}(I)\}\right),$$

where  $\widehat{K}^{\text{al}}$  denotes the algebraic closure of the completion of  $K$ , so that  $\nu$  extends uniquely to a valuation  $\widehat{\nu}^{\text{al}}$  on  $\widehat{K}^{\text{al}}$ ,  $V_{\widehat{K}^{\text{al}}}(I)$  denotes the affine variety of  $I$  over  $\widehat{K}^{\text{al}}$ , and  $\text{cl}(\cdot)$  is the closure in the euclidean topology.

In this article, our focus lies on zero-dimensional ideals  $I \subseteq K[\mathbf{x}^\pm]$ , in which case  $\text{Trop}(I)$  is a finite set of  $\deg(I)$  points if each point  $w \in \text{Trop}(I)$  is counted with the multiplicity corresponding to the number of solutions  $p \in V_{\widehat{K}^{\text{al}}}(I)$  with  $\widehat{\nu}^{\text{al}}(p) = w$ .

In the univariate case, the tropical variety of an ideal  $I = (f) \subseteq K[x_1^\pm]$  simply consists of the negated slopes in the Newton polygon of  $f$  [Neu99, Proposition II.6.3]. Our approach for computing zero-dimensional tropical varieties of multivariate ideals is based on reducing computations to the univariate case.

**Definition 2.3** We say that a zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$  is in *shape position* if the projection onto the last coordinate  $p_n: (K^*)^n \rightarrow K^*$ ,  $(a_1, \dots, a_n) \mapsto a_n$  defines a closed embedding  $p_n|_{V(I)}: V(I) \hookrightarrow K^*$ .

In this article, we will concentrate on ideals that are in shape position. Lemma 2.4 shows an easy criterion to decide whether a given ideal is in shape position, while Lemma 2.5 shows how to coax degenerate ideals into shape position.

**Lemma 2.4** ([CLO05, §4 Exercise 16]) *A zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$  is in shape position if and only if its (reduced) lexicographical Gröbner basis is of the form*

$$G = \{f_n, x_{n-1} - f_{n-1}, \dots, x_2 - f_2, x_1 - f_1\} \quad (\text{SP})$$

*for some univariate polynomials  $f_1, \dots, f_n \in K[x_n]$ . The polynomials  $f_i$  are unique.*

**Lemma 2.5** *Let  $I \subseteq K[\mathbf{x}^\pm]$  be a zero-dimensional ideal. Then there exists a euclidean dense open subset  $\mathcal{V} \subseteq \mathbb{R}^{n-1}$  such that for any  $(u_1, \dots, u_{n-1}) \in \mathcal{V} \cap \mathbb{Z}^{n-1}$  the unimodular transformation*

$$\Phi_u: K[\mathbf{x}^\pm] \rightarrow K[\mathbf{x}^\pm], \quad x_i \mapsto \begin{cases} x_i & \text{if } i < n, \\ x_n \prod_{i=1}^{n-1} x_i^{u_i} & \text{if } i = n \end{cases}$$

maps  $I$  into an ideal in shape position.

*Proof.* Without loss of generality, we may assume that the field  $K$  is algebraically closed. For any  $u = (u_1, \dots, u_{n-1}) \in \mathbb{Z}^{n-1}$ , let  $f_u: (K^*)^n \rightarrow (K^*)^n$  be the torus automorphism induced by  $\Phi_u$ , so that  $V(\Phi_u(I)) = f_u^{-1}(V(I))$ . Then the transformed ideal  $\Phi_u(I)$  is in shape position if and only if the map  $p_n \circ f_u^{-1}: (K^*)^n \rightarrow K^*$ ,  $(a_1, \dots, a_n) \mapsto a_n \cdot \prod_{i=1}^{n-1} a_i^{-u_i}$  is injective on the finite set  $V(I)$ .

For  $a \in (K^*)^n \setminus \{(1, \dots, 1)\}$ , the set  $N_a := \{w \in \mathbb{Z}^n \mid a_1^{w_1} \dots a_n^{w_n} = 1\}$  is a  $\mathbb{Z}^n$ -sublattice of positive corank. Hence,  $W_a := \{w \in \mathbb{R}^{n-1} \mid (w_1, \dots, w_{n-1}, -1) \in N_a \otimes_{\mathbb{Z}} \mathbb{R}\}$  is a proper affine subspace of  $\mathbb{R}^{n-1}$ . By definition, for any two elements  $b \neq c \in V(I)$ , we have

$$p_n \circ f_u^{-1}(b) = p_n \circ f_u^{-1}(c) \iff (u_1, \dots, u_{n-1}) \in W_{b^{-1}c}.$$

Thus,  $\Phi_u(I)$  is in shape position if and only if  $(u_1, \dots, u_{n-1}) \in \mathcal{V} \cap \mathbb{Z}^{n-1}$ , where  $\mathcal{V} := \mathbb{R}^{n-1} \setminus \bigcup_{b \neq c \in V(I)} W_{b^{-1}c}$  is a euclidean dense open subset of  $\mathbb{R}^{n-1}$ .  $\square$

### 3. UNIMODULAR TRANSFORMATIONS ON LEXICOGRAPHICAL GRÖBNER BASES

In this section, we introduce a special class of unimodular transformations and describe how they operate on lexicographical Gröbner bases in shape position.

**Definition 3.1** We will consider unimodular transformations indexed by the set

$$\mathcal{U} := \{u \in \mathbb{Z}^n \mid \exists 1 \leq \ell \leq n : u_\ell = -1 \text{ and } u_i \geq 0 \text{ for all } i \neq \ell\}.$$

For any  $u \in \mathcal{U}$ , we define a unimodular ring automorphism

$$\varphi_u: K[\mathbf{x}^\pm] \rightarrow K[\mathbf{x}^\pm], \quad x_i \mapsto \begin{cases} x_1^{u_1} \cdots x_{\ell-1}^{u_{\ell-1}} x_\ell x_{\ell+1}^{u_{\ell+1}} \cdots x_n^{u_n} & \text{if } i = \ell, \\ x_i & \text{otherwise,} \end{cases}$$

and a linear projection

$$\pi_u: \mathbb{R}^n \rightarrow \mathbb{R}, \quad (w_1, \dots, w_n) \mapsto - \sum_{i=1}^n u_i w_i.$$

We call such a  $\varphi_u$  a *slim (unimodular) transformation concentrated at  $\ell$* .

While our slim unimodular transformations might seem overly restrictive, the next lemma states that they are sufficient to compute arbitrary projections of tropical varieties, which is what we will need in Section 4.

**Lemma 3.2** *Let  $\varphi_u$  be a slim transformation concentrated at  $\ell$ . Then*

$$\pi_u(\text{Trop}(I)) = \text{Trop}(\varphi_u(I) \cap K[x_\ell^\pm]).$$

*Proof.* We may assume that  $K$  is algebraically closed. The ring automorphism  $\varphi_u$  induces a torus automorphism  $f_u: (K^*)^n \rightarrow (K^*)^n$  with  $f_u^{-1}(V(I)) = V(\varphi_u(I))$ , which in turn induces a linear transformation  $h_u: \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$  mapping  $\text{Trop}(\varphi_u(I))$  to  $\text{Trop}(I)$ :

$$\begin{array}{ccc}
 x_\ell \prod_{i \neq \ell} x_i^{u_i} & \longleftarrow & x_\ell \\
 K[\mathbf{x}^\pm] & \xleftarrow{\varphi_u} & K[\mathbf{x}^\pm] \\
 & \text{induces} & \\
 (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_\ell \cdot \prod_{i \neq \ell} z_i^{u_i}, \dots, z_n) \\
 (K^*)^n & \xrightarrow{f_u} & (K^*)^n \\
 \downarrow \nu & & \downarrow \nu \\
 \mathbb{R}^n & \xrightarrow{h_u} & \mathbb{R}^n \\
 (w_1, \dots, w_n) & \longmapsto & (w_1, \dots, w_\ell + \sum_{i \neq \ell} u_i w_i, \dots, w_n)
 \end{array}$$

Hence, with  $p_\ell: \mathbb{R}^n \rightarrow \mathbb{R}$  denoting the projection onto the  $\ell$ -th coordinate:

$$\text{Trop}(\varphi_u(I) \cap K[x_\ell^\pm]) = p_\ell(\text{Trop}(\varphi_u(I))) = (p_\ell \circ h_u^{-1})(\text{Trop}(I)) = \pi_u(\text{Trop}(I)). \quad \square$$

The following easy properties of slim unimodular transformations serve as a basic motivation for their inception. They map polynomials to polynomials, which is important when working with software which only supports polynomial data. Moreover, they preserve saturation and shape position for zero-dimensional ideals, which is valuable as saturating and restoring shape position as in Lemma 2.5 are two expensive operations.

**Lemma 3.3** *For any slim transformation  $\varphi_u$  and any zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$ , we have*

- (1)  $\varphi_u(K[\mathbf{x}]) \subseteq K[\mathbf{x}]$ ,
- (2)  $\varphi_u(I) \cap K[\mathbf{x}] = \varphi_u(I \cap K[\mathbf{x}])$ ,
- (3)  $I$  in shape position  $\Leftrightarrow \varphi_u(I)$  in shape position.

*Proof.* From the definition, it is clear that polynomials get mapped to polynomials under  $\varphi_u$ , showing (1). In particular,  $\varphi_u|_{K[\mathbf{x}]}: K[\mathbf{x}] \rightarrow K[\mathbf{x}]$  induces a morphism  $\hat{f}_u: K^n \rightarrow K^n$  which on the torus  $(K^*)^n$  restricts to an automorphism  $f_u$  with  $\hat{f}_u^{-1}(V(I)) = V(\varphi_u(I))$ . To show (2), we need to see that  $\hat{f}_u^{-1}(V(I \cap K[\mathbf{x}])) \subseteq K^n$  does not have irreducible components supported outside the torus  $(K^*)^n$ . Now,  $V(I \cap K[\mathbf{x}])$  is the closure of  $V(I) \subseteq (K^*)^n$  in  $K^n$ , so by zero-dimensionality of  $I$ , we have  $V(I \cap K[\mathbf{x}]) \subseteq (K^*)^n$ . Since  $\hat{f}_u^{-1}((K^*)^n) = (K^*)^n$ , this proves (2). Finally, we note that  $\varphi_u(x_n) = x_n$ , so we have  $p_n \circ \varphi_u = p_n$ , where  $p_n: (K^*)^n \rightarrow K^*$  denotes the projection onto the last coordinate. Hence,  $p_n|_{V(I)}$  is a closed embedding if and only if  $p_n|_{V(\varphi_u(I))}$  is, proving (3).  $\square$

The next Algorithm 3.4 allows us to efficiently transform a lexicographical Gröbner basis of  $I$  into a lexicographical Gröbner basis of  $\varphi_u(I)$ . This is the main advantage of slim unimodular transformations, which we will leverage to compute  $\pi_u(\text{Trop}(I))$ .

**Algorithm 3.4** (Slim unimodular transformation of a Gröbner basis)

**Input:**  $(\varphi_u, G)$ , where

- $\varphi_u$  is a slim transformation concentrated at  $\ell$ ,
- $G = \{f_n, x_{n-1} - f_{n-1}, \dots, x_2 - f_2, x_1 - f_1\}$  is the lexicographical Gröbner basis of an ideal  $I \subseteq K[\mathbf{x}^\pm]$  in shape position as in (SP).

**Output:**  $G'$ , the lexicographical Gröbner basis of  $\varphi_u(I)$ .

- 1: In the univariate polynomial ring  $K[x_n]$ , compute  $f'_\ell$  with  $\deg(f'_\ell) < \deg(f_n)$  and

$$f'_\ell \equiv \left( x_n^{u_n} \cdot \prod_{\substack{i=1 \\ i \neq \ell}}^{n-1} f_i^{u_i} \right)^{-1} \cdot f_\ell \pmod{f_n}.$$

- 2: **return**  $G' := \{f_n, x_{n-1} - f_{n-1}, \dots, x_\ell - f'_\ell, \dots, x_1 - f_1\}$ .

*Correctness of Algorithm 3.4.* The polynomial ideal  $I \cap K[\mathbf{x}]$  is saturated with respect to the product of variables  $x_1 \cdots x_n$ , and is by assumption generated by  $G$ . This implies that  $f_n$  is relatively prime to each  $f_i$  for  $i < n$  and to  $x_n$ . In particular, the inverse in  $K[x_n]/(f_n)$  showing up in the definition of  $f'_\ell$  is well-defined. The ideal  $\varphi_u(I) \subseteq K[\mathbf{x}^\pm]$  is generated by

$$\varphi_u(G) = \{f_n, x_{n-1} - f_{n-1}, \dots, \left( \prod_{\substack{i=1 \\ i \neq \ell}}^n x_i^{u_i} \right) x_\ell - f_\ell, \dots, x_1 - f_1\}.$$

Note that the expression  $(\prod_{i \neq \ell} x_i^{u_i}) x_\ell - f_\ell$  is equivalent to  $x_\ell - f'_\ell$  modulo the ideal  $(f_n, x_i - f_i \mid i \neq \ell, n)$ . It follows that  $\varphi_u(I)$  is generated by  $G'$ , and it is clear that  $G'$  is a lexicographical Gröbner basis.  $\square$

#### 4. COMPUTING ZERO-DIMENSIONAL TROPICAL VARIETIES VIA PROJECTIONS

In this section, we assemble our algorithm for computing  $\text{Trop}(I)$  from a zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$ . This is done in two stages, see Figure 1: In the first stage, we project  $\text{Trop}(I)$  onto all coordinate axes of  $\mathbb{R}^n$ . In the second stage, we iteratively glue the coordinate projections together until  $\text{Trop}(I)$  is fully assembled.

For the sake of simplicity, all algorithms contain some elements of ambiguity to minimise the level of technical detail. To see how these ambiguities are resolved in the actual implementation, see Section 5. Moreover, we will only consider  $\text{Trop}(I)$  as points in  $\mathbb{R}^n$  without multiplicities. It is straightforward to generalise the algorithms to work with  $\text{Trop}(I)$  as points in  $\mathbb{R}^n$  with multiplicities, which is how they are implemented in SINGULAR.

The following algorithm merges several small projections into a single large projection. For clarity, given a finite subset  $A \subseteq \{1, \dots, n\}$ , we use  $\mathbb{R}^A$  to denote the linear subspace of  $\mathbb{R}^n$  spanned by the unit vectors indexed by  $A$  and  $p_A$  to denote the projection  $\mathbb{R}^n \rightarrow \mathbb{R}^A$ . For  $w \in \mathbb{R}^n$  and  $I \subseteq K[\mathbf{x}^\pm]$ , we denote  $w_A := p_A(w) \in \mathbb{R}^A$  and  $\text{Trop}(I)_A := p_A(\text{Trop}(I)) \subseteq \mathbb{R}^A$ .



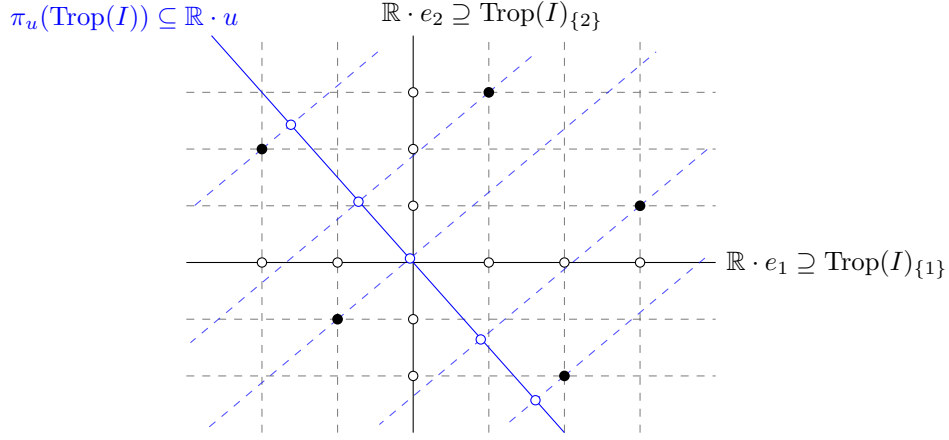


FIGURE 1. Computing zero-dimensional tropical varieties via projections.

**Algorithm 4.1** (gluing projections)

**Input:**  $(G, \text{Trop}(I)_{A_1}, \dots, \text{Trop}(I)_{A_k})$ , where

- $G$  is the lexicographical Gröbner basis of a zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$  in shape position as in (SP),
- $A_1, \dots, A_k \subseteq \{1, \dots, n\}$  are non-empty sets.

**Output:**  $\text{Trop}(I)_A \subseteq \mathbb{R}^A$ , where  $A := A_1 \cup \dots \cup A_k$ .

1: Construct the candidate set

$$T := \left\{ w \in \mathbb{R}^A \mid w_{A_i} \in \text{Trop}(I)_{A_i} \text{ for } i = 1, \dots, k \right\}.$$

2: Pick a slim transformation  $\varphi_u$  such that the following map is injective:

$$\pi_u|_T: T \rightarrow \mathbb{R}, \quad (w_i)_{i \in A} \mapsto - \sum_{i \in A} u_i w_i.$$

3: Using Algorithm 3.4, transform  $G$  into a Gröbner basis  $G'$  of  $\varphi_u(I)$ :

$$G' := \{f_n, x_{n-1} - f_{n-1}, \dots, x_\ell - f'_\ell, \dots, x_1 - f_1\}.$$

4: Compute the minimal polynomial  $\mu \in K[z]$  of  $\bar{f}'_\ell \in K[x_n]/(f_n)$  over  $K$  and read off  $\text{Trop}(\mu) \subseteq \mathbb{R}$  from its Newton polygon.

5: **return**  $\{w \in T \mid \pi_u(w) \in \text{Trop}(\mu)\}$ .

*Correctness of Algorithm 4.1.* First, we argue that Line 2 can be realised, i.e., we show the existence of a slim unimodular transformation  $\varphi_u$  such that  $\pi_u$  is injective on the candidate set  $T$ . Pick  $\ell \neq n$  and denote  $B := \{1, \dots, n\} \setminus \{\ell\}$ . It suffices to show that the set

$$Z := \{v \in \mathbb{R}_{\geq 0}^B \mid \pi_{v-e_\ell}|_T \text{ is injective}\} \subseteq \mathbb{R}^B$$

contains an integer point. By the definition of  $\pi_{v-e_\ell}$ , we see that

$$Z = \mathbb{R}_{\geq 0}^B \setminus \bigcup_{w \neq w' \in T} H_{w-w'}, \text{ where } H_{w-w'} := \left\{ v \in \mathbb{R}^B \mid \sum_{i \in B} (w_i - w'_i) v_i = z_\ell \right\}.$$

This describes  $Z$  as the complement of an affine hyperplane arrangement in  $\mathbb{R}^B$  inside the positive orthant. Therefore,  $Z$  must contain an integer point.

Next, we note that the candidate set  $T$  contains  $\text{Trop}(I)_A$  by construction, so injectivity of  $\pi_u|_T$  shows that  $\text{Trop}(I)_A = \{w \in T \mid \pi_u(w) \in \pi_u(\text{Trop}(I))\}$ . Therefore, the correctness of the output will follow from showing  $\pi_u(\text{Trop}(I)) = \text{Trop}(\mu)$ . By Lemma 3.2, it suffices to prove that  $\mu(x_\ell) \in K[x_\ell]$  generates the elimination ideal  $\varphi_u(I) \cap K[x_\ell^\pm]$ .

For this, we observe that reducing a univariate polynomial  $g \in K[x_\ell]$  with respect to the lexicographical Gröbner basis  $G'$  substitutes  $x_\ell$  by  $f'_\ell$  to obtain a univariate polynomial in  $K[x_n]$  and then reduces the result modulo  $f_n$ . In particular, this shows that such  $g \in K[x_\ell]$  lies in the ideal  $\varphi_u(I)$  if and only if  $g(\overline{f'_\ell}) = 0$  in  $K[x_n]/(f_n)$ . Hence, the elimination ideal  $\varphi_u(I) \cap K[x_\ell]$  is generated by  $\mu(x_\ell)$ .  $\square$

The next algorithm computes  $\text{Trop}(I)$  by projecting it onto all coordinate axes and gluing the projections together via Algorithm 4.1.

**Algorithm 4.2** (tropical variety via projections)

**Input:**  $G = \{f_n, x_{n-1} - f_{n-1}, \dots, x_2 - f_2, x_1 - f_1\}$ , the lexicographical Gröbner basis of a zero-dimensional ideal  $I \subseteq K[\mathbf{x}^\pm]$  in shape position as in (SP).

**Output:**  $\text{Trop}(I) \subseteq \mathbb{R}^n$

- 1: Compute the projection onto the last coordinate  $\text{Trop}(I)_{\{n\}} = \text{Trop}(f_n)$ .
- 2: **for**  $k \in \{1, \dots, n-1\}$  **do**
- 3:   Compute the minimal polynomial  $\mu_k \in K[z]$  of  $\overline{f_k} \in K[x_n]/(f_n)$  over  $K$  and read off the projection  $\text{Trop}(I)_{\{k\}} = \text{Trop}(\mu_k)$ .
- 4: Initialise a set of computed projections  $W := \{\text{Trop}(I)_{\{1\}}, \dots, \text{Trop}(I)_{\{n\}}\}$ .
- 5: **while**  $W \not\supseteq \text{Trop}(I)_{\{1, \dots, n\}}$  **do**
- 6:   Pick projections  $\text{Trop}(I)_{A_1}, \dots, \text{Trop}(I)_{A_k} \in W$  to be merged together such that  $\text{Trop}(I)_A \notin W$  for  $A := A_1 \cup \dots \cup A_k$ .
- 7:   Using Algorithm 4.1, compute  $\text{Trop}(I)_A$ .
- 8:    $W := W \cup \{\text{Trop}(I)_A\}$ .
- 9: **return**  $\text{Trop}(I)_{\{1, \dots, n\}}$ .

*Correctness of Algorithm 4.2.* Since  $G$  is the lexicographical Gröbner basis of  $I$ , the elimination ideal  $I \cap K[x_n^\pm]$  is generated by  $f_n$ , so we indeed have the equality  $\text{Trop}(I)_{\{n\}} = \text{Trop}(f_n)$  in Line 1. The equality  $\text{Trop}(I)_{\{k\}} = \text{Trop}(\mu_k)$  in Line 3 holds because  $\mu_k(x_\ell) \in K[x_\ell]$  generates the elimination ideal  $I \cap K[x_\ell^\pm]$  by the same argument as in the proof of correctness of Algorithm 4.1.

In every iteration of the **while** loop, the set  $W$  grows in size. Since there are only finitely many coordinate sets  $A \subseteq \{1, \dots, n\}$ , we will after finitely many iterations compute  $\text{Trop}(I) = \text{Trop}(I)_{\{1, \dots, n\}}$ , hence the while loop terminates.  $\square$

**Example 4.3** Consider  $K = \mathbb{Q}$  equipped with the 2-adic valuation and the ideal

$$I = \left( \underbrace{2 + x_3 + x_3^2 + x_3^3 + 2x_3^4}_{=:f_3}, x_2 - \underbrace{2x_3}_{=:f_2}, x_1 - \underbrace{4x_3}_{=:f_1} \right) \subseteq K[x_1^\pm, x_2^\pm, x_3^\pm].$$

This ideal is in shape position by Lemma 2.4. From the Newton polygon of  $f_3$ , see Figure 2 (left), it is not hard to see that

$$\begin{aligned} \text{Trop}(I)_{\{3\}} &= \text{Trop}(f_3) = \{-1, \mathbf{0}, 1\}, \\ \text{Trop}(I)_{\{2\}} &= \{\lambda + 1 \mid \lambda \in \text{Trop}(I)_{\{3\}}\} = \{0, \mathbf{1}, 2\}, \\ \text{Trop}(I)_{\{1\}} &= \{\lambda + 2 \mid \lambda \in \text{Trop}(I)_{\{3\}}\} = \{1, \mathbf{2}, 3\}, \end{aligned}$$

where points with multiplicity 2 are highlighted in bold. To merge  $\text{Trop}(I)_{\{1\}}$  and  $\text{Trop}(I)_{\{2\}}$ , we consider the following projection that is injective on the candidate set  $T := \text{Trop}(I)_{\{1\}} \times \text{Trop}(I)_{\{2\}}$ :

$$\pi_{(-1,3,0)} : T \longrightarrow \mathbb{R}, \quad (w_1, w_2) \longmapsto w_1 - 3w_2.$$

The corresponding unimodular transformation  $\varphi_{(-1,3,0)}$  sends  $x_1$  to  $x_1x_2^3$  and hence  $\varphi_{(-1,3,0)}(I)$  is generated by  $\{f_3, x_2 - f_2, x_1x_2^3 - 4x_3\}$ , which Algorithm 3.4 transforms into the following lexicographical Gröbner basis:

$$\varphi_{(-1,3)}(I) = \left( f_3, x_2 - f_2, x_1 - \underbrace{\left( \frac{1}{4}x_3^3 - \frac{3}{8}x_3^2 - \frac{1}{8}x_3 - \frac{1}{8} \right)}_{=:f'_1} \right).$$

The minimal polynomial of  $\overline{f'_1}$  in  $K[x_3]/(f_3)$  over  $K$  can be computed as the resultant

$$\text{Res}_{x_3}(f_3, x_1 - f'_1) = 8x_1^4 + 3x_1^3 + \frac{7}{2}x_1^2 + \frac{3}{4}x_1 + \frac{1}{2}.$$

Figure 2 (middle) shows the Newton polygon of the resultant, from which we see:

$$\text{Trop}(\text{Res}_{x_3}(f_3, x_1 - f'_1)) = \{-3, -\mathbf{1}, 1\}.$$

Thus,

$$\text{Trop}(I)_{\{1,2\}} = \{(3, 2), (\mathbf{2}, \mathbf{1}), (1, 0)\}.$$

To merge  $\text{Trop}(I)_{\{1,2\}}$  and  $\text{Trop}(I)_{\{3\}}$ , we consider the following projection that is injective on the candidate set  $T := \text{Trop}(I)_{\{1,2\}} \times \text{Trop}(I)_{\{3\}}$ :

$$\pi_{(-1,0,3)} : T \longrightarrow \mathbb{R}, \quad (w_1, w_2, w_3) \longmapsto w_1 - 3w_3.$$

The corresponding unimodular transformation  $\varphi_{(-1,0,3)}$  sends  $x_1$  to  $x_1x_3^3$  and hence  $\varphi_{(-1,0,3)}(I)$  is generated by  $\{f_3, x_2 - f_2, x_1x_3^3 - 4x_3\}$ , which Algorithm 3.4 transforms into the following lexicographical Gröbner basis:

$$\varphi_{(-1,0,3)}(I) = \left( f_3, x_2 - f_2, x_1 - \underbrace{(2x_3^3 - 3x_3^2 - x_3 - 1)}_{=:f''_1} \right).$$

Another resultant computation yields the minimal polynomial of  $\overline{f''_1} \in K[x_3]/(f_3)$  over  $K$ :

$$\text{Res}_{x_3}(f_3, x_1 - f''_1) = 8x_1^4 + 24x_1^3 + 224x_1^2 + 384x_1 + 2048.$$

Figure 2 (right) shows the Newton polygon of the resultant, from which we see:

$$\text{Trop}(\text{Res}_{x_3}(f_3, x_1 - f_1'')) = \{0, \mathbf{2}, 4\},$$

and thus

$$\text{Trop}(I) = \text{Trop}(I)_{\{1,2,3\}} = \{(3, 2, 1), (\mathbf{2}, \mathbf{1}, \mathbf{0}), (1, 0, -1)\}.$$

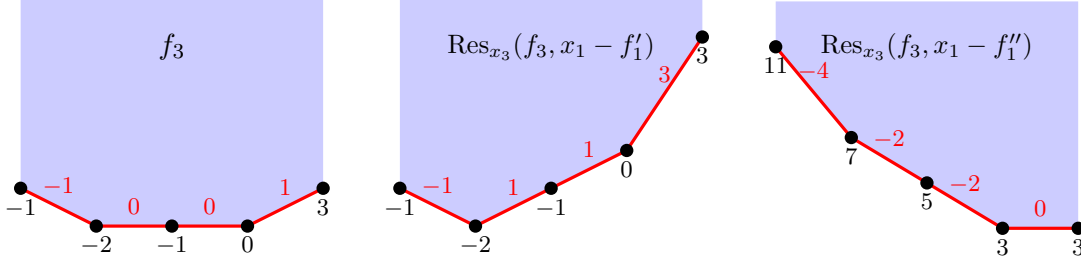


FIGURE 2. Newton polygons of  $f_3$  and the two resultants in Example 4.3. Below each vertex is its height, above each edge is its slope.

## 5. IMPLEMENTATION

In this section, we reflect on some design decisions that were made in the implementation of the algorithms in the SINGULAR library `tropicalProjection.lib`. While the reader who is only interested in the algorithms, their performance, and their complexity may skip this section without impeding their understanding, we thought it important to include this section for the reader who is interested in the actual implementation.

**5.1. Picking unimodular transformations in Algorithm 4.1 Line 2.** As  $\pi_u|_T$  is injective for generic  $u \in \mathcal{U}$ , it seems reasonable to sample random  $u \in \mathcal{U}$  until the corresponding projection is injective on the candidate set. Our implementation however iterates over all  $u \in \mathcal{U}$  in increasing  $\ell_1$ -norm until the smallest one with injective  $\pi_u|_T$  is found. This is made in an effort to keep the slim unimodular transformation  $\varphi_u(I)$  as simple as possible, since Lines 3–4 are the main bottlenecks of our algorithm.

**5.2. Transforming Gröbner bases in Algorithm 4.1 Line 3.** As mentioned before, Lines 3–4 are the main bottlenecks of our algorithm. Two common reasons why polynomial computations may scale badly are an explosion in degree or in coefficient size. The degree of the polynomials is not problematic in our algorithm, as using Algorithm 3.4 in Line 3 only incurs basic arithmetic operations in  $K[x_n]/(f_n)$  whose elements can be represented by polynomials of degree bounded by  $\deg(f_n)$ , while the degree of the minimal polynomial in Line 4 also is bounded by  $\deg(f_n)$ . Therefore, the only aspect that needs to be controlled in our computation is the size of the coefficients.

Coefficient explosion is a common problem for computing inverses in  $K[x_n]/(f_n)$  via the Extended Euclidean Algorithm [GG13, §6.1]. To make matters worse, the polynomial  $\bar{h} := \bar{x}_n^{-u_n} \cdot \prod_{i \neq \ell, n} \bar{f}_i^{u_i} \in K[x_n]/(f_n)$  to be inverted in Algorithm 3.4 usually already has large coefficients. However, we can exploit the fact that the minimal polynomial  $\sum_{i=0}^k a_i z^i$  of  $\bar{f}'_\ell \in K[x_n]/(f_n)$  is the reflection of the minimal polynomial  $\sum_{i=0}^k a_{k-i} z^i$  of  $(\bar{f}'_\ell)^{-1}$ . Instead of computing  $\bar{f}'_\ell = \bar{h}^{-1} \bar{f}_\ell$  in Algorithm 3.4, it therefore suffices to compute  $(\bar{f}'_\ell)^{-1} = \bar{h} \cdot (\bar{f}_\ell)^{-1} \in K[x_n]/(f_n)$ , which is easier as  $\bar{f}_\ell$  has generally smaller coefficients than  $\bar{h}$  and is independent of  $u$ , so its inversion modulo  $f_n$  is much faster.

**5.3. Computing minimal polynomials in Algorithm 4.1 Line 4.** The computation of minimal polynomials for elements in  $K[x_n]/(f_n)$  can be carried out in many different ways, for example using:

*Resultants:* We can compute the resultant of the two polynomials  $f_n$  and  $hx_\ell - f_\ell \in K[x_\ell, x_n]$  with respect to the variable  $x_n$  by standard resultant algorithms. The minimal polynomial  $\mu(x_\ell) \in K[x_\ell]$  is the squarefree part of the resultant.

*Linear Algebra:* Let  $k \in \mathbb{N}$  be minimal such that in the finite-dimensional  $K$ -vector space  $K[x_n]/(f_n)$  the set of polynomials  $\{\bar{h}^{d-i} \bar{f}_\ell^i \mid i = 0, \dots, k\}$  is linearly dependent, where  $d := \deg(f_n)$ . We can find a linear dependence  $\sum_{i=0}^k a_i \bar{h}^{d-i} \bar{f}_\ell^i = 0$  and conclude that  $\mu = \sum_{i=0}^k a_{k-i} z^i$ .

*Gröbner bases:* Note that  $\{f_n, x_\ell - f'_\ell\} \subseteq K[x_\ell, x_n]$  forms a Gröbner basis with respect to the lexicographical ordering with  $x_n \prec x_\ell$ . We can transform this to a Gröbner basis with respect to the lexicographical ordering with  $x_\ell \prec x_n$  using FGLM [FGLM93] and read off the eliminant  $\mu(x_\ell)$  as the generator of the elimination ideal  $(x_\ell - f'_\ell, f_n) \cap K[x_\ell]$ .

For polynomials with small coefficients, the implementation using SINGULAR's resultants seemed the fastest, but SINGULAR's FGLM seems to be best when dealing with very large coefficients.

For  $K = \mathbb{Q}$  however, we can use a modular approach thanks to the SINGULAR library `modular.lib` [Ste19]: It computes the minimal polynomial over  $\mathbb{F}_p$  for several primes  $p$  using any of the above methods, then lifts the results to  $\mathbb{Q}$ . This modular approach avoids problems caused by very large coefficients and works particularly well using the method based on linear algebra from above. We can check if the lifted  $\mu$  is correct by testing whether  $\mu(\bar{f}'_\ell) = 0$  in  $K[x_n]/(f_n)$ .

**5.4. Picking gluing strategies in Algorithm 4.2 Line 6.** Algorithm 4.2 is formulated in a flexible way: Different strategies of realising the choice of coordinate sets  $A_1, \dots, A_k$  in Line 6 can adapt to the needs of a specific tropicalization problem. The four gluing strategies that follow seem very natural and are implemented in our SINGULAR library. See Figure 3 for an illustration in the case  $n = 5$ .

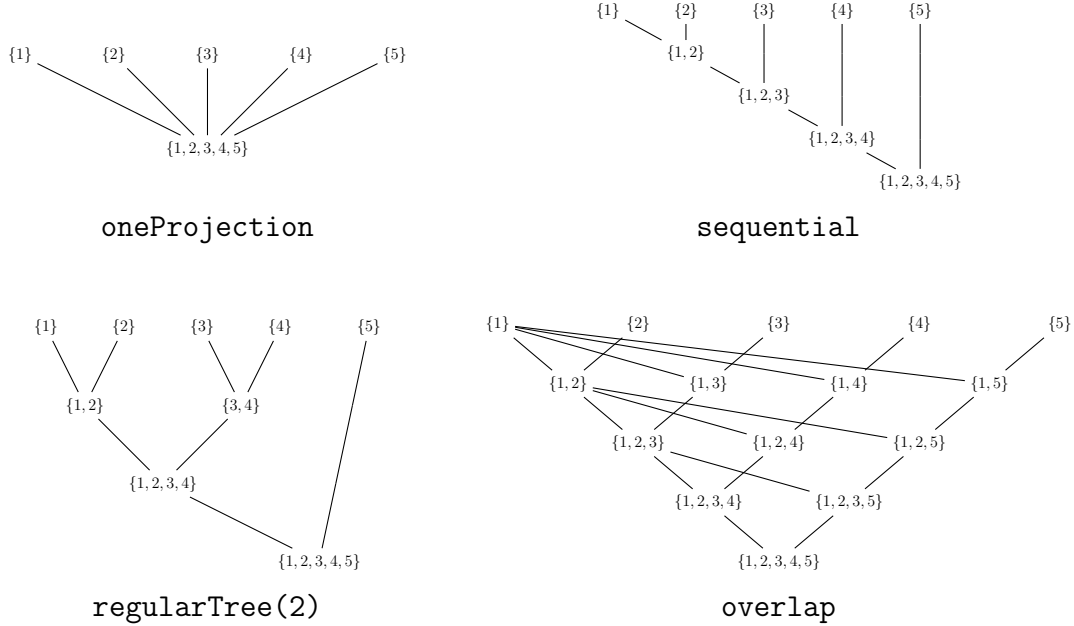


FIGURE 3. Visualisation of different gluing strategies.

**oneProjection:** Only a single iteration of the `while` loop, in which we pick  $k = n$  and  $A_i = \{i\}$  for  $i = 1, \dots, n$ .

**sequential:**  $n - 1$  iterations of the `while` loop, during which we pick  $k = 2$  and  $A_1 = \{1, \dots, i\}$  and  $A_2 = \{i + 1\}$  in the  $i$ -th iteration.

**regularTree( $k$ ):**  $n - 1$  iterations of the `while` loop, which can be partially run in parallel in  $\lceil \log_k n \rceil$  batches. In each batch we merge  $k$  of the previous projections.

**overlap:**  $(n - 1)n/2$  iterations of the `while` loop, which can be partially run in parallel in  $n - 1$  batches. During batch  $i$ , we pick  $k = 2$  and  $A_1 = \{1, \dots, i\}$ ,  $A_2 = \{1, \dots, i - 1, j\}$  for  $j > i$ .

**oneProjection** is the simplest strategy, requiring only one unimodular transformation. For examples of very low degree, it is the best strategy due to its minimal overhead. For examples of higher degree  $d$ , the candidate set  $T$  in Algorithm 4.1 can become quite large, at worst  $|T| = d^n$ . This generally leads to larger  $u \in \mathcal{U}$  in Line 2 and causes problems due to coefficient growth.

**sequential** avoids the problem of a large candidate set  $T$  by only gluing two projections at a time, guaranteeing  $|T| \leq d^2$ . This comes at the expense of computing  $n - 1$  unimodular transformations, but even for medium-sized instances we observe considerable improvements compared to **oneProjection**. In Section 7, we prove that **sequential** guarantees good complexity bounds on Algorithm 4.2.

**regularTree( $k$ )** can achieve considerable speed-up by parallelisation. Whereas every `while`-iteration in **sequential** depends on the output of the previous iteration,

`regularTree(k)` allows us to compute all gluings in parallel in  $\lceil \log_k n \rceil$  batches. The total number of gluings remains the same.

`overlap` further reduces the size of the candidate set  $T$  compared to `sequential`, while exploiting parallel computation like `regularTree(k)`. It glues projections two at a time, but only those  $A_1$  and  $A_2$  which overlap significantly. This can lead to much smaller candidate sets  $T$ , at best  $|T| = d$  which makes a unimodular transformation obsolete. The strategy `overlap` seems particularly successful in practice and is the one used for the timings in Section 6.

Our implementation in SINGULAR also allows for custom gluing strategies by means of specifying a graph as in Figure 3.

## 6. TIMINGS

In this section we present timings of our SINGULAR implementation of Algorithm 4.2 for  $K = \mathbb{Q}$  and the 2-adic valuation. We compare it to a MAGMA [BCP97] implementation which approximates the roots in the 2-adic norm. While SINGULAR is also capable of the same task, we chose to compare to MAGMA instead as it is significantly faster due to its finite precision arithmetic over  $p$ -adic numbers. Our SINGULAR timings use the `overlap` strategy, a modular approach and parallelisation with up to four threads. The SINGULAR times we report on are total CPU times across all threads (for reference, the longest example in SINGULAR required 118 seconds total CPU time, but only 32 seconds real time). All computations were run on a server with 2 Intel Xeon Gold 6144 CPUs, 384GB RAM and Debian GNU/Linux 9.9 OS. All examples and scripts are available at <https://software.mis.mpg.de>.

**6.1. Random lexicographical Gröbner bases in shape position.** Given natural numbers  $d$  and  $n$ , a random lexicographical Gröbner basis  $G$  of an ideal  $I \subseteq \mathbb{Q}[x_1, \dots, x_n]$  of degree  $d$  in shape position will be a Gröbner basis of the form

$$G = \{f_n, x_{n-1} - f_{n-1}, x_{n-2} - f_{n-2}, \dots, x_2 - f_2, x_1 - f_1\},$$

where  $f_n, f_{n-1}, f_{n-2}, \dots, f_1$  are univariate polynomials in  $x_n$  of degree  $d, d-1, d-1, \dots, d-1$  respectively with coefficients of the form  $2^\lambda \cdot (2k+1)$  for a random  $\lambda \in \{0, \dots, 99\}$  and a random  $k \in \{0, \dots, 4999\}$ .

Figure 4 shows timings for  $n = 5$  and varying  $d$ . Each computation was aborted if it failed to terminate within one hour. We see that MAGMA is significantly faster for small examples, while SINGULAR scales better with increasing degree.

For many of the ideals  $I$  however,  $\text{Trop}(I)$  has fewer than  $d$  distinct points. This puts our algorithm at an advantage, as it allows for easier projections in Algorithm 4.2 Line 2. Mathematically, it is not an easy task to generate non-trivial examples with distinct tropical points. Picking  $f_n$  to have  $d$  roots with distinct valuation for example would make all roots live in  $\mathbb{Q}_2$ , in which case MAGMA terminates instantly. Our next special family of examples has criteria which guarantee distinct points.

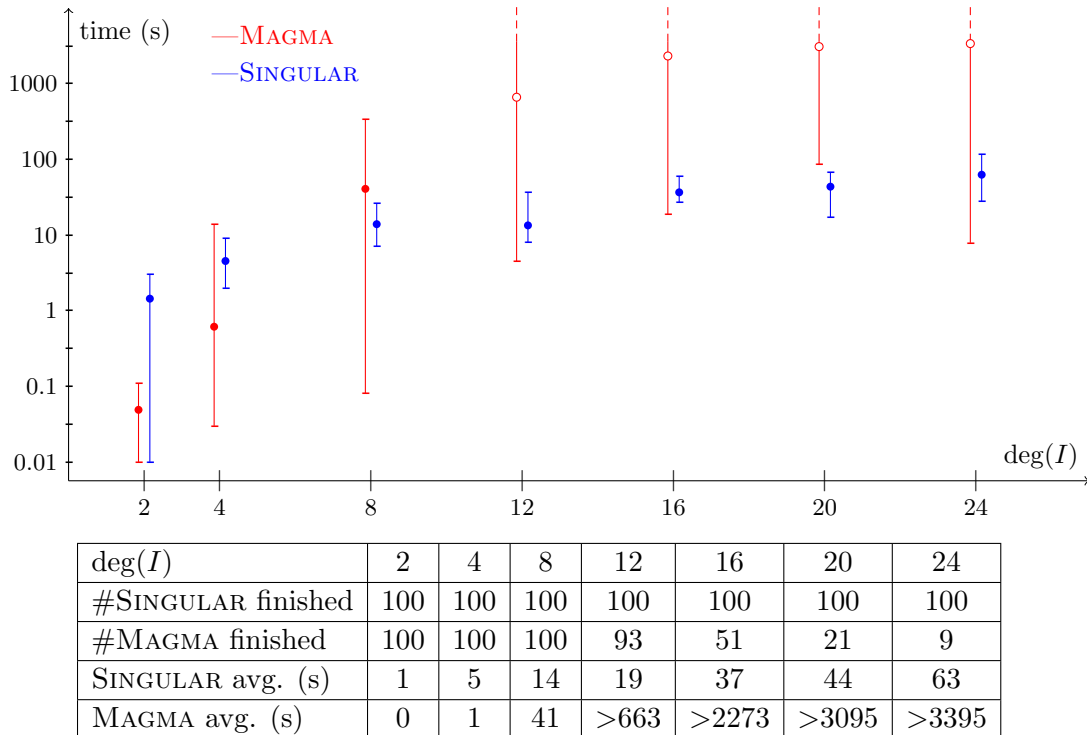


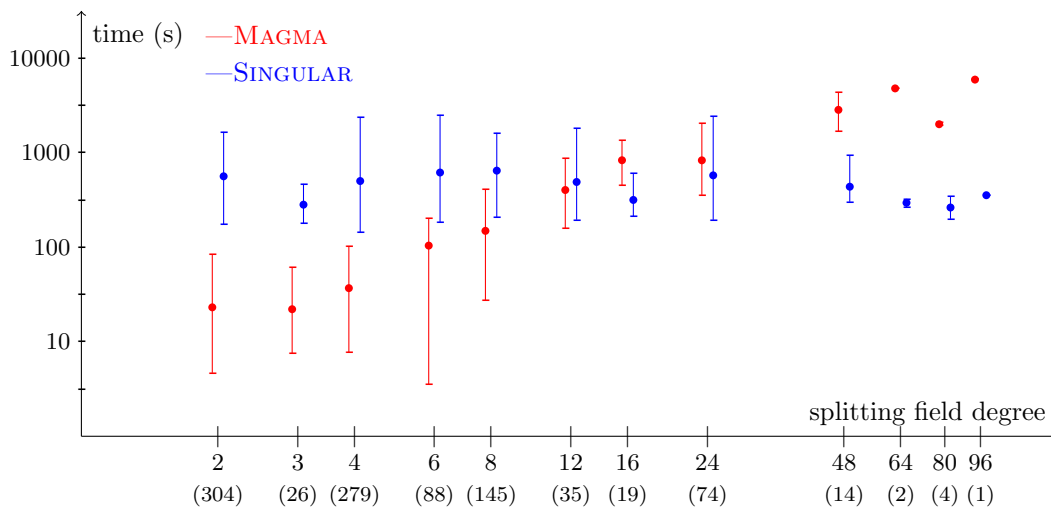
FIGURE 4. Timings for the randomly generated ideals in shape position.

**6.2. Tropical lines on a random honeycomb cubic.** Let  $V(f) \subseteq \mathbb{P}^3$  be a smooth cubic surface. In [PV19], it is shown that  $\text{Trop}(f) \subseteq \mathbb{R}^3$  may contain infinitely many tropical lines. However, for general  $f$  whose coefficient valuations induce a honeycomb subdivision of its Newton polytope,  $\text{Trop}(f)$  will always contain exactly 27 distinct tropical lines [PV19, Theorem 27], which must therefore be the tropicalizations of the 27 lines on  $V(f)$ .

We used POLYMAKE [GJ00] to randomly generate 1000 cubic polynomials with honeycomb subdivisions whose coefficients are pure powers of 2. For each cubic polynomial  $f$ , we constructed the one-dimensional homogeneous ideal  $\mathcal{L}_f \subseteq \mathbb{Q}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$  of degree 27 whose solutions are the lines on  $V(f)$  in Plücker coordinates. Figure 5 shows the timings for computing  $\text{Trop}(L_f)$ , where  $L_f := \mathcal{L}_f + (p_{34} - 1)$  is a zero-dimensional ideal of degree 27. Out of our 1000 random cubics, 8 had to be discarded because  $L_f$  was of lower degree, i.e.,  $V(f)$  contained lines with  $p_{34} = 0$ .

Unsurprisingly, the SINGULAR timings are relatively stable, while the MAGMA timings heavily depend on the degree of the splitting field of  $L_f$  over  $\mathbb{Q}_2$ . Over  $\mathbb{Q}$ , the generic splitting field degree would be 51840 [EJ12]. Over  $\mathbb{Q}_2$ , the distinct tropical points of  $\text{Trop}(L_f)$  severely restrict the Galois group of the splitting field.





splitting deg.	2	3	4	6	8	12	16	24	48	64	80	96
Frequency	304	26	279	88	145	35	19	74	14	2	4	1
SINGULAR avg.	556	281	505	610	651	490	313	580	440	294	261	352
MAGMA avg.	23	22	37	104	149	403	831	830	2840	4791	1998	5935

FIGURE 5. Timings for the 27 tropical lines on a tropical honeycomb cubic.

## 7. COMPLEXITY

In this section, we bound the complexity for computing a zero-dimensional tropical variety from a given Gröbner basis using Algorithm 4.2 with the `sequential` strategy. We show that the number of required arithmetic operations is polynomial in the degree of the ideal and the ambient dimension. Based on this, we argue that the complexity of computing a higher-dimensional tropical variety is dominated by the Gröbner walk required to traverse the Gröbner complex, as the computation of a tropical link is essentially polynomial time in the aforementioned sense.

**Convention 7.1** For the remainder of the section, consider a zero-dimensional ideal  $I \subseteq K[x_1^\pm, \dots, x_n^\pm]$  of degree  $d$  and assume  $\nu(K^*) \subseteq \mathbb{Q}$ , so that  $\text{Trop}(I) \subseteq \mathbb{Q}^n$ .

For the sake of convenience, we recall some results on the complexity of arithmetic operations over algebraic extensions, a well-studied topic in the area of computational algebra.

**Proposition 7.2** ([GG13, Corollary 4.6 + Section 4.3 + Exercise 12.10]) *Let  $f, g \in K[z]$  be two univariate polynomials of degree  $\leq d$ . Then:*

- (1) *Addition, multiplication and inversion in  $K[z]/(f)$  require  $\mathcal{O}(d^2)$  arithmetic operations in  $K$ .*
- (2) *Computing the  $k$ -th power of  $\bar{g} \in K[z]/(f)$  requires  $\mathcal{O}(d^2 \log k)$  arithmetic operations in  $K$ .*

(3) Computing the minimal polynomial of  $\bar{g} \in K[z]/(f)$  requires  $\mathcal{O}(d^2 \log d \log \log d)$  arithmetic operations in  $K$ .

**Proposition 7.3** *Algorithm 3.4, which computes the lexicographical Gröbner basis of  $\varphi_u(I)$  for some slim transformation  $\varphi_u$ , requires  $\mathcal{O}(d^2 \sum_{u_i > 0} (1 + \log u_i))$  arithmetic operations in  $K$ .*

*Proof.* We need to count the number of field operations in which the following polynomial  $f'_\ell \in K[x_n]$  can be computed:

$$f'_\ell \equiv \left( x_n^{u_n} \cdot \prod_{\substack{i=1 \\ i \neq \ell}}^{n-1} f_i^{u_i} \right)^{-1} \cdot f_\ell \equiv \left( x_n^{u_n} \cdot f_\ell^{-1} \cdot \prod_{\substack{i=1 \\ i \neq \ell}}^{n-1} f_i^{u_i} \right)^{-1} \pmod{f_n}.$$

Denoting  $k := |\{i \in \{1, \dots, n\} \mid u_i \neq 0\}|$ , this entails the following operations in  $K[x_n]/(f_n)$ :

- $k - 1$  exponentiations  $x_n^{u_n}$  and  $f_i^{u_i}$  for  $i \neq \ell, n$ .
- 1 inversion for  $f_\ell$ ,
- $k - 1$  multiplications for the product of  $f_\ell^{-1}$ ,  $x_n^{u_n}$  and all other  $f_i^{u_i}$ ,
- 1 final inversion.

An exponentiation to the power  $u_i$  requires  $\mathcal{O}(d^2 \log u_i)$  arithmetic operations in  $K$ , while every other operation requires  $\mathcal{O}(d^2)$  arithmetic operations in  $K$  by Proposition 7.2. In total, the number of required field operations in  $K$  is

$$\mathcal{O}\left(d^2 \sum_{u_i > 0} (1 + \log u_i) + d^2 + d^2(k - 1) + d^2\right) = \mathcal{O}\left(d^2 \sum_{u_i > 0} (1 + \log u_i)\right). \quad \square$$

**Lemma 7.4** *Let  $X, Y \subseteq \mathbb{Q}$  be finite sets of cardinality  $\leq d$ . Then there exists a non-negative integer  $m \leq \binom{d^2}{2}$  such that  $X \times Y \rightarrow \mathbb{Q}$ ,  $(a, b) \mapsto a - mb$  is injective. The smallest such  $m$  can be found in  $\mathcal{O}(d^4)$  arithmetic operations in  $\mathbb{Q}$ .*

*Proof.* The map  $(a, b) \mapsto a - mb$  will fail to be injective if and only if there exists a pair of points in  $X \times Y$  lying on an affine line with slope  $m$ . Since there are at most  $\binom{d^2}{2}$  pairs of points, the statement follows by the pigeonhole principle.

We can determine all integral slopes attained by a line between any two points of  $X \times Y$  with  $\mathcal{O}\left(\binom{d^2}{2}\right) = \mathcal{O}(d^4)$  arithmetic operations in  $\mathbb{Q}$ . Picking the smallest natural number not occurring among these slopes gives the desired  $m$ .  $\square$

**Proposition 7.5** *Let  $k \in \{2, \dots, n\}$  and assume that the following are known from a previous call of Algorithm 4.1 within Algorithm 4.2 running the **sequential** strategy:*

- $\text{Trop}(I)_{\{1, \dots, k-1\}}$  and  $\text{Trop}(I)_{\{k\}}$ ,
- a slim transformation  $\varphi_v$  concentrated at  $\ell$  with  $v_i = 0$  for  $i \geq k$  such that  $\pi_v$  is injective on  $\text{Trop}(I)_{\{1, \dots, k-1\}}$ ,
- the lexicographical Gröbner basis of  $\varphi_v(I)$ .

Then Algorithm 4.1 for gluing the two projections into  $\text{Trop}(I)_{\{1,\dots,k\}}$  requires only  $\mathcal{O}(d^2 \log d \log \log d)$  and  $\mathcal{O}(d^4)$  arithmetic operations in  $K$  and  $\mathbb{Q}$  respectively.

*Proof.* Applying Lemma 7.4 to  $X := \pi_v(\text{Trop}(I)_{\{1,\dots,k-1\}})$  and  $Y := \text{Trop}(I)_{\{k\}}$ , we can compute a minimal  $m \leq \binom{d^2}{2}$  such that  $(a, b) \mapsto a - mb$  is injective on  $X \times Y$  in  $\mathcal{O}(d^4)$  arithmetic  $\mathbb{Q}$ -operations. Setting  $w := v + me_k$ , this means that  $\pi_w$  is injective on  $\text{Trop}(I)_{\{1,\dots,k-1\}} \times \text{Trop}(I)_{\{k\}}$ .

Since  $\varphi_w(I) = \varphi_u(\varphi_v(I))$  for  $u := me_k - e_\ell$  and a lexicographical Gröbner basis of  $\varphi_v(I)$  is already known, we may compute the lexicographical Gröbner basis of  $\varphi_w(I)$  by applying Algorithm 3.4 to  $u$  and  $\varphi_v(I)$ . By Proposition 7.3, this requires  $\mathcal{O}(d^2 \log m) = \mathcal{O}(d^2 \log d)$  arithmetic operations in  $K$ .

By Proposition 7.2, computing the minimal polynomial of  $\bar{f}_\ell \in K[x_n]/(f_n)$  requires  $\mathcal{O}(d^2 \log d \log \log d)$  arithmetic operations in  $K$ , so the overall number of arithmetic  $K$ -operations in Algorithm 4.1 is also  $\mathcal{O}(d^2 \log d \log \log d)$ .  $\square$

**Theorem 7.6** *Algorithm 4.2, which computes the zero-dimensional tropical variety  $\text{Trop}(I)$ , with the **sequential** strategy requires  $\mathcal{O}(n d^2 \log d \log \log d)$  and  $\mathcal{O}(nd^4)$  arithmetic operations in  $K$  and  $\mathbb{Q}$  respectively.*

*Proof.* Algorithm 4.2 using the **sequential** strategy consists of the following operations:

- computing minimal polynomials of  $\bar{f}_k \in K[x_n]/(f_n)$  for  $k = 1, \dots, n-1$ ,
- applying Algorithm 4.1 to  $\text{Trop}(I)_{\{1,\dots,k-1\}}$  and  $\text{Trop}(I)_{\{k\}}$  for  $k = 2, \dots, n$ .

We may store the information on the unimodular transformation computed in iteration  $k-1$  during the computation of  $\text{Trop}(I)_{\{1,\dots,k-1\}}$  and this information may be used in the next iteration. Then Propositions 7.2 and 7.5 allow us to deduce the claimed bounds on arithmetic operations in Algorithm 4.2.  $\square$

**Remark 7.7** (Computing positive-dimensional tropical varieties) Currently, GFAN and SINGULAR are the only software systems capable of computing general tropical varieties, and both rely on a guided traversal of the Gröbner complex as introduced in [BJSST07]. Their frameworks roughly consist of two parts:

- (1) the Gröbner walk to traverse the tropical variety,
- (2) the computation of tropical links to guide the Gröbner walk.

While the computation of tropical links had been a major bottleneck of the original algorithm and in early implementations, experiments suggest that it has since been resolved by new approaches [Cha13; HR18]. However, the algorithm in [Cha13, §4.2] relies heavily on projections, while [HR18, Algorithm 2.10] relies on root approximations to an unknown precision, so neither approach has good complexity bounds. In fact, [HR18, Timing 3.9] shows that the necessary precision can be exponential in the number of variables.

Algorithm 4.2 was designed with [HR18, Algorithm 2.10] in mind, and with Theorem 7.6 we argue that the complexity of calculating tropical links as in [HR18, Algorithm 4.6] is dominated by the complexity of the Gröbner basis computations required for the Gröbner walk. In the following, let  $J \subseteq K[x_1^\pm, \dots, x_n^\pm]$  be a homogeneous ideal of codimension  $c$  and degree  $d$ .

- (1) The Gröbner walk requires Gröbner bases of initial ideals  $\text{in}_w(J)$  with respect to weight vectors  $w \in \text{Trop}(J)$  with  $\dim C_w(J) = \dim \text{Trop}(J) - 1$ , where  $C_w(J)$  denotes the Gröbner polyhedron of  $J$  around  $w$ . Note that  $\text{in}_w(J)$  is neither monomial since  $w \in \text{Trop}(J)$  nor binomial as  $\dim C_w(J) < \dim \text{Trop}(J)$ . Therefore, this is a general Gröbner basis computation which is commonly regarded as double exponential time.
- (2) Replacing [HR18, Algorithm 2.10] in [HR18, Algorithm 4.6] with our Algorithm 4.2 requires Gröbner bases of ideals of the form

$$\text{in}_w(J)|_{x_1=\dots=x_{c-1}=1, x_c=\lambda} \subseteq K[x_{c+1}, \dots, x_n],$$

where  $w$  is chosen as before and  $\lambda \in K$  is chosen to satisfy  $\nu(\lambda) = \pm 1$ . These ideals are zero-dimensional of degree at most  $d$ , and it is known that Gröbner bases of zero-dimensional ideals can be on average computed in polynomial time in the number of solutions [Lak91; LL91]. Thus the entire computation of tropical links can on average be done in polynomial time.

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