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ON THE ANISOTROPIC MOSER-TRUDINGER INEQUALITY FOR UNBOUNDED DOMAINS IN $\mathbb{R}^n$

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Abstract. In this paper, we investigate a sharp Moser-Trudinger inequality which involves the anisotropic Sobolev norm in unbounded domains. Under this anisotropic Sobolev norm, we establish the Lions type concentration-compactness alternative firstly. Then by using a blow-up procedure, we obtain the existence of extremal functions for this sharp geometric inequality. In particular, we combine the low dimension case of $n = 2$ and the high dimension case of $n \geq 3$ to prove the existence of the extremal functions, which is different from the arguments of isotropic case, see [BR, LR].

Key words: Moser-Trudinger inequality, anisotropic Sobolev norm, blow-up analysis, existence of extremal functions

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ denote a domain with $n \geq 2$. When Ω is a bounded domain, the classical Trudinger-Moser inequality states that for all functions $u \in W_0^{1,n}(\Omega)$ with Dirichlet norm $||u||_D = (\int_\Omega |\nabla u|^n dx)^{\frac{1}{n}}$ it holds that

$$\sup_{||u||_D \leq 1} \int_\Omega (e^{\alpha |u|^{\frac{n}{n-1}}} - 1)dx = C(\Omega, \alpha) \begin{cases} < +\infty & when \quad \alpha \leq \lambda_n, \\ = +\infty & when \quad \alpha > \lambda_n, \end{cases} \tag{1}$$

where $\lambda_n = n\omega_{n-1}^{\frac{n}{n-1}}$, and ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Moreover, when $\alpha \leq \lambda_n$, the supremum can be attained by some $u \in W_0^{1,n}(\Omega)$ with $||u||_D = 1$.

It is well known that whether the extremal functions exist or not is an interesting question about Moser-Trudinger inequalities. There are lots of contributions in this direction. The first result is due to Carleson and Chang [CC], who proved that the supremum is attained when Ω is a unit ball in $\mathbb{R}^n$. Then Struwe [S] got the existence of extremals for Ω close to a ball. Struwe's technique was then used and extended by

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Flucher [F] to Ω which is a more general bounded smooth domain in $\mathbb{R}^2$. Later, Lin [L2] generalized the existence result to a bounded smooth domain in dimension-n.

When Ω is an unbounded domain, the situation is different, i.e. Supremum (1) becomes infinity. Hence the Trudinger-Moser inequality is not available for such domains (and in particular for $\mathbb{R}^n$).

However, if Ω is an unbounded domain in $\mathbb{R}^2$, Ruf [BR] replaced the Dirichlet norm $||u||_D$ by the standard Sobolev norm $||u||_S = (\int_\Omega (|\nabla u|^2 + |u|^2)dx)^{\frac{1}{2}}$ on $W_0^{1,2}(\Omega)$ to show that

$$\sup_{||u||_S \le 1} \int_\Omega (e^{\alpha |u|^2} - 1)dx = C(\alpha) \begin{cases} < +\infty & when \quad \alpha \le 4\pi, \\ = +\infty & when \quad \alpha > 4\pi. \end{cases} \tag{2}$$

In particular when $\alpha \le 4\pi$ the supremum can be attained. For $n \ge 3$, Li and Ruf [LR] generalized the result, which states that the supremum

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n} (|\nabla u|^n + |u|^n)dx \le 1} \int_{\mathbb{R}^n} \phi(\alpha |u|^{\frac{n}{n-1}}), \tag{3}$$

is uniformly bounded and can be attained by some $u_0 \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} (|\nabla u_0|^n + |u_0|^n)dx = 1$, where $\alpha \le \lambda_n$, and

$$\phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

When $\alpha > \lambda_n$, the supremum is infinite.

Recently, due to a wide range of applications in geometric analysis and partial differential equations (see [AS, FOR, LL2] and reference therein), numerous generalizations, extensions and applications of the Moser-Trudinger inequality have been given. We recall in particular Lions concentration compactness principle obtained by Lions [L2], which says that if $\{u_k\}$ is a sequence of functions in $W_0^{1,n}(\Omega)$ with $||\nabla u_k||_{L^n(\Omega)} = 1$ such that $u_k \rightharpoonup u \ne 0$ weakly in $W_0^{1,n}(\Omega)$, then for any $0 < p < (1 - ||\nabla u||^n_{L^n(\Omega)})^{-1/(n-1)}$, one has

$$\sup_k \int_\Omega e^{p\lambda_n |u_k|^{\frac{n}{n-1}}} dx < +\infty.$$

Here Ω is a bounded domain. This conclusion gives more precise information than (1) when $u_k \rightharpoonup u \ne 0$ weakly in $W_0^{1,n}(\Omega)$.

A typical generalization is about the anisotropic Moser-Trudinger type inequality which involves a Finsler-Laplacian operator Q_n

$$Q_n u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{n-1}(\nabla u) F_{\xi_i}(\nabla u)).$$

Here the function $F(x)$ is convex, positive and homogeneous of degree 1, and its polar F^o represents a Finsler metric on $\mathbb{R}^n$. In particular, when Ω is a bounded domain, for $u \in W_0^{1,n}(\Omega)$, $(\int_\Omega F^n(\nabla u)dx)^{\frac{1}{n}}$ is an equivalent norm of u, which can be called the anisotropic Dirichlet norm, while $\Omega = \mathbb{R}^n$, $(\int_\Omega F^n(\nabla u) + |u|^n dx)^{\frac{1}{n}}$ is an equivalent Sobolev norm of $u \in W_0^{1,n}(\mathbb{R}^n)$, which can be called as the anisotropic Sobolev norm. In 2012, Wang and Xia [WX1] proved the anisotropic

Moser-Trudinger type inequality in a bounded domain Ω

$$\int_\Omega e^{\lambda u^{\frac{n}{n-1}}} dx \le C(n)|\Omega| \tag{4}$$

for all $u \in W_0^{1,n}(\Omega)$ with the anisotropic Dirichlet norm $\int_\Omega F(\nabla u)^n dx \le 1$. Here $\lambda \le \alpha_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$ and $\kappa_n = |\{x \in \mathbb{R}^n : F^o(x) \le 1\}|$. Moreover, α_n is optimal, that means that if $\lambda > \alpha_n$ we can find a sequence $\{u_k\}$ such that $\int_\Omega e^{\lambda u_k^{\frac{n}{n-1}}} dx$ diverges. Recently, Zhou and Zhou [ZZ] generalized Lions type concentration compactness principle to the anisotropic case and then showed that supremum of the anisotropic Moser-Trudinger functional can be attained.

In this paper, we continue to study the anisotropic Moser-Trudinger type inequality and its extremal functions in $\mathbb{R}^n$. We replace the isotropic Sobolev norm by the anisotropic Sobolve norm

$$||u||_F = (\int_{\mathbb{R}^n} F^n(\nabla u) + |u|^n dx)^{\frac{1}{n}}.$$

Our main results are

Theorem 1.1. *For any $\alpha \in (0, \alpha_n)$, there exist a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^n} \phi(\alpha(\frac{|u(x)|}{||\nabla u||_{L^n(\mathbb{R}^n)}})^{\frac{n}{n-1}}) dx \le C_\alpha \frac{||u(x)||^n_{L^n(\mathbb{R}^n)}}{||\nabla u||^n_{L^n(\mathbb{R}^n)}} \tag{5}$$

for any $u \in W^{1,n}(\mathbb{R}^n)\backslash\{0\}$.

Theorem 1.2. *There exists a constant $d > 0$ such that*

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), ||u||_F \le 1} \int_{\mathbb{R}^n} \phi(\alpha_n |u|^{\frac{n}{n-1}}) dx \le d. \tag{6}$$

Moreover, the inequality is sharp, i.e. for any $\alpha > \alpha_n$, the supremum is $+\infty$.

I we set

$$S = \sup_{u \in W^{1,n}(\mathbb{R}^n), ||u||_F \le 1} \int_{\mathbb{R}^n} \phi(\alpha_n |u|^{\frac{n}{n-1}}) dx.$$

Theorem 1.3. *S is attained. In other words, we can find a function $u \in W^{1,n}(\mathbb{R}^n)$, with $||u||_F = 1$, s.t.*

$$S = \int_{\mathbb{R}^n} \phi(\alpha_n |u|^{\frac{n}{n-1}}) dx.$$

We would like to point out that the second part of Theorem 1.2 is trival. In fact, for any fixed $\alpha > \alpha_n$, we take $\beta \in (\alpha_n, \alpha)$, we can find a positive sequence $\{u_k\}$ in

$$\{u \in W_0^{1,n}(\mathcal{W}_1) : \int_{\mathcal{W}_1} F^n(\nabla u) dx = 1\}$$

such that

$$\lim_{k \to +\infty} \int_{\mathcal{W}_1} e^{\beta u_k^{\frac{n}{n-1}}} dx = +\infty.$$

Here $\mathcal{W}_1 = \{x \in \mathbb{R}^n : F^o(x) \le 1\}$, which is defined in detail in the next section. By Anisotropic Lions type concentration compactness principle in [ZZ], we can get

$u_k \rightharpoonup 0$. Then by the compact embedding theorem, we may assume $||u_k||_{L^p(\mathcal{W}_1)} \to 0$ for any $p > 1$. Hence we have

$$\int_{\mathbb{R}^n} [F^n(\nabla u_k) + u_k^n] dx \to 1.$$

Since $\alpha(\frac{u_k}{||u_k||_F})^{\frac{n}{n-1}} > \beta u_k^{\frac{n}{n-1}}$ when k is sufficiently large, we can get

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi(\alpha(\frac{u_k}{||u_k||_F})^{\frac{n}{n-1}}) dx \geq \int_{\mathcal{W}_1} (e^{\beta u_k^{\frac{n}{n-1}}} - 1) dx = +\infty.$$

Theorem 1.1 will be proved by convex symmetry with respect to $F^o(x)$. And Theorem 1.2 and Theorem 1.3 will be proved by blow up analysis. We will use the ideas from [L1] and [LR]. The key step is to establish the anisotropic Lions type concentration compactness principle for unbounded domain by using convex symmetric rearrangement. The other key step is to give the asymptotic representation of the anisotropic Green function G . Once we have obtained the anisotropic Lions type concentration compactness principle and the asymptotic representation of the anisotropic Green function G, we can apply the blowing up analysis to analyze the asymptotic behavior of the maximizing sequence near and away from the blow up point, and then to give the proof of Theorem 1.2 and Theorem 1.3. Here it is worthy to mention that we need not to distinguish the low dimension case of $n = 2$ form the high dimension case of $n \geq 3$ to prove Theorem 1.3, which is different from the arguments in [BR, LR].

2. Anisotropic Lions type concentration compactness principle

In this section, we will give the notations and preliminaries.

Throughout this paper, let $F : \mathbb{R}^n \mapsto \mathbb{R}$ be a nonnegative convex function of class $C^2(\mathbb{R}^n\backslash\{0\})$ which is even and positively homogenous of degree 1, so that

$$F(t\xi) = |t|F(\xi) \qquad \text{for any} \qquad t \in R,\ \xi \in \mathbb{R}^n.$$

A typical example is $F(\xi) = (\sum_i |\xi|^q)^{\frac{1}{q}}$ for $q \in [1, \infty)$. We further assume that

$$F(\xi) > 0 \qquad \text{for any} \qquad \xi \neq 0.$$

Thanks to homogeneity of F, there exist two constants $0 < a \leq b < \infty$ such that

$$a|\xi| \leq F(\xi) \leq b|\xi|.$$

Usually, we shall assume that the $Hess(F^2)$ is positively definite in $\mathbb{R}^n\backslash\{0\}$. Then by R.L.Xie and H.J.Gong [XG], $Hess(F^n)$ is also positively definite in $\mathbb{R}^n\backslash\{0\}$. We consider the energy containing the expression

$$\int_\Omega F^n(\nabla u) dx$$

by replacing the usual energy. Its Euler equations contain operators of the form

$$Q_n u := \sum_{i=1}^n \frac{\partial}{\partial x_i}(F^{n-1}(\nabla u) F_{\xi_i}(\nabla u)).$$

Note that these operators are not linear unless F is the Euclidean norm in dimension two. We call this nonlinear operator as Finsler-Laplacian. This operator Q_n was studied by many mathematicians, see [WX, FK, WX1, AVP, BFK, XG] and the references therein.

Consider the map

$$\phi : S^{n-1} \to \mathbb{R}^n,\ \phi(\xi) = F_\xi(\xi).$$

Its image $\phi(S^{n-1})$ is a smooth, convex hypersurface in $\mathbb{R}^n$, which is called Wulff shape of F. Let F^o be the support function of $K := \{x \in \mathbb{R}^n : F(x) \le 1\}$, which is defined by

$$F^o(x) := \sup_{\xi \in K} \langle x, \xi \rangle.$$

It is easy to verify that $F^o : \mathbb{R}^n \mapsto [0, +\infty)$ is also a convex, homogeneous function of class of $C^2(\mathbb{R}^n \backslash \{0\})$. Actually F^o is dual to F in the sense that

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \qquad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^o(\xi)}.$$

One can see easily that $\phi(S^{n-1}) = \{x \in \mathbb{R}^n\ |F^o(x) = 1\}$. We denote $\mathcal{W}_F := \{x \in \mathbb{R}^n | F^o(x) \le 1\}$ and $\kappa_n := |\mathcal{W}_F|$, the Lebesgue measure of $\mathcal{W}_F$. We also use the notion $\mathcal{W}_r(x_0) := \{x \in \mathbb{R}^n | F^o(x - x_0) \le r\}$. we call $\mathcal{W}_r(x_0)$ a Wulff shape ball of radius r with center at x_0. For later use, we give some simple properties of the function F, which follows directly from the assumption on F, also see [WX, FK, BP]

Lemma 2.1. *We have*

(i) $|F(x) - F(y)| \le F(x + y) \le F(x) + F(y)$;
(ii) $\frac{1}{C} \le |\nabla F(x)| \le C$, *and* $\frac{1}{C} \le |\nabla F^o(x)| \le C$ *for some* $C > 0$ *and any* $x \neq 0$;
(iii) $\langle x, \nabla F(x) \rangle = F(x), \langle x, \nabla F^o(x) \rangle = F^o(x)$ *for any* $x \neq 0$;
(iv) $F(\nabla F^o(x)) = 1,\ F^o(\nabla F(x)) = 1$ *for any* $x \neq 0$;
(v) $F^o(x) F_\xi(\nabla F^o(x)) = x = F(x) F^o_\xi(\nabla F(x))$ *for any* $x \neq 0$;
(vi) $F_\xi(t\xi) = sgn(t) F_\xi(\xi)$ *for any* $\xi \neq 0$ *and* $t \neq 0$.

It is well known (also see [FM]) that the co-area formula

$$\int_\Omega |\nabla u|_F = \int_0^\infty P_F(|u| > t)dt \tag{7}$$

and the isoperimetric inequality

$$P_F(E) \ge n\kappa_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}} \tag{8}$$

holds.

In the sequel, we will use the convex symmetrization with respect to F^o. The convex symmetrization generalizes the Schwarz symmetrization (see [T3]). It was defined in [AVP] and will be an essential tool for this paper. Let us consider a measurable function u on $\Omega \subset \mathbb{R}^n$. The one dimensional decreasing rearrangement of u is

$$u^* = \sup\{s \ge 0 : |\{x \in \Omega : |u(x)| > s\}| > t\}, \qquad \text{for} \quad t \in \mathbb{R}.$$

The convex symmetrization of u with respect to F is defined as

$$u^\star(x) = u^*(\kappa_n F^o(x)^n), \qquad \text{for } x \in \Omega^*.$$

Here $\kappa_n F^o(x)^n$ is just the Lebesgue measure of a homothetic Wulff ball with radius $F^o(x)$ and Ω^* is the homothetic Wulff ball centered at the origin having the same measure as Ω. Next we will attain concentration compactness principle in unbounded domain with Finsler metric.

Lemma 2.2. *Let $u \in W^{1,n}(\mathbb{R}^n)$ and $u^\star$ the convex symmetrization of u with respect to $F^o(x)$. If $||u||_F \leq 1$, then for each $R > 0$ and $q > 0$, there exist a positive constant $C = C(q, R, n)$ such that*

$$\int_{F^o(x)\geq R} \phi(q|u^\star|^{\frac{n}{n-1}})dx \leq C(q, R, n).$$

Proof. By the monotone convergence theorem, we have

$$\begin{aligned}
\int_{F^o(x)\geq R} \phi(q|u^\star|^{\frac{n}{n-1}})dx &= \int_{F^o(x)\geq R} [\sum_{j=n-1}^{+\infty} \frac{(q|u^\star|^{\frac{n}{n-1}})^j}{j!}]dx \\
&= \sum_{j=n-1}^{+\infty} \int_{F^o(x)\geq R} \frac{(q|u^\star|^{\frac{n}{n-1}})^j}{j!}dx \\
&\leq \frac{q^{n-1}}{(n-1)!}||u^\star||^n_{L^n(\mathbb{R}^n)} + \sum_{j=n}^{+\infty} \frac{q^j}{j!} \int_{F^o(x)\geq R} |u^\star|^{\frac{jn}{n-1}}dx
\end{aligned}$$

In view of the radial symmetrization with respect to $F^o(x)$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} |u^\star(x)|^n dx \geq \int_{\mathcal{W}_r} |u^\star(x)|^n dx &= \int_0^r |u^*(\kappa_n t^n)|^n dt \int_{F^o(x)=t} \frac{1}{|\nabla F^o(x)|}ds \\
&= \int_0^r |u^*(\kappa_n t^n)|^n n\kappa_n t^{n-1} dt \\
&\geq (u^\star(x))^n)|_{F^o(x)=r}\kappa_n r^n.
\end{aligned}$$

Since $||u||_F \leq 1$ implies that $||u^\star||_{L^n(\mathbb{R}^n)} \leq 1$, we have

$$u^\star(x)|_{F^o(x)=r} \leq \frac{||u^\star||_{L^n(\mathbb{R}^n)}}{\kappa_n^{\frac{1}{n}}}\frac{1}{r} \leq \frac{1}{r\kappa_n^{\frac{1}{n}}}.$$

Thus for all $j \geq n$,

$$\begin{aligned}
\int_{F^o(x)\geq R} |u^\star|^{\frac{jn}{n-1}}dx &\leq \int_{F^o(x)\geq R} \frac{1}{F^o(x)^{\frac{jn}{n-1}}}(\frac{1}{\kappa_n})^{\frac{j}{n-1}}dx = (\frac{1}{\kappa_n})^{\frac{j}{n-1}} \int_{F^o(x)\geq R} \frac{1}{F^o(x)^{\frac{jn}{n-1}}}dx \\
&= (\frac{1}{\kappa_n})^{\frac{j}{n-1}} \int_R^{+\infty} \frac{1}{t^{\frac{jn}{n-1}}}dt \int_{F^o(x)=t} \frac{1}{|\nabla F^o(x)|}ds \\
&= (\frac{1}{\kappa_n})^{\frac{j}{n-1}} \int_R^{+\infty} \frac{1}{t^{\frac{jn}{n-1}}} n\kappa_n t^{n-1}dt = \frac{n-1}{j+1-n}\kappa_n^{1-\frac{j}{n-1}} R^{n-\frac{jn}{n-1}}.
\end{aligned}$$

From the equality above we can conclude that

$$\int_{F^o(x)\geq R} \phi(q|u^\star|^{\frac{n}{n-1}})dx \leq \frac{q^{n-1}}{(n-1)!} + \kappa_n R^n \sum_{j=n}^{+\infty} \frac{q^j}{j!}\frac{n-1}{j+1-n}(\frac{1}{\kappa_n R^n})^{\frac{j}{n-1}}.$$

The conclusion follows form the convergence of the series of $\sum_{j=n}^{+\infty} \frac{q^j}{j!}\frac{n-1}{j+1-n}(\frac{1}{\kappa_n R^n})^{\frac{j}{n-1}}$.

□

Lemma 2.3. *For any $p > 1$ and any $u \in W^{1,n}(\mathbb{R}^n)$, there holds*

$$\int_{\mathbb{R}^n} \phi(p|u|^{\frac{n}{n-1}})dx < +\infty.$$

Proof. Fix $p>1$ and $u\in W^{1,n}(\mathbb{R}^n)$, let $u^\star$ be the convex symmetric rearrangement of u with respect to $F^o(x)$, we have

$$\int_{\mathbb{R}^n}\phi(p|u|^{\frac{n}{n-1}})dx=\int_{\mathbb{R}^n}\phi(p|u^\star|^{\frac{n}{n-1}})dx=\int_{F^o(x)\geq R}\phi(p|u^\star|^{\frac{n}{n-1}})dx+\int_{F^o(x)\leq R}\phi(p|u^\star|^{\frac{n}{n-1}})dx.$$

Since $W^{1,n}(\mathcal{W}_R)$ is a continuous embedding in $L^q(\mathcal{W}_R)$ for $q\geq 1$, we obtain that

$$\int_{\mathcal{W}_R}\sum_{j=0}^{n-2}|u^\star(x)|^{\frac{jn}{n-1}}dx\leq C(R).$$

Define $v(x)=u^\star(x)-u^\star(R)$, $x\in\mathcal{W}_R$. Obvious, $v(x)\in W_0^{1,n}(\mathcal{W}_R)$. By calculating, we have, there exists a constant $A=A(n)$,

$$\begin{aligned}|u^\star(x)|^{\frac{n}{n-1}}&\leq(|v(x)|+|u^\star(R)|)^{\frac{n}{n-1}}\\&\leq|v|^{\frac{n}{n-1}}+A|v|^{\frac{1}{n-1}}|u^\star(R)|+|u^\star(R)|^{\frac{n}{n-1}},\end{aligned}$$

and

$$|v|^{\frac{1}{n-1}}|u^\star(R)|=(|v|^{\frac{n}{n-1}})^{\frac{1}{n}}(|u^\star|^{\frac{n}{n-1}})^{\frac{n-1}{n}}\leq\frac{\epsilon}{A}|v|^{\frac{n}{n-1}}+(\frac{\epsilon}{A})^{-\frac{1}{n-1}}|u^\star(R)|^{\frac{n}{n-1}}.$$

Thus,

$$|u^\star(x)|^{\frac{n}{n-1}}\leq(1+\epsilon)|v|^{\frac{n}{n-1}}+C(\epsilon,n)|u^\star(R)|^{\frac{n}{n-1}},$$

where $C(\epsilon,n)=A^{\frac{n}{n-1}}\epsilon^{-\frac{1}{n-1}}+1$. Choose $\epsilon>0$, by means of the Hölder inequality, we get

$$\int_{\mathcal{W}_R}e^{p|u^\star(x)|^{\frac{n}{n-1}}}dx\leq(\int_{\mathcal{W}_R}e^{ps(1+\varepsilon)|v|^{\frac{n}{n-1}}}dx)^{\frac{1}{s}}(\int_{\mathcal{W}_R}e^{ps'C(\epsilon,n)|u^\star(R)|^{\frac{n}{n-1}}})^{\frac{1}{s'}}<+\infty,$$

where $s>1, s'>1$ and $\frac{1}{s}+\frac{1}{s'}=1$. Together with Lemma 2.2, the calculation holds.

□

Now we establish the anisotropic Lions type concentration-compactness lemma in $\mathbb{R}^n$. Similar arguments under the isotropic Dirichlet norm can be seen in [CCH, OMS]. The anisotropic Lions type concentration-compactness lemma in bounded domain can be found in [ZZ].

Theorem 2.4. *Let $\{u_k\}$ be a nonnegative sequence in $W^{1,n}(\mathbb{R}^n)$ such that $||u_k||_F=1$ and $u_k\rightharpoonup u\not\equiv 0$ in $W^{1,n}(\mathbb{R}^n)$. If*

$$0<p<p_n(u)=\frac{1}{(1-||u||_F^n)^{\frac{1}{n-1}}},$$

then

$$\sup_k\int_{\mathbb{R}^n}\phi(p\alpha_n|u_k|^{\frac{n}{n-1}})dx<+\infty.$$

Furthermore, $p_n(u)$ is sharp in the sense that there exists a sequence u_k satisfying $||u_k||_S=1$ and $u_k\rightharpoonup u\not\equiv 0$ in $W^{1,n}(\mathbb{R}^n)$ such that the supremum is infinite for $p\geq p_n(u)$.

Proof. **Case 1:** $0<||u||_F<1$. Assume by contradiction that for some $p_1<p_n(u)$, we have

$$\sup_k\int_{\mathbb{R}^n}\phi(p_1\alpha_n|u_k|^{\frac{n}{n-1}})dx=+\infty.$$

This implies

$$\sup_k \int_{\mathbb{R}^n} \phi(p_1\alpha_n|u_k^\star|^{\frac{n}{n-1}})dx = +\infty,$$

where $u_k^\star$ is the convex symmetrization of u_k with respect to $F^o(x)$. For fixed $R > 0$, we write

$$\int_{\mathbb{R}^n} \phi(p_1\alpha_n|u_k^\star|^{\frac{n}{n-1}})dx = \int_{\mathcal{W}_R} \phi(p_1\alpha_n|u_k^\star|^{\frac{n}{n-1}})dx + \int_{F^o(x)\geq R} \phi(p_1\alpha_n|u_k^\star|^{\frac{n}{n-1}})dx.$$

Since $W^{1,n}(\mathcal{W}_R)$ is a continuous embedding in $L^q(\mathcal{W}_R)$ for $q \geq 1$, we infer that

$$\int_{\mathcal{W}_R} \sum_{j=0}^{n-2} |u_k^\star|^{\frac{jn}{n-1}} dx \leq C(R).$$

From this estimate and Lemma 2.2 with $q = p_1\alpha_n$, we can conclude that

$$\sup_k \int_{\mathcal{W}_R} e^{p_1\alpha_n|u_k^\star|^{\frac{n}{n-1}}} dx = +\infty.$$

Define $v_k(x) = u_k^\star(x) - u_k^\star(R)$, $x \in \mathcal{W}_R$. Obvious, $v_k(x) \in W_0^{1,n}(\mathcal{W}_R)$. By some similar arguments in Lemma 2.3, we have

$$|u_k^\star(x)|^{\frac{n}{n-1}} \leq (1+\epsilon)|v_k|^{\frac{n}{n-1}} + C(\epsilon, n)|u_k^\star(R)|^{\frac{n}{n-1}},$$

where $C(\epsilon, n) = A^{\frac{n}{n-1}}\epsilon^{-\frac{1}{n-1}} + 1$. Choose $s > 0$ and $\epsilon > 0$, such that $(1+\epsilon)sp_1 < p_n(u)$. By means of the Hölder inequality, we get

$$\int_{\mathcal{W}_R} e^{p_1\alpha_n|u_k^\star(x)|^{\frac{n}{n-1}}} dx \leq (\int_{\mathcal{W}_R} e^{(1+\epsilon)p_1 s\alpha_n|v_k(x)|^{\frac{n}{n-1}}} dx)^{\frac{1}{s}} (\int_{\mathcal{W}_R} e^{s' p_1\alpha_n C(\epsilon,n)|u_k^\star(R)|^{\frac{n}{n-1}}})^{\frac{1}{s'}},$$

113 which implies

$$\sup_k \int_{\mathcal{W}_R} e^{\overline{p}_1\alpha_n|v_k|^{\frac{n}{n-1}}} = +\infty, \qquad \overline{p}_1 = (1+\epsilon)p_1 s. \tag{9}$$

Since $v_k(x) = u_k^\star(x) - u_k^\star(R)$, in view of the Pólya-Szegö inequality, we have

$$||F(\nabla v_k^\star)||_{L^n(\mathcal{W}_R)} \leq ||F(\nabla v_k)||_{L^n(\mathcal{W}_R)} = ||F(\nabla u_k^\star)||_{L^n(\mathcal{W}_R)} \leq ||F(\nabla u_k)||_{L^n(\mathcal{W}_R)} \leq 1.$$

114 Denoting $r = F^o(x)$ and taking a change of variable for $t = \kappa_n r^n$, it follows that

$$\begin{aligned}
\int_{\mathcal{W}_R} F^n(\nabla v_k^\star)dx &= \int_{\mathcal{W}_R} F^n(\nabla v_k^*(\kappa_n F^o(x)^n))dx \\
&= \int_0^R F^n(\frac{dv_k^*(t)}{dt}\kappa_n n r^{n-1}\nabla F^o(x))dr \int_{F^o(x)=r} \frac{1}{|\nabla F^o|}dx \\
&= \int_0^R [(-\frac{dv_k^*(t)}{dt})n\kappa_n r^{n-1}]^n n\kappa_n r^{n-1} dr \\
&= \int_0^{|\mathcal{W}_R|} (n\kappa_n^{\frac{1}{n}}(-\frac{dv_k^*(t)}{dt}))^n t^{n-1} dt. \qquad (10)
\end{aligned}$$

115 Then for $k \in \mathbb{N}$ we have

$$(\int_0^{|\mathcal{W}_R|} (n\kappa_n^{\frac{1}{n}}(-\frac{dv_k^*(t)}{dt}))^n t^{n-1} dt)^{\frac{1}{n}} = ||F(\nabla v_k^\star)||_{L^n(\mathcal{W}_R)} \leq 1.$$

Since $v_k^*(|\mathcal{W}_R|)=0$, and v_k^* is locally absolutely continuous,

$$v_k^*(s)=\int_s^{|\mathcal{W}_R|}-\frac{dv_k^*}{dt}dt \qquad for\ s\in(0,|\mathcal{W}_R|). \tag{11}$$

Hölder inequality and (11) yield

$$\begin{aligned} v_k^*(s) &\le (\int_s^{|\mathcal{W}_R|}(n\kappa_n^{\frac{1}{n}}(-\frac{dv_k^*(t)}{dt}))^n t^{n-1}dt)^{\frac{1}{n}}(\int_s^{|\mathcal{W}_R|}\frac{1}{n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}}\frac{dt}{t})^{\frac{n-1}{n}} \\ &\le ||F(\nabla v_k)||_{L^n(\mathcal{W}_R)}(\frac{1}{n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}}\log(\frac{|\mathcal{W}_R|}{s}))^{\frac{n-1}{n}} \\ &\le (\frac{1}{n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}}\log(\frac{|\mathcal{W}_R|}{s}))^{\frac{n-1}{n}} \qquad for\ s\in(0,|\mathcal{W}_R|). \end{aligned} \tag{12}$$

Now we claim: for any $p_2\in(\overline{p_1},p_n(u))$ and every $k_0\in\mathbb{N}$ and every $s_0\in(0,|\mathcal{W}_R|)$ there exist $k\in\mathbb{N}$, $k>k_0$, and $s\in(0,s_0)$ such that

$$v_k^*(s)\ge(\frac{1}{p_2 n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}})^{\frac{n-1}{n}}\log^{\frac{n-1}{n}}(\frac{|\mathcal{W}_R|}{s}).$$

Indeed, by contradiction, suppose that there exist $k_0\in\mathbb{N}$ and $s_0\in(0,|\mathcal{W}_R|)$ such that

$$v_k^*(s)<(\frac{1}{p_2 n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}})^{\frac{n-1}{n}}\log^{\frac{n-1}{n}}(\frac{|\mathcal{W}_R|}{s}) \quad for\ every\ s\in(0,s_0),\ k\ge k_0.$$

By the latter estimate and inequality (12), one has that, if $\overline{p_1}<p_2$ and $k\ge k_0$, then

$$\begin{aligned} \int_{\mathcal{W}_R}exp(\alpha_n\overline{p_1}|v_k|^{\frac{n}{n-1}})dx &= \int_0^{|\mathcal{W}_R|}exp(\alpha_n\overline{p_1}|v_k^*|^{\frac{n}{n-1}})ds \\ &\le \int_0^{s_0}(\frac{|\mathcal{W}_R|}{s})^{\frac{\overline{p_1}}{p_2}}ds+\int_{s_0}^{|\mathcal{W}_R|}(\frac{|\mathcal{W}_R|}{s})^{\overline{p_1}}ds \\ &< +\infty, \end{aligned}$$

contradicting (9). Our claim is proved. Thus, possibly passing to a subsequence, there exist a sequence s_k, such that

$$v_k^*(s_k)\ge(\frac{1}{p_2 n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}})^{\frac{n-1}{n}}\log^{\frac{n-1}{n}}(\frac{|\mathcal{W}_R|}{s_k}) \quad and \quad s_k\le\frac{1}{k} \qquad k\in\mathbb{N}. \tag{13}$$

Now, given $L>0$, define the truncation operator T^L and T_L acting on any function $v:\mathcal{W}_R\to\mathbb{R}^+\cup\{0\}$ as

$$T^L(v)=\min\{v,L\} \quad and \quad T_L(v)=v-T^L(v).$$

Since $||T^L(u)||_F\to||u||_F$ as $L\to+\infty$, taking $p_3\in(p_2,p_n(u))$, and choose L so large that

$$\frac{1-||u||_F^n}{1-||T^L(u)||_F^n}>(\frac{p_3}{p_n(u)})^{n-1}. \tag{14}$$

It follows from (13) that $v_k^*(s_k)\to+\infty$ as $k\to+\infty$. Since $v_k^*(|\mathcal{W}_R|)=0$, by passing to a subsequence if necessary, we have that $v_k^*(s_k)>L$ for every $k\in\mathbb{N}$ large enough. Consequently, there exists $r_k\in(s_k,|\mathcal{W}_R|)$ such that $v_k^*(r_k)=L$ for every $k \in \mathbb{N}$. Owing to (13) and to Hölder inequality, via the same argument as in the proof of (12) we obtain

$$\begin{aligned}(\frac{1}{p_2 n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}})^{\frac{n-1}{n}} \log^{\frac{n-1}{n}}(\frac{|\mathcal{W}_R|}{s_k}) - L \le v_k^*(s_k) - v_k^*(r_k) = \int_{s_k}^{r_k} -\frac{dv_k^*(t)}{dt} dt \\ \le || -\frac{dv_k^*(t)}{dt}(n\kappa_n^{\frac{1}{n}})t^{\frac{n-1}{n}}||_{L^n(s_k,r_k)} \cdot \frac{1}{n\kappa_n^{\frac{1}{n}}}(\log(\frac{|\mathcal{W}_R|}{s_k}))^{\frac{n-1}{n}} \quad for\ k \in \mathbb{N}.\end{aligned}$$

Since $\log^{\frac{n-1}{n}}(\frac{|\mathcal{W}_R|}{s_k}) \to +\infty$, for k large enough, we have

$$\begin{aligned}(\frac{1}{p_3})^{\frac{n-1}{n}} &\le || -\frac{dv_k^*(t)}{dt}(n\kappa_n^{\frac{1}{n}})t^{\frac{n-1}{n}}||_{L^n(s_k,r_k)} + o_k(1) \\ &\le || -\frac{dv_k^*(t)}{dt}(n\kappa_n^{\frac{1}{n}})t^{\frac{n-1}{n}}||_{L^n(0,r_k)} + o_k(1). \end{aligned} \tag{15}$$

By the definition of T^L and T_L, we can get

$$\begin{aligned}\int_{\mathcal{W}_R} F^n(\nabla T^L(v_k))dx &+ \int_{\mathcal{W}_R} F^n(\nabla T_L(v_k))dx = \int_{\mathcal{W}_R} F^n(\nabla v_k)dx \\ &= \int_{\mathcal{W}_R} F^n(\nabla u_k^\star)dx \\ &= \int_{\mathcal{W}_R} F^n(\nabla T^L(u_k^\star))dx + \int_{\mathcal{W}_R} F^n(\nabla T_L(u_k^\star))dx\end{aligned}$$

and

$$\int_{\mathcal{W}_R} F^n(\nabla T^L(u_k^\star))dx \le \int_{\mathcal{W}_R} F^n(\nabla T^L(v_k))dx.$$

Thus

$$\int_{\mathcal{W}_R} F^n(\nabla T_L(v_k))dx \le \int_{\mathcal{W}_R} F^n(\nabla T_L(u_k^\star))dx.$$

By using this inequality and Pólya-Szegö inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} F^n(\nabla T_L(u_k))dx &\ge \int_{\mathbb{R}^n} F^n(\nabla (T_L(u_k))^\star)dx = \int_{\mathbb{R}^n} F^n(\nabla T_L(u_k^\star))dx \\ &\ge \int_{\mathcal{W}_R} F^n(\nabla T_L(v_k))dx \\ &= || -\frac{dv_k^*(t)}{dt}(n\kappa_n^{\frac{1}{n}})t^{\frac{n-1}{n}}||^n_{L^n(0,r_k)}.\end{aligned}$$

Combining with (15) yields

$$(\frac{1}{p_3})^{n-1} \le \int_{\mathbb{R}^n} F^n(\nabla T_L(u_k))dx + o_k(1). \tag{16}$$

As $u_k = T^L(u_k) + T_L(u_k)$ and $T^L(u_k) \le u_k$ one has that

$$\begin{aligned}1 &= ||u_k||_F^n = \int_{\mathbb{R}^n} F^n(\nabla T^L(u_k))dx + \int_{\mathbb{R}^n} F^n(\nabla T_L(u_k))dx + \int_{\mathbb{R}^n} |u_k|^n dx \\ &\ge \int_{\mathbb{R}^n} F^n(\nabla T_L(u_k))dx + ||T^L(u_k)||_F^n.\end{aligned} \tag{17}$$

In view of (16), we have

$$||T^L(u_k)||_F^n + (\frac{1}{p_3})^{n-1} + o_k(1) \le 1. \tag{18}$$

For $L>0$ fixed, $\{T^L(u_k)\}$ is also bounded in $W^{1,n}(\mathbb{R}^n)$. Hence, up to a subsequence, $T^L(u_k)\to T^L(u)$ almost everywhere in $\mathbb{R}^n$ and $T^L(u_k)\rightharpoonup T^L(u)$ in $W^{1,n}(\mathbb{R}^n)$. By the lower semicontinuity of the norm in $W^{1,n}(\mathbb{R}^n)$ and the inequality (14), we obtain

$$\begin{aligned} p_3 \quad &\geq \quad \frac{1}{(1-\liminf_{k\to\infty||T^L(u_k)||_F^n})^{\frac{1}{n-1}}} \geq \frac{1}{(1-||T^L(u)||_F^n)^{\frac{1}{n-1}}} \\ &> \quad \frac{p_3}{p_n(u)}\frac{1}{(1-||u||_F^n)^{\frac{1}{n-1}}} = p_3, \end{aligned} \tag{19}$$

which is a contradiction.

Case 2: $||u||_F=1$. In this case, since $u_k\rightharpoonup u$ weakly and $W^{1,n}(\mathbb{R}^n)$ is a uniformly convex Banach space, we have $u_k\to u$ strongly in $W^{1,n}(\mathbb{R}^n)$. Using Proposition 1 in [OMS], up to a subsequence, we have $|u(x)|\leq v(x)$ for almost $x\in\mathbb{R}^n$ and some $v\in W^{1,n}(\mathbb{R}^n)$. Hence, the proof follows from Lemma 2.3 and Lebesgue dominated convergence theorem.

We conclude by showing that the assumption $p<p_n(u)$ cannot be relaxed. For every $\alpha\in(0,1)$, we exhibit a sequence $\{u_k\}\subset W^{1,n}(\mathbb{R}^n)$ and a function $u\in W^{1,n}(\mathbb{R}^n)$ such that

$$\begin{aligned} &||u_k||_F=1, \quad u_k\rightharpoonup u \textit{ weakly in } W^{1,n}(\mathbb{R}^n),\\ &||u||_F=\alpha \quad \textit{and} \quad \int_{\mathbb{R}^n}\phi(\alpha_n p_n|u_k|^{\frac{n}{n-1}})dx\to+\infty. \end{aligned}$$

Actually, Let us consider the sequence $v_k\in W^{1,n}(\mathbb{R}^n)$ and defined for $r>0$, for $k\in\mathbb{N}$, as

$$v_k(x)=\begin{cases} 0, & F^o(x)\geq r,\\ \kappa_n^{-\frac{1}{n}}\log\left(\frac{r}{F^o(x)}\right)k^{-\frac{1}{n}}, & re^{-\frac{k}{n}}\leq F^o(x)<r,\\ \frac{1}{n}\kappa_n^{-\frac{1}{n}}k^{\frac{n-1}{n}}, & 0\leq F^o(x)\leq re^{-\frac{k}{n}}. \end{cases}$$

We have that

$$\int_{\mathbb{R}^n}F^n(\nabla v_k)dx=\int_{re^{-\frac{k}{n}}}^{r}\kappa_n^{-1}k^{-1}\frac{1}{t^n}n\kappa_n t^{n-1}dt=1.$$

Obvious $v_k(x)\rightharpoonup 0$ in $W^{1,n}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}|v_k|^pdx\to 0$ for $p\geq 1$. Next for $R=3r$, Next, define $u\in W^{1,n}(\mathbb{R}^n)$ by

$$u(x)=\begin{cases} 0, & F^o(x)\geq R,\\ 3A-\frac{3A}{R}F^o(x), & \frac{2}{3}R\leq F^o(x)<R,\\ A, & 0\leq F^o(x)\leq\frac{2}{3}R. \end{cases}$$

where $A>0$ is chosen in such a way that $||u||_F=\alpha$. Finally, set

$$w_k=u+(1-\alpha^n)^{\frac{1}{n}}v_k \qquad for\ k\in\mathbb{N}.$$

Since ∇u and ∇v_k have disjoint supports, we have

$$||F(\nabla w_k)||_{L^n}^n=||F(\nabla u)||_{L^n}^n+1-\alpha^n.$$

Combining with the fact

$$||w_k||_{L^n}^n=\int_{\mathbb{R}^n}|u+(1-\alpha^n)^{\frac{1}{n}}v_k|^ndx=||u||_{L^n}^n+\xi_k,$$

where $\xi_k \to 0$, we have $||w_k||_F = 1+\xi_k$. Finally, set $u_k = \frac{w_k}{1+\xi_k}$, we have

$$||u_k||_F = 1, \quad u_k \rightharpoonup u \text{ in } W^{1,n}(\mathbb{R}^n), \quad ||u||_F = \alpha.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} \phi(\alpha_n p_n |u_k|^{\frac{n}{n-1}})dx \\
\geq\ & \int_{\mathcal{W}_{re^{-\frac{k}{n}}}(0)} exp[n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}} \frac{|u_k|^{\frac{n}{n-1}}}{(1-\alpha^n)^{\frac{1}{n-1}}}]dx - \sum_{j=0}^{n-2} \frac{n^{\frac{nj}{n-1}} \kappa_n^{\frac{j}{n-1}}}{j!(1-\alpha^n)^{\frac{j}{n-1}}} \int_{\mathcal{W}_{re^{-\frac{k}{n}}}(0)} |u_k(x)|^{\frac{jn}{n-1}} dx \\
\geq\ & \int_{\mathcal{W}_{re^{-\frac{k}{n}}}(0)} exp[n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}} \frac{((1+\xi_k)^{-1}[A + (1-\alpha^n)^{\frac{1}{n}} v_k])^{\frac{n}{n-1}}}{(1-\alpha^n)^{\frac{1}{n-1}}}]dx + C(u) + O_k(1) \\
=\ & \int_{\mathcal{W}_{re^{-\frac{k}{n}}}(0)} exp[n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}} ((1+\xi_k)^{-1}[C + v_k])^{\frac{n}{n-1}}]dx + C(u) + O_k(1) \\
=\ & C_1 e^{-k} exp([(1+\xi_k)^{-1}(C_2 + k^{\frac{n-1}{n}})]^{\frac{n}{n-1}}) + C(u) + O_k(1) \to +\infty,
\end{aligned}$$

for some positive constants C, C_1, C_2, where

$$C(u) = -\sum_{j=0}^{n-2} \frac{n^{\frac{nj}{n-1}} \kappa_n^{\frac{j}{n-1}}}{j!(1-\alpha^n)^{\frac{j}{n-1}}} \int_{\mathcal{W}_{re^{-\frac{k}{n}}}(0)} |u(x)|^{\frac{jn}{n-1}} dx.$$

This concludes the proof. □

3. The maximizing sequence

Let $\{R_k\}$ be an increasing sequence which diverges to infinity, and $\{\beta_k\}$ an increasing sequence which converges to α_n.

Setting

$$S_{\beta_k}(u) = \int_{\mathcal{W}_{R_k}} \phi(\beta_k |u|^{\frac{n}{n-1}})dx.$$

and

$$H = \{u \in W_0^{1,n}(\mathcal{W}_{R_k}) : \int_{\mathcal{W}_{R_k}} (F^n(\nabla u) + |u|^n)dx = 1\}.$$

We have

Lemma 3.1. *For any fixed k, there exists an extremal functional function $u_k \in H$ and $u_k \geq 0$ such that*

$$S_{\beta_k}(u_k) = \sup_{u \in H} S_{\beta_k}(u).$$

Proof. There exists a sequence of $\{v_i\} \in H$ such that

$$\lim_{i \to +\infty} S_{\beta_k}(v_i) = \sup_{u \in H} S_{\beta_k}(u).$$

We set $v_i = 0$ in $\mathbb{R}^n \setminus \mathcal{W}_{R_k}$. Since v_i is bounded in $W^{1,n}(\mathbb{R}^n)$, there exist a subsequence, which will still be denoted by v_i, such that

$$\begin{aligned}
v_i &\rightharpoonup u_k \quad \text{weakly in} \quad W^{1,n}(\mathbb{R}^n), \\
v_i &\to u_k \quad \text{strongly in} \quad L^s(\mathbb{R}^n),
\end{aligned}$$

for any $1 < s < \infty$ as $i \to \infty$. Hence $v_i \to u_k$ $a.e$ in $\mathbb{R}^n$, and

$$g_i = \phi(\beta_k |v_i|^{\frac{n}{n-1}}) \to g_k = \phi(\beta_k |u_k|^{\frac{n}{n-1}}) \ a.e \text{ in } \mathbb{R}^n.$$

We claim that $u_k \not\equiv 0$. If not, then g_i is bounded in $L^r(\mathcal{W}_{R_k})$ for some $r > 1$, thus $g_i \to 0$ strongly in $L^1(\mathcal{W}_{R_k})$. Therefore, $\sup_{u\in H} S_{\beta_k}(u) = 0$, which is impossible. By Theorem 2.4, we have for any $p < p_n = \frac{1}{(1-||u_k||_F^n)^{\frac{1}{n-1}}}$,

$$\sup_i \int_{\mathbb{R}^n} \phi(p\alpha_n |v_i|^{\frac{n}{n-1}})dx < +\infty.$$

So $g_i \to g_k$ strongly in $L^1(\mathcal{W}_{R_k})$, as $i \to +\infty$. Therefore, the extremal function is attained for the case $\beta_k < \alpha_n$ and $||u_k||_F = 1$. From the form of $S_{\beta_k}(u_k)$, we can choose the function $u_k \geq 0$. □

Similar as in [LR, LZ], we give the following

Lemma 3.2. *Let u_k as above. Then u_k is a maximizing sequence for S and u_k may be chosen to be radially symmetric and decreasing with respect to $F^o(x)$.*

Proof. Let η be a cut-off function which is 1 on $\mathcal{W}_1$ and 0 on $\mathbb{R}^n\backslash\mathcal{W}_2$. Then given any $\varphi \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}(F(\nabla\varphi)^n + |\varphi|^n)dx = 1$, we have

$$\tau^n(L) := \int_{\mathbb{R}^n} (F^n(\nabla(\eta(\frac{x}{L})\varphi)) + |\eta(\frac{x}{L})\varphi|^n)dx \to 1, \qquad as \quad L \to +\infty.$$

Hence for a fixed L and $R_k > 2L$

$$\begin{aligned}\int_{\mathcal{W}_L} \phi(\beta_k|\frac{\varphi}{\tau(L)}|^{\frac{n}{n-1}})dx &\leq \int_{\mathcal{W}_{2L}} \phi(\beta_k|\frac{\eta(\frac{x}{L})\varphi}{\tau(L)}|^{\frac{n}{n-1}})dx \\ &\leq \int_{\mathcal{W}_{R_k}} \phi(\beta_k u_k^{\frac{n}{n-1}})dx.\end{aligned}$$

By the Levi Lemma, we can have

$$\int_{\mathcal{W}_L} \phi(\alpha_n|\frac{\varphi}{\tau(L)}|^{\frac{n}{n-1}})dx \leq \lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}})dx.$$

Then, Letting $L \to +\infty$, we get

$$\int_{\mathbb{R}^n} \phi(\alpha_n|\varphi|^{\frac{n}{n-1}})dx \leq \lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}})dx.$$

Hence, we get

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}})dx = \sup_{||v||_F\leq 1, v\in W^{1,n}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(\alpha_n|v|^{\frac{n}{n-1}})dx.$$

Let $u_k^\star$ be the convex symmetric rearrangement of u_k with respect to $F^o(x)$, then we have

$$\tau_k^n := \int_{\mathcal{W}_{R_k}} (F^n(\nabla u_k^\star) + {u_k^\star}^n)dx \leq \int_{\mathcal{W}_{R_k}} (F^n(\nabla u_k) + u_k^n)dx = 1.$$

It is well known that $\tau_k = 1$ if and only if u_k is radial with respect to $F^o(x)$. Since

$$\int_{\mathcal{W}_{R_k}} \phi(\beta_k {u_k^\star}^{\frac{n}{n-1}})dx = \int_{\mathcal{W}_{R_k}} \phi(\beta_k u_k^{\frac{n}{n-1}})dx,$$

we have

$$\int_{\mathcal{W}_{R_k}} \phi(\beta_k (\frac{u_k^\star}{\tau_k})^{\frac{n}{n-1}})dx \geq \int_{\mathcal{W}_{R_k}} \phi(\beta_k u_k^{\frac{n}{n-1}})dx.$$

Hence $\tau_k = 1$ and

$$\int_{\mathcal{W}_{R_k}} \phi(\beta_k u_k^{\star\frac{n}{n-1}})dx = \sup_{\int_{\mathcal{W}_{R_k}}(F^n(\nabla v)+|v|^n)dx=1, v\in W_0^{1,n}(\mathcal{W}_{R_k})} \int_{\mathcal{W}_{R_k}} \phi(\beta_k |v|^{\frac{n}{n-1}})dx.$$

So, we can assume $u_k = u_k(r)$ and $r = F^o(x)$, $u_k(r)$ is decreasing. $\square$

Assume now $u_k \rightharpoonup u$ in $W_0^{1,n}(\mathcal{W}_{R_k})$. Then, to prove Theorem 1.2 and Theorem 1.3, we only need to show that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}})dx = \int_{\mathbb{R}^n} \phi(\alpha_n u^{\frac{n}{n-1}})dx.$$

4. BLOW UP ANALYSIS

In this section, the method of blow-up analysis will be used to analyze the asymptotic behavior of the maximizing sequence $\{u_k\}$.

After a direct computation, the Euler-Lagrange equation for the extremal function $u_k \in W_0^{1,n}(\mathcal{W}_{R_k})$ can be written as

$$-Q_n(u_k) + u_k^{n-1} = \frac{u_k^{\frac{1}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}})}{\lambda_k}, \tag{20}$$

where λ_k is the constant satisfying

$$\lambda_k = \int_{\mathcal{W}_{R_k}} u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}})dx. \tag{21}$$

First, we need to prove the following result.

Lemma 4.1. $\liminf\limits_{k\to+\infty} \lambda_k > 0$.

Proof. We show this lemma by contradiction. Without loss of generality, we assume $\lambda_k \to 0$.

When $n = 2$, since $e^t - 1 \le te^t$ for any $t \ge 0$, we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}^2} (e^{\beta_k u_k^2} - 1)dx \le \alpha_n \lim_{k\to+\infty} \int_{\mathbb{R}^2} u_k^2 e^{\beta_k u_k^2} dx = \alpha_n \lim_{k\to+\infty} \lambda_k \to 0,$$

this is a contradiction.

When $n \ge 3$, Since

$$\begin{aligned} \lambda_k &= \int_{\mathbb{R}^n} u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}})dx = \int_{\mathbb{R}^n} u_k^{\frac{n}{n-1}} \sum_{j=n-2}^{\infty} \frac{(\beta_k u_k^{\frac{n}{n-1}})^j}{j!} dx \\ &= \int_{\mathbb{R}^n} (\frac{\beta_k^{n-2} u_k^n}{(n-2)!} + \cdots)dx \ge \frac{\beta_k^{n-2}}{(n-2)!} \int_{\mathbb{R}^n} u_k^n dx, \end{aligned} \tag{22}$$

we have

$$\int_{\mathbb{R}^n} u_k^n dx \le C \int_{\mathbb{R}^n} u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}})dx \le C\lambda_k \to 0.$$

Since $u_k = u_k(r)$ is decreasing, we have $u_k^n(L)|\mathcal{W}_L| \le \int_{\mathcal{W}_L} u_k^n dx \le 1$, and then

$$u_k(L) \le \frac{1}{\kappa_n^{\frac{1}{n}} L}. \tag{23}$$

Set $\epsilon = \frac{1}{\kappa_n^{\frac{1}{n}} L}$. Then $u_k(x) \leq \epsilon$ for any $x \notin \mathcal{W}_L$, and hence we have, by using the form of the function $\phi(x)$, that

$$\int_{\mathbb{R}^n \setminus \mathcal{W}_L} \phi(\beta_k u_k^{\frac{n}{n-1}}) dx \leq C \int_{\mathbb{R}^n \setminus \mathcal{W}_L} u_k^n dx \leq C\lambda_k \to 0.$$

Since

$$\phi(\beta_k u_k^{\frac{n}{n-1}}) = \sum_{j=n-1}^{\infty} \frac{(\beta_k u_k^{\frac{n}{n-1}})^j}{j!} = \sum_{j=n-2}^{\infty} \frac{\beta_k u_k^{\frac{n}{n-1}} (\beta_k u_k^{\frac{n}{n-1}})^j}{(j+1)j!} \leq \beta_k u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}}),$$

we have

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{\mathcal{W}_L} \phi(\beta_k u_k^{\frac{n}{n-1}}) dx = \lim_{k\to+\infty} \Big(\int_{\mathcal{W}_L \cap \{u_k \geq 1\}} + \int_{\mathcal{W}_L \cap \{u_k \leq 1\}}\Big) \phi(\beta_k u_k^{\frac{n}{n-1}}) dx \\
\leq\ & \lim_{k\to+\infty} \Big[C \int_{\mathcal{W}_L} u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx + \int_{\{x \in \mathcal{W}_L | u_k(x) \leq 1\}} \phi(\beta_k u_k^{\frac{n}{n-1}}) dx\Big] \\
\leq\ & \lim_{k\to+\infty} \Big(C\lambda_k + C \int_{\mathcal{W}_L} u_k^n dx\Big) = 0.
\end{aligned}$$

This is impossible. Thus we get a contradiction. □

We denote $c_k = \max_{x\in\mathbb{R}^n} u_k(x) = u_k(0)$. It is clear $\sup_k c_k$ can be infinite. However $\sup_k c_k$ can be finite, we have the following result.

Lemma 4.2. *If* $\sup_k c_k < +\infty$*, then Theorem 1.2 and Theorem 1.3 hold.*

Proof. By Lemma 4.1 and Theorem 1 in [L3], then $u_k \to u$ in $C^1_{loc}(\mathbb{R}^n)$. For any $\epsilon > 0$, by (23), we are able to find L such that $u_k(x) \leq \epsilon$ for $x \notin \mathcal{W}_L$. Since

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \mathcal{W}_L} \Big(\phi(\beta_k u_k^{\frac{n}{n-1}}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!}\Big) dx \\
\leq\ & C \int_{\mathbb{R}^n \setminus \mathcal{W}_L} u_k^{\frac{n^2}{n-1}} dx \leq C\epsilon^{\frac{n^2}{n-1} - n} \int_{\mathbb{R}^n} u_k^n dx \leq C\epsilon^{\frac{n^2}{n-1} - n},
\end{aligned}$$

we have

$$\int_{\mathbb{R}^n} \Big(\phi(\beta_k u_k^{\frac{n}{n-1}}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!}\Big) dx = \int_{\mathcal{W}_L} \Big(\phi(\beta_k u_k^{\frac{n}{n-1}}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!}\Big) dx + O(\epsilon^{\frac{n^2}{n-1} - n}).$$

It follows from $\sup_k c_k < +\infty$ that

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}}) dx &= \int_{\mathcal{W}_L} \Big(\phi(\beta_k u_k^{\frac{n}{n-1}}) - \frac{\beta_k^{n-1} u_k^n}{(n-1)!}\Big) dx + \int_{\mathbb{R}^n} \frac{\beta_k^{n-1} u_k^n}{(n-1)!} dx + O(\epsilon^{\frac{n^2}{n-1} - n}) \\
&\leq C(L). \qquad (24)
\end{aligned}$$

Thus, Theorem 1.2 holds.

Next we show Theorem 1.3. We proceed by dividing two cases.

Case 1: $u \neq 0$.

In this case, we first show that $\int_{\mathbb{R}^n} u_k^n dx \to \int_{\mathbb{R}^n} u^n dx$. By (24) we have

$$\begin{aligned} S &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \phi(\beta_k u_k^{\frac{n}{n-1}})dx \\ &= \int_{\mathbb{R}^n} \phi(\alpha_n u^{\frac{n}{n-1}})dx + \frac{\alpha_n^{n-1}}{(n-1)!} \lim_{k\to\infty} \int_{\mathbb{R}^n} (u_k^n - u^n)dx. \end{aligned} \tag{25}$$

Set

$$\tau^n = \lim_{k\to+\infty} \frac{\int_{\mathbb{R}^n} u_k^n dx}{\int_{\mathbb{R}^n} u^n dx}.$$

By the Levi Lemma, we have $\tau \geq 1$.

Let $\tilde{u} = u(\frac{x}{\tau})$. Then, we have

$$\begin{aligned} \int_{\mathbb{R}^n} F^n(\nabla \tilde{u})dx &= \int_{\mathbb{R}^n} F^n(\nabla u)dx \leq \lim_{k\to+\infty} \int_{\mathbb{R}^n} F^n(\nabla u_k)dx, \\ \int_{\mathbb{R}^n} \tilde{u}^n dx &= \tau^n \int_{\mathbb{R}^n} u^n dx = \lim_{k\to+\infty} \int_{\mathbb{R}^n} u_k^n dx. \end{aligned}$$

Then

$$\int_{\mathbb{R}^n} (F^n(\nabla \tilde{u}) + \tilde{u}^n)dx \leq \lim_{k\to+\infty} \int_{\mathbb{R}^n} (F^n(\nabla u_k) + u_k^n)dx = 1.$$

Hence, we have by (25)

$$\begin{aligned} S &\geq \int_{\mathbb{R}^n} \phi(\alpha_n \tilde{u}^{\frac{n}{n-1}})dx \\ &= \tau^n \int_{\mathbb{R}^n} \phi(\alpha_n u^{\frac{n}{n-1}})dx \\ &= [\int_{\mathbb{R}^n} \phi(\alpha_n u^{\frac{n}{n-1}})dx + (\tau^n - 1) \int_{\mathbb{R}^n} \frac{\alpha_n^{n-1}}{(n-1)!} u^n dx] \\ &+ (\tau^n - 1) \int_{\mathbb{R}^n} (\phi(\alpha_n u^{\frac{n}{n-1}}) - \frac{\alpha_n^{n-1}}{(n-1)!} u^n)dx \\ &= \lim_{k\to+\infty} \int_{\mathbb{R}^n} (\phi(\beta_k u_k^{\frac{n}{n-1}})dx \\ &+ (\tau^n - 1) \int_{\mathbb{R}^n} (\phi(\alpha_n u^{\frac{n}{n-1}}) - \frac{\alpha_n^{n-1}}{(n-1)!} u^n)dx \\ &= S + (\tau^n - 1) \int_{\mathbb{R}^n} (\phi(\alpha_n u^{\frac{n}{n-1}}) - \frac{\alpha_n^{n-1}}{(n-1)!} u^n)dx. \end{aligned}$$

Since $\phi(\alpha_n u^{\frac{n}{n-1}}) - \frac{\alpha_n^{n-1}}{(n-1)!} u^n > 0$, we have $\tau = 1$, and then

$$S = \int_{\mathbb{R}^n} \phi(\alpha_n u^{\frac{n}{n-1}})dx.$$

Thus we obtain that u is an extremal function.

Case 2: $u = 0$.

In this case, since $u_k \to 0$ in $C^1_{loc}(\mathbb{R}^n)$, we have

$$\lim_{k\to+\infty} \int_{\mathcal{W}_L} \phi(\alpha_n u_k^{\frac{n}{n-1}})dx = \int_{\mathcal{W}_L} \phi(\lim_{k\to+\infty} \alpha_n u_k^{\frac{n}{n-1}})dx = 0.$$

By (24) and letting $L \to +\infty$, we obtain

$$\begin{aligned}\lim_{k\to+\infty}\int_{\mathbb{R}^n}\phi(\alpha_n u_k^{\frac{n}{n-1}})dx &= \lim_{k\to+\infty}\Big(\frac{\alpha_n^{n-1}}{(n-1)!}\int_{\mathbb{R}^n} u_k^n dx + o_k(1)\Big)\\ &\le \frac{\alpha_n^{n-1}}{(n-1)!}.\end{aligned}$$

In the following, we show that $u = 0$ will not happen. Indeed, for any fixed $v \in W^{1,n}(\mathbb{R}^n)$ with $v \neq 0$, we can introduce a family of functions v_t for $t > 0$ that

$$v_t(x) = t^{\frac{1}{n}} v(t^{\frac{1}{n}} x).$$

We easily verify that

$$||F(\nabla v_t)||^n_{L^n(\mathbb{R}^n)} = t||F(\nabla v)||^n_{L^n(\mathbb{R}^n)},\ ||v_t||^p_{L^p(\mathbb{R}^n)} = t^{\frac{p-n}{n}}||v||^p_{L^p(\mathbb{R}^n)}.$$

Hence, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\phi(\alpha_n(\frac{v_t}{||v_t||_F})^{\frac{n}{n-1}})dx &\ge \frac{\alpha_n^{n-1}||v_t||^n_{L^n(\mathbb{R}^n)}}{(n-1)!||v_t||_F^n} + \frac{\alpha_n^n||v_t||^{n^2/(n-1)}_{L^{n^2/(n-1)}(\mathbb{R}^n)}}{n!||v_t||_F^{n^2/(n-1)}}\\ &= \frac{\alpha_n^{n-1}}{(n-1)!} + \frac{\alpha_n^{n-1}}{(n-1)!}g_v(t),\end{aligned}$$

where

$$\begin{aligned}g_v(t) &= \frac{\alpha_n}{n}\Big(\frac{1}{t||F(\nabla v)||^n_{L^n(\mathbb{R}^n)} + ||v||^n_{L^n(\mathbb{R}^n)}}\Big)^{\frac{n}{n-1}} t^{\frac{1}{n-1}}||v||^{n^2/(n-1)}_{L^{n^2/(n-1)}(\mathbb{R}^n)}\\ &\quad - \frac{t||F(\nabla v)||^n_{L^n(\mathbb{R}^n)}}{t||F(\nabla v)||^n_{L^n(\mathbb{R}^n)} + ||v||^n_{L^n(\mathbb{R}^n)}}\\ &= \frac{\alpha_n||v||^{n^2/(n-1)}_{L^{n^2/(n-1)}(\mathbb{R}^n)}}{n||v||^{n^2/(n-1)}_{L^n(\mathbb{R}^n)}}t^{\frac{1}{n-1}}(1+O(t)) - \frac{||F(\nabla v)||^n_{L^n(\mathbb{R}^n)}}{||v||^n_{L^n(\mathbb{R}^n)}}t(1+O(t)).\end{aligned}$$

Note that $g_v(0) = 0$. Once we show that $g_v(t) > 0$ for small $t > 0$ for some v, it leads to $S > \frac{\alpha_n^{n-1}}{(n-1)!}$, which is a contradiction. Thus we finish the proof of Theorem.

Indeed, when $n \ge 3$, it is clear that $g_v(t) > 0$ for some v when t is small enough. When f $n = 2$, we know that

$$\begin{aligned}g_v(t) &= \Big(\frac{\alpha_2||v||^4_{L^4(\mathbb{R}^2)}}{2||v||^4_{L^2(\mathbb{R}^2)}} - \frac{||F(\nabla v)||^2_{L^2(\mathbb{R}^2)}}{||v||^2_{L^2(\mathbb{R}^2)}}\Big)t(1+O(t))\\ &= \frac{||F(\nabla v)||^2_{L^2(\mathbb{R}^2)}}{||v||^2_{L^2(\mathbb{R}^2)}}\Big(\frac{\alpha_2}{2}\frac{||v||^4_{L^4(\mathbb{R}^2)}}{||v||^2_{L^2(\mathbb{R}^2)}||F(\nabla v)||^2_{L^2(\mathbb{R}^2)}} - 1\Big)(t+O(t)).\end{aligned}$$

We claim that $\overline{B}_2 := \sup\limits_{u\in W^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||F(\nabla u)||^2_{L^2(\mathbb{R}^2)}}$ is attained by some function $g(F^o(x)) \in W^{1,2}(\mathbb{R}^n)$, and $\overline{B}_2 > \frac{2}{\alpha_2}$. Thus we can take $v = g(F^o(x))$, and hence

$$g_v(t) = \frac{||F(\nabla v)||^2_{L^2(\mathbb{R}^2)}}{||v||^2_{L^2(\mathbb{R}^2)}}\Big(\frac{\alpha_2}{2}\overline{B}_2 - 1\Big)(t+O(t)) > 0,$$

for some small $t > 0$.

Next we show the above claim. By using Pólya-Szëgo principle, we have

$$\begin{aligned}\int_{\mathbb{R}^2} F^2(\nabla u^\star)dx &\le \int_{\mathbb{R}^2} F^2(\nabla u)dx,\\ \int_{\mathbb{R}^2} |u^\star|^2 dx &= \int_{\mathbb{R}^2} |u|^2 dx,\\ \int_{\mathbb{R}^2} |u^\star|^4 dx &= \int_{\mathbb{R}^2} |u|^4 dx.\end{aligned}$$

Set $E = \{u \in W^{1,2}(\mathbb{R}^2) : u(x)$ is radially symmetric and decreasing with respect to $F^o(x)\}$, then we have

$$\sup_{u\in W^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||F(\nabla u)||^2_{L^2(\mathbb{R}^2)}} = \sup_{u\in E\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||F(\nabla u)||^2_{L^2(\mathbb{R}^2)}}.$$

For any $u \in E\setminus\{0\}$, Due to

$$\begin{aligned}\int_{\mathbb{R}^2} |u^\star|^2 dx &= \int_{\mathbb{R}^2} |u^\#|^2 dx,\\ \int_{\mathbb{R}^2} |u^\star|^4 dx &= \int_{\mathbb{R}^2} |u^\#|^4 dx,\\ \int_{\mathbb{R}^2} F^2(\nabla u^\star)dx &= \frac{\kappa_2}{\pi}\int_{\mathbb{R}^2} |\nabla u^\#|^2 dx,\end{aligned}$$

where $u^\#$ is the Schwarz symmetric rearrangement of $u(x)$, we have

$$\sup_{u\in E\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||F(\nabla u)||^2_{L^2(\mathbb{R}^2)}} = \frac{\pi}{\kappa_2}\sup_{u\in H\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||\nabla u||^2_{L^2(\mathbb{R}^2)}}.$$

Here $H = \{u \in W^{1,2}(\mathbb{R}^n) : u$ is the Schwarz symmetric function$\}$. Recall that [I, W], there is some function $g(x) \in H$ and

$$\sup_{u\in H\setminus\{0\}} \frac{||u||^4_{L^4(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}||\nabla u||^2_{L^2(\mathbb{R}^2)}} = \frac{||g||^4_{L^4(\mathbb{R}^2)}}{||g||^2_{L^2(\mathbb{R}^2)}||\nabla g||^2_{L^2(\mathbb{R}^2)}} > \frac{1}{2\pi}.$$

It implies $\overline{B}_2 > \frac{1}{2\kappa_2}$. Therefore the claim is proved. □

From now on, we assume $c_k \to +\infty$ as $k \to +\infty$. We define

$$r_k^n = \frac{\lambda_k}{c_k^{\frac{n}{n-1}} e^{\beta_k c_k^{\frac{n}{n-1}}}}.$$

By (23) we can find a sufficiently large L such that $u_k \le 1$ on $\mathbb{R}^n\setminus\mathcal{W}_L$, and

$$\int_{\mathcal{W}_L} F^n(\nabla(u_k - u_k(L))^+)dx \le 1.$$

Hence, by Moser-Trudinger inequality involving the anisotropic Dirichlet Norm in [ZZ], we have

$$\int_{\mathcal{W}_L} e^{\alpha_n[(u_k-u_k(L))^+]^{\frac{n}{n-1}}}dx \le C(L).$$

Clearly, for any $p < \alpha_n$ we can find a constant $C(p)$, such that

$$pu_k^{\frac{n}{n-1}} \le \alpha_n[(u_k - u_k(L))^+]^{\frac{n}{n-1}} + C(p),$$

and then we get

$$\int_{\mathcal{W}_L} e^{pu_k^{\frac{n}{n-1}}}dx < C = C(L,p).$$

Hence,

$$\begin{aligned}\lambda_k e^{-\frac{\beta_k}{2}c_k^{\frac{n}{n-1}}} &= e^{-\frac{\beta_k}{2}c_k^{\frac{n}{n-1}}}[\int_{\mathbb{R}^n\setminus\mathcal{W}_L} u_k^{\frac{n}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx + \int_{\mathcal{W}_L} u_k^{\frac{n}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx] \\ &\le C\int_{\mathbb{R}^n\setminus\mathcal{W}_L} u_k^{\frac{n}{n-1}}dx e^{-\frac{\beta_k}{2}c_k^{\frac{n}{n-1}}} + \int_{\mathcal{W}_L} e^{\frac{\beta_k}{2}u_k^{\frac{n}{n-1}}}u_k^{\frac{n}{n-1}}dx.\end{aligned}$$

Since u_k converges strongly in $L^q(\mathcal{W}_L)$ for any $q>1$, we get

$$\lambda_k \le Ce^{\frac{\beta_k}{2}c_k^{\frac{n}{n-1}}},$$

and hence

$$r_k^n \le Ce^{-\frac{\beta_k}{2}c_k^{\frac{n}{n-1}}}.$$

Now, we set

$$\begin{aligned} v_k(x) &= \frac{u_k(r_k x)}{c_k}, \\ w_k(x) &= c_k^{\frac{1}{n-1}}(v_k(x)-c_k),\end{aligned}$$

where v_k and w_k are defined on $\Omega_k = \{x\in\mathbb{R}^n | r_k x \in \mathcal{W}_1\}$.

By a direct calculation we obtain that

$$-div(F^{n-1}(\nabla v_k)F_\xi(\nabla v_k)) + \frac{u_k^{n-1}(r_k x)r_k^n}{c_k^{n-1}} = \frac{v_k^{\frac{1}{n-1}}}{c_k^n}e^{\beta_k(u_k^{\frac{n}{n-1}}(r_k x)-c_k^{\frac{n}{n-1}})} + O(r_k^n c_k^n).$$

Since $0\le v_k\le 1$ and $\frac{v_k^{\frac{1}{n-1}}}{c_k^n}e^{\beta_k(u_k^{\frac{n}{n-1}}(r_k x)-c_k^{\frac{n}{n-1}})}\to 0$ in $\mathcal{W}_r(0)$ for any $r>0$, which implies $\frac{v_k^{\frac{1}{n-1}}}{c_k^n}e^{\beta_k(u_k^{\frac{n}{n-1}}(r_k x)-c_k^{\frac{n}{n-1}})}$ is uniformly bounded in $L^\infty(\overline{\mathcal{W}_r(0)})$, by Theorem 1 in [T2], v_k is uniformly bounded in $C^{1,\alpha}(\overline{\mathcal{W}_{\frac{r}{2}}(0)})$. By Ascoli-Arzela's theorem, we can find a sequence $k_j\to+\infty$ such that $v_{k_j}\to v$ in $C^1_{loc}(\mathbb{R}^n)$, where $v\in C^1(\mathbb{R}^n)$ and satisfies

$$-div(F^{n-1}(\nabla v)F_\xi(\nabla v)) = 0 \qquad \text{in } \mathbb{R}^n.$$

Furthermore, we have $0\le v\le 1$ and $v(0)=1$, and the Liouville theorem (see [HKM]) leads to $v\equiv 1$.

Also we have

$$-div(F^{n-1}(\nabla w_k)F_\xi(\nabla w_k)) = v_k^{\frac{1}{n-1}}e^{\beta_k(u_k^{\frac{n}{n-1}}(r_k x)-c_k^{\frac{n}{n-1}})} + O(r_k^n c_k^n) \ \text{ in } \Omega_k. \quad (26)$$

For any $r>0$, since $0\le u_k(r_k x)\le c_k$ we have $-div(F^{n-1}(\nabla w_k)F_\xi(\nabla w_k)) = O(1)$ in $\mathcal{W}_r(0)$ for large k. Then form $w_k(0)=0$ and Theorem 1 in [T2] and Ascoli-Arzela's theorem, there exist $w\in C^1(\mathbb{R}^n)$ such that w_k converges to w in $C^1_{loc}(\mathbb{R}^n)$. Therefore we have

$$\begin{aligned} u_k^{\frac{n}{n-1}}(r_k x) - c_k^{\frac{n}{n-1}} &= c_k^{\frac{n}{n-1}}(v_k^{\frac{n}{n-1}}(x)-1) \\ &= \frac{n}{n-1}w_k(x)(1+O((v_k(x)-1)^2)). \end{aligned} \quad (27)$$

By taking $\epsilon\to 0$, we know that w satisfies

$$-div(F^{n-1}(\nabla w)F_\xi(\nabla w)) = e^{\frac{n}{n-1}\alpha_n w}. \quad (28)$$

in the distributional sense. We also have the facts $w(0) = 0 = \max_{x\in\mathbb{R}^n} w(x)$.

Since w is radially symmetric and non-increasing with respect to $F^o(x)$, it is easy to see that (28) has only one solution. We can check that

$$w(r) = -\frac{n-1}{\alpha_n}\log(1+\kappa_n^{\frac{1}{n-1}} r^{\frac{n}{n-1}}), \text{where } r = F^o(x). \tag{29}$$

Thus we get that

$$\int_{\mathbb{R}^n} e^{\frac{n}{n-1}\alpha_n w} dx = 1.$$

and

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_{Lr_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx = \lim_{L\to+\infty}\int_{\mathcal{W}_L} e^{\frac{n}{n-1}\alpha_n w} dx = 1. \tag{30}$$

For $A > 1$, let $u_k^A = \min\{u_k, \frac{c_k}{A}\}$. We have the following result

Lemma 4.3. *For any $A > 1$, there holds*

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^n} (F^n(\nabla u_k^A) + |u_k^A|^n) dx \le \frac{1}{A}.$$

Proof. Since $|\{x|u_k \ge \frac{c_k}{A}\}||\frac{c_k}{A}|^n \le \int_{u_k\ge\frac{c_k}{A}} u_k^n dx \le 1$, we can find a sequence $\rho_k \to 0$ such that

$$\{x|u_k \ge \frac{c_k}{A}\} \subset \mathcal{W}_{\rho_k}.$$

Since u_k converges in $L^p(\mathcal{W}_1)$ for any $p > 1$, we have

$$\lim_{k\to+\infty}\int_{u_k\ge\frac{c_k}{A}} |u_k^A|^p dx \le \lim_{k\to+\infty}\int_{u_k\ge\frac{c_k}{A}} u_k^p dx = 0,$$

and

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ u_k^p dx = 0,$$

for any $p > 0$.

We chose $(u_k - \frac{c_k}{A})^+$ as a test function of (20) to get

$$\begin{aligned} &-\int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ div(F^{n-1}(\nabla u_k)F_\xi(\nabla u_k))dx + \int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ u_k^{n-1} dx \\ = &\int_{\mathbb{R}^n} \frac{(u_k - \frac{c_k}{A})^+ u_k^{\frac{1}{n-1}}}{\lambda_k} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx. \end{aligned} \tag{31}$$

For any $L > 0$, the estimation of (31) is

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{(u_k - \frac{c_k}{A})^+ u_k^{\frac{1}{n-1}}}{\lambda_k} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx \\
\geq\ & \int_{\mathcal{W}_{Lr_k}} \frac{(u_k - \frac{c_k}{A})^+ u_k^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}} dx + o_k(1) \\
=\ & \int_{\mathcal{W}_L(0)} (u_k(r_k x) - \frac{c_k}{A})^+ \frac{r_k^n u_k(r_k x)^{\frac{1}{n-1}}}{\lambda_k} e^{\beta_k u_k^{\frac{n}{n-1}}(r_k x)} dx + o_k(1) \\
=\ & \int_{\mathcal{W}_L(0)} (v_k - \frac{1}{A})^+ v_k^{\frac{1}{n-1}} e^{\beta_k (u^{\frac{n}{n-1}}(r_k x) - c_k^{\frac{n}{n-1}})} dx + o_k(1) \\
\to\ & \int_{\mathcal{W}_L(0)} (1 - \frac{1}{A}) e^{\frac{n}{n-1}\alpha_n w} dx. \qquad (32)
\end{aligned}$$

Notice that

$$\begin{aligned}
&- \int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ div(F^{n-1}(\nabla u_k) F_\xi(\nabla u_k)) dx + \int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ u_k^{n-1} dx \\
=\ & -\int_{\mathbb{R}^n} (u_k - \frac{c_k}{A})^+ div(F^{n-1}(\nabla (u_k - \frac{c_k}{A})^+) F_\xi(\nabla (u_k - \frac{c_k}{A})^+)) dx + o_k(1) \\
=\ & \int_{\mathbb{R}^n} F^n(\nabla (u_k - \frac{c_k}{A})^+) dx + o_k(1). \qquad (33)
\end{aligned}$$

Now we put (31)(32)(33) together, and take $L \to \infty$ first and then $k \to \infty$, we obtain

$$\liminf_{k \to +\infty} \int_{\mathbb{R}^n} F^n(\nabla (u_k - \frac{c_k}{A})^+) dx \geq 1 - \frac{1}{A}.$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^n} (F^n(\nabla u_k^A) + |u_k^A|^n) dx \\
=\ & \int_{u_k \leq \frac{c_k}{A}} (F^n(\nabla u_k) + |u_k|^n) dx + \int_{u_k \geq \frac{c_k}{A}} (\frac{c_k}{A})^n dx \\
=\ & 1 - \int_{u_k \geq \frac{c_k}{A}} (F^n(\nabla u_k) + |u_k|^n) dx + \int_{u_k \geq \frac{c_k}{A}} (\frac{c_k}{A})^n dx \\
=\ & 1 - \int_{\mathbb{R}^n} F^n(\nabla (u_k - \frac{c_k}{A})^+) dx \\
\leq\ & 1 - (1 - \frac{1}{A}) + o_k(1).
\end{aligned}$$

Thus the conclusion holds. □

Lemma 4.4. *We have*

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} (F^n(\nabla u_k) + |u_k|^n) dx = 0$$

for any $\delta > 0$, and then $u = 0$.

Proof. Since $\{x | u_k \leq c\} \subset \{x | u_k \leq \frac{c_k}{A}\}$ for any constant c, we have

$$\int_{u_k \leq c} (F^n(\nabla u_k) + |u_k|^n) dx \leq \int_{\mathbb{R}^n} (F^n(\nabla u_k^A) + |u_k^A|^n) dx,$$

Taking $k \to \infty$ first and then take $A \to +\infty$, the result follows from Lemma 4.3 and (23).

$\square$

Lemma 4.5. *There holds*

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\phi(\beta_k u_k^{\frac{n}{n-1}})dx \le \lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_{Lr_k}}(e^{\beta_k|u_k|^{\frac{n}{n-1}}}-1)dx = \limsup_{k\to+\infty}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}, \quad (34)$$

and consequently

$$\lim_{k\to+\infty}\frac{c_k}{\lambda_k}=0 \quad \textit{and} \quad \sup_k\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}<+\infty. \quad (35)$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\phi(\beta_k u_k^{\frac{n}{n-1}})\\
\le\ &\int_{\{u_k\le\frac{c_k}{A}\}}\phi(\beta_k u_k^{\frac{n}{n-1}})dx+\int_{\{u_k>\frac{c_k}{A}\}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx\\
\le\ &\int_{\mathbb{R}^n}\phi(\beta_k(u_k^A)^{\frac{n}{n-1}})dx + A^{\frac{n}{n-1}}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}\int_{\{u_k>\frac{c_k}{A}\}}\frac{u_k^{\frac{n}{n-1}}}{\lambda_k}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx.
\end{aligned}$$

Applying (23), we can find L such that $u_k \le 1$ on $\mathbb{R}^n\backslash\mathcal{W}_L$. Then by Lemma 4.4 and the form of ϕ, we have

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n\backslash\mathcal{W}_L}\phi(p\beta_k(u_k^A)^{\frac{n}{n-1}})dx \le C(p)\lim_{k\to+\infty}\int_{\mathbb{R}^n\backslash\mathcal{W}_L}u_k^n dx = 0 \quad (36)$$

for any $p > 0$.

Since by Lemma 4.3, it follows from the anisotropic Moser-Trudinger inequality in [ZZ] to get

$$\sup_k\int_{\mathcal{W}_L}e^{p'\beta_k((u_k^A-u_k(L))^+)^{\frac{n}{n-1}}}dx<+\infty$$

for any $p' < A^{\frac{1}{n-1}}$. Since for any $p < p'$

$$p(u_k^A)^{\frac{n}{n-1}} \le p'((u_k^A-u_k(L))^+)^{\frac{n}{n-1}}+C(p,p'),$$

we have

$$\sup_k\int_{\mathcal{W}_L}\phi(p\beta_k(u_k^A)^{\frac{n}{n-1}})dx<+\infty. \quad (37)$$

for any $p < A^{\frac{1}{n-1}}$. Then on $\mathcal{W}_L$, we get

$$\lim_{k\to+\infty}\int_{\mathcal{W}_L}\phi(\beta_k(u_k^A)^{\frac{n}{n-1}})dx=\int_{\mathcal{W}_L}\phi(0)dx=0.$$

Hence, by (21), we have

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{\mathbb{R}^n}\phi(\beta_k u_k^{\frac{n}{n-1}})dx\\
\le\ &\lim_{L\to+\infty}\lim_{k\to+\infty}A^{\frac{n}{n-1}}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}\int_{\mathcal{W}_L}\frac{u_k^{\frac{n}{n-1}}}{\lambda_k}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx\\
=\ &\lim_{k\to+\infty}A^{\frac{n}{n-1}}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}.
\end{aligned}$$

In view of (27), we obtain

$$\begin{aligned}\int_{\mathcal{W}_{Lr_k}} (e^{\beta_k|u_k|^{\frac{n}{n-1}}}-1)dx &= r_k^n\int_{\mathcal{W}_L} e^{\beta_k|u_k(r_k y)|^{\frac{n}{n-1}}}dy + o_k(1)\\ &= \frac{\lambda_k}{c_k^{\frac{n}{n-1}}}\Big(\int_{\mathcal{W}_L} e^{\frac{n}{n-1}\alpha_n w}dy+o_k(1)\Big)+o_k(1)\\ &= \frac{\lambda_k}{c_k^{\frac{n}{n-1}}}(1+o_L(1)+o_k(1))+o_k(1).\end{aligned}$$

Therefore

$$\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_{Lr_k}} (e^{\beta_k|u_k|^{\frac{n}{n-1}}}-1)dx = \limsup_{k\to+\infty}\frac{\lambda_k}{c_k^{\frac{n}{n-1}}}. \tag{38}$$

Then taking $A\to 1$, we obtain (34).

If $\frac{\lambda_k}{c_k}$ is bounded or $\limsup_{k\to+\infty}\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}=+\infty$, it would follow from (34) and Lemma 3.2 that

$$\sup_{||v||_F\le 1, v\in W^{1,n}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(\alpha_n|v|^{\frac{n}{n-1}})dx=0,$$

which is impossible. □

Now we claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\frac{c_k}{\lambda_k}u_k^{\frac{1}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx=1. \tag{39}$$

To this purpose, we denote $\varphi_k=\frac{c_k}{\lambda_k}u_k^{\frac{1}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})$. Clearly

$$\int_{\mathbb{R}^n}\varphi_k dx=\int_{\{u_k<\frac{c_k}{A}\}}\varphi_k dx+\int_{\{u_k\ge\frac{c_k}{A}\setminus\mathcal{W}_{r_kL}\}}\varphi_k dx+\int_{\mathcal{W}_{r_kL}}\varphi_k dx.$$

We estimate the three integrates on the right hands respectively. By (35) (36) (37) and Lemma (4.4), for any $1<p<A^{\frac{1}{n-1}}$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}0\le\int_{\{u_k<\frac{c_k}{A}\}}\varphi_k dx &= \frac{c_k}{\lambda_k}\int_{\{u_k<\frac{c_k}{A}\}}u_k^{\frac{1}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx\\ &\le \frac{c_k}{\lambda_k}||u_k^{\frac{1}{n-1}}||_{L^q(\mathbb{R}^n)}||e^{\beta_k|u_k^A|^{\frac{n}{n-1}}}||_{L^p(\mathbb{R}^n)}\to 0,\end{aligned} \tag{40}$$

and

$$\begin{aligned}\int_{\{u_k\ge\frac{c_k}{A}\setminus\mathcal{W}_{r_kL}\}}\varphi_k dx &\le A\int_{\{\mathbb{R}^n\setminus\mathcal{W}_{r_kL}\}}\frac{u_k^{\frac{n}{n-1}}}{\lambda_k}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx\\ &= A\Big(1-\int_{\mathcal{W}_{r_kL}}\frac{u_k^{\frac{n}{n-1}}}{\lambda_k}e^{\beta_k u_k^{\frac{n}{n-1}}}dx+o_k(1)\Big)\\ &= A\Big(1-\int_{\mathcal{W}_L}e^{\frac{n}{n-1}\alpha_n w}dx+o_k(1)\Big),\end{aligned}$$

and

$$\int_{\mathcal{W}_{r_kL}}\varphi_k dx=\int_{\mathcal{W}_L}e^{\frac{n}{n-1}\alpha_n w}dx+o_k(1).$$

Letting $k\to+\infty$ first and then letting $L\to+\infty$, we conclude (39).

Lemma 4.6. *On any domain $\Omega \subset\subset \mathbb{R}^n \backslash \{0\}$, we have that $c_k^{\frac{1}{n-1}} u_k$ converges to G in $C^1(\Omega)$, where $G \in C^{1,\alpha}_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfies the following equation:*

$$-Q_n G + G^{n-1} = \delta_0 \ in\ \mathbb{R}^n. \tag{41}$$

Proof. Define $U_k = c_k^{\frac{1}{n-1}} u_k$, which satisfies the equations:

$$-Q_n U_k + U_k^{n-1} = \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \phi'(\beta_k u_k^{\frac{n}{n-1}}). \tag{42}$$

For our purpose, we need to prove that

$$\int_{\mathcal{W}_R} |\nabla U_k|^q dx \le C(q, R), \quad 1 < q < n, \tag{43}$$

where $C(q, R)$ does not depend on k.

Set $\Omega_t = \{0 \le U_k \le t\}$, $U_k^t = \min\{U_k, t\}$. Testing Eq.(42) with U_k^t, we get from Lemma 2.1 and (39) that

$$\begin{aligned}
\int_{\Omega_t} (F^n(\nabla U_k^t) + |U_k^t|^n) dx &\le \int_{\mathcal{W}_{R_k}} (F^n(\nabla U_k^t) + |U_k^t|^n) dx \\
&\le \int_{\mathcal{W}_{R_k}} (F^{n-1}(\nabla U_k) F_\xi(\nabla U_k) \nabla U_k^t + U_k^t U_k^{n-1}) dx \\
&= \int_{\partial \mathcal{W}_{R_k}} U_k^t \big(F^{n-1}(\nabla U_k) F_\xi(\nabla U_k) \cdot \vec{n}\big) d\sigma + \int_{\mathcal{W}_{R_k}} (-Q_n U_k + U_k^{n-1}) U_k^t dx \\
&= \int_{\mathcal{W}_{R_k}} (-Q_n U_k + U_k^{n-1}) U_k^t dx \\
&= \int_{\mathbb{R}^n} (-Q_n U_k + U_k^{n-1}) U_k^t dx \\
&= \int_{\mathbb{R}^n} U_k^t \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx \le 2t,
\end{aligned}$$

where $\vec{n}$ is the unit external normal vector of $\partial \mathcal{W}_{R_k}$.

Let η be a radially symmetric cut off function with respect to $F^o(x)$ which satisfies that $\eta = 1$ in $\mathcal{W}_{\frac{R}{2}}$, $\eta = 0$ in $\mathcal{W}_R^c$, $F(\nabla \eta) \le \frac{C}{R}$. Hence, when R large enough, we have

$$\int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \le \int_{\mathcal{W}_R} |U_k^t|^n F^n(\nabla \eta) dx + \int_{\mathcal{W}_R} |\eta|^n F^n(\nabla U_k^t) dx \le C(R)t + C_0(R).$$

Taking t large enough such that $C(R)t > C_0(R)$, then we have

$$\int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \le 2C(R)t.$$

Set $|\mathcal{W}_\rho| = |\{x \in \mathcal{W}_R : U_k \ge t\}|$. We have

$$\inf_{\psi \in W_0^{1,n}(\mathcal{W}_R), \psi|_{\mathcal{W}_\rho} = t} \int_{\mathcal{W}_R} F^n(\nabla \psi) dx \le \int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \le 2C(R)t. \tag{44}$$

The above infimum can be attained (see [Y, ZZ]) by

$$\psi_1(x) = \begin{cases} t \log \frac{R}{F^o(x)} / \log \frac{R}{\rho} & in \quad \mathcal{W}_R \backslash \mathcal{W}_\rho, \\ t & in \quad \mathcal{W}_\rho. \end{cases}$$

By calculating $||F(\nabla\psi_1)||^n_{L^n(\mathcal{W}_R)}$, we have by (44), $\rho \le Re^{-C_1 t}$ for some constant $C_1 > 0$. Hence

$$|\{x \in \mathcal{W}_R : U_k \ge t\}| = |\mathcal{W}_\rho| \le \kappa_n R^n e^{-nC_1 t}.$$

For any $0 < \delta < nC_1$, we obtain

$$\begin{aligned}\int_{\mathcal{W}_R} e^{\delta U_k^+} dx &\le e^\delta |\mathcal{W}_R| + \sum_{m=1}^{\infty} e^{(m+1)\delta} |\{x \in \mathcal{W}_R :\ m \le U_k \le m+1\}| \\ &\le e^\delta |\mathcal{W}_R| + \kappa_n R^n e^\delta \sum_{m=1}^{\infty} e^{-(nC_1-\delta)m} \le C_2\end{aligned}$$

for some constant C_2. Testing Eq.(42) with $\log \frac{1+2U_k}{1+U_k}$, we have

$$\begin{aligned}&\int_{\mathcal{W}_R} \frac{F^n(\nabla U_k)}{(1+U_k)(1+2U_k)} dx \\ \le\ & \log 2 \int_{\mathcal{W}_R} \frac{c_k u_k^{\frac{1}{n-1}}}{\lambda_k} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx - \int_{\mathcal{W}_R} U_k^{n-1} \log \frac{1+2U_k}{1+U_k} dx \le C_3.\end{aligned}$$

By the Young inequality, we have for any $1 < q < n$,

$$\begin{aligned}\int_{\mathcal{W}_R} F^q(\nabla U_k) dx &\le \int_{\mathcal{W}_R} \frac{F^n(\nabla U_k)}{(1+U_k^+)(1+2U_k)} dx + \int_{\mathcal{W}_R} ((1+U_k)(1+2U_k))^{\frac{q}{n-q}} dx \\ &\le C_4 (1 + \int_{\mathcal{W}_R} e^{\delta U_k} dx) \le C_5,\end{aligned}$$

for some constants C_3 and C_5 depending only on q, n and $\mathcal{W}_R$. Then the (43) holds.

Hence U_k is bounded in $L^q(\Omega)$ for any $q > 0$. By Lemma 4.4 and Theorem 1.1, we can get $e^{\beta_k u_k^{\frac{n}{n-1}}}$ is also bounded in $L^q(\Omega)$ for any $q > 0$. Thanks to theorem 2 in [J] and theorem 1 in [T2], $||U_k||_{C^{1,\alpha}(\Omega)} \le C$, then by Ascoli-Arzela's theorem, U_k converges to G in $C^1(\Omega)$. □

For the Green function G, we have the following results:

Lemma 4.7. *$G \in C^{1,\alpha}_{loc}(\mathbb{R}^n \backslash \{0\})$ and near 0 we can write*

$$G = -\frac{n}{\alpha_n} \log r + C_G + o_r(1); \tag{45}$$

where C_G is a constant and $r = F^o(x)$. Moreover, for any $\delta > 0$, we have

$$\begin{aligned}&\lim_{k\to+\infty} \int_{\mathbb{R}^n \backslash \mathcal{W}_\delta} (F^n(\nabla(c_k^{\frac{1}{n-1}} u_k)) + (c_k^{\frac{1}{n-1}} u_k)^n) dx \\ =\ & \int_{\mathbb{R}^n \backslash \mathcal{W}_\delta} (F^n(\nabla G) + G^n) dx = G(\delta)(1 - \int_{\mathcal{W}_\delta} G^{n-1} dx).\end{aligned} \tag{46}$$

Proof. We will prove (45) in section 6. Here we will use (45) to prove (46). Firstly, we have

$$\int_{\mathbb{R}^n \backslash \mathcal{W}_\delta} u_k^{\frac{n}{n-1}} \phi'(\beta_k u_k^{\frac{n}{n-1}}) dx \le C \int_{\mathbb{R}^n \backslash \mathcal{W}_\delta} u_k^n dx \to 0. \tag{47}$$

Recall that $U_k = c_k^{\frac{1}{n-1}} u_k \in W_0^{1,n}(\mathcal{W}_{R_k})$, by Equation (42) we get

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\mathcal{W}_\delta} (F^n(\nabla U_k) + U_k^n)dx \\
= &\frac{c_k^{\frac{n}{n-1}}}{\lambda_k}\int_{\mathbb{R}^n\setminus\mathcal{W}_\delta} u_k^{\frac{n}{n-1}}\phi'(\beta_k u_k^{\frac{n}{n-1}})dx - \int_{\partial\mathcal{W}_\delta}\frac{\partial U_k}{\partial n}F^{n-1}(\nabla U_k)U_k dS.
\end{aligned}$$

By (35) and (47) we then get

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus\mathcal{W}_\delta}(F^n(\nabla U_k) + U_k^n)dx \\
= &-\lim_{k\to+\infty}\int_{\partial\mathcal{W}_\delta}\frac{\partial U_k}{\partial n}F^{n-1}(\nabla U_k)U_k dS \\
= &-G(\delta)\int_{\partial\mathcal{W}_\delta}\frac{\partial G}{\partial n}F^{n-1}(\nabla G)dS = G(\delta)(1 - \int_{\mathcal{W}_\delta} G^{n-1}dx).
\end{aligned}$$

□

Proof of Theorem 1.2: From (36) we have

$$\int_{\mathbb{R}^n\setminus\mathcal{W}_R}\phi(\beta_k u_k^{\frac{n}{n-1}})dx \le C.$$

So, we only need to prove on $\mathcal{W}_R$,

$$\int_{\mathcal{W}_R} e^{\beta_k u_k^{\frac{n}{n-1}}}dx \le C = C(R).$$

By Lemma 4.6, for any fixed $R > 0$, we have $c_k^{\frac{1}{n-1}} u_k(R) \to G(R)$ as $k \to +\infty$, i.e. $u_k(R) = O(\frac{1}{c_k^{\frac{1}{n-1}}})$. Hence we have

$$\begin{aligned}
u_k^{\frac{n}{n-1}} &\le ((u_k - u_k(R))^+ + u_k(R))^{\frac{n}{n-1}} \\
&\le ((u_k - u_k(R))^+)^{\frac{n}{n-1}} + C_1.
\end{aligned}$$

Then, we get

$$\int_{\mathcal{W}_R} e^{\beta_k u_k^{\frac{n}{n-1}}}dx \le C.$$

Proof of Theorem 1.1: To prove Theorem 1.1, we use an idea of [SK]. By means of symmetrization, it suffices to show the desired inequality (5) for functions $u(x) = u(F^o(x))$, which are non-negative, radially symmetric with respect to $F^o(x)$ and decreasing.

Define

$$w(t) = n\kappa_n^{\frac{1}{n}} u(e^{-\frac{t}{n}}), \qquad F^o(x) = e^{-\frac{t}{n}}. \tag{48}$$

Then $w(t)$ is defined on $(-\infty, +\infty)$, and we have

$$\int_{\mathbb{R}^n} F^n(\nabla u)dx = \int_{-\infty}^{+\infty}|\dot{w}(t)|^n dt, \tag{49}$$

$$\int_{\mathbb{R}^n}\phi(\alpha u^{\frac{n}{n-1}}) = \kappa_n\int_{-\infty}^{+\infty}\phi(\frac{\alpha}{\alpha_n}w(t)^{\frac{n}{n-1}})e^{-t}dt, \tag{50}$$

$$\int_{\mathbb{R}^n}|u(x)|^n dx = \frac{1}{n^n}\int_{-\infty}^{+\infty}|w(t)|^n e^{-t}dt. \tag{51}$$

For the following proof, one can refer to [SK] for details.

5. Existence of the extremal function

In this section, we will show that the existence of the extremal functions. For this purpose, it is sufficient to show that the maximizing sequence u_k does not blow up. To this point, we argue by contradiction. We assume the maximizing sequence u_k blows up, i.e. $c_k \to +\infty$ as $k \to \infty$, then we will establish the upper bound of S which is the supremum of our Moser-Trudinger functional. On the other hand, we can construct an explicit test function, which provides a lower bound of S, which is a contradiction.

To get the upper bound of S, we will use the following Carleson-Chang type inequality which is shown in [ZZ].

Lemma 5.1. *Assume that u_k is a normalized concentrating sequence in $W_0^{1,n}(\mathcal{W}_1)$ with a blow up point at the origin, i.e. $\int_{\mathcal{W}_1} F^n(\nabla u_k)dx = 1$, $u_k \rightharpoonup 0$ weakly in $W_0^{1,n}(\mathcal{W}_1)$, and $\lim_{k\to\infty} \int_{\mathcal{W}_1\setminus\mathcal{W}_r} F^n(\nabla u_k)dx = 0$ for any $0 < r < 1$, then*

$$\limsup_{k\to+\infty} \int_{\mathcal{W}_1} (e^{\alpha_n|u_k|^{\frac{n}{n-1}}} - 1)dx \le \kappa_n e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}. \tag{52}$$

Lemma 5.2. *If S cannot be attained, then*

$$S \le \kappa_n e^{\alpha_n C_G + 1 + \frac{1}{2}+\cdots+\frac{1}{n-1}},$$

where C_G is the constant in (45) .

Proof. Set $\tilde{u}_k = \frac{(u_k(x)-u_k(\delta))^+}{||F(\nabla u_k)||_{L^n(\mathcal{W}_\delta)}}$ which is in $W_0^{1,n}(\mathcal{W}_\delta)$. Then by Lemma 5.1, we have

$$\limsup_{k\to+\infty} \int_{\mathcal{W}_\delta} e^{\beta_k \tilde{u}_k^{\frac{n}{n-1}}} dx \le |\mathcal{W}_\delta|(1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}).$$

By Lemma 4.7 we have

$$\int_{\mathbb{R}^n\setminus\mathcal{W}_\delta} (F^n(\nabla c_k^{\frac{1}{n-1}} u_k) + (c_k^{\frac{1}{n-1}} u_k)^n)dx \to G(\delta)(1 - \int_{\mathcal{W}_\delta} G^{n-1}dx).$$

Hence we get

$$\begin{aligned} \int_{\mathcal{W}_\delta} F^n(\nabla u_k)dx &= 1 - \int_{\mathbb{R}^n\setminus\mathcal{W}_\delta} (F^n(\nabla u_k) + u_k^n)dx - \int_{\mathcal{W}_\delta} u_k^n dx \\ &= 1 - \frac{G(\delta) + \epsilon_k(\delta)}{c_k^{\frac{n}{n-1}}}, \end{aligned} \tag{53}$$

where $\lim_{\delta\to 0} \lim_{k\to+\infty} \epsilon_k(\delta) = 0$.

By (36) and Lemma 4.5 we have

$$\lim_{L\to+\infty} \lim_{k\to+\infty} \int_{\mathcal{W}_\rho\setminus\mathcal{W}_{Lr_k}} e^{\beta_k u_k^{\frac{n}{n-1}}} dx = |\mathcal{W}_\rho|,$$

for any $\rho < \delta$. Furthermore, on $\mathcal{W}_\rho$ we have by (53)

$$
\begin{aligned}
(\tilde{u}_k)^{\frac{n}{n-1}} &\leq \frac{u_k^{\frac{n}{n-1}}}{(1-\frac{G(\delta)+\epsilon_k(\delta)}{c_k^{\frac{n}{n-1}}})^{\frac{1}{n-1}}} \\
&= u_k^{\frac{n}{n-1}}(1+\frac{1}{n-1}\frac{G(\delta)+\epsilon_k(\delta)}{c_k^{\frac{n}{n-1}}}+O(\frac{1}{c_k^{\frac{2n}{n-1}}})) \\
&= u_k^{\frac{n}{n-1}}+\frac{1}{n-1}G(\delta)(\frac{u_k}{c_k})^{\frac{n}{n-1}}+O(c_k^{-\frac{n}{n-1}}) \\
&\leq u_k^{\frac{n}{n-1}}-\frac{n\log\delta}{(n-1)\alpha_n}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_\rho\backslash\mathcal{W}_{Lr_k}} e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}dx \\
\leq\ & O(\delta^{-n})\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_\rho\backslash\mathcal{W}_{Lr_k}} e^{\beta_k u_k^{\frac{n}{n-1}}}dx \to |\mathcal{W}_\rho|O(\delta^{-n}).
\end{aligned}
$$

Since $\tilde{u}_k\to 0$ on $\mathcal{W}_\delta\backslash\mathcal{W}_\rho$, we get $\lim\limits_{k\to+\infty}\int_{\mathcal{W}_\delta\backslash\mathcal{W}_\rho}(e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}-1)dx=0$, then

$$
0\leq\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_\delta\backslash\mathcal{W}_{Lr_k}}(e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}-1)dx\leq|\mathcal{W}_\rho|O(\delta^{-n}).
$$

Letting $\rho\to 0$, we get

$$
\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_\delta\backslash\mathcal{W}_{Lr_k}}(e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}-1)dx=0.
$$

This implies

$$
\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_{Lr_k}}(e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}-1)dx\leq|\mathcal{W}_\delta|e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.
$$

It is easy to check that

$$
\frac{\tilde{u}_k(r_kx)}{c_k}\to 1 \qquad \textit{and} \qquad (\tilde{u}_k(r_kx))^{\frac{1}{n-1}}u_k(\delta)\to G(\delta).
$$

By using that $u_k(\delta)=O(\frac{1}{c_k^{\frac{1}{n-1}}})$ and $||F(\nabla u_k)||_{L^n(\mathcal{W}_\delta)}=1+O(\frac{1}{c_k^{\frac{n}{n-1}}})$, for a fixed L and any $x\in\mathcal{W}_{Lr_k}$, we have

$$\begin{aligned}
\beta_k u_k^{\frac{n}{n-1}} &= \beta_k\Big(\frac{u_k}{||F(\nabla u_k)||_{L^n(\mathcal{W}_\delta)}}\Big)^{\frac{n}{n-1}}\Big(\int_{\mathcal{W}_\delta}F^n(\nabla u_k)dx\Big)^{\frac{1}{n-1}} \\
&= \beta_k\Big(\tilde{u}_k+\frac{u_k(\delta)}{||F(\nabla u_k)||_{L^n(\mathcal{W}_\delta)}}\Big)^{\frac{n}{n-1}}\Big(\int_{\mathcal{W}_\delta}F^n(\nabla u_k)dx\Big)^{\frac{1}{n-1}} \\
&= \beta_k\Big(\tilde{u}_k+u_k(\delta)+O\big(\frac{1}{c_k^{(n+1)/(n-1)}}\big)\Big)^{\frac{n}{n-1}}\Big(\int_{\mathcal{W}_\delta}F^n(\nabla u_k)dx\Big)^{\frac{1}{n-1}} \\
&= \beta_k\tilde{u}_k^{\frac{n}{n-1}}\Big(1+\frac{u_k(\delta)}{\tilde{u}_k}+O\big(\frac{1}{c_k^{2n/(n-1)}}\big)\Big)^{\frac{n}{n-1}}\Big(1-\frac{G(\delta)+\epsilon_k(\delta)}{c_k^{n/(n-1)}}\Big)^{\frac{1}{n-1}} \\
&= \beta_k\tilde{u}_k^{\frac{n}{n-1}}\Big[1+\frac{n}{n-1}\frac{u_k(\delta)}{\tilde{u}_k}-\frac{1}{n-1}\frac{G(\delta)+\epsilon_k(\delta)}{c_k^{n/(n-1)}}+O\big(\frac{1}{c_k^{2n/(n-1)}}\big)\Big].
\end{aligned}$$

So, we get

$$\begin{aligned}
&\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\mathcal{W}_{Lr_k}}(e^{\beta_k u_k^{\frac{n}{n-1}}}-1)dx \\
=\ &\lim_{L\to+\infty}\lim_{k\to+\infty}e^{\alpha_n G(\delta)}\int_{\mathcal{W}_{Lr_k}}(e^{\beta_k\tilde{u}_k^{\frac{n}{n-1}}}-1)dx \\
\le\ &e^{\alpha_n((-\frac{n}{\alpha_n})\log\delta+C_G+O_\delta(1))}\delta^n\kappa_n e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.
\end{aligned}$$

Letting $\delta\to 0$, then together with Lemma 4.5 implies Lemma 5.2. $\square$

Next we will construct a function $u_\epsilon\in W^{1,n}(\mathbb{R}^n)$ with $||u_\epsilon||_F=1$ which satisfies

$$\int_{\mathbb{R}^n}\phi(\alpha_n|u_\epsilon|^{\frac{n}{n-1}})dx>\kappa_n e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}},$$

for $\epsilon>0$ sufficiently small. To this purpose we set

$$u_\epsilon=\begin{cases} C+C^{-\frac{1}{n-1}}\big(-\frac{n-1}{\alpha_n}\log(1+\kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}})+b\big), & x\in\overline{\mathcal{W}_{R\epsilon}(0)},\\ C^{-\frac{1}{n-1}}G, & x\in\mathcal{W}^c_{R\epsilon}(0),\end{cases}\tag{54}$$

where $R=-\log\epsilon$, b,C are functions of ϵ (which will be defined later). In order to assure that $u_\epsilon\in W^{1,n}(\mathbb{R}^n)$, we set

$$C+C^{-\frac{1}{n-1}}\Big(-\frac{n-1}{\alpha_n}\log(1+\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}})+b\Big)=C^{-\frac{1}{n-1}}G(R\epsilon),\tag{55}$$

Next we make sure that $\int_{\mathbb{R}^n}F^n(\nabla u_\epsilon)+u_\epsilon^n dx=1$. By the coarea formula (8), we have

$$\begin{aligned}
\int_{\mathcal{W}_{R\epsilon}(0)}\frac{(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}}\frac{1}{\epsilon^n}}{(1+\kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}})^n}dx &= n\kappa_n\int_0^{R\epsilon}\frac{(\frac{s}{\epsilon})^{\frac{n}{n-1}}\frac{1}{\epsilon^n}}{(1+\kappa_n^{\frac{1}{n-1}}(\frac{s}{\epsilon})^{\frac{n}{n-1}})^n}s^{n-1}ds \\
&= \frac{n-1}{\kappa_n^{\frac{1}{n-1}}}\int_0^{\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}}}\frac{t^{n-1}}{(1+t)^n}dt,
\end{aligned}$$

which leads to

$$\begin{aligned}
\int_{\mathcal{W}_{R\epsilon}(0)} F^n(\nabla u_\epsilon)dx &= C^{-\frac{n}{n-1}}\frac{n-1}{\alpha_n}\int_0^{\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}}}\frac{t^{n-1}}{(1+t)^n}dt\\
&= C^{-\frac{n}{n-1}}\frac{n-1}{\alpha_n}\int_0^{\kappa_N^{\frac{1}{n-1}}R^{\frac{n}{n-1}}}\frac{(t+1-1)^{n-1}}{(1+t)^n}dt\\
&= C^{-\frac{n}{n-1}}\frac{n-1}{\alpha_n}(\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1}\\
&\quad +\log(1+\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}})+O(R^{-\frac{n}{n-1}}))\\
&= C^{-\frac{n}{n-1}}\frac{n-1}{\alpha_n}(-(1+\frac{1}{2}+\cdots+\frac{1}{n-1})\\
&\quad +\log(1+\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}})+O(R^{-\frac{n}{n-1}})), \qquad (56)
\end{aligned}$$

where we have used the fact that

$$-\sum_{k=0}^{n-2}\frac{C_{n-1}^k(-1)^{n-1-k}}{n-k-1}=1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

It is easy to check that

$$\int_{\mathcal{W}_{R\epsilon}}|u_\epsilon|^n dx=O((R\epsilon)^n C^n\log R). \qquad (57)$$

Moreover, we have

$$\begin{aligned}
\int_{\mathcal{W}_{R\epsilon}^c}(F^n(\nabla u_\epsilon)+u_\epsilon^n)dx &= \frac{1}{C^{n/(n-1)}}(\int_{\mathcal{W}_{R\epsilon}^c}F^n(\nabla G)dx+\int_{\mathcal{W}_{R\epsilon}^c}G^n dx)\\
&= \frac{1}{C^{n/(n-1)}}\int_{\partial\mathcal{W}_{R\epsilon}}G(R\epsilon)F^{n-1}(\nabla G)\frac{\partial G}{\partial n}dS\\
&= \frac{G(R\epsilon)}{C^{n/(n-1)}}(1-\int_{\mathcal{W}_{R\epsilon}}G^{n-1}dx), \qquad (58)
\end{aligned}$$

Putting (56),(57),(58) together, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}(F^n(\nabla u_\epsilon)+u_\epsilon^n)dx=\frac{1}{\alpha_n C^{\frac{n}{n-1}}}\{&-(n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1})+\alpha_n C_G\\
&+(n-1)\log(1+\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}})-\log(R\epsilon)^n+\varphi_\varepsilon(C)\},
\end{aligned}$$

where $\varphi_\varepsilon(C)=O((R\epsilon)^n C^n\log R+(R\epsilon)^n\log^n(R\epsilon)+R^{-\frac{n}{n-1}})$. Since $\int_{\mathbb{R}^n}(F^n(\nabla u_\epsilon)+u_\epsilon^n)dx=1$, we have

$$\alpha_n C^{\frac{n}{n-1}}=-(n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1})+\alpha_n C_G+\log\kappa_n-\log\epsilon^n+\varphi_\varepsilon(C). \qquad (59)$$

By (55) we have

$$\alpha_n C^{\frac{n}{n-1}}-(n-1)\log(1+\kappa_n^{\frac{1}{n-1}}R^{\frac{n}{n-1}})+\alpha_n b=\alpha_n G(R\epsilon),$$

and hence

$$-(n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1})+\alpha_n C_G-\log(R\epsilon)^n+\varphi_\varepsilon(C)+\alpha_n b=\alpha_n G(R\epsilon),$$

This implies that

$$b = -\frac{n-1}{\alpha_n}(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) + \varphi_\varepsilon(C). \tag{60}$$

In view of (55) and (60), there holds on $\mathcal{W}_{R\epsilon}(0)$,

$$\begin{aligned} \alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}} &\geq \alpha_n C^{\frac{n}{n-1}} - n\log(1 + \kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}}) + \frac{n\alpha_n}{n-1} b + O(R^{-\frac{2n}{n-1}}) \\ &\geq -n\log\epsilon + \log\kappa_n + \alpha_n C_G + (1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) \\ &\quad -n\log(1 + \kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}}) + \varphi_\varepsilon(C) \end{aligned}$$

where we use the inequality $|1-t|^{\frac{n}{n-1}} \geq 1 - \frac{n}{n-1}t + O(t^3)$ for $|t| < 1$. Since by using the fact that

$$\sum_{k=0}^{n-2} \frac{C_{n-2}^k (-1)^{n-k-2}}{n-k-1} = \frac{1}{n-1}$$

we have

$$\begin{aligned} &\int_{\mathcal{W}_{R\epsilon}(0)} e^{-n\log\epsilon - n\log(1+\kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}})} dx \\ &= \frac{1}{\epsilon^n}\int_{\mathcal{W}_{R\epsilon}(0)} \frac{1}{(1+\kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\epsilon})^{\frac{n}{n-1}})^n} dx \\ &= (n-1)\int_0^{\kappa_n^{\frac{1}{n-1}} R^{\frac{n}{n-1}}} \frac{t^{n-2}}{(1+t)^n} dt \\ &= (n-1)\int_0^{\kappa_n^{\frac{1}{n-1}} R^{\frac{n}{n-1}}} \frac{(t+1-1)^{n-2}}{(1+t)^n} dt \\ &\geq (n-1)(\frac{1}{n-1} + O(R^{-\frac{n}{n-1}})) = 1 + O(R^{-\frac{n}{n-1}})), \end{aligned}$$

we obtain that

$$\int_{\mathcal{W}_{R\epsilon}(0)} e^{\alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}}} dx \geq \kappa_n e^{\alpha_n C_G + (1+\frac{1}{2}+\cdots+\frac{1}{n-1})} + \varphi_\varepsilon(C),$$

and further to get that

$$\int_{\mathcal{W}_{R\epsilon}(0)} \phi(\alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}}) dx \geq \kappa_n e^{\alpha_n C_G + (1+\frac{1}{2}+\cdots+\frac{1}{n-1})} + \varphi_\varepsilon(C).$$

Moreover, on $\mathbb{R}^n \backslash \mathcal{W}_{R\epsilon}$ we have the estimate

$$\int_{\mathbb{R}^n\backslash\mathcal{W}_{R\epsilon}} \phi(\alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}}) dx \geq \frac{\alpha_n^{n-1}}{(n-1)!}\int_{\mathbb{R}^n\backslash\mathcal{W}_{R\epsilon}} |\frac{G(x)}{C^{1/(n-1)}}|^n dx,$$

and thus we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \phi(\alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}}) dx \\ &\geq \kappa_n e^{\alpha_n C_G + (1+\frac{1}{2}+\cdots+\frac{1}{n-1})} + \frac{\alpha_n^{n-1}}{(n-1)! C^{n/(n-1)}}\int_{\mathbb{R}^n\backslash\mathcal{W}_{R\epsilon}} |G(x)|^n dx + \varphi_\varepsilon(C). \end{aligned} \tag{61}$$

Next we show that that there exists a $C = C(\epsilon)$ which solves Equation (59). To this point, we set

$$f(t) = -\alpha_n t^{n/(n-1)} - (n-1)(1+\frac{1}{2}+\cdots+\frac{1}{n-1}) + \alpha_n C_G + \log\kappa_n - \log\epsilon^n + \varphi_\varepsilon(t).$$

Since for sufficient small ε we have

$$f((-\frac{2}{\alpha_n}\log\epsilon^n)^{(n-1)/n}) = \log\epsilon^n + O(1) + \varphi_\varepsilon((-\frac{2}{\alpha_n}\log\epsilon^n)^{(n-1)/n}) < 0$$

and

$$f((-\frac{1}{2\alpha_n}\log\epsilon^n)^{(n-1)/n}) = -\frac{1}{2}\log\epsilon^n + O(1) + \varphi_\varepsilon((-\frac{2}{\alpha_n}\log\epsilon^n)^{(n-1)/n}) > 0$$

then $f(t)$ has a zero point in

$$((-\frac{1}{2\alpha_n}\log\epsilon^n)^{(n-1)/n}, (-\frac{2}{\alpha_n}\log\epsilon^n)^{(n-1)/n}).$$

We denote this zero point by C, then it satisfies $\alpha_n C^{n/(n-1)} = -\log\epsilon^n + O(1)$. Therefore, as $\epsilon \to 0$, we have

$$\frac{\log R}{C^{n/(n-1)}} \to 0,$$

and

$$(R\epsilon)^n C^n \log R + (R\epsilon)^n \log^n(R\epsilon) + R^{-\frac{n}{n-1}} \to 0.$$

Therefore, we can conclude from (62) that for $\epsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^n} \phi(\alpha_n |u_\epsilon(x)|^{\frac{n}{n-1}})dx > \kappa_n e^{\alpha_n C_G + (1+\frac{1}{2}+\cdots+\frac{1}{n-1})}.$$

6. Asymptotic representation of G

In this section we will give the asymptotic representation of the anisotropic Green function G by using similar arguments in [Y, WX1, KV].

The proof of Lemma 4.7: Since $c_k^{\frac{n}{n-1}} u_k \geq 0$ in $\mathbb{R}^n\backslash\{0\}$, we have $G \geq 0$ in $\mathbb{R}^n\backslash\{0\}$. Theorem 1 in [S1] gives

$$\frac{1}{K} \leq \frac{G}{-\log r} \leq K \quad \text{in} \quad \mathbb{R}^n\backslash\{0\} \tag{62}$$

for some constant $K > 0$. Assume $\Gamma(r) = -c(n)\log r,\ c(n) = (n\kappa_n)^{-\frac{1}{n-1}}$. Set $G_k(x) = \frac{G(r_k x)}{\Gamma(r_k)}$, which is defined in $\{x \in \mathbb{R}^n\backslash\{0\}, r_k x \in \mathcal{W}_\delta\}$ for some small $\delta > 0$. Here $r_k \to 0$ as $k \to +\infty$. Then G_k satisfies the equation

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}(F^{n-1}(\nabla G_k)F_\xi(\nabla G_k)) + r_k^n G_k^{n-1} = 0.$$

By theorem 1 in [T2], when $r_k \to 0$, G_k converges to G^* in $C^1_{loc}(\mathbb{R}^n\backslash\{0\})$ and G^* is bounded, where G^* satisfies

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}(F^{n-1}(\nabla G^*)F_\xi(\nabla G^*)) = 0.$$

From serrin's result (see [S1]) and (62), 0 is a removable singularity and G^* can be extended to $\hat{G} \in C^1(\mathbb{R}^n)$. Consequently, form Liouville type theorem (see [HKM]),

$\hat{G}$ must be a constant. Let $\gamma_k = \sup_{\mathcal{W}_\delta\backslash\mathcal{W}_{r_k}} \frac{G(x)}{\Gamma(x)}$, and $\gamma = \lim_{k\to+\infty} \gamma_k, (\gamma > 0)$. This means the constant function $\hat{G} = \gamma$.

Set

$$G_\eta^+(x) = (\gamma+\eta)(\Gamma(x) - \Gamma(\delta)) - c(n)(\gamma+\eta)(F^o(x) - \delta) + \sup_{\partial\mathcal{W}_\delta} G,$$
$$G_\eta^-(x) = (\gamma-\eta)(\Gamma(x) - \Gamma(\delta)) + c(n)(\gamma-\eta)(F^o(x) - \delta) + \inf_{\partial\mathcal{W}_\delta} G.$$

A straightforward calculation shows

$$-Q_n G_\eta^+(x) = c^{n-1}(n)(\gamma+\eta)^{n-1}\frac{n-1}{F^o(x)}(\frac{1}{F^o(x)}+1)^{n-2},$$
$$-Q_n G_\eta^-(x) = -c^{n-1}(n)(\gamma-\eta)^{n-1}\frac{n-1}{F^o(x)}(\frac{1}{F^o(x)}-1)^{n-2}.$$

It is clear that, for any fixed $0 < \eta < \gamma$, we have

$$-Q_n G_\eta^+(x) \geq -Q_n G \qquad in \quad \mathcal{W}_\delta\backslash\mathcal{W}_{r_k},$$
$$G_\eta^+|_{\partial\mathcal{W}_\delta} \geq G|_{\partial\mathcal{W}_\delta}, \qquad G_\eta^+|_{\partial\mathcal{W}_{r_k}} \geq G|_{\partial\mathcal{W}_{r_k}},$$

provided that δ are sufficiently small and $r_k < \delta$. By the comparison principle (see [XG]), we have

$$G \leq (\gamma+\eta)\Gamma(x) + C_\delta \qquad in \quad \mathcal{W}_\delta\backslash\mathcal{W}_{r_k} \tag{63}$$

for some constant C_δ. Letting $\eta \to 0$ first, then $k \to \infty$, one has

$$G \leq \gamma\Gamma(x) + C_\delta \qquad in \quad \mathcal{W}_\delta\backslash\{0\}.$$

A similar argument gives $G \geq \gamma\Gamma(x) + C_\delta^{'}$ in $\mathcal{W}_\delta\backslash\{0\}$ for some constant $C_\delta^{'}$. Hence $G - \gamma\Gamma(x)$ is bounded in $L^\infty(\mathcal{W}_\delta)$.

Next we prove the continuity of $G - \gamma\Gamma(x)$ at 0. To this point, we consider the points where the bounded function $G - \gamma\Gamma(x)$ achieves its supremum in $\overline{\mathcal{W}_\delta}$. We set $\lambda = \sup_{\overline{\mathcal{W}_\delta}}(G - \gamma\Gamma(x))$.

If λ achieves at some point in $\mathcal{W}_\delta\backslash\{0\}$, then $G - \gamma\Gamma(x) - \gamma c(n)F^o(x)$ also achieves at some point in $\mathcal{W}_\delta\backslash\{0\}$. It follows from comparison principle (see [D1]) that $G - \gamma\Gamma(x) - \gamma c(n)F^o(x)$ is a constant. This implies the continuity of $G - \gamma\Gamma(x)$ at 0.

Next we assume that λ achieves at 0. We can set

$$w_r(x) = G(rx) - \gamma\Gamma(r) \qquad in \quad \mathcal{W}_{\frac{\delta}{r}}\backslash\{0\}.$$

It is clear that w_r satisfies

$$-Q_n(w_r(x)) + r^n G^{n-1}(rx) = 0.$$

We also have

$$r^n G^{n-1}(rx) \in L^\infty(\mathcal{W}_R), \quad |w_r - \gamma\Gamma(x)| \leq C_0$$

for $C_0 = \sup_{\mathcal{W}_\delta\backslash\{0\}} |G - \gamma\Gamma(x)|$ and $R > 0$. By Theorem 1 in [T2], when $r \to 0$, $w_r \to w$ in $C^1_{loc}(\mathbb{R}^n\backslash\{0\})$, where $w \in C^1(\mathbb{R}^n\backslash\{0\})$ satisfies $-Q_n(w) = 0$. For the sequence $\xi_j = \frac{x_{r_j}}{r_j}, F^o(\xi_j) = 1$, which maybe assumed to converge to $\xi^0 \in \partial\mathcal{W}_1$, we have

$$w_{r_j}(\xi_j) - \gamma\Gamma(\xi_j) = G(x_{r_j}) - \gamma\Gamma(x_{r_j}) \to \lambda.$$

Hence

$$w(x) \leq \gamma\Gamma(x) + \lambda \qquad and \qquad w(\xi^0) = \gamma\Gamma(\xi^0) + \lambda.$$

By maximum principle (see [D1]), $w(x) = \gamma\Gamma(x) + \lambda$ and hence $w_r \to \gamma\Gamma(x) + \lambda$ in $C^1_{loc}(\mathbb{R}^n \backslash \{0\})$. This implies

$$\lim_{r\to 0}(G(rx) - \gamma\Gamma(rx)) = \lambda, \qquad \lim_{r\to 0} \nabla_x (G(rx) - \Gamma(rx)) = 0. \tag{64}$$

The above equalities lead to the continuity of $G - \gamma\Gamma$ and $\lim_{x\to 0} F^o(x)\nabla(G - \gamma\Gamma) = 0$.

Finally, we assume that λ achieves at some point on $\partial\mathcal{W}_\delta$, i.e. $\sup_{x\in\mathcal{W}_\delta}(G - \gamma\Gamma) = \sup_{F^o(x)=\delta}(G - \gamma\Gamma)$. We define w_r as the above, then $w_r \to w$ in $C^1_{loc}(\mathbb{R}^n \backslash \{0\})$ and $|w - \gamma\Gamma| \le C_0$. We now look at the points where $w - \gamma\Gamma$ achieves its supremum in $\mathbb{R}^n$. Set $\tilde{\lambda} = \sup_{\mathbb{R}^n}(w - \gamma\Gamma)$.

If $\tilde{\lambda}$ is achieved at some point in $\mathbb{R}^n \backslash \{0\}$, then $w - \gamma\Gamma$ equals to some constant by strong maximum principle (see [D1]), which implies $G(rx) - \gamma\Gamma(rx) \to \tilde{\lambda}$ in $C^1_{loc}(\mathbb{R}^n \backslash \{0\})$ as $r \to 0$. For any fixed $\epsilon > 0$, there exists n_0 such that $n \ge n_0$ and $x \in \partial\mathcal{W}_1$, we have

$$\gamma\Gamma(r_n x) + \tilde{\lambda} - \epsilon \le G(r_n x) \le \gamma\Gamma(r_n x) + \tilde{\lambda} + \epsilon.$$

Applying maximum principle in $\mathcal{W}_{r_{n_0}} \backslash \mathcal{W}_{r_n}$ we obtain

$$\gamma\Gamma(x) + \tilde{\lambda} - \epsilon \le G(x) \le \gamma\Gamma(x) + \tilde{\lambda} + \epsilon,$$

which leads to (64) with λ replaced by $\tilde{\lambda}$.

If $\tilde{\lambda}$ is achieved at 0, we can use the similar arguments as above to deduce

$$\lim_{x\to 0}(w - \gamma\Gamma) = \tilde{\lambda} \qquad \textit{and} \quad \textit{hence} \qquad \lim_{x\to 0}\lim_{r_n\to 0}(G(r_n x) - \gamma\Gamma(r_n x)) = \tilde{\lambda}. \tag{65}$$

If $\tilde{\lambda}$ is achieved at ∞, the same idea can be applied when we defined $\lambda(R) = \max_{\delta \le F^o(x) \le R}(w - \gamma\Gamma) = \max_{\partial\mathcal{W}_R}(w - \gamma\Gamma)$. Letting R tend to ∞, we can obtain

$$\lim_{x\to\infty}(w - \gamma\Gamma) = \tilde{\lambda}, \qquad \lim_{x\to\infty}\lim_{r_n\to 0}(G(r_n x) - \gamma\Gamma(r_n x)) = \tilde{\lambda}. \tag{66}$$

As long as we have (65)and (66), we can have use maximum principle again to conclude (64) as before.

Integrating by parts on both sides of over $\mathcal{W}_\delta$, we have

$$-\int_{\mathcal{W}_\delta} div(F^{n-1}(\nabla G)F_\xi(\nabla G))dx + \int_{\mathcal{W}_\delta} G^{n-1}dx = 1. \tag{67}$$

Since $G(x) = \gamma\Gamma(x) + o(1)$ and $\nabla G(x) = \gamma\nabla\Gamma(x) + o(\frac{1}{F^o(x)})$ as $x \to 0.$, we insert the above two equalities into (67), then let $\delta \to 0$ to obtain $\gamma = 1$.

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