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Stability estimates for the conformal group of $\mathbb{S}^{n-1}$ in dimension $n \geq 3$

by

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Abstract

The purpose of this paper is to exhibit a quantitative stability result for the class of Möbius transformations of $\mathbb{S}^{n-1}$ when $n \geq 3$. The main estimate is of local nature and asserts that for a Lipschitz map that is apriori close to a Möbius transformation, an average conformal-isoperimetric type of deficit controls the deviation (in an average sense) of the map in question from a particular Möbius map. The optimality of the result together with its link with the geometric rigidity of the special orthogonal group are also discussed.

1 Introduction

One of the most classical rigidity theorems in differential geometry is Liouville's theorem which in modern terms can be stated as follows.

Theorem 1.1. ***(Liouville)*** *Let $n \geq 3$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $u \in W^{1,n}(U;\mathbb{R}^n)$ is a generalized conformal map, that is a map whose gradient satisfies the differential inclusion*

$$\nabla u \in CO_+(n) \quad \text{a.e. in } U,$$

where $CO_+(n) := \{\lambda R \in \mathbb{R}^{n\times n};\ \lambda > 0,\ R \in SO(n)\}$. Then u is the restriction of a Möbius transformation on U, that is

$$u(x) = Ax + b \quad \text{or} \quad u(x) = AB\frac{x-a}{|x-a|^2} + b,$$

where $b \in \mathbb{R}^n, a \in \mathbb{R}^n \setminus U, A \in CO_+(n)$ and $B = \mathrm{diag}(1, ..., 1, -1)$.

Liouville was the first one to prove the theorem around 1850 for C^3-regular maps. Subsequently Gehring proved it for homeomorphisms belonging to the Sobolev class $W^{1,n}(U;\mathbb{R}^n)$ in [1] and Reshetnyak removed the injectivity assumption in [2]. Later in [3], Iwaniec proved that there exists a critical threshold $p_n < n$ such that Liouville's theorem holds for maps in $W^{1,p}(U;\mathbb{R}^n)$ whenever $p \geq p_n$ and together with Martin in [4] they proved that the optimal value is $p_n = \frac{n}{2}$ in case n is even while conjecturing that this is also true when n is odd.

Liouville's theorem does not hold in dimension 2. According to the famous Riemann mapping theorem in complex analysis every simply connected domain in $\mathbb{C}$ that is not $\mathbb{C}$ itself is conformally equivalent to the open unit disk, so the class of conformal maps defined on a fixed open subdomain of the plane does not admit a simple characterization as before. This dichotomy between two dimensions and higher ones does not happen for example for the special orthogonal group where Liouville's theorem can be stated in the following way.

Theorem 1.2. ***(Liouville)*** *Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $u \in W^{1,2}(U;\mathbb{R}^n)$ is such that*

$$\nabla u \in SO(n) \quad \text{a.e. in } U.$$

Then u is a rigid motion, that is $u(x) = Rx + b$, for some $b \in \mathbb{R}^n$ and $R \in SO(n)$.

A simple modern proof of this theorem, as can be found in [5], can be carried out along the following lines.

$$\nabla u \in SO(n) \text{ a.e. in } U \implies \operatorname{cof}\nabla u = \nabla u \text{ a.e. in } U.$$

By Piola's identity we have that at least in the sense of distributions $\operatorname{div}\operatorname{cof}\nabla u = 0$, hence $\Delta u = 0$ in U. Using the Bochner formula for harmonic functions we infer that

$$\frac{1}{2}\Delta(|\nabla u|^2 - n) = \nabla u \cdot \Delta \nabla u + |\nabla^2 u|^2 = |\nabla^2 u|^2 \text{ in } U$$

and since $\nabla u \in SO(n)$ we obtain $\nabla^2 u \equiv 0$ in U, concluding that u is affine with gradient in $SO(n)$.

A natural question would then be whether this rigidity theorem is stable, meaning that if for a map u its gradient is close to $SO(n)$ in an average sense, whether it is close to a single rotation in average. Among several results in this direction (see for example [2], [6], [7], [8], [5]) let us highlight two of them, the first one being a qualitative stability statement while the second one is its optimal quantitative analogue.

Theorem 1.3. ***(Reshetnyak, [2])*** *Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. If $(u_j)_{j\in\mathbb{N}} \in W^{1,2}(U;\mathbb{R}^n)$ is a weakly convergent sequence in $W^{1,2}(U;\mathbb{R}^n)$ such that*

$$\operatorname{dist}(\nabla u_j, SO(n)) \to 0 \text{ as } j \to \infty$$

in measure, then there exists $R \in SO(n)$ such that $\nabla u_j \to R$ in $L^2(U)$.

Modern proofs of this theorem can also be found in [9] or [10]. In their pioneering work in [5] Friesecke, James and Müller proved a sharp, scaling invariant quantitative estimate for the above theorem which has been used widely since then, for example in questions related to dimension reduction in nonlinear elasticity theory. Their geometric rigidity estimate, which is the nonlinear counterpart of the classical Korn's inequality appearing in linearized elasticity, is the following one.

Theorem 1.4. ***(Friesecke, James, Müller, [5])*** *Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. There exists $C := C(U) > 0$ so that for every $u \in W^{1,2}(U;\mathbb{R}^n)$ there exists an associated $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(U)} \leq C \left\|\operatorname{dist}(\nabla u, SO(n))\right\|_{L^2(U)}. \tag{1.1}$$

The same estimate holds in $L^p(U)$ for any $p \in (1,\infty)$ and apart from translation and rotation invariant it is also scaling invariant, meaning that if $C := C(U) > 0$ stands for the optimal constant for which (1.1) holds, then $C(\lambda RU + x_0) = C(U)$ for every $\lambda > 0$ $R \in SO(n)$ and $x_0 \in \mathbb{R}^n$. The exponent with which the norm on the right hand side appears is sharp, a fact that can easily be checked by considering a sequence of affine maps with gradients approaching $SO(n)$.

Returning back to Liouville's theorem, we have a similar dichotomy phenomenon (but in one dimension less) when one wants to study the structure of the group of conformal diffeomorphisms of $\mathbb{S}^{n-1}$ (the *round* sphere embedded in $\mathbb{R}^n$). Recall that a regular map $u \in C^1(\mathbb{S}^{n-1};\mathbb{R}^n)$ is called conformal at a point $x \in \mathbb{S}^{n-1}$ iff its tangential gradient is a nonsingular map at x and u preserves the angle between any two tangent vectors at that point, that is

$$\frac{\langle \nabla_T u(x)\xi, \nabla_T u(x)\eta\rangle}{|\nabla_T u(x)\xi||\nabla_T u(x)\eta|} = \frac{\langle \xi,\eta\rangle}{|\xi||\eta|} \text{ for every } \xi,\eta \in T_x\mathbb{S}^{n-1}\backslash\{0\}.$$

Here $\langle a,b\rangle$ denotes the Euclidean inner product between $a,b \in \mathbb{R}^n$ and ∇_T stands for the tangential gradient, computed with respect to a local orthonormal frame field $\{\tau_1(x),\tau_2(x),...,\tau_{n-1}(x)\}$ of

$T_x\mathbb{S}^{n-1}$ and the standard orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $\mathbb{R}^n$, so that it can be represented by the $n \times (n-1)$ matrix with entries

$$(\nabla_T u(x))_{ij} := \langle \nabla_T u^i(x), \tau_j(x) \rangle \text{ for every } i = 1, 2, ..., n \text{ and } j = 1, 2, ..., n-1.$$

If we also denote by I_x the identity map acting on $T_x\mathbb{S}^{n-1}$ the above definition of conformality at a point $x \in \mathbb{S}^{n-1}$ implies that

$$\left(\nabla_T u^t \nabla_T u\right)(x) = \left(\frac{|\nabla_T u(x)|^2}{n-1}\right) I_x. \tag{1.2}$$

A map $u \in W^{1,\infty}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ is then called *generalized conformal* on $\mathbb{S}^{n-1}$ iff it satisfies equation (1.2) at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$. With this definition we can state Liouville's theorem on $\mathbb{S}^{n-1}$ as follows.

Theorem 1.5. ***(Liouville)*** *If $n \geq 3$ and u is a conformal diffeomorphism of $\mathbb{S}^{n-1}$, then u is a Möbius transformation of $\mathbb{S}^{n-1}$, that is there exist $O \in O(n)$, $\xi \in \mathbb{S}^{n-1}$ and $\lambda > 0$ such that*

$$u(x) = O\phi_{\xi,\lambda}(x). \tag{1.3}$$

Here $\phi_{\xi,\lambda} := \sigma_\xi^{-1} \circ i_\lambda \circ \sigma_\xi$, where $\sigma_\xi : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is the stereograpic projection of $\mathbb{S}^{n-1}$ onto the tangent plane $T_\xi\mathbb{S}^{n-1}$ and $i_\lambda : T_\xi\mathbb{S}^{n-1} \mapsto T_\xi\mathbb{S}^{n-1}$ is the dilation in $T_\xi\mathbb{S}^{n-1}$ by factor λ. Analytically this yields,

$$\phi_{\xi,\lambda}(x) := \frac{-\lambda^2(1 - \langle x, \xi \rangle)\xi + 2\lambda(x - \langle x, \xi \rangle \xi) + (1 + \langle x, \xi \rangle)\xi}{\lambda^2(1 - \langle x, \xi \rangle) + (1 + \langle x, \xi \rangle)}.$$

We have stated the theorem in such a way that it includes both the orientation preserving and the orientation reversing Möbius transformations. The statement is of course void for circles in the plane since conformality is a trivial notion in one dimension, but in higher dimensions Liouville's theorem asserts that the only conformal diffeomorphisms of $\mathbb{S}^{n-1}$ are exactly its Möbius transformations. This rigidity theorem naturally motivates the question of stability of the conformal group of $\mathbb{S}^{n-1}$ that in loose terms can be described as follows.

Question. If $n \geq 3$ and $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is a map which is "almost conformal" with $u(\mathbb{S}^{n-1})$ "close" to being a round sphere, can we have a quantitative statement concerning its deviation from a particular Möbius transformation of $\mathbb{S}^{n-1}$?

As it is common in many questions regarding the stability of geometric/functional inequalities or the stability of absolute minimizers in geometric variational problems, the notions in which we measure the above mentioned deviations are an important feature of the problem. As it is well known, Möbius transformations are not the only conformal maps that one can define on the sphere (they are just the only ones that map the sphere onto itself). In general, there exist bijective conformal maps from the sphere to other closed embedded hypersurfaces (for example from the sphere to an ellipsoid) and actually in the case $n = 3$ the Uniformization Theorem asserts that any smooth two-dimensional closed surface that is topologically equivalent to the Riemann sphere is actually conformally equivalent to it. Therefore there is a big variety of conformal maps defined on $\mathbb{S}^2$ if we do not impose any restrictions on their image. This is the reason why we focus our attention on "almost conformal" maps which map the sphere to "almost a round sphere".

Here is a brief motivation for the introduction of the deficit in terms of which the main estimate is going to be stated. First of all if $u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n)$ is a generalized conformal map then

$$\sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} = \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} \quad \mathcal{H}^{n-1} - \text{ a.e. on } \mathbb{S}^{n-1}. \tag{1.4}$$

One can average over $\mathbb{S}^{n-1}$ this pointwise equality and use the area-formula to obtain

$$\frac{\mathcal{H}^{n-1}\left(u(\mathbb{S}^{n-1})\right)}{n\omega_n} \leq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1},$$

with equality in the first inequality holding iff the map u is injective.

Suppose now that u is an arbitrary Lipschitz map. Let $0 \leq a_1^2 \leq a_2^2 \leq ... \leq a_{n-1}^2$ be the eigenvalues of the symmetric positive-definite matrix $\nabla_T u^t \nabla_T u$, where $a_i \geq 0$ for every $i = 1, 2, ..., n-1$. In view of the arithmetic mean-geometric mean inequality we have ($\mathcal{H}^{n-1}$-a.e.)

$$\sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} = \left(\prod_{i=1}^{n-1} a_i^2\right)^{\frac{1}{2}} \leq \left(\frac{\sum_{i=1}^{n-1} a_i^2}{n-1}\right)^{\frac{n-1}{2}} = \left(\frac{\mathrm{Tr}\left(\nabla_T u^t \nabla_T u\right)}{n-1}\right)^{\frac{n-1}{2}} = \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}$$

and by averaging both expressions again, we have that in general

$$\frac{\mathcal{H}^{n-1}(u(\mathbb{S}^{n-1}))}{n\omega_n} \leq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1} \leq \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1}.$$

Equalities in the above inequalities hold iff u is injective and $0 \leq a_1(x) = ... = a_{n-1}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$, implying that u is a generalized conformal map according to the definition we gave above.

Let $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ be a Lipschitz embedding. By this we mean that u is injective and $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ the vectors $(\partial_{\tau_i} u)_{i=1,...,n-1}$ are linearly independent. Let $|V_n(u)|$ stand for the volume enclosed by the closed Lipschitz hypersurface $u(\mathbb{S}^{n-1})$ in $\mathbb{R}^n$ normalized by the volume of the unit ball and $V_n(u)$ stand for the normalized signed volume, which in the case of a general immersion is given by the formula

$$V_n(u) := \fint_{\mathbb{S}^{n-1}} \langle u, \nu_u \rangle g_u \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1}. \tag{1.5}$$

Here $\nu_u := \frac{\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u}{|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u|}$ using the identification between the normalized wedge product of $(n-1)$ linearly independent vectors in $\mathbb{R}^n$ and the unit vector normal to the hyperplane they span. We have also denoted by $g_u := |\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u| = \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}$ the area element induced by u. When u is an embedding the actual enclosed volume (normalized by ω_n) is really given by the absolute value of the signed volume, justifying the notation we use.
In view of the arithmetic mean-geometric mean inequality and the isoperimetric inequality we have that in general

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1} \geq \frac{\mathcal{H}^{n-1}(u(\mathbb{S}^{n-1}))}{n\omega_n} \geq |V_n(u)|^{\frac{n-1}{n}}.$$

In this setting, equality in the above chain of inequalities would hold iff the map u is generalized conformal, injective and $u(\mathbb{S}^{n-1})$ is a round sphere in $\mathbb{R}^n$, which up to a scaling factor can be taken

to be the unit sphere. In other words u has to be a conformal self-transformation of $\mathbb{S}^{n-1}$ and hence a Möbius transformation (up to scaling factor). The previous chain of inequalities can be rewritten as

$$D_n(u) \geq P_n(u) \geq V_n(u), \tag{1.6}$$

where

$$D_n(u) := \left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \text{ and } P_n(u) := \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}$$

and $V_n(u)$ is as defined before. Notice moreover that these three quantities are all invariant under the group of conformal reparametrizations of $\mathbb{S}^{n-1}$, which we denote by $\mathrm{Conf}(\mathbb{S}^{n-1})$, meaning that for all $\psi \in \mathrm{Conf}(\mathbb{S}^{n-1})$

$$D_n(u \circ \psi) = D_n(u),\ P_n(u \circ \psi) = P_n(u) \text{ and } V_n(u \circ \psi) = V_n(u).$$

With these considerations in mind our main stability result is of local nature. It concerns maps that are apriori close (in a certain sense) to a Möbius transformation of $\mathbb{S}^{n-1}$, which without loss of generality we can take to be the identity, at least as long as we focus on compact subsets of Möbius transformations, whose gradient is bounded from below and above by fixed positive constants. From now on for $M > 0$, $\theta > 0$ and $\varepsilon > 0$ we define

$$\mathcal{A}_{M,\theta,\varepsilon} := \left\{ u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \begin{array}{l} (i)\ \ \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M, \\ (ii)\ \|u - \mathrm{id}_{\mathbb{S}^{n-1}}\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq \theta, \\ (iii)\ D_n(u) \leq (1+\varepsilon) V_n(u). \end{array} \right\}$$

Denoting by P_T the tangential gradient of the identity map on $\mathbb{S}^{n-1}$, our main Theorem can be stated as follows.

Theorem 1.6. *Let $n \geq 3$ and $M > 0$ be given. There exist positive constants $\delta_0 := \delta_0(n, M)$, $\theta_0 := \theta_0(n, M)$, $\varepsilon_0 := \varepsilon_0(n)$ and $C := C(n) > 0$ such that for every $0 < \theta \leq \theta_0$ and $0 < \varepsilon \leq \varepsilon_0$ the following statements are true:*

(i) In the case $n = 3$, for every $u \in \mathcal{A}_{M,\theta,\varepsilon}$ there exist $\lambda_u > 0$ and $\phi_u \in \mathrm{Conf}(\mathbb{S}^2)$ such that

$$\left\| \nabla_T \left(\frac{u \circ \phi_u}{\lambda_u} \right) - P_T \right\|_{L^2(\mathbb{S}^2)} \leq C\sqrt{\varepsilon}. \tag{1.7}$$

.

(ii) In the case $n \geq 4$, for every $u \in \mathcal{A}_{M,\theta,\varepsilon}$ there exist $\lambda_u > 0$ and $\phi_u \in \mathrm{Conf}(\mathbb{S}^{n-1})$ with the following property. If $\left\| \nabla_T \left(\frac{u \circ \phi_u}{\lambda_u} \right) - P_T \right\|_{L^\infty(\mathbb{S}^{n-1})} \leq \delta_0$ then again

$$\left\| \nabla_T \left(\frac{u \circ \phi_u}{\lambda_u} \right) - P_T \right\|_{L^2(\mathbb{S}^{n-1})} \leq C\sqrt{\varepsilon}. \tag{1.8}$$

Notice that the exponent $\frac{1}{2}$ with which the ε-deficit appears on the right hand side is optimal. This can be easily checked by considering the sequence of affine maps $(u_\sigma)_{\sigma \downarrow 0^+} : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ where $u_\sigma(x) = A_\sigma x$ with $A_\sigma := \mathrm{diag}(1, ..., 1, 1+\sigma)$.

Let us explain the main steps in the proof of this Theorem. First of all, the assumptions of a global Lipschitz bound for u and that it is apriori close to the identity in $W^{1,2}(\mathbb{S}^{n-1})$ allows us to center and scale our map properly and perform a formal Taylor expansion of both the D_n- and the V_n- term around the identity, thus moving our focus of attention on the resulting quadratic form. In particular we have the following.

Proposition 1.7. *There exist constants $C_1, C_2 > 0$ depending only on n and M such that for every $u \in \mathcal{A}_{M,\theta,\varepsilon}$ and after setting $w := u - \mathrm{id}|_{\mathbb{S}^{n-1}}$,*

(a) The "$(n-1)$-Dirichlet energy" $D_n(u)$ has the formal Taylor expansion

$$D_n(u) \;=\; 1 + Q_{D_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{1,n}(\nabla_T w)\; d\mathcal{H}^{n-1}$$

where $Q_{D_n}(w)$ is a quadratic form defined through

$$Q_{D_n}(w) := \frac{1}{2}\frac{n}{n-1}\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\; d\mathcal{H}^{n-1} + \frac{n-3}{n-1}\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\; d\mathcal{H}^{n-1}\right). \tag{1.9}$$

Regarding the growth behaviour of the remainder term we have the following two cases

$$\begin{cases} \text{If } n = 3, \quad \fint_{\mathbb{S}^2} |R_{1,3}(\nabla_T w)|\; d\mathcal{H}^2 \le C_1 \left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\; d\mathcal{H}^2\right)^2. \\ \text{If } n \ge 4, \quad \fint_{\mathbb{S}^{n-1}} |R_{1,n}(\nabla_T w)|\; d\mathcal{H}^{n-1} \le C_1 \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^3\; d\mathcal{H}^{n-1}. \end{cases} \tag{1.10}$$

(b) The (signed) volume term $V_n(u)$ has the formal Taylor expansion

$$V_n(u) \;=\; 1 + Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{2,n}(w, \nabla_T w)\; d\mathcal{H}^{n-1},$$

where $Q_{V_n}(w)$ is a quadratic form which is defined through the following (equivalent) formulas:

$$Q_{V_n}(w) := \begin{cases} \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^n x_j \nabla_T w^j\rangle\; d\mathcal{H}^{n-1} \\ \frac{n}{2}\fint_{\mathbb{S}^{n-1}} (2\; \mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2)\; d\mathcal{H}^{n-1} \\ \frac{1}{2}\fint_B \left((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2\right) dx, \end{cases} \tag{1.11}$$

where $w_h : \overline{B} \mapsto \mathbb{R}^n$ stands for the (componentwise) harmonic continuation of w in the interior of the unit ball. Regarding the growth behaviour of the remainder term we again have two cases

$$\begin{cases} \text{If } n = 3, \; \fint_{\mathbb{S}^2} |R_{2,3}(w, \nabla_T w)|\; d\mathcal{H}^2 \le C_2 \left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\; d\mathcal{H}^2\right)^{1+\gamma} \;\; \forall\, \gamma \in (0, \tfrac{1}{2}). \\ \text{If } n \ge 4, \; \fint_{\mathbb{S}^{n-1}} |R_{2,n}(w, \nabla_T w)|\; d\mathcal{H}^{n-1} \le C_2 \left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\; d\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}. \end{cases} \tag{1.12}$$

We just mention that the above expansions hold after we have scaled our map u properly so that $\fint_{\mathbb{S}^{n-1}} \langle u, x\rangle\; d\mathcal{H}^{n-1} = 1$, a fact that we explain more in detail later. The last assertion of the proposition follows from the algebraic structure of the remainder term in the expansion of V_n and the Sobolev inequality on the sphere. Let us also mention that the difference between $n = 3$ and $n \ge 4$ in the growth of $\fint_{\mathbb{S}^{n-1}} |R_{1,n}(\nabla_T w)|\; d\mathcal{H}^{n-1}$ is the one that forces us to put an extra apriori condition in our main Theorem in the case $n \ge 4$.

The next step is to examine the coercivity of the quadratic form appearing in the expansion in a purely L^2-setting. We emphasize that the main ingredient here is the nice interplay between the Fourier decomposition of an $L^2(\mathbb{S}^{n-1})$-vector field into spherical harmonics and the invariance properties of the first order differential operator associated to Q_{V_n}. To be more precise, let us define

$$H := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0 \ , \ \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} = 0 \right\}.$$

For every $k \geq 1$ we define H_k to be the subspace of H consisting of maps in H whose components are purely k-th order spherical harmonics. As we mention in the Appendix A it is well known that each $w \in H_k$ is the restriction on $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial of degree k, the subspaces $(H_k)_{k=1}^{\infty}$ are pairwise orthogonal to each other with respect to the L^2-inner product and each one is finite dimensional. Hence, one has the L^2-orthogonal decomposition $H = \bigoplus_{k=1}^{\infty} H_k$. We also define the subspaces

$$\tilde{H}_k := \left\{ w_h : \overline{B} \mapsto \mathbb{R}^n : \begin{array}{rcl} \Delta w_h & = & 0 \text{ in } B \\ w_h|_{\mathbb{S}^{n-1}} & \in & H_k \end{array} \right\}$$

so that $\bigoplus_{k=1}^{\infty} \tilde{H}_k$ is an L^2-decomposition of the space of harmonic functions $w_h : \overline{B} \mapsto \mathbb{R}^n$ for which $w_h(0) = 0$ and $\mathrm{Tr}\nabla w_h(0) = 0$. Furthermore, for each $k \geq 1$ we consider the L^2-orthogonal decomposition

$$\tilde{H}_k = \tilde{H}_{k,\mathrm{sol}} \bigoplus \tilde{H}_{k,\mathrm{sol}}^{\perp}, \text{ where } \tilde{H}_{k,\mathrm{sol}} := \{ w_h \in \tilde{H}_k : \mathrm{div} w_h \ \equiv \ 0 \}$$

or equivalently the $L^2(\mathbb{S}^{n-1})-$decomposition

$$H_k = H_{k,\mathrm{sol}} \bigoplus H_{k,\mathrm{sol}}^{\perp}, \text{ where } H_{k,\mathrm{sol}} := \{ w \in H_k : w_h \in \tilde{H}_{k,\mathrm{sol}} \}. \tag{1.13}$$

We now consider the bilinear form

$$Q_{V_n}(w,v) := \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \langle w, A(v) \rangle \, d\mathcal{H}^{n-1} \ \text{ for any } w, v \in H,$$

where the associated linear first order differential operator A is defined for every $w \in H$ as

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j. \tag{1.14}$$

The main observation is that A is a symmetric operator with respect to the L^2-inner product with trivial kernel in H that leaves each one of the subspaces $(H_k)_{k \geq 1}$ invariant, but even more specifically, A leaves $H_{k,\mathrm{sol}}$ and thus also $H_{k,\mathrm{sol}}^{\perp}$ invariant. Each one of these subspaces has therefore an eigenvalue decomposition with respect to A.

Theorem 1.8. *The following statements are true.*

(i) For every $k \geq 1$, the subspace $H_{k,\mathrm{sol}}$ has an eigenvalue decomposition with respect to A as

$$H_{k,\mathrm{sol}} = H_{k,1} \bigoplus H_{k,2},$$

where $H_{k,1}$ is the eigenspace of A corresponding to the eigenvalue $\sigma_{k,1} := -k$ and $H_{k,2}$ is the one corresponding to the eigenvalue $\sigma_{k,2} := 1$.

(ii) For every $k \geq 1$, the subspace $H_{k,3} := H_{k,\mathrm{sol}}^{\perp}$ is an eigenspace with respect to A corresponding to the eigenvalue $\sigma_{k,3} := k + n - 2$.

It is then easy to see that for the eigenvalue decomposition $H_k = H_{k,1} \bigoplus H_{k,2} \bigoplus H_{k,3}$ the quadratic forms Q_{D_n} and Q_{V_n} diagonalize completely in each one of the eigenspaces $(H_{k,i})_{k\geq 1, i=1,2,3}$ and moreover these subspaces are pairwise both $Q_{V_n}-$ and $Q_{D_n}-$ orthogonal. As a result we have the following coercivity estimate.

Theorem 1.9. *There exists an explicit constant $C_n > 0$ such that for every $w \in H$ one has*

$$Q_n(w) \geq C_n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T w - \nabla_T(\Pi_n w)\right|^2 \, d\mathcal{H}^{n-1}, \tag{1.15}$$

where $Q_n := Q_{D_n} - Q_{V_n}$ and $\Pi_n : H \mapsto H_0 := H_{1,2} \bigoplus H_{2,3}$ is the L^2-orthogonal projection on the kernel of Q_n.

The "Q_n-degenerate space H_0" is in one to one correspondence with the Lie algera of infinitesimal Möbius transformations, which allows us to use the Inverse Function Theorem and a topological argument in order to find a Möbius transformation ϕ_u such that $u \circ \phi_u$ has trivial projection in H_0, enabling us to conclude our main Theorem.

The plan of the paper is the following. In Section 2 we introduce some notation and terminology, give a short and to our knowledge new proof of Liouville's theorem regarding the conformal group of $\mathbb{S}^{n-1}$ and set up a bit more in detail the framework in which our quantitative stability estimate will be stated. The coercivity inequality satisfied by the quadratic form in the Taylor expansion of the geometric deficit, together with another Korn-type inequality satisfied by the quadratic form in the expansion of the "purely" conformal deficit are presented in Section 3 . The latter one is analogous to the Korn-type inequality satisfied by the deviatoric part of the gradient in domains of $\mathbb{R}^n$ that are starshaped with respect to a ball, first proven by Reshetnyak (see [11], Chapter 3). The completion of the proof of Theorem (1.6) is presented in Section 4. In Section 5 we discuss how the technique used to prove the coercivity estimates of the third section can be adapted to give an alternative proof of the geometric rigidity estimate (1.4) of Friesecke, James and Müller regarding the stability of the special orthogonal group $SO(n)$. In the appendices we include some basic facts from the theory of spherical harmonics that we are using, as well as some details on the derivation of the formulas that appear in earlier chapters.

The results of this paper will be included in the second author's Ph.D. thesis at the Max Planck Institute for Mathematics in the Sciences and the University of Leipzig.

2 Notation and preliminaries

In what follows $n \geq 3$ will be a natural number denoting the dimension of the ambient Euclidean space. For any two vectors $a, b \in \mathbb{R}^n$ we denote by $\langle a, b\rangle$ their Euclidean inner product while for any two matrices $A, B \in \mathbb{R}^{n\times m}$ their Euclidean inner product will be denoted by $A : B$, i.e. $A : B = \mathrm{Tr}(A^t B)$. The Euclidean norm of vectors or matrices is denoted by $|\cdot|$, being clear from the context to which one we refer each time. For a matrix field A, $A_{\mathrm{sym}} := \frac{A+A^t}{2}$ and $A_{\mathrm{skew}} := \frac{A-A^t}{2}$ will denote its symmetric and antisymmetric part respectively and $\overline{A}$ will denote its mean value on its corresponding domain of definition. For every $\rho > 0$ and $x_0 \in \mathbb{R}^n$ we denote as usual $B_\rho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ and its boundary by $\mathbb{S}^{n-1}_\rho(x_0)$. In the special case $\rho = 1$, $x_0 = 0$ we omit the corresponding subscripts. Without mentioning it further, $\mathbb{S}^{n-1}$

will be regarded as the standard embedding of the round unit sphere in $\mathbb{R}^n$ given by the identity map and equipped with the restriction of the Euclidean metric onto its tangent space at each point.

By ω_n we denote the volume of the unit ball in $\mathbb{R}^n$. $\mathcal{H}^k$ stands as usual for the k-dimensional Hausdorff measure, while for an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^n$, $|E|$ is used to denote its volume and $\mathrm{Per}(E)$ its perimeter (in the measure-theoretic sense of De Giorgi, Cacciopoli).

We also use the standard notation $O(n)$, $SO(n)$ and $CO_+(n)$ for the orthogonal, the special orthogonal and the conformal group (or actually its positive cone) in n dimensions respectively.

The unit normal vector field to the sphere $\mathbb{S}^{n-1}_r$ will be denoted by $\vec{\nu}_r$, i.e. $\vec{\nu}_r(x) = \frac{x}{r}$ for every $x \in \mathbb{S}^{n-1}_r$. The Euclidean gradient, divergence and Laplace operators will be denoted as standard by ∇, div and Δ. The tangential gradient, divergence and the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ will be denoted by ∇_T, $\mathrm{div}_{\mathbb{S}^{n-1}}$ and $\Delta_{\mathbb{S}^{n-1}}$ respectively and on any other sphere $\mathbb{S}^{n-1}_r$ by ∇_T (abusing notation), $\mathrm{div}_{\mathbb{S}^{n-1}_r}$ and $\Delta_{\mathbb{S}^{n-1}_r}$. The radial derivative of a function $f : \overline{B}_1 \mapsto \mathbb{R}$ on that intermediate sphere will be denoted by $\partial_{\vec{\nu}_r} f$, with the index r being omitted when $r = 1$.

Finally, for convenience of notation we also use the symbols $\sim_{M_1,M_2}$, $\lesssim_{M_1,M_2,\dots}$ to indicate that the corresponding equality or inequality respectively is valid up to a constant that is allowed to vary from line to line but depends only on the parameters M_1, M_2 etc.

2.1 Conformal maps defined on $\mathbb{S}^{n-1}$ and a proof of Liouville's theorem

As we mentioned in the Introduction a map $u \in C^1(\mathbb{S}^{n-1};\mathbb{R}^n)$ is called conformal at a point $x \in \mathbb{S}^{n-1}$ iff its tangential gradient is a nonsingular map at x and u preserves the angle between any two tangent vectors at that point, that is

$$\frac{\langle \nabla_T u(x)\xi, \nabla_T u(x)\eta\rangle}{|\nabla_T u(x)\xi||\nabla_T u(x)\eta|} = \frac{\langle \xi,\eta\rangle}{|\xi||\eta|} \text{ for every } \xi,\eta \in T_x\mathbb{S}^{n-1}\backslash\{0\}$$

a condition that is equivalent to requiring

$$\left(\nabla_T u^t \nabla_T u\right)(x) = \left(\frac{|\nabla_T u(x)|^2}{n-1}\right) I_x, \tag{2.1}$$

where I_x is the identity map acting on $T_x\mathbb{S}^{n-1}$.

Definition 2.1. A map $u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n)$ is called generalized conformal on $\mathbb{S}^{n-1}$ it satisfies equation (2.1) at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$.

We now present a short proof of Liouville's theorem (1.5) regarding the group of conformal diffeomorphisms of $\mathbb{S}^{n-1}$, which to our knowledge has not appeared in the literature so far.

Proof of Theorem 1.5. First of all the maps $(\phi_{\xi.\lambda})_{\xi\in\mathbb{S}^{n-1},\ \lambda>0}$ are conformal diffeomorphisms of $\mathbb{S}^{n-1}$. Let now $u \in \mathrm{Conf}(\mathbb{S}^{n-1})$. Since u maps bijectively (and conformally) $\mathbb{S}^{n-1}$ onto itself, we can use Jensen's inequality and Poincare's inequality (A.5) to obtain

$$\begin{aligned} 1 &= \frac{\mathcal{H}^{n-1}(u(\mathbb{S}^{n-1}))}{n\omega_n} = \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \\ &\geq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1}\right)^{\frac{n-1}{2}} \geq \left(\fint_{\mathbb{S}^{n-1}} \left|u - \fint_{\mathbb{S}^{n-1}} u\right|^2 d\mathcal{H}^{n-1}\right)^{\frac{n-1}{2}}. \end{aligned} \tag{2.2}$$

If we assume for the moment that $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0$ then the last integral is exactly equal to 1 since $|u(x)| = 1$ for every $x \in \mathbb{S}^{n-1}$ and therefore equalities must hold at each step in the above chain of inequalities. The equality case in Poincare's inequality implies that in the Fourier expansion of u in spherical harmonics no other spherical harmonics except the first order ones should appear, hence $u(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$. Actually it is easy to see that $A = \nabla u_h(0)$, with $u_h : \overline{B}_1 \mapsto \mathbb{R}^n$ denoting the harmonic extension of u in the interior of the unit ball. But this linear map transforms $\mathbb{S}^{n-1}$ to an ellipsoid which after possibly an orthogonal change of coordinates is

$$u(\mathbb{S}^{n-1}) := \Big\{ y = (y_1, y_2, ...y_n) \in \mathbb{R}^n : \frac{y_1^2}{\beta_1^2} + \frac{y_2^2}{\beta_2^2} + ... + \frac{y_n^2}{\beta_n^2} = 1 \Big\},$$

where $0 < \beta_1^2 \leq \beta_2^2 \leq ... \leq \beta_n^2$ are the eigenvalues of the symmetric matrix A^tA. But since $u(\mathbb{S}^{n-1}) \equiv \mathbb{S}^{n-1}$ this forces $\beta_1^2 = \beta_2^2 = ... = \beta_n^2 = 1$ hence $A \in O(n)$. We remark that for the above argument to work we need the convexity of the function $t \mapsto t^{\frac{n-1}{2}}$ which holds iff $n \geq 3$.

If $b_u := \fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} \neq 0$ it is easy to show that there always exist $\xi_0 \in \mathbb{S}^{n-1}$ and $\lambda_0 > 0$ such that $\fint_{\mathbb{S}^{n-1}} \phi_{\xi_0,\lambda_0} \circ u \, d\mathcal{H}^{n-1} = 0$. Indeed, consider the map $F : \mathbb{S}^{n-1} \times [0,1] \mapsto \overline{B}_1$ defined as

$$F(\xi, \lambda) := \fint_{\mathbb{S}^{n-1}} \phi_{\xi,\lambda} \circ u \, d\mathcal{H}^{n-1} \quad \text{for } \lambda \in (0,1] \ \text{ and } \ F(\xi, 0) := \lim_{\lambda \downarrow 0^+} F(\xi, \lambda).$$

Obviously F is continuous. Since $\phi_{\xi,\lambda} \circ u \to \xi$ in $L^2(\mathbb{S}^{n-1})$ as $\lambda \downarrow 0^+$ we see that $F(\xi, 0) = \xi$ for every $\xi \in \mathbb{S}^{n-1}$, whereas $F(\xi, 1) = b_u$. Hence F is a continous homotopy between $\mathbb{S}^{n-1}$ and the point $b_u \in \overline{B}_1 \setminus \{0\}$. Therefore there must exist $\lambda_0 \in (0,1)$ such that $0 \in F(\mathbb{S}^{n-1}, \lambda_0)$, i.e. there exists also a $\xi_0 \in \mathbb{S}^{n-1}$ such that $F(\xi_0, \lambda_0) = \fint_{\mathbb{S}^{n-1}} \phi_{\xi_0,\lambda_0} \circ u \, d\mathcal{H}^{n-1} = 0$, as claimed.

If we apply the previous argument to the conformal map $\phi_{\xi_0,\lambda_0} \circ u$ that now has zero mean, we conclude that there exists $O \in O(n)$ such that

$$(\phi_{\xi_0,\lambda_0} \circ u)(x) = Ox \implies u(x) = \phi^{-1}_{\xi_0,\lambda_0}(Ox) = \phi_{\xi_0,\frac{1}{\lambda_0}}(Ox) = O\phi_{\xi,\lambda}(x),$$

where $\xi := O^t\xi_0 \in \mathbb{S}^{n-1}$ and $\lambda := \frac{1}{\lambda_0} > 0$. The proof is now complete. □

In the Introduction we had mentioned that if $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is a Lipschitz embedding then $D_n(u) \geq V_n(u)$ with equality in this case iff u is a Möbius transformation of $\mathbb{S}^{n-1}$ up to a scaling factor. Keeping in mind the conformal invariance of both quantities and their natural scaling, we can restate the question we posed in the Introduction as follows.

Question. Let $n \geq 3$, $\varepsilon > 0$ and $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ be a Lipschitz map such that $D_n(u) \leq (1+\varepsilon)V_n(u)$. Can we describe in a quantitative way the deviation of u from a Möbius transformation of $\mathbb{S}^{n-1}$ in terms of ε?

Once again we remark that this $\varepsilon > 0$ should be thought of as a measure of deviation both from conformality and from isoperimetry (for the image of u) and provides a natural candidate for such an average geometric deficit.

2.2 Setup of the local stability estimate

We start the discussion of our local stability result, where we assume at first place that the map in question is apriori close to the identity. As in the Introduction for $M > 0$, $\theta > 0$ and $\varepsilon > 0$ we

consider the following set of mappings

$$\mathcal{A}_{M,\theta,\varepsilon} := \left\{ u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \begin{array}{l} (i) \ \ \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M, \\ (ii) \ \|u - \mathrm{id}_{\mathbb{S}^{n-1}}\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq \theta, \\ (iii) \ D_n(u) \leq (1+\varepsilon)V_n(u). \end{array} \right\}$$

Remark 2.2. Even though our basic result is a "close to the identity statement", the same conlusions would hold if we were to assume that the map u is apriori close to a Möbius transfomation whose gradient is bounded by above and below by some fixed positive constants. To be more precise, for any $0 < \mu_1 \leq \mu_2$ (which can be thought of as initial parameters in our problem) consider the following subset of Möbius transformations of $\mathbb{S}^{n-1}$

$$\mathrm{Conf}(\mathbb{S}^{n-1};\mu_1,\mu_2) := \{\phi \in \mathrm{Conf}(\mathbb{S}^{n-1}) : \mu_1 \leq |\nabla_T \phi(x)| \leq \mu_2 \text{ for } H^{n-1} - \text{a.e. } x \in \mathbb{S}^{n-1}\}.$$

Given now $M > 0, \theta > 0, \varepsilon > 0$ consider the set

$$\mathcal{A}_{\mu_1,\mu_2,M,\theta,\varepsilon} := \left\{ \begin{array}{l} (i) \ \ \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M, \\ (ii) \ \exists \ \phi_u \in \mathrm{Conf}(\mathbb{S}^{n-1};\mu_1,\mu_2) : \|u - \phi_u\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq \theta, \\ (iii) \ D_n(u) \leq (1+\varepsilon)V_n(u). \end{array} \right\}$$

It is then easy to show that there exist $\tilde{M} := \tilde{M}(n, M, \mu_1) > 0$ and $\tilde{\theta} := \tilde{\theta}(n, \theta, \mu_2) > 0$ such that whenever $u \in \mathcal{A}_{\mu_1,\mu_2,M,\theta,\varepsilon}$ we have that $u \circ \phi_u^{-1} \in \mathcal{A}_{\tilde{M},\tilde{\theta},\varepsilon}$. We could then apply all our subsequent arguments to $u \circ \phi_u^{-1}$ instead of u. Indeed, by the chain rule (with the gradients seen as linear maps between the corresponding tangent spaces) and the conformality of ϕ_u we have

$$\begin{aligned} &\left(\nabla_T \phi_u^{-1}(x)\right)^t \circ \left(\nabla_T \phi_u(\phi_u^{-1}(x))\right)^t \circ \nabla_T \phi_u(\phi_u^{-1}(x)) \circ \nabla_T \phi_u^{-1}(x) = I_x \\ \implies &\frac{|\nabla_T \phi_u^{-1}(x)|^2}{n-1} \cdot \frac{|\nabla_T \phi_u(\phi_u^{-1}(x))|^2}{n-1} = 1. \end{aligned}$$

If we now consider $u \in \mathcal{A}_{\mu_1,\mu_2,M,\theta,\varepsilon}$ then for the map $u \circ \phi_u^{-1}$ we can estimate

$$\left|\nabla_T (u \circ \phi_u^{-1})(x)\right| \lesssim_n \left|\nabla_T u(\phi_u^{-1}(x))\right| \left|\nabla_T \phi_u^{-1}(x))\right| \lesssim_n \frac{M}{\mu_1} \implies \left\|\nabla_T (u \circ \phi_u^{-1})\right\|_{L^\infty(\mathbb{S}^{n-1})} \leq \tilde{M},$$

where $\tilde{M} \sim_n \frac{M}{\mu_1} > 0$. Moreover,

$$\fint_{\mathbb{S}^{n-1}} |(u \circ \phi_u^{-1}) - \mathrm{id}_{\mathbb{S}^{n-1}}|^2 \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} |u(y) - \phi_u(y)|^2 \, g_u(y) \, d\mathcal{H}^{n-1}(y),$$

where $g_u(y) := \left(\det\left(\left(\nabla_T \phi_u^{-1}(x)\right)^t \nabla_T \phi_u^{-1}(x)\right)\right)^{-\frac{1}{2}}\Big|_{x=\phi_u(y)} = \left(\frac{|\nabla_T \phi_u^{-1}(x)|^2}{n-1}\right)^{-\frac{n-1}{2}}\Big|_{x=\phi_u(y)}$. Similarly

$$\begin{aligned} \fint_{\mathbb{S}^{n-1}} |\nabla_T (u \circ \phi_u^{-1}) - P_T|^2 d\mathcal{H}^{n-1} &\lesssim_n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T u(\phi_u^{-1}(x)) - \nabla_T \phi_u(\phi_u^{-1}(x))\right|^2 \left|\nabla_T \phi_u^{-1}(x)\right|^2 \, d\mathcal{H}^{n-1}(x) \\ &= \fint_{\mathbb{S}^{n-1}} |\nabla_T u(y) - \nabla_T \phi_u(y)|^2 \, |\nabla_T \phi_u^{-1}(x)|^2|_{x=\phi_u(y)} g_u(y) \, d\mathcal{H}^{n-1}(y). \end{aligned}$$

Hence,

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} |(u\circ\phi_u^{-1}) - \mathrm{id}_{\mathbb{S}^{n-1}}|^2 \, d\mathcal{H}^{n-1} &\sim_n \fint_{\mathbb{S}^{n-1}} |u(y)-\phi_u(y)|^2 \left|\nabla_T\phi_u^{-1}(x)\right|^{1-n}\Big|_{x=\phi_u(y)} \, d\mathcal{H}^{n-1}(y)\\ &\lesssim_n \fint_{\mathbb{S}^{n-1}} |u(y)-\phi_u(y)|^2 \, \|\nabla_T\phi_u\|^{n-1}_{L^\infty(\mathbb{S}^{n-1})} \, d\mathcal{H}^{n-1}(y)\\ &\lesssim_n \mu_2^{n-1}\theta^2\end{aligned}$$

and

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} |\nabla_T(u\circ\phi_u^{-1}) - P_T|^2 d\mathcal{H}^{n-1} &\lesssim_n \fint_{\mathbb{S}^{n-1}} |\nabla_T u(y) - \nabla_T\phi_u(y)|^2 \, |\nabla_T\phi_u^{-1}(x)|^{3-n}|_{x=\phi_u(y)} \, d\mathcal{H}^{n-1}(y)\\ &\lesssim_n \fint_{\mathbb{S}^{n-1}} |\nabla_T u(y) - \nabla_T\phi_u(y)|^2 \, \|\nabla_T\phi_u\|^{n-3}_{L^\infty(\mathbb{S}^{n-1})} \, d\mathcal{H}^{n-1}(y)\\ &\lesssim_n \mu_2^{n-3}\theta^2.\end{aligned}$$

We have therefore obtained $\left\|u\circ\phi_u^{-1} - \mathrm{id}_{\mathbb{S}^{n-1}}\right\|_{W^{1,2}(\mathbb{S}^{n-1})} \le \tilde\theta$, where $\tilde\theta \sim_n \sqrt{\mu_2^{n-1}+\mu_2^{n-3}}\theta > 0$.

Similar conditions have appeared for example in [12] and [13], where the authors prove quantitative stability estimates for compact subsets of the conformal group $CO_+(n)$ in bounded domains of $\mathbb{R}^n$. The conditions are basically imposed to avoid degeneracy issues at the origin and at infinity of the "cone" $CO_+(n)$.

Let us also set $w := u - \mathrm{id}|_{\mathbb{S}^{n-1}}$, as if at first place the optimal candidate for being the closest Möbius map to u in terms of the conformal-isoperimetric deficit is really the identity map and as always denote by $w_h : \overline{B} \mapsto \mathbb{R}^n$ the harmonic extension of w. We first start with a Lemma that although trivial, allows us to fix the center and the scale of the map u and will be of use later.

Lemma 2.3. *Given $M > 0$, $\theta > 0$ (sufficiently small) and $\varepsilon > 0$ there exists $\tilde\theta = \tilde\theta(n,\theta) > 0$ such that after possibly replacing θ with $\tilde\theta$ we can assume that every $u \in \mathcal{A}_{M,\tilde\theta,\varepsilon}$ has the following additional properties:*

(i) $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0 \;\Leftrightarrow\; \fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0$,

(ii) $\fint_{\mathbb{S}^{n-1}} \langle u, x\rangle \, d\mathcal{H}^{n-1} = 1 \;\Leftrightarrow\; \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} = 0$.

Proof. The first property follows trivially by considering $u - \fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1}$ instead of u (and replacing θ by $\sqrt{2}\theta$) if necessary. Regarding the second one, by the mean value property of harmonic functions

$$\fint_{\mathbb{S}^{n-1}} \langle u, x\rangle \, d\mathcal{H}^{n-1} = \frac{1}{n}\fint_B \mathrm{div} u_h \, dx = \frac{1}{n}\mathrm{Tr}\left(\fint_B \nabla u_h \, dx\right) = \frac{\mathrm{Tr}\nabla u_h(0)}{n}. \tag{2.3}$$

Applying Lemma A.3 we can estimate

$$\left|\frac{\mathrm{Tr}\nabla u_h(0)}{n} - 1\right|^2 \le \frac{|\nabla u_h(0) - I_n|^2}{n} \le \fint_B \frac{|\nabla u_h - I_n|^2}{n} \, dx \le \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u - P_T|^2}{n-1} d\mathcal{H}^{n-1} \le \frac{\theta^2}{n-1}.$$

If $\theta > 0$ is sufficiently small (depending on the dimension) we have that

$$0 < 1 - \frac{\theta}{\sqrt{n-1}} < \frac{\mathrm{Tr}\nabla u_h(0)}{n} < 1 + \frac{\theta}{\sqrt{n-1}}$$

and by setting $\lambda_u := \frac{\mathrm{Tr}\nabla u_h(0)}{n}$ we can replace u with $\frac{u}{\lambda_u}$ if necessary to achieve the second property stated in the Lemma. Observe also that

$$\left\| \left(\frac{u}{\lambda_u}\right) - \mathrm{id}_{\mathbb{S}^{n-1}} \right\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{1}{\lambda_u} \left\| u - \mathrm{id}_{\mathbb{S}^{n-1}} \right\|_{L^2(\mathbb{S}^{n-1})} + \left| \frac{1}{\lambda_u} - 1 \right| \leq \tilde{\theta}$$

and

$$\left\| \nabla_T \left(\frac{u}{\lambda_u}\right) - P_T \right\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{1}{\lambda_u} \left\| \nabla_T u - P_T \right\|_{L^2(\mathbb{S}^{n-1})} + \left| \frac{1}{\lambda_u} - 1 \right| \left\| P_T \right\|_{L^2(\mathbb{S}^{n-1})} \leq \tilde{\theta},$$

where $\tilde{\theta} := 2\left(1 - \frac{\theta}{\sqrt{n-1}}\right)^{-1} \theta \ll 1$. □

According to the previous Lemma, after possibly replacing the constant $\theta > 0$ with the constant $\tilde{\theta} > 0$ (which we do not relabel here) we can focus our attention on the set of maps

$$\tilde{\mathcal{A}}_{M,\theta,\varepsilon} := \left\{ u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \begin{array}{l} (i)\ \ \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M \\ (ii)\ \|u - \mathrm{id}_{\mathbb{S}^{n-1}}\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq \theta \\ (iii)\ \fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0 \\ (iv)\ \fint_{\mathbb{S}^{n-1}} \langle u, x\rangle \, d\mathcal{H}^{n-1} = 1 \\ (v)\ D_n(u) \leq (1+\varepsilon) V_n(u) \end{array} \right\} \tag{2.4}$$

We can now perform a formal Taylor expansion of the deficit around the identity and calculate the quadratic term appearing in the expansion, as we have already stated in Proposition 1.7. For the most part, the proof relies on standard computations which are given for completeness and for the convenience of the reader in the appendices. Here we just give the proof of the last part of the Proposition regarding the growth behaviour of the remainder term in the expansion of $V_n(u)$ around the identity. It is basically a consequence of its algebraic structure and the Sobolev inequality on the sphere which we recall here and refer the reader to [14], [15] for its proof and more details.

Theorem 2.4. ***(Sobolev inequality)*** *If $n \geq 3$ and $m \geq 1$, for every $w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^m)$ the following interpolation inequality holds on the sphere*

$$\left(\fint_{\mathbb{S}^{n-1}} |w|^p \, d\mathcal{H}^{n-1} \right)^{\frac{2}{p}} \leq \frac{p-2}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} \tag{2.5}$$

for every $p \in (2, 2^*]$ with $2^* := \frac{2n-2}{n-3}$ if $n \geq 4$ and for any $p \in (2,\infty)$ if $n = 3$.

Proof of Proposition 1.7. As mentioned, we give here only the proof of (1.12) and include the derivation of the other formulas in Appendix B. We remark there that under our assumptions, the expansion of $V_n(u)$ around the identity is given by

$$V_n(u) \ = \ 1 + Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{2,n}(w, \nabla_T w) \, d\mathcal{H}^{n-1},$$

and the remainder term in the expansion can be written as

$$\fint_{\mathbb{S}^{n-1}} R_{2,n}(w, \nabla_T w) \, d\mathcal{H}^{n-1} = \sum_{k=3}^{n} \fint_{\mathbb{S}^{n-1}} R_{2,n,k}(w, \nabla_T w) \, d\mathcal{H}^{n-1}.$$

For every $k = 3, ..., n$, the algebraic structure of the k-th summand in the remainder term is

$$\fint_{\mathbb{S}^{n-1}} R_{2,n,k}(w, \nabla_T w)\ d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \langle w, A_k(w)\rangle\ d\mathcal{H}^{n-1},$$

where A_k is a nonlinear first order differential operator that is a "homogeneous polynomial" of order $k-1$ in the first derivatives of w. In particular, since in our case $\|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \leq M + \sqrt{n-1}$,

$$|R_{2,n}(w, \nabla_T w)| \lesssim_{n,M} |w||\nabla_T w|^2.$$

We can now apply the Sobolev inequality on w. Having assumed that $\fint_{\mathbb{S}^{n-1}} w\ d\mathcal{H}^{n-1} = 0$ we can couple it with Poincare's inequality (A.5) to write it in the form

$$\left(\fint_{\mathbb{S}^{n-1}} |w|^p\ d\mathcal{H}^{n-1}\right)^{\frac{2}{p}} \leq \frac{p-1}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1}$$

for $p \in (2, 2^*]$ with $2^* := \frac{2n-2}{n-3}$ if $n \geq 4$ and for any $p \in (2, \infty)$ if $n = 3$.

So let first $n = 3$ and $\gamma \in (0, \frac{1}{2})$. For any $p > 2$ with q being its Hölder conjugate exponent, i.e. $q = \frac{p}{p-1} \in (1, 2)$, we can use Hölder's inequality and then the Sobolev inequality to obtain

$$\begin{aligned}\fint_{\mathbb{S}^2} |R_{2,3}(w, \nabla_T w)|\ d\mathcal{H}^2 &\lesssim \fint_{\mathbb{S}^2} |w||\nabla_T w|^2\ d\mathcal{H}^2 \lesssim \left(\fint_{\mathbb{S}^2} |w|^p\ d\mathcal{H}^2\right)^{\frac{1}{p}} \left(\fint_{\mathbb{S}^2} |\nabla_T w|^{2q}\ d\mathcal{H}^2\right)^{\frac{1}{q}} \\ &\lesssim_p M^{\frac{2}{p}} \left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\ d\mathcal{H}^2\right)^{\frac{1}{2}+\frac{1}{q}}.\end{aligned}$$

This estimate holds for any $p > 2$ so we can choose $\frac{1}{q} = \frac{1}{2} + \gamma$ or equivalently $p = \frac{2}{1-2\gamma} > 2$ to obtain the desired estimate.

If $n \geq 4$, we perform the same estimates but at the critical Sobolev exponent $2^* = \frac{2n-2}{n-3}$ whose Hölder conjugate exponent is $2^{**} := \frac{2n-2}{n+1}$ and we arrive at the second estimate stated in (1.12). □

As a direct consequence of Proposition (1.7) we obtain the following.

Lemma 2.5. *There exist constants $c_1 := c_1(M, \theta, \varepsilon) > 0$ and $c_2 := c_2(n, M, \theta, \varepsilon) > 0$ such that for every $u := w + \mathrm{id}_{\mathbb{S}^{n-1}} \in \tilde{\mathcal{A}}_{M,\theta,\varepsilon}$, and for $Q_n(w)$ as defined in Theorem 1.9 the following estimates hold*

$$\begin{cases} \text{If } n = 3,\ Q_3(w) \leq \varepsilon + c_1(M, \theta, \varepsilon) \fint_{\mathbb{S}^2} |\nabla_T w|^2\ d\mathcal{H}^2. \\ \text{If } n \geq 4,\ Q_n(w) \leq \varepsilon + c_2(n, M, \theta, \varepsilon) \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1} + C_1(n, M) \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^3\ d\mathcal{H}^{n-1}. \end{cases}$$

The precise expressions for c_1, c_2 are $c_1(M, \theta, \varepsilon) := c_3\varepsilon + C_2(1+\varepsilon)\theta^{2\gamma} + C_1\theta^2$ where C_1, C_2 are the constants appearing in Proposition 1.7, $\gamma \in (0, \frac{1}{2})$, $c_3 > 0$ is an absolute constant and $c_2(n, M, \theta, \varepsilon) := c_n\varepsilon + C_2(1+\varepsilon)\theta^{\frac{2}{n-1}}$ with $c_n > 0$ a dimensional constant. The important thing is that both c_1 and c_2 tend to 0 when θ and ε tend to 0.

Proof. If $u := w + \mathrm{id}_{\mathbb{S}^{n-1}} \in \tilde{\mathcal{A}}_{M,\theta,\varepsilon}$ and $Q_n(w) := Q_{D_n}(w) - Q_{V_n}(w)$, by the last property in the definition of the set $\tilde{\mathcal{A}}_{M,\theta,\varepsilon}$ we obtain

$$Q_n(w) \leq \varepsilon + (1+\varepsilon)\fint_{\mathbb{S}^{n-1}} R_2(w, \nabla_T w)\ d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} R_1(\nabla_T w)\ d\mathcal{H}^{n-1} + \varepsilon Q_{V_n}(w).$$

The Lemma now follows easily by Proposition 1.7 after some algebraic manipulations. One of those is the fact that the quadratic form $Q_{V_n}(w)$ is bounded by above by the Dirichlet integral. Indeed, since we have assumed that $\fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0$, we can use the Cauchy-Schwarz inequality and Poincare's inequality to estimate

$$
\begin{aligned}
|Q_{V_n}(w)| &\leq \left(\fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^{n-1}} |(\mathrm{div}_{\mathbb{S}^{n-1}} w)x|^2 + \left| \sum_{j=1}^{n} x_j \nabla_T w^j \right|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \\
&\leq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T w|^2}{n-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w : P_T|^2 + \left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{j=1}^{n} |\nabla_T w^j|^2 \right) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \\
&\lesssim_n \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}. \qquad \square
\end{aligned}
$$

As a conclusion of these technical Lemmata, it is important to examine the coercivity properties of the quadratic form Q_n, which will be treated in the next section. This can be thought of as a "linearization" of the nonlinear deficit, a procedure that is common in geometric rigidity estimates of this flavour (see for example [12], [5], [13]).

3 On the coercivity of the quadratic form Q_n.

We embark on examining the coercivity properties of the quadratic form Q_n defined on the space

$$
H := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0 \ , \ \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} = 0 \right\}.
$$

Our goal is to prove Theorems 1.8 and 1.9. For every $k \geq 1$ we define the subspaces $H_k, \tilde{H}_k, \tilde{H}_{k,\mathrm{sol}}$ and $H_{k,\mathrm{sol}}$ as well as their L^2-orthogonal complements as in page 7 in the Introduction, where we have also introduced the first order linear differential operator

$$
A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j
$$

so that

$$
Q_{V_n}(w, v) := \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \langle w, A(v) \rangle \, d\mathcal{H}^{n-1} \quad \text{for any } w, v \in H.
$$

Proof of Theorem 1.8. As we outlined in the Introduction, the important observation is that A is a symmetric operator with respect to the L^2-inner product with trivial kernel in H and leaves each one of the subspaces $(H_k)_{k \geq 1}$ invariant, but moreover A leaves the subspaces $H_{k,\mathrm{sol}}$ and also $H_{k,\mathrm{sol}}^{\perp}$ invariant, so that all of them have an eigenvalue decomposition with respect to A.

For every $k \geq 1$ we set $N(n,k) := \dim H_k < \infty$, $N_1(n,k) := \dim H_{k,\mathrm{sol}} < \infty$ and also $N_2(n,k) := \dim H_{k,\mathrm{sol}}^{\perp} < \infty$ so that $N(n,k) = N_1(n,k) + N_2(n,k)$. By the above observations there exists an L^2-orthonormal basis of eigenfunctions $w_{k,1}, ..., w_{k,N_1(n,k)}$ of $H_{k,\mathrm{sol}}$ and $w_{k,N_1(n,k)+1}, ..., w_{k,N(n,k)}$ of $H_{k,\mathrm{sol}}^{\perp}$ such that for every $i = 1, ..., N(n,k)$ $w_{k,i}$ satisfies the eigenvalue equation

$$
A(w_{k,i}) := (\mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i})x - \sum_{j=1}^{n} x_j \nabla_T w_{k,i}^j = \sigma_{k,i} w_{k,i} \quad \text{on } \mathbb{S}^{n-1}. \tag{3.1}
$$

For each such eigenvalue $\sigma_{k,i}$ we denote its corresponding eigenspace by $H_{k,i}$. If we take the inner product with the unit normal vector field on $\mathbb{S}^{n-1}$ we obtain that each eigenfunction $w_{k,i}$ satisfies the equation

$$\mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} = \sigma_{k,i}\langle w_{k,i}, x\rangle \ \text{ on } \mathbb{S}^{n-1}. \tag{3.2}$$

As we already know the harmonic extension of $w_{k,i}$ in $\overline{B}$ is a homogeneous harmonic polynomial of degree k and therefore all its derivatives will be polynomials again, thus having a continuous (and even analytic) trace up to the boundary. In particular,

$$\mathrm{div} w_{k,i,h} \ = \ \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} + \langle \partial_{\bar{\nu}} w_{k,i,h}, x\rangle \ = \ (\sigma_{k,i}+k)\ \langle w_{k,i}, x\rangle \text{ on } \ \mathbb{S}^{n-1}. \tag{3.3}$$

We now fix the index $k \geq 1$ and consider different cases that will allow us to find the eigenvalues of A in the invariant subspaces $H_{k,\mathrm{sol}}$ and $H_{k,\mathrm{sol}}^{\perp}$ respectively.

(a_1) Let w be an eigenfunction of A in $H_{k,\mathrm{sol}}$. By definition $\mathrm{div} w_h \equiv 0$ in $\overline{B}$ and by homogeneity this is equivalent to $\mathrm{div} w_h \equiv 0$ on $\mathbb{S}^{n-1}$. By (3.3) we see that one possibility for this is that $\sigma = -k$. We thus set $\sigma_{k,1} := -k$ and $H_{k,1} := \mathrm{span}\{w_{k,1}, ..., w_{k,p_k}\}$ its corresponding eigenspace.

(a_2) Let w be an eigenfunction of A in $H_{k,\mathrm{sol}}$ but $w \in H_{k,1}^{\perp}$. The only possibility is then if $\langle w, x\rangle \equiv 0$ on $\mathbb{S}^{n-1}$. In that case w is a tangential vector field and by (3.2) we have that $\mathrm{div}_{\mathbb{S}^{n-1}} w \equiv 0$ as well. Equation (3.1) then gives $\sigma w = -\sum_{j=1}^{n} x_j \nabla_T w^j$ on $\mathbb{S}^{n-1}$. Testing this equation with the vector field w itself and integrating by parts implies that

$$\begin{aligned}
\sigma \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} \ &= \ -\sum_{j=1}^{n} \fint_{\mathbb{S}^{n-1}} \langle \nabla_T w^j, x_j w\rangle \, d\mathcal{H}^{n-1} = \sum_{j=1}^{n} \fint_{\mathbb{S}^{n-1}} w^j \mathrm{div}_{\mathbb{S}^{n-1}}(x_j w) \, d\mathcal{H}^{n-1} \\
&= \ \sum_{j=1}^{n} \fint_{\mathbb{S}^{n-1}} w^j \langle \nabla_T x_j, w\rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Therefore this eigenvalue is labeled as $\sigma_{k,2} := 1$ and $H_{k,2} := \mathrm{span}\{w_{k,p_k+1}, ..., w_{k,N_1(n,k)}\}$ will be its corresponding eigenspace. Finally, $H_{k,\mathrm{sol}} = H_{k,1} \bigoplus H_{k,2}$.

(b) Let us now look at eigenfunctions of A in the subspace $H_{k,\mathrm{sol}}^{\perp}$, where the divergence of $w_h \in \tilde{H}_k$ does not vanish identically in $\overline{B}$. Since w_h is a k-homogeneous harmonic polynomial we have that $\mathrm{div} w_h$ is a $(k-1)$-homogeneous harmonic polynomial and therefore its restriction on $\mathbb{S}^{n-1}$ is a scalar valued $(k-1)$-spherical harmonic. We can then take the Laplace-Beltrami operator on both sides of (3.3) to obtain the following equations on $\mathbb{S}^{n-1}$

$$\begin{aligned}
(k-1)(k+n-3)\mathrm{div} w_h \ &= \ -\Delta_{\mathbb{S}^{n-1}}(\mathrm{div} w_h) \ = \ -(\sigma+k)\Delta_{\mathbb{S}^{n-1}}\big(\langle w, x\rangle\big) \\
&= \ (\sigma+k)\Big(\langle -\Delta_{\mathbb{S}^{n-1}} w, x\rangle - 2\nabla_T w : P_T + \langle w, -\Delta_{\mathbb{S}^{n-1}} x\rangle\Big) \\
&= \ \Big(k(k+n-2) - 2\sigma + n - 1\Big)(\sigma+k)\langle w, x\rangle \\
&= \ \Big(k(k+n-2) - 2\sigma + n - 1\Big)\mathrm{div} w_h.
\end{aligned}$$

Since in this case $\mathrm{div} w_h$ does not vanish identically on $\mathbb{S}^{n-1}$ we conclude that

$$k(k+n-2) - 2\sigma + n - 1 \ = \ (k-1)(k+n-3) \implies \sigma \ = \ k+n-2.$$

We label this eigenvalue as $\sigma_{k,3} := k+n-2$ and its corresponding eigenspace as $H_{k,3}$. In particular we find that $H_{k,\mathrm{sol}}^{\perp} = H_{k,3}$.

In total we have the L^2-orthogonal decomposition of our space of interest into eigenspaces of A as

$$H := \bigoplus_{k=1}^{\infty} \left(H_{k,1} \bigoplus H_{k,2} \bigoplus H_{k,3} \right).$$

It is easy to construct examples showing that none of these eigenspaces are trivial, except for $H_{1,3}$. Indeed let $w(x) := Bx \in H_{1,3}$ for some $B \in \mathbb{R}^{n\times n}$. Then by assumption

$$0 = \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \langle Bx, x\rangle \, d\mathcal{H}^{n-1} = \frac{1}{n}\mathrm{Tr}B$$

and therefore also $\mathrm{div} w_h \equiv \mathrm{Tr}B \equiv 0$, i.e. $w \in H_{1,\mathrm{sol}} = H_{1,3}^{\perp}$ forcing $w \equiv 0$. □

The next lemma is a direct consequence of this eigenvalue decomposition.

Lemma 3.1. *For every $k \geq 1$ the quadratic forms Q_{V_n} and Q_n diagonalize on each one of the subspaces $H_{k,1}$, $H_{k,2}$, $H_{k,3}$, that is there exist constants $(c_{n,k,i})_{i=1,2,3}$ and $(C_{n,k,i})_{i=1,2,3}$ such that*

$$Q_{V_n}(w) \;=\; c_{n,k,i} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} \text{ for every } w \in H_{k,i} \tag{3.4}$$

and

$$Q_n(w) \;=\; C_{n,k,i} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} \text{ for every } w \in H_{k,i}. \tag{3.5}$$

Proof. The proof is basically a straightforward computation. By (A.4) we know that

$$\fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} = \frac{1}{\lambda_k} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} \text{ for all } w \in H_k, \text{ where } \lambda_k := k(k+n-2).$$

If $w \in H_{k,1}$ we have $Q_{V_n}(w) = c_{n,k,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}$, where $c_{n,k,1} := \frac{n\sigma_{k,1}}{2\lambda_k} = \frac{-n}{2(k+n-2)} < 0$. By (3.2) and the formulas that equivalently define $Q_{V_n}(w)$ we obtain

$$\begin{aligned} Q_{V_n}(w) &= \frac{n}{2} \fint_{\mathbb{S}^{n-1}} (2\mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2) \, d\mathcal{H}^{n-1} \\ &= \left(\frac{n}{\sigma_{k,1}} - \frac{n^2}{2\sigma_{k,1}^2} \right) \fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \, d\mathcal{H}^{n-1} + \frac{n}{2\lambda_k} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}, \end{aligned}$$

and therefore

$$\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \, d\mathcal{H}^{n-1} = \alpha_{n,k,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}, \tag{3.6}$$

where $\alpha_{n,k,1} := \frac{c_{n,k,1} - \frac{n}{2\lambda_k}}{\frac{n}{\sigma_{k,1}} - \frac{n^2}{2\sigma_{k,1}^2}} = \frac{k(k+1)}{(k+n-2)(2k+n)} > 0$. In total we have obtained

$$Q_n(w) = C_{n,k,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}, \text{ where } C_{n,k,1} := \frac{1}{2}\frac{n}{n-1} + \frac{1}{2}\frac{n(n-3)}{(n-1)^2}\alpha_{n,k,1} - c_{n,k,1}$$

and can explicitely calculate that $C_{n,k,1} = \frac{n}{2}\left(\frac{1}{n-1} + \frac{1}{k+n-2} + \frac{(n-3)k(k+1)}{(n-1)^2(k+n-2)(2k+n)}\right) > 0$.

The same formulas hold true of course for Q_{V_n} and Q_n in $H_{k,2}$ and $H_{k,3}$, with the constants $c_{n,k,2}, \alpha_{n,k,2}, C_{n,k,2}$ and $c_{n,k,3}, \alpha_{n,k,3}, C_{n,k,3}$ satisfying the same relations as above with $\sigma_{k,2}, \sigma_{k,3}$ in place of $\sigma_{k,1}$ respectively. The actual values of the constants are important in this case since we

will need to "sum up" the identities to obtain an estimate on the full space H. We list them in the following tables.

$$\begin{cases} c_{n,k,1} = \frac{-n}{2(k+n-2)} \\ \alpha_{n,k,1} = \frac{k(k+1)}{(k+n-2)(2k+n)} \\ C_{n,k,1} = \frac{n}{2}\left(\frac{1}{n-1} + \frac{1}{k+n-2} + \frac{(n-3)k(k+1)}{(n-1)^2(k+n-2)(2k+n)}\right) \end{cases} \tag{3.7}$$

$$\begin{cases} c_{n,k,2} = \frac{n}{2k(k+n-2)} \\ \alpha_{n,k,2} = 0 \\ C_{n,k,2} = \frac{n}{2}\frac{(k-1)(k+n-1)}{(n-1)k(k+n-2)} \end{cases} \tag{3.8}$$

$$\begin{cases} c_{n,k,3} = \frac{n}{2k} \\ \alpha_{n,k,3} = \frac{(k+n-2)(k+n-3)}{k(2k+n-4)} \\ C_{n,k,3} = \frac{n(k-2)\left((3n-5)k+(n^2-6n+7)\right)}{2(n-1)^2k(2k+n-4)}. \end{cases} \tag{3.9}$$

□

Lemma 3.2. *The quadratic forms Q_{V_n} and Q_n additionally satisfy that*

(i) For every $k \geq 1$ and every $i, j = 1, 2, 3$ with $i \neq j$ the subspaces $H_{k,i}$ and $H_{k,j}$ are both Q_{V_n}- and Q_n-orthogonal, that is,

$$Q_{V_n}(w_{k,i}, w_{k,j}) = 0 \quad \text{and} \quad Q_n(w_{k,i}, w_{k,j}) = 0 \quad \text{for every } w_{k,i} \in H_{k,i} \text{ and } w_{k,j} \in H_{k,j}.$$

(ii) For every $k, l \geq 1$ with $k \neq l$ and every $i, j = 1, 2, 3$ the subspaces $H_{k,i}$ and $H_{l,j}$ are also Q_{V_n}- and Q_n-orthogonal.

Proof. The proof is a calculation similar to the previous one, making use of the fact that the subspaces $(H_{k,i})_{k\geq 1, i=1,2,3}$ are mutually orthogonal in $L^2(\mathbb{S}^{n-1})$.

(i) Let us fix $k \geq 1$. For any $w_{k,i} \in H_{k,i}$ and $w_{k,j} \in H_{k,j}$ with $i \neq j$ we have

$$Q_{V_n}(w_{k,i}, w_{k,j}) = \frac{n\sigma_{k,j}}{2} \fint_{\mathbb{S}^{n-1}} \langle w_{k,i}, w_{k,j} \rangle \, d\mathcal{H}^{n-1} = 0$$

and thus also

$$\fint_{\mathbb{S}^{n-1}} \langle \nabla_T w_{k,i}, \nabla_T w_{k,j} \rangle \, d\mathcal{H}^{n-1} = \lambda_k \fint_{\mathbb{S}^{n-1}} \langle w_{k,i}, w_{k,j} \rangle \, d\mathcal{H}^{n-1} = 0.$$

Using these, the bilinear form associated to Q_n becomes

$$Q_n(w_{k,i}, w_{k,j}) = \frac{1}{2}\frac{n(n-3)}{(n-1)^2} \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,j} \, d\mathcal{H}^{n-1}.$$

We are thus left with showing that the last term vanishes. It suffices to consider the case $i = 1$ and $j = 3$ since $\mathrm{div}_{\mathbb{S}^{n-1}} w \equiv 0$ on $\mathbb{S}^{n-1}$ when $w \in H_{k,2}$. We use again the different formulas we have at hand for Q_{V_n} and (3.2) to find that

$$\begin{aligned} 0 &= \fint_{\mathbb{S}^{n-1}} 2\mathrm{div}_{\mathbb{S}^{n-1}} w_{k,1} \langle w_{k,3}, x \rangle - n \langle w_{k,1}, x \rangle \langle w_{k,3}, x \rangle \, d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \langle w_{k,1}, w_{k,3} \rangle \, d\mathcal{H}^{n-1} \\ \Longrightarrow 0 &= \frac{2k+n}{k(k+n-2)} \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,1} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,3} \, d\mathcal{H}^{n-1} \\ \Longrightarrow 0 &= \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,1} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,3} \, d\mathcal{H}^{n-1}, \text{ as wanted.} \end{aligned}$$

(*ii*) Let us consider now $k, l \geq 1$ with $k \neq l$, $i, j = 1, 2, 3$ and $w_{k,i} \in H_{k,i}$, $w_{l,j} \in H_{l,j}$. As before it is immediate that $Q_{V_n}(w_{k,i}, w_{l,j}) = 0$. Since $w_{k,i}$ and $w_{l,j}$ are spherical harmonics of different order we again have that

$$Q_n(w_{k,i}, w_{l,j}) = \frac{1}{2}\frac{n(n-3)}{(n-1)^2}\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} \mathrm{div}_{\mathbb{S}^{n-1}} w_{l,j} \; d\mathcal{H}^{n-1}.$$

and as before

$$\begin{aligned} 0 &= \fint_{\mathbb{S}^{n-1}} 2\mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} \langle w_{l,j}, x\rangle - n\langle w_{k,i}, x\rangle\langle w_{l,j}, x\rangle \; d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \langle w_{k,1}, w_{l,j}\rangle \; d\mathcal{H}^{n-1} \\ \Longrightarrow 0 &= (2\sigma_{k,i} - n)\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} \mathrm{div}_{\mathbb{S}^{n-1}} w_{l,j} \; d\mathcal{H}^{n-1} \\ \Longrightarrow 0 &= \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{k,i} \mathrm{div}_{\mathbb{S}^{n-1}} w_{l,j} \; d\mathcal{H}^{n-1}, \end{aligned}$$

since $\sigma_{k,i} \in \{-k, 1, k+n-2\}$ and $n \geq 3$ imply that $2\sigma_{k,i} \neq n$. □

The estimate in Theorem 1.9 now follows immediately with the constant C_n being explicitely defined as $C_n := \min\limits_{\substack{k\geq 1,\ i\in\{1,2,3\} \\ (k,i)\neq(1,2),(2,3)}} C_{n,k,i} > 0$.

3.1 A Korn-type inequality for the purely conformal deficit

In this subsection we want to mention that the quadratic form appearing in the expansion of the "conformal deficit" also enjoys a coercivity estimate similar to the one that Q_n does. To be precise, let again $u := w + \mathrm{id}|_{\mathbb{S}^{n-1}} \in \mathcal{A}_{M,\theta,\varepsilon}$. If we perform a formal Taylor expansion around the identity we find that (see the Appendix B)

$$\begin{aligned} &\fint_{\mathbb{S}^{n-1}} \left(\left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} - \sqrt{\det(\nabla_T u^t \nabla_T u)} \right) \; d\mathcal{H}^{n-1} \\ = &\fint_{\mathbb{S}^{n-1}} \left| (P_T^t \nabla_T w)_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} w}{n-1} I_x \right|^2 \; d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right) \; d\mathcal{H}^{n-1} \\ = &\fint_{\mathbb{S}^{n-1}} \left| (P_T^t \nabla_T u)_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} u}{n-1} I_x \right|^2 \; d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right) \; d\mathcal{H}^{n-1}. \end{aligned}$$

Notice that here we have not yet scaled the map u so that $\fint_{\mathbb{S}^{n-1}} \langle u, x\rangle \; d\mathcal{H}^{n-1} = 1$. The situation is analogous to the case of conformal maps which are defined on subdomains of $\mathbb{R}^n$, $n \geq 3$. In that case if U is a connected domain in $\mathbb{R}^n$ (starshaped with respect to a ball) and $u \in W^{1,n}(U; \mathbb{R}^n)$ we again have

$$\int_U \left(\frac{|\nabla_T u|^2}{n} \right)^{\frac{n}{2}} \; dx \geq \int_U \det \nabla u \; dx$$

with equality iff $\nabla u \in CO_+(n)$ a.e. in U and hence u is the restriction on U of a Möbius transformation. Setting again $u := v + \mathrm{id}|_U$ and formally expanding the deficit we get again

$$\int_U \left(\frac{|\nabla u|^2}{n} \right)^{\frac{n}{2}} \; dx - \int_U \det \nabla u \; dx = \int_U \left| (\nabla u)_{\mathrm{sym}} - \frac{\mathrm{div} u}{n} I_n \right|^2 \; dx + \int_U \mathcal{O}\left(|\nabla_T v|^3\right) \; dx.$$

As it is well known the last quadratic form has an intimate connection to the geometry of $CO_+(n)$ (see [11], Chapters 2 and 3 or [13] for more details). If $TCO_+(n)$ stands for the tangent space to $CO_+(n)$ at the identity matrix (which is a finite dimensional vector space of dimension $\frac{(n+1)(n+2)}{2}$) it easy to see that

$$A \in TCO+(n) \iff A_{\text{sym}} = \frac{\text{Tr}A}{n} I_n$$

so the function $A \mapsto d(A) := \left|A_{\text{sym}} - \frac{\text{Tr}A}{n} I_n\right|$ is equivalent to the distance from A to $TCO_+(n)$. If now

$$\Sigma_n := \{u \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^n) : \nabla u \in TCO_+(n)\},$$

then Σ_n can be seen as the Lie algebra of the Möbius group $\mathcal{M}_n$ of $\overline{\mathbb{R}^n}$ which as a group is isomorphic to the connected component of the indefinite special orthogonal group $SO(n+2,1)$, i.e. Σ_n is isomorphic to $\mathfrak{so}(n+2,1)$. If $\Pi_{\Sigma_n} : W^{1,2}(U;\mathbb{R}^n) \mapsto \Sigma_n$ is the L^2-projection then the following Korn-type inequality firstly established by Reshetnyak holds

Theorem 3.3. ***(Reshetnyak)*** *If $n \geq 3$ and U is a subdomain of R^n starshaped with respect to a ball there exists a constant $C := C(n,U) > 0$ such that for every $u \in W^{1,2}(U;\mathbb{R}^n)$ we have*

$$\left\|(\nabla u)_{\text{sym}} - \frac{\text{div}u}{n} I_n\right\|_{L^2(U)} \geq C \left\|\nabla u - \nabla(\Pi_{\Sigma_n} u)\right\|_{L^2(U)}. \tag{3.10}$$

We refer the reader to [11], Chapter 3, for an extensive treatment of such estimates which hold true also in $L^p(U)$ for any $p \in (1,\infty)$. Here we want to present the analogous estimate for the quadratic form of the conformal deficit on the sphere. The interesting thing about the estimate on the sphere is the apparent role of the previous decomposition in spherical harmonics, which also makes the calculation of the optimal constant evident. We use the same eigenvalue decomposition as in the proof of Theorem 1.9, together with the following identity that is interesting on its own, and whose derivation is a simple computation which is also included in Appendix B.

Proposition 3.4. ***(Korn's identity)*** *For every $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ the following identity holds*

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u)_{\text{sym}}\right|^2 \, d\mathcal{H}^{n-1} = \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left(|P_T^t \nabla_T u|^2 + (\text{div}_{\mathbb{S}^{n-1}} u)^2\right) \, d\mathcal{H}^{n-1} - \frac{n-2}{n} Q_{V_n}(u). \tag{3.11}$$

The interesting point of this identity is that when $n \geq 3$ the quadratic form Q_{V_n} of the expansion of the "volume term" appears in the right hand side and it is really a "surface identity" in the following sense. The corresponding identity in the bulk is

$$\int_U |(\nabla u)_{\text{sym}}|^2 \, dx = \frac{1}{2}\int_U \left(|\nabla u|^2 + (\text{div}u)^2\right) \, dx - \frac{1}{2}\int_U \left((\text{div}u)^2 - \text{Tr}(\nabla u)^2\right) \, dx$$

but the last term on the right hand side is a null-Lagrangian hence depends only on the boundary values of u.

Theorem 3.5. *There exists an explicit constant $\tilde{C}_n > 0$ such that for every $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ one has*

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u)_{\text{sym}} - \frac{\text{div}_{\mathbb{S}^{n-1}} u}{n-1} I_x\right|^2 \, d\mathcal{H}^{n-1} \geq \tilde{C}_n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T u - \nabla_T(\tilde{\Pi}_n u)\right|^2 \, d\mathcal{H}^{n-1}, \tag{3.12}$$

where $\tilde{\Pi}_n : W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) \mapsto \tilde{H}_0 := H_{1,2} \bigoplus H_{1,3} \bigoplus H_{2,3}$ is the L^2-orthogonal projection on the kernel of the operator $u \mapsto (P_T^t \nabla_T u)_{\text{sym}} - \frac{\text{div}_{\mathbb{S}^{n-1}} u}{n-1} I_x$.

Proof. By Korn's identity we can write the left hand side as

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u)_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} u}{n-1} I_x\right|^2 \, d\mathcal{H}^{n-1} = \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T u|^2 + \frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} u)^2\right) d\mathcal{H}^{n-1}$$
$$- \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left|\sum_{j=1}^n x_j \nabla_T u^j\right|^2 \, d\mathcal{H}^{n-1} - \frac{n-2}{n} Q_{V_n}(u).$$

If $u_{k,i} \in H_{k,i}$ we have $\left|\sum_{j=1}^n x_j \nabla_T u_{k,i}^j\right|^2 = |\sigma_{k,i} u_{k,i} - (\mathrm{div}_{\mathbb{S}^{n-1}} u_{k,i})x|^2 = \sigma_{k,i}^2 |u_{k,i}|^2 - (\mathrm{div}_{\mathbb{S}^{n-1}} u_{k,i})^2$ and therefore

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u_{k,i})_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} u_{k,i}}{n-1} I_x\right|^2 \, d\mathcal{H}^{n-1} = \tilde{C}_{n,k,i} \fint_{\mathbb{S}^{n-1}} |\nabla_T u_{k,i}|^2 \, d\mathcal{H}^{n-1},$$

where $\tilde{C}_{n,k,i} = \frac{1}{2} + \frac{(n-2)\alpha_{n,k,i}}{n-1} - \frac{(n-2)\sigma_{k,i}}{2\lambda_k} - \frac{\sigma_{k,i}^2}{2\lambda_k}$ and in particular we have the precise formulas

$$\begin{cases} \tilde{C}_{n,k,1} = \frac{(k+n)(k+n-1)}{2(n-1)(k+n-2)(2k+n)} \\ \tilde{C}_{n,k,2} = \frac{(k-1)(k+n-1)}{2k(k+n-2)} \\ \tilde{C}_{n,k,3} = \frac{(n-2)(k-1)(k-2)}{(n-1)k(2k+n-4)}. \end{cases} \tag{3.13}$$

Since $\tilde{C}_{n,1,2} = \tilde{C}_{n,1,3} = \tilde{C}_{n,2,3} = 0$, we obtain the desired estimate as in Theorem 1.9 with the sharp constant $\tilde{C}_n$ being explicitely defined as $\tilde{C}_n := \min_{\substack{k\geq 1,\ i\in\{1,2,3\} \\ (k,i)\neq(1,2),(1,3),(2,3)}} \tilde{C}_{n,k,i} > 0$. □

We will nevertheless stick to the quadratic form Q_n and the mixed conformal-isoperimetric deficit in terms of which we stated our main Theorem for basically two reasons. Firstly because if $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ we have discussed that this implies that u is a Möbius transformation up to a scaling factor. On the other hand if we just have $D_n(u) = P_n(u)$, u is a generalized conformal map from $\mathbb{S}^{n-1}$ with image possibly another closed hypersurface, hence not necessarily Möbius. But also for the local stability estimate the advantage is that the remainder term in the expansion of $V_n(u)$ is "compact" in the $W^{1,2}$-topology thanks to the Sobolev embedding, while the remainder term in the expansion of $\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \, d\mathcal{H}^{n-1}$ is "qubic" in the gradient. In particular, in dimension 3 (where there is an abundance of conformal maps defined on $\mathbb{S}^2$) the local stability with respect to the conformal-isoperimetric deficit requires to assume only that u is close to the identity in the $W^{1,2}$-topology.

4 The local stability of $\mathrm{Conf}(\mathbb{S}^{n-1})$

The presence of H_0 is a small obstacle to overcome in order to prove Theorem 1.6. It basically means that although the map u is apriori supposed to be "θ-close" to the identity, there might be another Möbius transformation of $\mathbb{S}^{n-1}$ that is also "θ-close" to the identity and is a better candidate for the "nearest" Möbius map to u in terms of its conformal-isoperimetric deficit. As in [12], [13] where the authors prove quantitative estimates for conformal maps on subdomains of $\mathbb{R}^n$ in terms of average conformal deficits, a topological argument allows us to identify this more suitable candidate. Before doing this, let us present a useful fact about the structure of the subspace H_0 in the next Lemma.

Lemma 4.1. *The following statements are true.*

(i) $H_{1,2} = \{w(x) = Ax : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n,$ where $A \in \mathbb{R}^{n\times n}$ such that $A^t = -A\}$ *and therefore* $\dim H_{1,2} = \frac{n(n-1)}{2}$. *If* $\Pi_{H_{1,2}} : H \mapsto H_{1,2}$ *is the* L^2*-orthogonal projection we have*

$$\Pi_{H_{1,2}} w = 0 \iff \nabla w_h(0) = \nabla w_h(0)^t.$$

(ii) $H_{2,\mathrm{sol}} = \big\{w : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n : w^k(x) = \langle A^k x, x\rangle; A^k \in \mathbb{R}^{n\times n}_{\mathrm{sym}},\ \mathrm{Tr}A^k = 0,\ \sum_{l=1}^n A^l_{lk} = 0$ for every $k = 1, ...n\big\}$. *In particular,*

$$\dim H_{2,3} = \dim H_2 - \dim H_{2,\mathrm{sol}} = n.$$

If $\Pi_{H_{2,3}} : H \mapsto H_{2,3}$ *is the corresponding* L^2*-orthogonal projection we have*

$$\Pi_{H_{2,3}} w = 0 \iff \fint_{\mathbb{S}^{n-1}} (\mathrm{div} w_h(x))x \ d\mathcal{H}^{n-1}(x) = 0.$$

Proof. For the first part, if $w \in H_{1,2}$ then $w(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$ and by definition of this space

$$\langle w, x\rangle \equiv 0 \iff \sum_{1\le i\le j\le n} (A_{ij} + A_{ji})x_i x_j \equiv 0 \iff A^t = -A.$$

The characterization on the projection $\Pi_{H_{1,2}}$ is then immediate. For the second part, let $w \in H_{2,\mathrm{sol}}$. Its harmonic extension is a homogeneous solenoidal harmonic polynomial of degree 2, so for each $k = 1, ..., n$ there exists $A^k \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ such that $w^k_h(x) = \langle A^k x, x\rangle = \sum_{i,j=1}^n A^k_{ij} x_i x_j$. In particular, for each $l = 1, ..., n$ we have

$$\partial_l w^k_h(x) = 2\sum_{i=1}^n A^k_{li} x_i \implies \begin{cases} 0 \equiv \frac{1}{2}\Delta w^k_h = \mathrm{Tr}A^k \\ 0 \equiv \frac{1}{2}\mathrm{div} w_h = \sum_{k=1}^n \left(\sum_{l=1}^n A^l_{lk}\right) x_k \iff \sum_{l=1}^n A^l_{lk} = 0. \end{cases}$$

For the last characterization, by the mean value property of harmonic functions again

$$\Pi_{H_{2,3}} w = 0 \iff \Pi_{H_2} w \in H_{2,\mathrm{sol}} \iff 0 = \sum_{l=1}^n \left(\nabla^2 w^l_h(0)\right)_{lk} = \fint_{\mathbb{S}^{n-1}} (\mathrm{div} w_h(x))x^k \ d\mathcal{H}^{n-1}. \qquad \square$$

Notice that $\dim H_0 = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ which coincides with the dimension of $\mathrm{Conf}(\mathbb{S}^{n-1})$ when seen as a finite-dimensional Lie group. The next Proposition (see for example [13], Proposition 4.7 and the references therein for analogous statements) follows from a suitable application of the Inverse Function Theorem and is the final ingredient for the completion of the proof.

Proposition 4.2. *Given* $M > 0$, $\theta > 0$ *(sufficiently small) and* $\varepsilon > 0$ *there exist positive parameters* $\tilde{M} := \tilde{M}(n, M)$, $\tilde{\theta} := \tilde{\theta}(n, \theta)$ *so that for every* $u \in \mathcal{A}_{M,\theta,\varepsilon}$ *there exist* $\lambda_u > 0$ *and* $\phi_u \in \mathrm{Conf}(\mathbb{S}^{n-1})$ *such that* $\frac{u\circ\phi_u}{\lambda_u} \in \tilde{\mathcal{A}}_{\tilde{M},\tilde{\theta},\varepsilon}$ *and the map* $w_u := \frac{u\circ\phi_u}{\lambda_u} - \mathrm{id}_{\mathbb{S}^{n-1}}$ *satisfies* $\Pi_n w_u \equiv 0$.

Proof. Given $u \in \mathcal{A}_{M,\theta,\varepsilon}$ we define the map $\Psi_u : \mathrm{Conf}(\mathbb{S}^{n-1}) \mapsto \mathbb{R}^{\frac{n(n+1)}{2}}$ as follows: For every $\phi \in \mathrm{Conf}(\mathbb{S}^{n-1})$

$$\begin{aligned} \Psi_u(\phi) :\ &= \left(\fint_{\mathbb{S}^{n-1}} (\mathrm{div}(u\circ\phi)_h(x))x \ d\mathcal{H}^{n-1}, \left(\partial_j (u\circ\phi)^i_h(0) - \partial_i (u\circ\phi)^j_h(0)\right)_{1\le i<j\le n}\right) \\ &= \left(\fint_{\mathbb{S}^{n-1}} (\mathrm{div}(u\circ\phi)_h(x))x \ d\mathcal{H}^{n-1}, \left(\fint_{\mathbb{S}^{n-1}} \left((u\circ\phi)^i x_j - (u\circ\phi)^j x_i\right) \ d\mathcal{H}^{n-1}\right)_{1\le i<j\le n}\right). \end{aligned}$$

According to Lemma (4.1) what we would like to show is that $0 \in \mathrm{Im}(\Psi_u)$. For convenience of notation we set $\Psi := \Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}$ and then clearly $\Psi(\mathrm{id}_{\mathbb{S}^{n-1}}) = 0$. In order to apply the Inverse Function Theorem we look at the differential $d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}} : T_{\mathrm{id}_{\mathbb{S}^{n-1}}}\mathrm{Conf}(\mathbb{S}^{n-1}) \mapsto \mathbb{R}^{\frac{n(n+1)}{2}}$, which is a non-degenerate linear map. Indeed, a standard calculation shows that

$$T_{\mathrm{id}_{\mathbb{S}^{n-1}}}\mathrm{Conf}(\mathbb{S}^{n-1}) \equiv \left\{Y(x) := Sx + \mu\big(\langle x,\xi\rangle x - \xi\big) : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n; S^t = -S,\ \xi \in \mathbb{S}^{n-1},\ \mu \in \mathbb{R}\right\}.$$

The differential of Ψ at the identity is easy to compute. Given $Y \in T_{\mathrm{id}_{\mathbb{S}^{n-1}}}\mathrm{Conf}(\mathbb{S}^{n-1})$ as above, by linearity of the operations involved we have

$$\begin{aligned} d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}(Y) :\ &= \frac{d}{dt}\Big|_{t=0} \Psi\left(\exp_{\mathrm{id}_{\mathbb{S}^{n-1}}}(tY)\right) \\ &= \left(\fint_{\mathbb{S}^{n-1}} \mathrm{div} Y_h(x) x \ d\mathcal{H}^{n-1},\ \left(\fint_{\mathbb{S}^{n-1}} \left(Y^i(x)x_j - Y^j(x)x_i\right)\ d\mathcal{H}^{n-1}\right)_{1\le i<j\le n}\right). \end{aligned}$$

Now for $Y(x) = Sx + \mu\big(\langle x,\xi\rangle x - \xi\big) : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ with $S^t = -S,\ \xi \in \mathbb{S}^{n-1},\ \mu \in \mathbb{R}$ it is clear that $Y_h(x) = Sx + \mu\left(\langle x,\xi\rangle x - \left(\frac{|x|^2+n-1}{n}\right)\xi\right)$ in $\overline{B}$ and therefore $\mathrm{div} Y_h(x) = \frac{(n+2)(n-1)}{n}\mu\langle x,\xi\rangle$. Thus

$$\fint_{\mathbb{S}^{n-1}} \mathrm{div} Y_h(x) x \ d\mathcal{H}^{n-1} = \frac{(n+2)(n-1)}{n^2}\mu\xi,$$

while for every $1 \le i < j \le n$

$$\fint_{\mathbb{S}^{n-1}} \left(Y^i(x)x_j - Y^j(x)x_i\right)\ d\mathcal{H}^{n-1} = \frac{2}{n}S_{ij}.$$

It is therefore immediate that $\ker(d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}) = \{0\}$, i.e. $d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}$ is a linear isomorphism between $T_{\mathrm{id}_{\mathbb{S}^{n-1}}}\mathrm{Conf}(\mathbb{S}^{n-1})$ and $\mathbb{R}^{\frac{n(n+1)}{2}}$. Since the exponential mapping $\exp_{\mathrm{id}_{\mathbb{S}^{n-1}}}(\cdot)$ is also a local diffeomorphism between a neighbourhood of 0 in $T_{\mathrm{id}_{\mathbb{S}^{n-1}}}\mathrm{Conf}(\mathbb{S}^{n-1})$ and a neighbourhood of $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $\mathrm{Conf}(\mathbb{S}^{n-1})$ we can use the Inverse Function Theorem to find an open neighbourhood $\mathcal{U}_0$ of $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $\mathrm{Conf}(\mathbb{S}^{n-1})$ such that the map $\Psi : \mathcal{U}_o \subseteq \mathrm{Conf}(\mathbb{S}^{n-1}) \mapsto \Psi(\mathcal{U}_0) \subseteq \mathbb{R}^{\frac{n(n+1)}{2}}$ is a C^1-diffeomorphism. In particular $\deg(\Psi; 0; \mathcal{U}_0) = 1$. As a next step we justify that Ψ is homotopic to Ψ_u in $\mathcal{U}_0$. Indeed, for every $\phi \in \mathcal{U}_0$ we can estimate

$$\begin{aligned} \left|\Psi_u(\phi) - \Psi(\phi)\right|^2 &= \sum_{k=1}^n \left(\fint_{\mathbb{S}^{n-1}} \mathrm{div}\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]_h x_k \ d\mathcal{H}^{n-1}\right)^2 \\ &\quad + \sum_{1\le i<j\le n} \left(\fint_{\mathbb{S}^{n-1}} \left(\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]^i x_j - \left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]^j x_i\right)\ d\mathcal{H}^{n-1}\right)^2 \\ &\lesssim_n \fint_{\mathbb{S}^{n-1}} \left(\mathrm{div}\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]_h\right)^2 d\mathcal{H}^{n-1} + \sum_{i\neq j}\fint_{\mathbb{S}^{n-1}} \left(\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]^i\right)^2 x_j^2 \\ &\overset{A.3}{\lesssim}_n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right]\right|^2 d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \left|(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi\right|^2 d\mathcal{H}^{n-1} \end{aligned}$$

Since all topologies in the finite dimensional Lie group $\mathrm{Conf}(\mathbb{S}^{n-1})$ are equivalent, with a compu-

tation we have already done in Remark 2.2 we get

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} \left|\nabla_T \left[(u-\mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi(x)\right]\right|^2 \, d\mathcal{H}^{n-1} &\lesssim_n \fint_{\mathbb{S}^{n-1}} |\nabla_T(u-\mathrm{id}_{\mathbb{S}^{n-1}})(y)|^2 \left|\nabla_T\phi(x)\right|^{3-n}\Big|_{x=\phi^{-1}(y)} \, d\mathcal{H}^{n-1}(y)\\ &\le C_1(\mathcal{U}_0,n)\fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \, d\mathcal{H}^{n-1}\\ &\le C_1(\mathcal{U}_0,n)\theta^2.\end{aligned}$$

where $C_1(\mathcal{U}_0,n)\sim_n \sup_{\phi\in\mathcal{U}_0}\inf_{x\in\mathbb{S}^{n-1}}\left|\nabla_T\phi(x)\right|^{3-n}>0$ and actually depends only on the dimension once we fix the neighbourhood $\mathcal{U}_0$ around $\mathrm{id}_{\mathbb{S}^{n-1}}$. Similarly,

$$\fint_{\mathbb{S}^{n-1}} |((u-\mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi)(x)|^2 \, d\mathcal{H}^{n-1}(x) \le C_2(\mathcal{U}_0,n)\fint_{\mathbb{S}^{n-1}} |u-\mathrm{id}_{\mathbb{S}^{n-1}}|^2 \, d\mathcal{H}^{n-1} \le C_2(\mathcal{U}_0,n)\theta^2$$

where $C_2(\mathcal{U}_0,n)\sim_n \sup_{\phi\in\mathcal{U}_0}\inf_{x\in\mathbb{S}^{n-1}}\left|\nabla_T\phi(x)\right|^{1-n}>0$. We therefore conclude that

$$\|\Psi_u-\Psi\|_{L^\infty(\mathcal{U}_0)}\le C(\mathcal{U}_0,n)\theta,$$

for a dimensional constant $C(\mathcal{U}_0,n)>0$. We can now continue as in Proposition 4.7 of [13]. We present the argument here to make the proof self-contained.

Let $(\Gamma_s)_{s\in[0,1]}$ be a foliation of $\mathcal{U}_0$ by "compact hypersurfaces" in the connected component of orientation preserving transformations of $\mathrm{Conf}(\mathbb{S}^{n-1})$ such that $\Gamma_0=\{\mathrm{id}_{\mathbb{S}^{n-1}}\}$ and Γ_1 is the topological boundary of $\mathcal{U}_0$. For every $s\in[0,1]$ we set $m(s):=\min_{\phi\in\Gamma_s}|\Psi(\phi)|$, which is a continuous function of s. Since $\Psi|_{\Gamma_0}\equiv 0$ and $\Psi|_{\mathcal{U}_0}$ is a homeomorphism onto its image we infer that

$$m(s)>0 \text{ for all } s\in(0,1] \quad\text{with}\quad \lim_{s\to 0^+} m(s)=0.$$

We can choose $\theta>0$ small so that $\big(C(\mathcal{U}_0,n)+1\big)\theta\le\frac{m(1)}{2}$ (notice that $m(1)>0$ depends only on n and $\mathcal{U}_0$) and then define

$$s_\theta:=\inf\left\{s\in[0,1]: m(s)\ge\big(C(\mathcal{U}_0,n)+1\big)\theta\right\}.$$

Clearly $\lim_{\theta\to 0^+}s_\theta=0$. Then for every $t\in[0,1]$ and $\phi\in\Gamma_{s_\theta}\subseteq\mathcal{U}_0\subseteq\mathrm{Conf}(\mathbb{S}^{n-1})$ we have

$$\begin{aligned}\left|\big((1-t)\Psi+t\Psi_u\big)(\phi)\right| &\ge |\Psi(\phi)|-t|(\Psi_u-\Psi)(\phi)| \ \ge\ \min_{\phi\in\Gamma_{s_\theta}}|\Psi(\phi)|-\|\Psi_u-\Psi\|_{L^\infty(\mathcal{U}_0)}\\ &\ge m_{s_\theta}-C(\mathcal{U}_0,n)\theta \ \ge\ \theta>0.\end{aligned}$$

In particular $\big((1-t)\Psi+t\Psi_u\big)(\phi)\neq 0$ for every $t\in[0,1]$ and $\phi\in\Gamma_{s_\theta}$. Since the degree around 0 remains constant through this linear homotopy, if $\mathcal{U}_{s_\theta}$ is the open neighbourhood around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $\mathrm{Conf}(\mathbb{S}^{n-1})$ such that $\partial\mathcal{U}_{s_\theta}=\Gamma_{s_\theta}$ then $\deg(\Psi_u,0;\ \mathcal{U}_{s_\theta})=\deg(\Psi,0;\ \mathcal{U}_{s_\theta})=1$. In other words, there exists $\phi_u\in\mathcal{U}_{s_\theta}\subseteq\mathrm{Conf}(\mathbb{S}^{n-1})$ such that $\Psi_u(\phi_u)=0 \iff \Pi_n(u\circ\phi_u)\equiv 0$. In the same fashion as have estimated before,

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} |u\circ\phi_u-\mathrm{id}_{\mathbb{S}^{n-1}}|^2 \, d\mathcal{H}^{n-1} &\le\ 2\left(\fint_{\mathbb{S}^{n-1}} |(u-\mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi_u|^2 \, d\mathcal{H}^{n-1}+\fint_{\mathbb{S}^{n-1}} |\phi_u-\mathrm{id}_{\mathbb{S}^{n-1}}|^2 \, d\mathcal{H}^{n-1}\right)\\ &\le 2\Big(C_2(\mathcal{U}_0,n)\theta^2+C_3(\mathcal{U}_0,n)\Big)\end{aligned}$$

and

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T(u\circ\phi_u) - P_T|^2 \; d\mathcal{H}^{n-1} \leq \; 2\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T(u - \mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi_u|^2 + \fint_{\mathbb{S}^{n-1}} |\nabla_T\phi_u - P_T|^2\right)$$
$$\leq 2\Big(C_1(\mathcal{U}_0,n)\theta^2 + C_3(\mathcal{U}_0,n)\Big).$$

Using again the fact that all topologies in the finite dimensional manifold $\mathrm{Conf}(\mathbb{S}^{n-1})$ are equivalent, the positive constant $C_3(\mathcal{U}_0,n)$ can be made arbitrarily small as the neighbourhood $\mathcal{U}_0$ is shrunk around the identity. We can therefore take θ and the neighbourhood $\mathcal{U}_0$ small enough so that $u\circ\phi_u \in \mathcal{A}_{M_1,\theta_1,\varepsilon}$, where M_1 is proportional to M (up to a constant that depends on the size of $\mathcal{U}_0$) and $\theta_1^2 := 2\Big(\max\{C_1(\mathcal{U}_0,n), C_2(\mathcal{U}_0,n)\}\theta^2 + C_3(\mathcal{U}_0,n)\Big)$ is small enough. We can then apply Lemma 2.3 to obtain that for $\lambda_u := \mathrm{Tr}[\nabla(u\circ\phi_u)_h(0)]$ the map $\frac{u\circ\phi_u}{\lambda_u}$ belongs to the set $\tilde{\mathcal{A}}_{\tilde{M},\tilde{\theta},\varepsilon}$, where again $\tilde{M}$ is proportional to M up to a dimensional constant, $\tilde{\theta}$ is obtained from θ_1 as in 2.3 and $w_u := \frac{u\circ\phi_u}{\lambda_u} - \mathrm{id}_{\mathbb{S}^{n-1}}$ has trivial projection on H_0. □

The proof of Theorem 1.6 now follows immediately.

Proof of Theorem 1.6. If $n = 3$, $0 < \theta \leq \theta_0$, $0 < \varepsilon \leq \varepsilon_0$ for some $\theta_0 > 0$, $\varepsilon_0 > 0$ to be defined in a moment and $u \in \mathcal{A}_{M,\theta,\varepsilon}$, the conformal invariance of the deficit allows us to pick w_u as in Proposition 4.2 so that by using Theorem 1.9 and Lemma 2.5 (with $\gamma = \frac{1}{4}$ for example) we have

$$C_3\fint_{\mathbb{S}^2} |\nabla_T w_u|^2 \; d\mathcal{H}^2 \leq Q_3(w_u) \leq \varepsilon + \Big(c_3\varepsilon_0 + C_2(1+\varepsilon_0)\sqrt{\theta_0} + C_1\theta_0^2\Big) \fint_{\mathbb{S}^2} |\nabla_T w_u|^2 \; d\mathcal{H}^2.$$

We can thus choose ε_0 sufficiently small (with respect to the absolute constant c_3) and then θ_0 sufficiently small (with respect to M, ε_0) to absorb the last term of the right hand side in the left hand side of the estimate and conclude. In this case, this could be done without assuming apriori that w_u is close to 0 in $W^{1,\infty}(\mathbb{S}^2)$.
If $n \geq 4$, the presence of the qubic term on the right hand side of the second estimate in Lemma 2.5 forces the extra condition in the statement, under which we can conclude as before. □

5 Connection with the Geometric Rigidity of $SO(n)$

In this section we discuss a slightly different way of obtaining Theorem 1.4 of Friesecke, James and Müller using the method we presented before. The original proof of the Theorem consists of several steps. Firstly, the corresponding interior estimate when U is the unit qube is established, namely

Theorem 5.1. *Let Q be the n-dimensional unit cube and Q' be the n-dimensional cube of half the side length (both centered at the origin). There exists a dimensional constant $C(n) > 0$ such that for every $u \in W^{1,2}(Q;\mathbb{R}^n)$ there exists an associated $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(Q')} \leq C(n)\,\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(Q)}\,. \tag{5.1}$$

Since the above estimate is scaling, rotation and translation invariant with respect to the domain, the authors in [5] use a covering argument combined with a weighted Poincare inequality to obtain the global estimate. In order to prove the interior estimate the first observation is that by the compactness of $SO(n)$ and a truncation argument one may restrict to the case of Lipschitz mappings with Lipschitz constant uniformely bounded by a dimensional constant. By a harmonic replacement argument one can restrict further to the case of proving the interior estimate for

harmonic mappings. In this framework Bochner's identity and standard regularity estimates for harmonic functions are used to obtain first a suboptimal result, in which the right hand side appears with the exponent $\frac{1}{2}$. Although being suboptimal in terms of scaling, this estimate still allows to linearize the $\mathrm{dist}(\cdot, SO(n))$ around I_n, to get an optimal estimate for the L^2-norm of the symmetric part of the gradient in terms of the right hand side of (5.1). Then classical Korn's inequality is used to control the L^2-norm of the skew-symmetric part of the gradient as well.

What we want to present here is an alternative way of obtaining the interior estimate in a ball, which despite being longer, provides a combination of some ideas in the original work of Friesecke, James and Müller (the truncation argument, harmonic replacement) and the technique we presented before. We will prove the interior estimate in the unit ball and then again the general case would follow from the covering argument as in Section 3 of the original paper [5].

Theorem 5.2. *Let B be the n-dimensional unit ball and B' be the n-dimensional ball of half the radius (both centered at the origin). There exists a dimensional constant $C(n) > 0$ such that for every $u \in W^{1,2}(B;\mathbb{R}^n)$ there exists an associated $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(B')} \leq C(n)\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(B)}. \tag{5.2}$$

We discuss the alternative approach in several steps. First of all given a map $u \in W^{1,2}(B;\mathbb{R}^n)$ let us set

$$\varepsilon^2 := \int_B \mathrm{dist}^2(\nabla u, SO(n))\ dx.$$

We may without loss of generality suppose that $\varepsilon > 0$ is sufficiently small, which will be specified later. As in [5] we may further suppose that u is Lipschitz with $\|\nabla u\|_{L^\infty(\overline{B})} \leq M_n$, where $M_n > 0$ is a dimensional constant. As before let $u_h : \overline{B} \mapsto \mathbb{R}^n$ be the harmonic replacement of u. It was proven in [5] that

$$\int_B |\nabla u - \nabla u_h|^2\ dx \lesssim_n \varepsilon^2$$

and therefore also

$$\int_B \mathrm{dist}^2(\nabla u_h, SO(n))\ dx \lesssim_n \varepsilon^2.$$

Hence it suffices to prove that there exists $R \in SO(n)$ such that

$$\int_{B'} |\nabla u_h - R|^2\ dx \lesssim_n \int_B \mathrm{dist}^2(\nabla u_h, SO(n))\ dx \lesssim_n \varepsilon^2.$$

We thus also turn our attention to harmonic mappings but follow a different route than that in [5]. First of all we observe that due to the compactness result by Reshetnyak (Theorem 1.3) it suffices to consider the case that the map in question is apriori close to a fixed rotation.

Lemma 5.3. *Without loss of generality we may assume that there exists $Q_0 \in SO(n)$ and also $r_0 \in \left[\frac{8}{10}, \frac{9}{10}\right]$ such that*

$$\int_{\mathbb{S}^{n-1}_{r_0}} |\nabla_T u_h - Q_0 P_T|^2\ d\mathcal{H}^{n-1} \leq \theta^2, \tag{5.3}$$

where $\theta := \theta(n) > 0$ is a sufficiently small constant to be specified later.

Proof. What we want to prove in Theorem 5.2 is essentially that there exists a dimensional constant $C > 0$ with the following property. For every harmonic mapping $u_h : \overline{B} \mapsto \mathbb{R}^n$ with boundary conditions $u_h|_{\mathbb{S}^{n-1}} \equiv u|_{\mathbb{S}^{n-1}}$, where $u : \overline{B} \mapsto \mathbb{R}^n$ is a Lipschitz map such that $\|\nabla u\|_{L^\infty(B)} \leq M_n$, one has

$$E(u_h) := \frac{\int_B \operatorname{dist}^2(\nabla u_h, SO(n))\ dx}{\min\limits_{Q\in SO(n)} \int_{B'} |\nabla u_h - Q|^2\ dx} \geq C > 0, \tag{5.4}$$

whenever the denominator is positive. Observe that for the denominator we always have the upper bound

$$\min_{Q\in SO(n)} \int_{B'} |\nabla u_h - Q|^2\ dx \quad \leq 2\int_B |\nabla u_h|^2\ dx + 2n|B'| \leq 2\int_B |\nabla u|^2\ dx + 2^{1-n} n\omega_n \leq c_n < \infty,$$

where $c_n := 2\omega_n(M_n^2 + n2^{-n})$. In particular, for such harmonic mappings we have the lower bound

$$E(u_h) \geq c_n^{-1} \int_B \operatorname{dist}^2(\nabla u_h, SO(n))\ dx.$$

Suppose that we have shown the estimate for every such u_h that additionally satisfies the assumption (5.3) and for the sake of contradiction suppose that there exists a sequence of harmonic mappings $(u_{h,k})_{k\in\mathbb{N}}$ as above, for which $\lim_{k\to\infty} E(u_{h,k}) = 0$. By the lower bound, this further implies that

$$\lim_{k\to\infty} \int_B \operatorname{dist}^2(\nabla u_{h,k}, SO(n))\ dx = 0.$$

By the compactness result of Reshetnyak (Theorem 1.3) there exists $Q_0 \in SO(n)$ such that (up to a non-relabeled subsequence) $\lim\limits_{k\to\infty} \int_B |\nabla u_{h,k} - Q_0|^2 dx = 0$. Therefore, given the θ of the condition (5.3) there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$\int_B |\nabla u_{h,k} - Q_0|^2\ dx \leq \frac{n}{10(n-1)}\theta^2.$$

In particular,

$$\int_{\frac{8}{10}}^{\frac{9}{10}} \int_{\mathbb{S}_r^{n-1}} |\nabla u_{h,k} - Q_0|^2\ d\mathcal{H}^{n-1}\ dr \leq \frac{n}{10(n-1)}\theta^2.$$

Thus, there must exist $r_0 \in \left[\frac{8}{10}, \frac{9}{10}\right]$ such that

$$\frac{n-1}{n} \int_{\mathbb{S}_{r_0}^{n-1}} |\nabla u_{h,k} - Q_0|^2\ d\mathcal{H}^{n-1} \leq \theta^2,$$

which according to A.3 implies that

$$\int_{\mathbb{S}_{r_0}^{n-1}} |\nabla_T u_{h,k} - Q_0 P_T|^2\ d\mathcal{H}^{n-1} \leq \theta^2.$$

But this would mean that $(u_{h,k})_{k\in\mathbb{N}}$ also satisfies the condition (5.3) for large k. By assumption we would then have that $\lim_{k\to\infty} E(u_{h,k}) \geq C > 0$, yielding the desired contradiction. □

With this in mind we give now a rigidity estimate on $\mathbb{S}^{n-1}$, which is similar in flavour to 3.12 and is the analogue on the sphere of the classical Korn's inequality. The main ingredient in its proof is again Korn's identity (3.11), but the estimate also holds true in the case $n = 2$ in this context.

Theorem 5.4. *Let $n \geq 2$. There exists a constant $B_n > 0$ such that for every $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$*

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u)_{\rm sym}\right|^2 \, d\mathcal{H}^{n-1} \geq B_n \fint_{\mathbb{S}^{n-1}} |\nabla_T u - (\nabla u_h(0))_{\rm skew} P_T|^2 \, d\mathcal{H}^{n-1}. \tag{5.5}$$

Proof. If $n \geq 3$, by Korn's identity (3.11) and the same eigenvalue decomposition as in Section 3 we obtain that whenever $u_{k,i} \in H_{k,i}$, with $k \geq 1$ and $i = 1, 2, 3$

$$\fint_{\mathbb{S}^{n-1}} \left|(P_T^t \nabla_T u_{k,i})_{\rm sym}\right|^2 \, d\mathcal{H}^{n-1} = B_{n,k,i} \fint_{\mathbb{S}^{n-1}} |\nabla_T u_{k,i}|^2 \, d\mathcal{H}^{n-1},$$

with $B_{n,k,i} := \frac{1}{2} + \alpha_{n,k,i} - \frac{\sigma_{k,i}^2}{2\lambda_k} - \frac{(n-2)}{n} c_{n,k,i}$ and in particular

$$\begin{cases} B_{n,k,1} = \frac{1}{2} + \frac{(n-2)(k+n)}{2(2k+n)(k+n-2)} \\ B_{n,k,2} = \frac{(k-1)(k+n-1)}{2k(k+n-2)} \\ B_{n,k,3} = \frac{(k^2-k+n-2)}{k(2k+n-4)}. \end{cases}$$

The estimate would then follow again as before with $B_n := \min\limits_{\substack{k\geq 1,\ i\in\{1,2,3\} \\ (k,i)\neq(1,2)}} B_{n,k,i} > 0$.

In dimension $n = 2$ the procedure is the same, with the only difference that the last formula for $B_{2,1,3}$ does not make sense. But in this case for the first order spherical harmonics actually we can compute directly that

$$\begin{aligned}
\fint_{\mathbb{S}^1} \left|(P_T^t \nabla u_h(0) P_T)_{\rm sym}\right|^2 &= \fint_{\mathbb{S}^1} \Big(\sum_{k,l=1}^{2} (\nabla u_h(0))_{kl} (P_T)_{k1} (P_T)_{l1} \Big)^2 \, d\mathcal{H}^1 \\
&= \fint_{\mathbb{S}^1} \Big(\sum_{k,l=1}^{2} (\nabla u_h(0))_{kl} \langle e_k, (-x_2, x_1)\rangle \langle e_l, (-x_2, x_1)\rangle \Big)^2 \, d\mathcal{H}^1 \\
&= \fint_{\mathbb{S}^1} \left(\partial_1 u_h^1(0) x_2^2 + \partial_2 u_h^2(0) x_1^2 - (\partial_1 u_h^2(0) + \partial_2 u_h^1(0)) x_1 x_2\right)^2 d\mathcal{H}^1 \\
&\geq \frac{1}{4}\left((\partial_1 u_h^1(0))^2 + (\partial_2 u_h^2(0))^2 + \partial_1 u_h^2(0)\partial_2 u_h^1(0)\right) + \frac{1}{8}\left((\partial_1 u_h^2(0))^2 + (\partial_2 u_h^1(0))^2\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
\fint_{\mathbb{S}^1} |(\nabla u_h(0))_{\rm sym} P_T|^2 &= \fint_{\mathbb{S}^1} \left(\partial_1 u_h^1(0) x_2 - \left(\frac{\partial_2 u_h^1(0) + \partial_1 u_h^2(0)}{2}\right) x_1\right)^2 \, d\mathcal{H}^1 \\
&\quad + \fint_{\mathbb{S}^1} \left(\left(\frac{\partial_1 u_h^2(0) + \partial_2 u_h^1(0)}{2}\right) x_2 - \partial_2 u_h^2(0) x_1\right)^2 \, d\mathcal{H}^1 \\
&= \frac{1}{2}\left((\partial_1 u_h^1(0))^2 + (\partial_2 u_h^2(0))^2 + \partial_1 u_h^2(0)\partial_2 u_h^1(0)\right) + \frac{1}{4}\left((\partial_1 u_h^2(0))^2 + (\partial_2 u_h^1(0))^2\right).
\end{aligned}$$

In particular

$$\fint_{\mathbb{S}^1} \left|(P_T^t \nabla u_h(0) P_T)_{\rm sym}\right|^2 \, d\mathcal{H}^1 \geq \frac{1}{2} \fint_{\mathbb{S}^1} |\nabla u_h(0) P_T - (\nabla u_h(0))_{\rm skew} P_T|^2 \, d\mathcal{H}^1.$$

For the higher order spherical harmonics the identities are the same as in the case $n \geq 3$ and since all these eigenspaces are orthogonal with respect to this quadratic form as well, the proof can be completed as before. □

Proposition 5.5. *Let $u_h : \overline{B} \mapsto \mathbb{R}^n$ be a mapping satisfying the following conditions:*

(*i*) $\Delta u_h = 0$ *in* B *and* $u_h|_{\mathbb{S}^{n-1}} \equiv u|_{\mathbb{S}^{n-1}}$ *where* $u : \overline{B} \mapsto \mathbb{R}^n$ *is a Lipschitz mapping with* $\|\nabla u\|_{L^\infty(\overline{B})} \leq M_n$,

(*ii*) $\int_B \operatorname{dist}^2(\nabla u_h, SO(n))\, dx \lesssim_n \varepsilon^2$ *for an* $\varepsilon > 0$ *sufficiently small,*

(*iii*) $\int_{\mathbb{S}^{n-1}_{r_0}} |\nabla_T u_h - Q_0 P_T|^2\, d\mathcal{H}^{n-1} \lesssim_n \theta^2$ *for some* $Q_0 \in SO(n)$, $r_0 \in \left[\frac{8}{10}, \frac{9}{10}\right]$ *and a sufficiently small* $\theta := \theta(n) > 0$,

Under these assumptions, there exists $R_0 \in SO(n)$ *such that* $\int_{B'} |\nabla u_h - R_0|^2\, dx \lesssim_n \varepsilon^2$.

Proof. Without loss of generality we can suppose that $u_h(0) = 0$. By assumption (*ii*) we have that

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \int_{\mathbb{S}^{n-1}_r} \operatorname{dist}^2(\nabla u_h, SO(n))\, d\mathcal{H}^{n-1}\, dr \lesssim_n \varepsilon^2$$

and therefore there exists a $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$ for which

$$\int_{\mathbb{S}^{n-1}_r} \operatorname{dist}^2(\nabla u_h, SO(n))\, d\mathcal{H}^{n-1} \lesssim_n \varepsilon^2. \tag{5.6}$$

The mean value property of harmonic functions and assumption (*iii*) implies that for every $y \in \overline{B}_{\frac{3}{4}}$

$$\begin{aligned} |\nabla u_h(y) - Q_0| &\leq \fint_{B\left(y, \operatorname{dist}\left(y,\, \partial B_{\frac{4}{5}}\right)\right)} |\nabla u_h - Q_0|\, dx \lesssim_n \left(\fint_{B_{\frac{4}{5}}} |\nabla u_h - Q_0|^2\, dx\right)^{\frac{1}{2}} \\ &\lesssim_n \left(\int_{\mathbb{S}^{n-1}_{r_0}} |\nabla_T u_h - Q_0 P_T|^2\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \lesssim_n \theta. \end{aligned}$$

That is $\|\nabla u_h - Q_0\|_{L^\infty\left(\overline{B}_{\frac{3}{4}}\right)} \lesssim_n \theta \ll 1$, provided that θ is chosen sufficiently small. Since the determinant is a Lipschitz function we also have that $\|\det \nabla u_h - 1\|_{L^\infty\left(\overline{B}_{\frac{3}{4}}\right)} \lesssim_n \theta \ll 1$ and in particular $\det \nabla u_h > 0$ in $\overline{B}_{\frac{3}{4}}$.
We also remark that if $K \subseteq \mathbb{R}^{n\times n}$ with $\operatorname{diam} K \leq C_n < \infty$ then for every $A \in K$ all the quantities

$$|A^t A - I_n|,\ |\sqrt{A^t A} - I_n|,\ |\det A - I_n|,\ \left|\frac{|A|^2}{n} - 1\right| \lesssim_n \operatorname{dist}(A, SO(n)).$$

By polar decomposition, whenever $\det A > 0$, the distance of A to the special orthogonal group is given by $\operatorname{dist}(A, SO(n)) = |\sqrt{A^t A} - I_n|$ and therefore in our case we can write $\nabla u_h(0) = R_0 A_0$, where $R_0 \in SO(n)$ and $A_0 := \sqrt{\nabla u_h(0)^t \nabla u_h(0)}$. Notice that

$$\begin{aligned} \operatorname{dist}^2(\nabla u_h(0), SO(n)) &\leq |\nabla u_h(0) - Q_0|^2 \leq \fint_{\mathbb{S}^{n-1}_{r_0}} |\nabla u_h - Q_0|^2\, d\mathcal{H}^{n-1} \\ &\lesssim_n \int_{\mathbb{S}^{n-1}_{r_0}} |\nabla_T u_h - Q_0 P_T|^2\, d\mathcal{H}^{n-1} \lesssim_n \theta^2. \end{aligned}$$

Consider now the map $\tilde{u} := R_0^t u : \overline{B} \mapsto \mathbb{R}^n$ and set $w := \tilde{u} - \mathrm{id}$. With the same reasoning as above $\|\nabla w_h\|_{L^\infty\left(\overline{B}_{\frac{3}{4}}\right)} \lesssim_n \theta$. The following estimate is important for the completion of the proof.

$$\fint_{\mathbb{S}_r^{n-1}} \left|P_T^t \nabla_T w_h + (P_T^t \nabla_T w_h)^t\right|^2 \, d\mathcal{H}^{n-1} \lesssim_n \varepsilon^2 + \theta^2 \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \, d\mathcal{H}^{n-1}. \tag{5.7}$$

This follows from the pointwise identity $\nabla_T \tilde{u}_h^t \nabla_T \tilde{u}_h - I_x = P_T^t \nabla_T w_h + (P_T^t \nabla_T w_h)^t + \nabla_T w_h^t \nabla_T w_h$ on $\mathbb{S}_r^{n-1}$, since

$$\begin{aligned}
\fint_{\mathbb{S}_r^{n-1}} \left|P_T^t \nabla_T w_h + (P_T^t \nabla_T w_h)^t\right|^2 &\lesssim_n \fint_{\mathbb{S}_r^{n-1}} \left|\nabla_T \tilde{u}_h^t \nabla_T \tilde{u}_h - I_x\right|^2 d\mathcal{H}^{n-1} + \fint_{\mathbb{S}_r^{n-1}} \left|\nabla_T w_h^t \nabla_T w_h\right|^2 \, d\mathcal{H}^{n-1} \\
&\lesssim_n \fint_{\mathbb{S}_r^{n-1}} \left(\left|\nabla \tilde{u}_h^t \nabla \tilde{u}_h - I_n\right|^2 + \|\nabla_T w_h\|^2_{L^\infty(\mathbb{S}_r^{n-1})} |\nabla_T w_h|^2\right) \, d\mathcal{H}^{n-1} \\
&\lesssim_n \fint_{\mathbb{S}_r^{n-1}} \mathrm{dist}^2(\nabla \tilde{u}_h, SO(n)) \, d\mathcal{H}^{n-1} + \|\nabla w_h\|^2_{L^\infty\left(\overline{B}_{\frac{3}{4}}\right)} \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \\
&\lesssim_n \varepsilon^2 + \theta^2 \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Here we have used the pointwise estimate $\left|\nabla_T \tilde{u}_h^t \nabla_T \tilde{u}_h - I_x\right|^2 \lesssim_n \left|\nabla \tilde{u}_h^t \nabla \tilde{u}_h - I_n\right|^2$ on $\mathbb{S}_r^{n-1}$. Indeed,

$$\begin{aligned}
\left|\nabla_T \tilde{u}_h^t \nabla_T \tilde{u}_h - I_x\right|^2 &= \sum_{i,j=1}^{n-1} \left(\sum_{k=1}^{n} \langle \nabla_T \tilde{u}_h^k, \tau_i\rangle \langle \nabla_T \tilde{u}_h^k, \tau_j\rangle - \delta_{ij}\right)^2 \\
&= \sum_{i,j=1}^{n-1} \left(\sum_{k=1}^{n} \langle \nabla \tilde{u}_h^k, \tau_i\rangle \langle \nabla \tilde{u}_h^k, \tau_j\rangle - \delta_{ij}\right)^2 \\
&= \sum_{i,j=1}^{n-1} \left(\sum_{l,m=1}^{n} (\langle \partial_l \tilde{u}_h, \partial_m \tilde{u}_h\rangle - \delta_{ml}) \langle \tau_i, e_l\rangle \langle \tau_j, e_m\rangle + \sum_{l=1}^{n} \langle \tau_i, e_l\rangle \langle \tau_j, e_l\rangle - \delta_{ij}\right)^2 \\
&= \sum_{i,j=1}^{n-1} \left(\sum_{l,m=1}^{n} \left(\nabla \tilde{u}_h^t \nabla \tilde{u}_h - I_n\right)_{lm} \langle \tau_i, e_l\rangle \langle \tau_j, e_m\rangle\right)^2 \\
&\leq \sum_{i,j=1}^{n-1} \left(\sum_{l,m=1}^{n} \left(\left(\nabla \tilde{u}_h^t \nabla \tilde{u}_h - I_n\right)_{lm}\right)^2\right) \left(\sum_{l,m=1}^{n} \langle \tau_i, e_l\rangle^2 \langle \tau_j, e_m\rangle^2\right) \\
&= (n-1)^2 \left|\nabla \tilde{u}_h^t \nabla \tilde{u}_h - I_n\right|^2.
\end{aligned}$$

Since $\nabla w_h(0) = A_0 - I_n \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we can now couple (5.5) and (5.7) to obtain

$$4B_n \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \, d\mathcal{H}^{n-1} \leq \fint_{\mathbb{S}_r^{n-1}} \left|P_T^t \nabla_T w_h + (P_T^t \nabla_T w_h)^t\right|^2 \lesssim_n \varepsilon^2 + \theta^2 \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \, d\mathcal{H}^{n-1}.$$

If we choose $\theta > 0$ sufficiently small depending on the dimension (for example $\theta = \sqrt{3B_n} > 0$) by A.3 we finally obtain

$$\int_{B'} |\nabla u_h - R_0|^2 \, d\mathcal{H}^{n-1} \leq \fint_{\mathbb{S}_r^{n-1}} |\nabla_T u_h - R_0 P_T|^2 \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}_r^{n-1}} |\nabla_T w_h|^2 \, d\mathcal{H}^{n-1} \lesssim_n \varepsilon^2. \qquad \square$$

The proof of Theorem 5.2 now follows immediately as a combination of Lemma 5.3 and Proposition 5.5. A similar argument using Reshetnyak's compactness result and classical Korn's inequality on an intermediate ball could also yield the result, but we wanted to stress out the fact that in the end this can be casted as a rigidity property on an intermediate sphere only.

Appendices

A Spherical Harmonics

It is well known that the Hilbert space $L^2(\mathbb{S}^{n-1})$ admits an orthonormal basis consisting of eigenfunctions of the Laplace-Beltrami operator. In particular, for every $k \in \mathbb{N}$ there exists a finite number (denoted by $G(n,k)$) of linearly independent L^2-functions $(\psi_{k,j})_{j=1,2,...,G(n,k)}$ with the property that

$$\int_{\mathbb{S}^{n-1}} \psi_{k,j}\psi_{k',j'}\, d\mathcal{H}^{n-1} = \delta^{kk'}\delta^{jj'} \text{ for every } k,k' \in \mathbb{N},\ j = 1,2,...,G(n,k),\ j' = 1,2,...,G(n,k').$$

The functions $(\psi_{k,j})_{j=1,2,...,G(n,k)}$ are called k-th order spherical harmonics and are restrictions on $\mathbb{S}^{n-1}$ of homogeneous harmonic polynomials in $\mathbb{R}^n$ of degree k respectively. As already mentioned they are eigenfunctions of the Laplace-Beltrami operator so that for $k \in \mathbb{N}$ and $j = 1,2,...,G(n,k)$

$$-\Delta_{\mathbb{S}^{n-1}}\psi_{k,j} = \lambda_k\psi_{k,j}, \text{ where } \lambda_k := k(k+n-2). \tag{A.1}$$

In distributional formulation this eigenvalue equation can of course be rewritten as

$$\int_{\mathbb{S}^{n-1}} \langle \nabla_T\psi_{k,j}, \nabla_T\phi\rangle\, d\mathcal{H}^{n-1} = \lambda_k \int_{\mathbb{S}^{n-1}} \psi_{k,j}\phi\, d\mathcal{H}^{n-1} \text{ for every } \phi \in W^{1,2}(\mathbb{S}^{n-1}). \tag{A.2}$$

The dimension of each eigenspace is actually precisely known to be $G(n,0) = 1,\ G(n,1) = n$ and for $k \geq 2$ it is $G(n,k) = \binom{n+k-1}{k}$-$\binom{n+k-3}{k-2}$. The reader can refer to [16] for more information on spherical harmonics.

Remark A.1. For every vector field $u := (u^1, u^2, ..., u^n) \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ we have a formal expansion of each one of its components into a Fourier series as

$$u^i = \sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)} a^i_{k,j}\psi_{k,j}, \text{ where } a^i_{k,j} := \int_{\mathbb{S}^{n-1}} u^i\psi_{k,j}\, d\mathcal{H}^{n-1} \ \ \forall\, i = 1,...,n,\ k \in \mathbb{N},\ j = 1,...,G(n,k).$$

Let $P_{k,j}$ denote the k-th order homogeneous harmonic polynomial in $\mathbb{R}^n$ whose restriction on $\mathbb{S}^{n-1}$ is exactly $\psi_{k,j}$. In polar coordinates $(r,\theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ one can write $P_{k,j}(r,\theta) = r^k\psi_{k,j}(\theta)$. For each $i = 1,...,n$ the harmonic extension u^i_h has the same power series expansion in the interior of the ball, namely

$$u^i_h = \sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)} a^i_{k,j}P_{k,j} \text{ in } B.$$

If the vector field u has zero mean then $u^i_h(0) = \fint_{\mathbb{S}^{n-1}} u^i\, d\mathcal{H}^{n-1} = 0$ for every $i = 1,2,...,n$. In view of the homogeneity of the polynomials $P_{k,j}$ this is equivalent to $a^i_0 = 0$ for all $i = 1,2,...,n$. Another immediate but useful observation is that the linear part of u is given by the linear map

$x \mapsto \nabla u_h(0)x$, which is based on the fact that $\psi_{1,j}(\theta) = \frac{\theta_j}{\sqrt{\omega_n}}$.

The following Parseval identities hold true: If $\phi \in W^{1,2}(\mathbb{S}^{n-1})$ with its Fourier expansion in spherical harmonics being $\phi = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} a_{k,j}\psi_{k,j}$ then

$$\int_{\mathbb{S}^{n-1}} |\phi|^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} (a_{k,j})^2 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} |\nabla_T \phi|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \lambda_k (a_{k,j})^2. \tag{A.3}$$

In particular for every $k \geq 0$ and every $j = 1, 2, ..., G(n,k)$ we have the identity

$$\int_{\mathbb{S}^{n-1}} |\nabla_T \psi_{k,j}|^2 \, d\mathcal{H}^{n-1} = \lambda_k \int_{\mathbb{S}^{n-1}} |\psi_{k,j}|^2 \, d\mathcal{H}^{n-1}. \tag{A.4}$$

Remark A.2. The *sharp Poincare inequality* for functions $f \in W^{1,2}(\mathbb{S}^{n-1})$ is then easily deduced. Let $f = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} f_{k,j}\psi_{k,j}$. Since $\lambda_k \geq n-1$ for every $k \geq 1$, we obtain

$$\int_{\mathbb{S}^{n-1}} |\nabla_T f|^2 \, d\mathcal{H}^{n-1} \geq (n-1) \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} (f_{k,j})^2 = (n-1) \int_{\mathbb{S}^{n-1}} \left| f - \fint_{\mathbb{S}^{n-1}} f \right|^2 \, d\mathcal{H}^{n-1}. \tag{A.5}$$

Of course, depending on the number of vanishing first Fourier modes in the expansion of f, the constant in the above inequality can be improved in an obvious way. The same Poincare inequality holds true obviously also for vector-valued maps $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^m)$.

By expanding a function in spherical harmonics one can often obtain useful estimates. In the next Lemma we mention two of them that we have used earlier.

Lemma A.3. *If $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^m)$ and $u_h : \overline{B} \mapsto \mathbb{R}^m$ is its harmonic extension, the following estimates hold true:*

(i) $\fint_B |\nabla u_h|^2 \, dx \leq \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1}$,

(ii) $\frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} \leq \fint_{\mathbb{S}^{n-1}} |\nabla u_h|^2 \, d\mathcal{H}^{n-1} \leq 2 \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1}$.

Proof. We give the proof of the Lemma in the case that u is scalar-valued, since the case of vector-valued u would then follow immediately. Let us write again $u = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} a_{k,j}\psi_{k,j}$ and its harmonic extension in polar coordinates as $u_h(r,\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} r^k a_{k,j}\psi_{k,j}(\theta)$. For the first estimate

$$\begin{aligned}
\fint_B |\nabla u_h|^2 \, dx &= \fint_B \operatorname{div}(u_h \nabla u_h) \, dx = n \fint_{\mathbb{S}^{n-1}} u \, \partial_{\vec{\nu}} u_h \, d\mathcal{H}^{n-1} \\
&= \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} nk (a_{k,j})^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \frac{n\lambda_k}{k+n-2} (a_{k,j})^2 \\
&\leq \frac{n}{n-1} \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \lambda_k (a_{k,j})^2 = \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1},
\end{aligned}$$

while for the second one we can again write

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} |\nabla u_h|^2 \, d\mathcal{H}^{n-1} &= \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} |\partial_{\vec{\nu}} u_h|^2 \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} k^2 (a_{k,j})^2 \\ &= \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \frac{k}{k+n-2} \lambda_k (a_{k,j})^2\end{aligned}$$

and since $\frac{1}{n-1} \leq \frac{k}{k+n-2} \leq 1$ for every $k \geq 1$ we obtain the desired inequalities. □

B Proof of Proposition 1.7 and of Korn's identity

Let $u \in \tilde{\mathcal{A}}_{M,\tilde{\theta},\varepsilon}$ and as always let $w := u - \mathrm{id}_{\mathbb{S}^{n-1}}$. In that set of maps by the divergence theorem

$$\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = (n-1) \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} = 0.$$

The geometric quantities appearing in the deficit can be Taylor-expanded in terms of w as follows:

The "$(n-1)-$ *Dirichlet energy*" can be expanded as

$$\begin{aligned}D_n(u) := &\left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \\ = &\left(1 + \fint_{\mathbb{S}^{n-1}} \frac{1}{2} |\nabla_T w|^2 + \frac{1}{2} \frac{n-3}{n-1} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} + \fint_{\mathbb{S}^{n-1}} R_1(\nabla_T w) \, d\mathcal{H}^{n-1} \\ = &\, 1 + \frac{1}{2} \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right) \, d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} R_1(\nabla_T w) \, d\mathcal{H}^{n-1}.\end{aligned}$$

The quadratic term appearing in the expansion of $D_n(u)$ around the identity is

$$Q_{D_n}(w) := \frac{1}{2} \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right) d\mathcal{H}^{n-1}, \tag{B.1}$$

By the definition of the set $\tilde{\mathcal{A}}_{M,\tilde{\theta},\varepsilon}$ we have that $\|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \leq M + \sqrt{n-1}$. The remainder term has therefore qubic growth, that is $\fint_{\mathbb{S}^{n-1}} |R_1(\nabla_T w)| \, d\mathcal{H}^{n-1} \leq C_1 \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^3 \, d\mathcal{H}^{n-1}$ for a constant $C_1 := C_1(n, M) > 0$.

Regarding the "*perimeter − term*", we have that

$$\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \sqrt{\det(I_x + A)} \, d\mathcal{H}^{n-1},$$

where $A := P_T^t \nabla_T w + (P_T^t \nabla_T w)^t + \nabla_T w^t \nabla_T w$. The Taylor expansion of the determinant around the identity matrix gives

$$\det(I + A) = 1 + \mathrm{Tr} A + \frac{1}{2} \Big((\mathrm{Tr} A)^2 - \mathrm{Tr}(A^2) \Big) + \mathcal{O}(|A|^3)$$

and since in our case,

(a) $\mathrm{Tr}A = 2\mathrm{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2$

(b) $(\mathrm{Tr}A)^2 = 4(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 + \mathcal{O}(|\nabla_T w|^3)$

(c) $\mathrm{Tr}(A^2) = |P_T^t \nabla_T w + (P_T^t \nabla_T w)^t|^2 + \mathcal{O}(|\nabla_T w|^3)$

we obtain the formal expansion

$$\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \sqrt{1 + \Theta(w) + \mathcal{O}(|\nabla_T w|^3)}\, d\mathcal{H}^{n-1},$$

where

$$\Theta(w) := 2\mathrm{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2 + 2(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 - 2\left|\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2}\right|^2.$$

Since $(\Theta(w))^2 = 4(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 + \mathcal{O}(|\nabla_T w|^3)$ we can perform a Taylor expansion of the square root inside the integral to get

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\, d\mathcal{H}^{n-1} &= \fint_{\mathbb{S}^{n-1}} \left(1 + \frac{1}{2}\Theta(w) - \frac{1}{8}(\Theta(w))^2 + \mathcal{O}\left(|\nabla_T w|^3\right)\right) d\mathcal{H}^{n-1} \\
&= 1 + \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 - 2|D|^2\right) d\mathcal{H}^{n-1} \\
&\quad + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right) d\mathcal{H}^{n-1},
\end{aligned}$$

where D stands for the trace-free matrix field $D := \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} w}{n-1} I_x$. A final Taylor expansion of the function $t \mapsto t^{\frac{n}{n-1}}$ gives

$$\begin{aligned}
P_n(u) :&= \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}} \\
&= 1 + \frac{1}{2}\,\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 - 2|D|^2\right) d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right).
\end{aligned}$$

The quadratic term appearing in the expansion of $P_n(u)$ around the identity is therefore

$$Q_{P_n}(w) := \frac{1}{2}\,\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 - 2|D|^2\right) d\mathcal{H}^{n-1}, \tag{B.2}$$

with D being defined as above.

Regarding the "*volume − term*" the computations are again straightforward, but a bit more lengthy. In particular,

$$\begin{aligned}
V_n(u) :&= \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1} \;=\; \fint_{\mathbb{S}^{n-1}} \left\langle w + x, \bigwedge_{i=1}^{n-1} (\partial_{\tau_i} w + \partial_{\tau_i} x) \right\rangle d\mathcal{H}^{n-1} \\
&= \fint_{\mathbb{S}^{n-1}} \sum_{k=0}^{n-1} \sum_{|\alpha|=k} \sigma(\alpha, \bar{\alpha}) \left\langle w + x, \left(\bigwedge_{\alpha} \partial_{\tau_\alpha} w\right) \wedge \left(\bigwedge_{\bar{\alpha}} \partial_{\tau_{\bar{\alpha}}} x\right) \right\rangle d\mathcal{H}^{n-1} \\
&= I_0(w) + I_1(w) + I_2(w) + I_3(w).
\end{aligned}$$

Here we have used multiindex notation. For every $k \in \{0, 1, ..., n-1\}$ and for every multiindex $\alpha := (\alpha_1, ..., \alpha_k)$, where $(a_i)_{i=1}^k \in \mathbb{N}$ such that $1 \leq \alpha_1 < ... < \alpha_k \leq n-1$ we use the notation $\bar{\alpha}$ for its complementary multiindex (with its entries also in increasing order), $\sigma(\alpha, \bar{\alpha})$ denotes the sign of the permutation that maps $(\alpha, \bar{\alpha})$ to the standard ordering $(1, ..., n)$ and $\partial_{\tau_\alpha} w := \partial_{\tau_{\alpha_1}} w \wedge ... \wedge \partial_{\tau_{\alpha_k}} w$. We have also denoted by $(I_i(w))_{i=0,1,2}$ the zeroth, first and second order terms with respect to w and $\nabla_T w$ in the expansion of $V_n(u)$ around the identity respectively, and by $I_3(w)$ the remaining term which is a "polynomial" of order at least 3 and at most n in w and its first derivatives. Keeping in mind that $\partial_{\tau_i} x = \tau_i$ for each $i = 1, ..., n-1$ and that by an abuse of notation, $\tau_1 \wedge \tau_2 \wedge ... \wedge \tau_{n-1} \equiv x$, we can compute each term separately.

$$I_0(w) : = \fint_{\mathbb{S}^{n-1}} \langle x, \partial_{\tau_1} x \wedge ... \wedge \partial_{\tau_{n-1}} x \rangle d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} |x|^2 \, d\mathcal{H}^{n-1} = 1.$$

$$\begin{aligned} I_1(w) : &= \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} + \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle x, \left(\bigwedge_{l=1}^{i-1} \tau_l \right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{m=i+1}^{n-1} \tau_m \right) \right\rangle \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} \sum_{i=1}^{n} \langle \partial_{\tau_i} w, \tau_i \rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0. \end{aligned}$$

For the quadratic term we observe that we can write it as $I_2(w) := I_{2,1}(w) + I_{2,2}(w)$, where

$$\begin{aligned} I_{2,1}(w) : &= \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle w, \left(\bigwedge_{l=1}^{i-1} \partial_{\tau_l} x \right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{m=i+1}^{n-1} \partial_{\tau_m} x \right) \right\rangle \, d\mathcal{H}^{n-1} \\ &= \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle w, \left(\bigwedge_{l=1}^{i-1} \tau_l \right) \wedge \left(\sum_{j=1}^{n-1} \langle \partial_{\tau_i} w, \tau_j \rangle \tau_j + \langle \partial_{\tau_i} w, x \rangle x \right) \wedge \left(\bigwedge_{m=i+1}^{n-1} \tau_m \right) \right\rangle \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x \rangle \, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} \sum_{i=1}^{n-1} \langle w, \tau_i \rangle \langle \partial_{\tau_i} w, x \rangle \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^{n-1}. \end{aligned}$$

The change of sign in the one before the last equality is due to orientation reasons, since we have taken the local orthonormal basis $\{\tau_1, ..., \tau_{n-1}\}$ of $T_x \mathbb{S}^{n-1}$ in such a way that at every $x \in \mathbb{S}^{n-1}$, $\{\tau_1(x), ..., \tau_{n-1}(x), x\}$ is a positively oriented frame of $\mathbb{R}^n$. Moreover,

$$\begin{aligned} I_{2,2}(w) : &= \fint_{\mathbb{S}^{n-1}} \sum_{1 \leq i < j \leq n-1} \left\langle x, \left(\bigwedge_{k=1}^{i-1} \partial_{\tau_k} x \right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{l=i+1}^{j-1} \partial_{\tau_l} x \right) \wedge \partial_{\tau_j} w \wedge \left(\bigwedge_{m=j+1}^{n-1} \partial_{\tau_m} x \right) \right\rangle \\ &= \frac{1}{2} \fint_{\mathbb{S}^{n-1}} \sum_{1 \leq i,j \leq n-1} \left(\langle \partial_{\tau_i} w, \tau_i \rangle \langle \partial_{\tau_j} w, \tau_j \rangle - \langle \partial_{\tau_i} w, \tau_j \rangle \langle \partial_{\tau_j} w, \tau_i \rangle \right) \, d\mathcal{H}^{n-1}. \end{aligned}$$

After integrating by parts it is easy to see that the first term is

$$\fint_{\mathbb{S}^{n-1}} \sum_{1 \leq i,j \leq n-1} \langle \partial_{\tau_i} w, \tau_i \rangle \langle \partial_{\tau_j} w, \tau_j \rangle d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \langle w, (n-1)(\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \nabla_T \mathrm{div}_{\mathbb{S}^{n-1}} w \rangle \, d\mathcal{H}^{n-1},$$

while

$$\fint_{\mathbb{S}^{n-1}} \sum_{1\le i,j\le n-1} \langle \partial_{\tau_i} w, \tau_j\rangle \langle \partial_{\tau_j} w, \tau_i\rangle d\mathcal{H}^{n-1} \;=\; \fint_{\mathbb{S}^{n-1}} \langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \nabla_T \mathrm{div}_{\mathbb{S}^{n-1}} w + (n-2)\sum_{j=1}^n x_j \nabla_T w^j\rangle$$

and subtracting these two identities we arrive at

$$I_{2,2}(w) := \left(\frac{n}{2}-1\right)\fint_{\mathbb{S}^{n-1}} \left\langle w,\ (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^n x_j \nabla_T w^j\right\rangle\, d\mathcal{H}^{n-1}.$$

Finally,

$$\begin{aligned} Q_{V_n}(w) := I_2(w) \;&= \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^n x_j \nabla_T w^j\right\rangle d\mathcal{H}^{n-1} \\ &= \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left(2\ \mathrm{div}_{\mathbb{S}^{n-1}} w\langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2\right) d\mathcal{H}^{n-1} \\ &= \frac{1}{2}\fint_B \left((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2\right) dx. \end{aligned}$$

The identity between the first and the second line above follows from a simple integration by parts. The one between the second and the third line can also be checked by a straightforward calculation using Stokes' theorem or by observing that $\frac{1}{2}\fint_B \left((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2\right) dx$ is the quadratic term appearing in the Taylor expansion of $\fint_B \det(I_n + \nabla w_h)\ dx$ around I_n and using the fact that the determinant is a null Lagrangian.

Exploiting the same fact or just following the same procedure we followed to calculate $I_2(w)$, we can easily see that $I_3(w) = \sum_{k=3}^n I_{3,k}(w)$, where for every $k = 3, ..., n$, the algebraic structure of the k-th summand in the remainder term is

$$I_{3,k}(w) = \fint_{\mathbb{S}^{n-1}} \langle w, A_k(w)\rangle\ d\mathcal{H}^{n-1},$$

where A_k is a nonlinear first order differential operator that is a "homogeneous polynomial" of order $k-1$ in the first derivatives of w. The precise structure of the remaining term is not of great importance, since this observation is enough to infer that $I_3(w) = \fint_{\mathbb{S}^{n-1}} R_{2,n}(w, \nabla_T w)\ d\mathcal{H}^{n-1}$, where

$$|R_2(w, \nabla_T w)| \le C_2 |w||\nabla_T w|^2, \text{ whenever } \|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \le M + \sqrt{n-1},$$

with the constant $C_2 > 0$ depending only on n, M.

Proof of Proposition 3.4. With the notation we introduced before, we can easily see that

$$\begin{aligned} \fint_{\mathbb{S}^{n-1}} \mathrm{Tr}\big((P_T^t \nabla_T w)^2\big)\ d\mathcal{H}^{n-1} \;&= \sum_{i,j=1,\ k,l=1}^{n-1,n} \fint_{\mathbb{S}^{n-1}} \langle e_k, \tau_i\rangle\langle e_l, \tau_j\rangle\langle \nabla_T w^k, \tau_j\rangle\langle \nabla_T w^l, \tau_i\rangle\ d\mathcal{H}^{n-1} \\ &= \sum_{i,j=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \langle \partial_{\tau_i} w, \tau_j\rangle\langle \partial_{\tau_j} w, \tau_i\rangle\ d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\ d\mathcal{H}^{n-1} - 2I_{2,2}(w) \\ &= \fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\ d\mathcal{H}^{n-1} - \frac{2(n-2)}{n} Q_{V_n}(w), \end{aligned}$$

from where the desired identity follows . □

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