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by

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Abstract We study the average skew information-based coherence for both random pure and mixed states. The explicit formulae of the average skew information-based coherence are derived and shown to be the functions of the dimension N of the state space. We demonstrate that as N approaches to infinity, the average coherence is 1 for random pure states, and a positive constant less than $\frac{1}{2}$ for random mixed states. We also explore the typicality of average skew information-based coherence of random quantum states. Furthermore, we identify a coherent subspace such that the amount of the skew information based coherence for each pure state in this subspace can be bounded from below almost always by a fixed number that is arbitrarily close to the typical value of coherence.

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Key Words: Average coherence; skew information; random quantum states; typicality

1. Introduction

Quantum coherence is a fundamental issue in quantum mechanics. The rigorous framework for quantifying quantum coherence had been proposed not long ago [1], which attracted much attention during the past few years. Many distance based coherence measures and information related quantities, such as relative entropy, l_1 norm, max-relative entropy, modified trace distance, fidelity, skew information, affinity, generalized α-z-relative Rényi entropy and logarithmic coherence number, have been exploited to quantify quantum coherence [1]-[15]. Quantum coherence from other resource-theoretical perspectives, such as coherence distillation and coherence dilution [16]-[23], no-broadcasting of quantum coherence [37]-[25], certification of quantum memories [26] and quantum coherence among nondegenerate energy subspaces (CANES) [27], interconversion between

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quantum coherence and quantum entanglement or quantum correlations [28]-[35] have also been studied.

On the other hand, random pure quantum states provide new perspectives for various phenomena in quantum physics and quantum information processing by exploiting probability theory and random matrix theory [36]. These states possess many important properties including the concentration of measure phenomenon or typicality [37], which allow one to get more information on the structures of the quantum system [36]-[39]. The entanglement properties of pure bipartite quantum states sampled from the uniform Haar measure have been studied extensively [38]-[49], among which the average entropy of a subsystem has been investigated [40]-[42]. New analytical formulae describing the levels of entanglement expected in random pure states has been given in [50].

Average coherence based on relative entropy of coherence and its typicality for random pure states and random mixed states were derived in [51, 52], and average subentropy, coherence and entanglement of random mixed quantum states were discussed in [53]. The average of uncertainty product for bounded observables has been also investigated [54].

With respect to the basis-dependent coherence [1], the average coherence with respect to mutually unbiased bases has been discussed [55]-[56]. The coherence-generating power of quantum channels has been also calculated [57]-[61].

In this paper, by using integral theory over unitary groups, we present explicit formulae of the average skew information-based coherence for both random pure states and random mixed states. In particular, we depict the scatter plot of the average skew information-based coherence for random mixed states. Furthermore, utilizing Lévy's Lemma, we study the typicality of the average coherence for random pure and mixed states.

2. Preliminaries

Let $\mathcal{H} = \mathbb{C}^N$ be a Hilbert space of dimension N, $\mathrm{U(N)}$ be the group of all $N \times N$ unitary matrices, $\mathrm{M_N}(\mathbb{C})$ be the set of all $N \times N$ complex matrices, and $\mathrm{D}(\mathbb{C}^{\mathrm{N}})$ be the set of all density matrices on $\mathbb{C}^N$. The set of pure states on $\mathbb{C}^N$ is the complex projective space $\mathbb{C}P^{N-1}$. For the space of pure states $|\psi\rangle$ there exists a unique measure $d(\psi)$ induced from the uniform Haar measure $d\mu(U)$ on the unitary group $U(N)$, which implies that any random pure state $|\psi\rangle$ can be obtained via a unitary operation on a fixed pure state $|\psi_0\rangle$: $|\psi\rangle = U|\psi_0\rangle$. The average value of a function $g(\psi)$ of pure states $|\psi\rangle$ is defined as

$$\mathbb{E}_\psi[g(\psi)] = \int d(\psi) g(\psi) = \int_{\mathrm{U(N)}} d\mu(U) g(U\psi_0).$$

For a fixed reference basis $\{|k\rangle\}$, the coherence measure based on skew information is defined by [7]

$$C_I(\rho) = \sum_{k=1}^{N} I(\rho, |k\rangle\langle k|), \tag{1}$$

where $I(\rho, |k\rangle\langle k|) = -\frac{1}{2}\mathrm{Tr}\{[\rho, |k\rangle\langle k|]\}^2$ is the skew information of the state ρ with respect to the projection $|k\rangle\langle k|$. Direct calculations show that $C_I(\rho)$ can be further written as [7]

$$C_I(\rho) = 1 - \sum_{k=1}^{N} \langle k|\sqrt{\rho}|k\rangle^2. \tag{2}$$

It can be seen from the definition that $\max_\rho C_I(\rho) = 1 - \frac{1}{N}$, and the sate $\rho = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\theta_j}|j\rangle$ attains the maximum.

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, one has

$$C_I(\psi) = 1 - \sum_{k=1}^{N} |\langle k|\psi\rangle|^4. \tag{3}$$

Let (X, d_1) and (Y, d_2) be two metric spaces and $T : X \to Y$ is a mapping. T is called a Lipschitz continuous function on X with the Lipschitz constant η, if there exists $\eta > 0$ such that

$$d_2(T(x), T(y)) \leq \eta d_1(x, y)$$

holds for all $x, y \in X$ [62].

Lévy's Lemma (see [37] and [39]). Let $T : \mathbb{S}^k \to \mathbb{R}$ be a Lipschitz continuous functions from the k-sphere to the real line with a Lipschitz constant η (with respect to the Hilbert-Schmidt norm). Let $z \in \mathbb{S}^k$ be a chosen uniformly at random. Then for any $\epsilon > 0$, we have

$$\Pr\{|T(z) - \mathbb{E}[T]| > \epsilon\} \leq 2\exp\left(-\frac{(k+1)\epsilon^2}{9\pi^3\eta^2 \ln 2}\right), \tag{4}$$

where $\mathbb{E}[T]$ is the expected value of T.

Note that the average over the Haar distributed N-dimensional pure states is equivalent to the average over the k sphere with $k = 2N - 1$.

To prove the existence of concentrated subspaces with a fixed amount of coherence, we need the notion of small nets [38]. Given a Hilbert space $\mathcal{H}$ of dimension N and $0 < \epsilon_0 < 1$, there exists a set $\mathcal{N}$ of pure states in $\mathcal{H}$ with $|\mathcal{N}| \leq (5/\epsilon_0)^{2N}$ such that for every pure state $|\psi\rangle \in \mathcal{H}$, there exists $|\tilde{\psi}\rangle \in \mathcal{N}$ such that $\||\psi\rangle - |\tilde{\psi}\rangle\|_2 \leq \frac{\epsilon_0}{2}$, where $\|A\|_2 := \sqrt{\mathrm{Tr}A^\dagger A}$ is the Hilbert-Schmidt norm of a matrix A [63]. This set $\mathcal{N}$ is called an ϵ_0 net.

One assumes naturally that the distributions of the eigenvalues and the eigenvectors of a quantum state ρ, via the spectral decomposition $\rho = U\Lambda U^\dagger$, are independent. Thus any probability measure μ on $\mathrm{D}(\mathbb{C}^\mathrm{N})$ can be factorized as $\mathrm{d}\mu(\rho) = \mathrm{d}\nu(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U})$, where $\mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U})$ is the unique Haar measure on the unitary group and ν defines the distribution of eigenvalues without unique choice for it [53, 64].

The measures used frequently over $\mathrm{D}(\mathbb{C}^{\mathrm{N}})$ can be obtained by partial tracing the Haar-distributed pure states in the higher dimensional Hilbert space $N \otimes M$. For the sake of convenience, we suppose that $N \leq M$. The joint probability density function of spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ of ρ is [64]

$$\mathrm{d}\nu_{N,M}(\Lambda) = C_{N,M}\delta\left(1 - \sum_{j=1}^{N}\lambda_j\right)\prod_{1\leq i<j\leq N}(\lambda_i - \lambda_j)^2\prod_{j=1}^{N}\lambda_j^{M-N}\theta(\lambda_j)d\lambda_j, \tag{5}$$

where δ is the Dirac delta function, the theta function θ ensures that ρ is positive definite, $C_{N,M}$ is the normalization constant,

$$C_{N,M} = \frac{\Gamma(NM)}{\prod_{j=0}^{N-1}\Gamma(N-j+1)\Gamma(M-j)}.$$

In particular, in this paper we will focus on the case $N = M$, which corresponds to the Hilbert-Schmidt measure, a flat metric over the $\mathrm{D}(\mathbb{C}^{\mathrm{N}})$, denoted by $\mathrm{d}\mu_{\mathrm{HS}}(\rho)$. We also denote $\mathrm{d}\nu_{\mathrm{N,M}} = \mathrm{d}\nu$ and $C_{\mathrm{HS}}^{N} = C_{N,M}$ for $N = M$. Thus we have [54]

$$\mathrm{d}\mu_{\mathrm{HS}}(\rho) = \mathrm{d}\nu(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U}) \tag{6}$$

for $\rho = U\Lambda U^{\dagger}$. Here $\mathrm{d}\nu(\Lambda)$ is given by

$$\mathrm{d}\nu(\Lambda) = C_{\mathrm{HS}}^{N}\delta\left(1 - \sum_{j=1}^{N}\lambda_i\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}d\lambda_j, \tag{7}$$

where $|\Delta(\lambda)|^2 = \prod_{1\leq i<j\leq N}(\lambda_i - \lambda_j)^2$, and C_{HS}^{N} is the normalization constant,

$$C_{\mathrm{HS}}^{N} = \frac{\Gamma(N^2)}{\Gamma(N+1)\prod_{j=1}^{N}\Gamma(j)^2}. \tag{8}$$

3. Average skew information-based coherence and its typicality for random pure states

We first calculate the average skew information-based coherence for random pure states.

Theorem 1 The average skew information-based coherence for a random pure state $|\psi\rangle \in \mathrm{D}(\mathbb{C}^N)$ is given by

$$\mathbb{E}_{\psi}[C_I(\psi)] = \frac{N-1}{N+1}. \tag{9}$$

Proof. From Eq. (3) the expected value of the coherence based on skew information is given by

$$\mathbb{E}_{\psi}[C_I(\psi)] := \int d\mu(\psi)\left(1 - \sum_{k=1}^{N}|\langle k|\psi\rangle|^4\right), \tag{10}$$

where μ is a unitarily invariant uniform probability measure.

Take $|\psi\rangle = U|1\rangle$, where U is sampled from the Haar distribution and $|1\rangle$ is a fixed state. Noting that the Haar measure is left-invariant, we obtain

$$\begin{aligned}\mathbb{E}_\psi[C_I(\psi)] &= 1-\sum_{k=1}^{N}\int d\mu(U)|\langle k|U|1\rangle|^4\\ &= 1-N\int d\mu(U)|U_{11}|^4,\end{aligned}$$

where $U_{11} = \langle 1|U|1\rangle$. The distribution of $|U_{11}|^2$ is given by $(N-1)(1-r)^{N-2}dr$, where $0 \leq r \leq 1$ [51]. Therefore, we get

$$\mathbb{E}_\psi[C_I(\psi)] = 1-N(N-1)\int_0^1 r^2(1-r)^{d-2}dr = 1-N(N-1)B(3,N-1), \tag{11}$$

where $B(\alpha,\beta)$ is the β function

$$B(\alpha,\beta) := \int_0^1 r^{\alpha-1}(1-r)^{\beta-1}dr = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{12}$$

Noting that

$$B(3,N-1) = \frac{\Gamma(3)\Gamma(N-1)}{\Gamma(N+2)} = \frac{2}{(N+1)N(N-1)},$$

we obtain from Eq. (11) the formula (9). □

By Theorem 1, it is easy to see that $\mathbb{E}_\psi[C_I(\psi)] = \frac{1}{3}$ for qubit pure states and $\mathbb{E}_\psi[C_I(\psi)] = \frac{1}{2}$ for qutrit pure states. The limit of the average coherence is $\lim_{N\to\infty}\mathbb{E}_\psi[C_I(\psi)] = 1$ for $N \to \infty$.

Moreover, it is easy to see that $(1-\frac{1}{N}) - (\frac{N-1}{N+1}) < 1-\frac{N-1}{N+1}$ for all integers $N \geq 2$, i.e., $\max_\psi C_I(\psi) - \mathbb{E}_\psi[C_I(\psi)] < \mathbb{E}_\psi[C_I(\psi)] - \min_\psi C_I(\psi)$, which means that the average skew information-based coherence is always closer to the maximum coherence than the minimum coherence. One can also find that as N approaches to infinity, the limit of the gap between the average skew information-based coherence and the maximum coherence is $\lim_{N\to\infty}(\max_\psi C_I(\psi) - \mathbb{E}_\psi[C_I(\psi)]) = \lim_{N\to\infty}\frac{N-1}{N(N+1)} = 0$. Therefore, we can give the following theorem about the concentration of measure phenomenon for quantum coherence with respect to random pure states.

Theorem 2 (Typicality of skew information-based coherence for random pure states) Let $|\psi\rangle \in \mathrm{D}(\mathbb{C}^N)$ be a random pure state. Then for all $\epsilon > 0$, we have

$$\Pr\left\{\left|C_I(\psi) - \frac{N-1}{N+1}\right| > \epsilon\right\} \leq 2\exp\left(-\frac{N^3\epsilon^2}{72\pi^3\ln 2}\right). \tag{13}$$

Proof. Consider the map $T : |\psi\rangle \to T(\psi) := C_I(\psi)$. It follows from Eq. (9) that $\mathbb{E}_\psi[T(\psi)] = \frac{N-1}{N+1}$. Set $k = 2N-1$ in Eq. (4). We need to fix the Lipschitz constant η for T satisfying $|T(\psi) - T(\phi)| \leq \eta\|\psi-\phi\|_2$. Suppose that $|\psi\rangle = \sum_{k=1}^N \psi_k|k\rangle$ with

$\sum_{k=1}^{N} |\psi_k|^2 = 1$. Denote $p_k = |\psi_k|^2$. Since $T(\psi) = 1 - \sum_{k=1}^{N} |\langle k|\psi\rangle|^4 = 1 - \sum_{k=1}^{N} |\psi_k|^4$, we have

$$\begin{aligned}\eta^2 := \sup_{\langle\psi|\psi\rangle\leq 1} \nabla T \cdot \nabla T &= \sum_{k=1}^{N}(4|\psi_k|^3)^2 = 16\sum_{k=1}^{N}|\psi_k|^6 = 16\sum_{k=1}^{N}p_k^3 \\ &\leq 16N(\frac{1}{N})^3 = \frac{16}{N^2}. \end{aligned} \tag{14}$$

Therefore, $\eta \leq \frac{4}{N}$. By definition, we can take $\eta = \frac{4}{N}$ as the Lipschitz constant. This completes the proof. □

The inequality (13) implies that, similar to the relative entropy of coherence, for large N, the number of pure states with the skew information-based coherence not very close to $\frac{N-1}{N+1}$ is exponentially small. Namely, most randomly-chosen pure states have almost $\frac{N-1}{N+1}$ amount of skew information-based coherence. This is just the so-called concentration of skew information-based coherence around its expected value, i.e., the typicality of the skew information-based coherence.

Next, we shall identify a coherent subspace, i.e., a large subspace of the Hilbert space such that the amount of the skew information-based coherence for each pure state in this subspace can be bounded from below almost always by a fixed number that is arbitrarily close to the typical value of coherence.

Theorem 3 (Coherent subspaces) Let $\mathcal{H} = \mathbb{C}^N$ be a Hilbert space of dimension N. Then for any $0 < \epsilon < \frac{1}{N}$, there exists a subspace $\mathcal{S} \subset \mathcal{H}$ of dimension

$$s = \left\lfloor \frac{N^3\epsilon^2 - 1}{3095(3 - \ln\epsilon N)} \right\rfloor, \tag{15}$$

such that all the pure states $|\psi\rangle \in \mathcal{S}$ almost always satisfy $C_I(\psi) \leq \frac{N-1}{N+1} - \epsilon$. Here $\lfloor\rfloor$ denotes the floor function.

Proof. Let $\mathcal{S}$ be a random s-dimensional subspace of $\mathcal{H}$. Let $\mathcal{N}_S$ be an ϵ_0 net for states on $\mathcal{S}$, where $\epsilon_0 = \frac{\epsilon}{4/N}$. It follows from the definition that $|\mathcal{N}_S| \leq (5/\epsilon_0)^{2s}$. Identify $\mathcal{S}$ with $U\mathcal{S}_0$, where $\mathcal{S}_0$ is fixed, and U is a unitary distributed according to the Haar measure. Endow the net $\mathcal{N}_{S_0}$ on $\mathcal{S}_0$ and let $\mathcal{N}_S = U\mathcal{N}_{S_0}$. Given $|\psi\rangle \in \mathcal{S}$, we can choose $|\tilde{\psi}\rangle \in \mathcal{N}_S$ such that $\||\psi\rangle - |\tilde{\psi}\rangle\|_2 \leq \frac{\epsilon_0}{2}$. Since $C_I(\psi)$ is a Lipschitz continuous function with the Lipschitz constant $\eta = \frac{4}{N}$, by the definition of the ϵ_0 set, we have

$$|C_I(\psi) - C_I(\tilde{\psi})| \leq \eta\||\psi\rangle - |\tilde{\psi}\rangle\|_2 \leq \eta\frac{\epsilon_0}{2} = \epsilon/2.$$

Define $\mathbb{P} = \Pr\{\min_{|\psi\rangle\in\mathcal{S}} C_I(\psi) < \frac{N-1}{N+1} - \epsilon\}$. From Theorem 2 we have

$$\begin{aligned}\mathbb{P} &\leq \Pr\left\{\min_{|\psi\rangle\in\mathcal{S}} C_I(\psi) < \frac{N-1}{N+1} - \frac{\epsilon}{2}\right\} \\ &\leq |\mathcal{N}_S|\Pr\left\{C_I(\psi) < \frac{N-1}{N+1} - \frac{\epsilon}{2}\right\} \\ &\leq 2\left(\frac{20}{\epsilon N}\right)^{2s}\exp\left(-\frac{N^3\epsilon^2}{72\pi^3\ln 2}\right). \end{aligned} \tag{16}$$

If the probability $\mathbb{P} < 1$, a subspace with the properties mentioned in the theorem will exist. This fact holds if

$$s < \frac{N^3\epsilon^2 - 1}{3095(3 - \ln\epsilon N)}.$$

Noting that $\epsilon < \frac{1}{N}$, for $s \geq 2$, we require that $N \geq 32941$. Therefore, we get $s = \left\lfloor \frac{N^3\epsilon^2-1}{3095(3-\ln\epsilon N)} \right\rfloor$. This completes the proof. □

4. Average skew information-based coherence and its typicality for random mixed states

We now turn to the average skew information-based coherence and its typicality for random mixed quantum states. We first present the following lemma.

Lemma 1 Denote $|\Delta(\mu)|^2 = \prod_{1\leq i<j\leq N}(\mu_i - \mu_j)^2$. It holds that

$$\begin{aligned}
&\int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right) |\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j \\
&= (N-2)! \prod_{j=1}^N \Gamma(j)^2 \left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right],
\end{aligned} \tag{17}$$

where $I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)} (-1)^{k+l}\binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r}\frac{\Gamma(\frac{3}{2}+r)}{r!}$.

The proof of Lemma 1 is given in Appendix A. Based on the above Lemma, we can give the analytical formula of average skew information-based coherence for random mixed states in terms of the dimension N.

Theorem 4 The average skew information-based coherence for a random mixed state $\rho \in \mathrm{D}(\mathbb{C}^N)$ is given by

$$\begin{aligned}
\mathbb{E}_\rho[C_I(\rho)] &:= \int_{\mathrm{D(\mathbb{C}^N)}} C_I(\rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho) \\
&= 1 - \frac{1}{N+1}\left(2 + \frac{1}{N^2}\left[\left(\sum_{k=0}^{N-1} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=0}^{N-1}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right),
\end{aligned} \tag{18}$$

where $\mathrm{d}\mu_{\mathrm{HS}}$ is a normalized Hilbert-Schmidt measure, i.e., $\int_{\mathrm{D(\mathbb{C}^N)}} \mathrm{d}\mu_{\mathrm{HS}}(\rho) = 1$, and $I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)} (-1)^{k+l}\binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r}\frac{\Gamma(\frac{3}{2}+r)}{r!}$.

The proof of Theorem 4 is given in Appendix B. Setting $N = 2$ and $N = 3$ in Theorem 4, we obtain the explicit values of the average coherence for qubit states and qutrit states,

$$\mathbb{E}_\rho[C_I(\rho)] = 1 - \frac{1}{3}\left(2 + \frac{3\pi}{16}\right) = \frac{1}{3} - \frac{\pi}{16} \approx 0.137$$

and

$$\mathbb{E}_\rho[C_I(\rho)] = 1 - \frac{1}{4}\left(2 + \frac{103\pi}{256}\right) = \frac{1}{2} - \frac{103\pi}{1024} \approx 0.184,$$

respectively.

In Figure 1, we plot the average skew information-based coherence for random mixed states. The A-axis shows the value of $\mathbb{E}_\rho[C_I(\rho)]$ given by Eq.(24). It shows that as the dimension N increases, the expectation value $\mathbb{E}_\rho[C_I(\rho)]$ approaches to 0.28. Numerical computation shows that unlike the random pure states, for random mixed states, the average skew information-based coherence is closer to the minimal coherence 0 than the maximum coherence $1-\frac{1}{N}$.

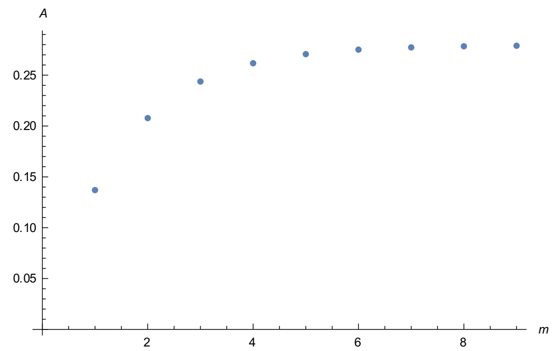

Figure 1: The average skew information-based coherence $A = \mathbb{E}_\rho[C_I(\rho)]$ as a function of $N = 2^m$.

Based on the above result, we can similarly discuss the typicality of quantum coherence $C_I(\rho)$ for random mixed states.

Theorem 5 (Typicality of skew information-based coherence for random mixed states) Let $\rho_A \in \mathrm{D}(\mathbb{C}^N)$ be a random mixed quantum state obtained via partial tracing over a Haar distributed pure state $|\psi\rangle_{AB}$ in $\mathbb{C}^N \otimes \mathbb{C}^N$. Then for all $\epsilon > 0$, we have

$$\Pr\{|C_I(\rho_A) - \mathbb{E}_\rho[C_I(\rho_A)]| > \epsilon\} \leq 2\exp\left(-\frac{N\epsilon^2}{72\pi^3 \ln 2}\right), \tag{19}$$

where $\mathbb{E}_\rho[C_I(\rho_A)]$ is given by Eq. (18).

Proof. Define the map $T : \mathbb{S}^{N^2} \mapsto \mathbb{R}$ as $T(\psi_{AB}) = C_I(\rho_A)$. Let $|\psi\rangle_{AB} = \sum_{k,l=1}^N \psi_{kl}|k\rangle_A|l\rangle_B$. Then $\rho_A = \sum_{k,k'=1}^N p_{kk'}|k\rangle_A\langle k'|$, where $p_{kk'} = \sum_{l=1}^N \psi_{kl}\overline{\psi_{k'l}}$. For a bipartitie pure state $|\psi\rangle_{AB}$, it has been shown that $1 - C_I(\psi_{AB}) \leq [1 - C_I(\rho_A)][1 - C_I(\rho_B)]$ [7]. Since $0 \leq C_I(\rho_B) \leq 1 - \frac{1}{N}$, we have $C_I(\rho_A) \leq C_I(\psi_{AB}) = 1 - \sum_{k,l=1}^N |\langle k \otimes l|\psi\rangle|^4 = 1 - \sum_{k,l=1}^N |\psi_{kl}|^4$. Denote $\tilde{T}(\psi_{AB}) = C_I(\psi_{AB})$. Noting that $p_{kk} = \sum_{l=1}^N |\psi_{kl}|^2$ with $\sum_{k=1}^N p_{kk} = 1$, we have

$$\begin{aligned}\eta^2 := \sup_{\langle\psi|\psi\rangle\leq 1} \nabla\tilde{T}\cdot\nabla\tilde{T} &= \sum_{k,l=1}^N (4|\psi_{kl}|^3)^2 = 16\sum_{k,l=1}^N |\psi_{kl}|^6 = 16\sum_{k,l=1}^N (|\psi_{kl}|^2)^3 \\ &\leq 16\left(\sum_{k,l=1}^N |\psi_{kl}|^2\right)^3 = 16, \end{aligned}\tag{20}$$

which implies that $\eta \leq 4$. Now, the Lipschitz constant for T can be obtained in the following way. Suppose that σ_A is the reduced state of another pure state $|\phi\rangle_{AB}$. Without loss of generality, assume that $C_I(\sigma_A) \leq C_I(\rho_A)$. We can choose $|\phi\rangle_{AB}$ such that $C_I(\sigma_A) = C_I(\phi_{AB})$. Then

$$C_I(\rho_A) - C_I(\sigma_A) \leq C_I(\psi_{AB}) - C_I(\phi_{AB}) \leq \eta \| |\psi\rangle_{AB} - |\phi\rangle_{AB} \|_2,$$

Thus the Lipschitz constant of T is bounded by that of $\tilde{T}$ and can be chosen to be 4. This completes the proof. □

5. Conclusions and discussions

We have deduced the explicit formulae for the skew information-based coherence for both random pure states and random mixed states. It is found that as N approaches to infinity, the limit of the average coherence for random pure states is 1, while this limit for random mixed states is a positive constant less than 1 by numerical computation. The average skew information-based coherence is always closer to the maximum coherence than the minimum coherence for random pure states, while the average skew information-based coherence is always closer to the minimum coherence than the maximum coherence for random mixed states, which demonstrate that for a randomly-chosen state, a quantum pure state may give rise to more coherence as a resource compared with a quantum mixed state.

Comparing Eq. (9) with the counterpart in [51], we found that the quantity given in Eq. (9) is uniformly bounded, while the average skew information-based coherence for random pure states is $\mathbb{E}_\psi[C_r(\psi)] = H_N - 1$, which approaches to infinity as the dimension N increases, where $H_N = \sum_{k=1}^{N} 1/k$ is the Nth harmonic number. Surprisingly, as is given in [52], the average relative entropy of coherence for random mixed states is $\mathbb{E}_\rho[C_r(\rho)] = \frac{N-1}{2N}$, combining this with the quantity in Eq. (24), we conclude that in the mixed state case, the average coherence for skew information-based coherence and relative entropy of coherence are both uniformly bounded. Furthermore, in pure state case, it is interesting to note that for skew information-based coherence, the gap between the maximal coherence and the average coherence is $1 - \frac{1}{N} - \frac{N-1}{N+1} = \frac{N-1}{N(N+1)} > 0$, while for the relative entropy of coherence, it is found that this gap $\ln N - H_N + 1 \gg 0$.

Furthermore, we have shown that the average skew information-based coherence of pure quantum states (mixed quantum states) sampled randomly from the uniform Haar measure is typical, i.e., the probability that the skew information-based coherence of a randomly chosen pure state (mixed state) is not equal to the average relative entropy of coherence (within an arbitrarily small error) is exponentially small in the dimension of the Hilbert space.

We have also identified a coherent subspace, a large subspace of the Hilbert space such that the amount of the skew information-based coherence for each pure state in this subspace can be bounded from below almost always by a fixed number that is arbitrarily close to the typical value of coherence. The obtained results in this paper complement the corresponding results for relative entropy of coherence, and may shed new light on the study of quantum coherence from the probabilistic perspective.

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Appendix A: Proof of Lemma 1

Proof of Lemma 1. Note that $\prod_{1\leq i<j\leq N}(\mu_i-\mu_j)$ is the classical Vandermonde determinant

$$\prod_{1\leq i<j\leq N}(\mu_i-\mu_j)=\begin{vmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_N \\ \vdots & \ddots & \vdots \\ \mu_1^{N-1} & \cdots & \mu_N^{N-1} \end{vmatrix}.$$

It can be seen that if $P_0, P_1, \cdots, P_{N-1}$ are polynomials of respective degrees $0, 1, \cdots, N-1$ and respective dominant coefficients $a_0, a_1, \cdots, a_{N-1}$, one has

$$\prod_{1\leq i<j\leq N}(\mu_i-\mu_j)=\frac{1}{\prod_{k=0}^{N-1}a_k}\begin{vmatrix} P_0(\mu_1) & \cdots & P_0(\mu_N) \\ P_1(\mu_1) & \cdots & P_1(\mu_N) \\ \vdots & \ddots & \vdots \\ P_{N-1}(\mu_1) & \cdots & P_{N-1}(\mu_N) \end{vmatrix}$$

Now choose $P_k(x)$ to be Laguerre polynomials $L_k(x)$:

$$L_k(x)=\sum_{j=0}^{k}(-1)^k\tbinom{k}{k-j}\frac{x^j}{j!}.$$

Note that $L_k(x)$ have the orthogonality property

$$\int_0^\infty L_k(x)L_l(x)e^{-x}dx=\delta_{kl}, \tag{21}$$

and the coefficient of the term with the highest degree is $a_k = \frac{(-1)^k}{k!}$. We have

$$
\begin{aligned}
\prod_{1\le i<j\le N} (\mu_i - \mu_j)^2 &= \frac{1}{\prod_{k=0}^{N-1} a_k^2} \begin{vmatrix} L_0(\mu_1) & \cdots & L_0(\mu_N) \\ L_1(\mu_1) & \cdots & L_1(\mu_N) \\ \vdots & \ddots & \vdots \\ L_{N-1}(\mu_1) & \cdots & L_{N-1}(\mu_N) \end{vmatrix} \\
&= \prod_{k=0}^{N-1} (k!)^2 \sum_{\sigma,\tau\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) L_{\sigma(k)-1}(\mu_k) L_{\tau(k)-1}(\mu_k), \quad (22)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right) |\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j \\
&= \prod_{k=0}^{N-1} (k!)^2 \sum_{\sigma,\tau\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \left(\int_0^\infty \sqrt{\mu_1} e^{-\mu_1} L_{\sigma(1)-1}(\mu_1) L_{\tau(1)-1}(\mu_1) d\mu_1\right) \\
&\left(\int_0^\infty \sqrt{\mu_2} e^{-\mu_2} L_{\sigma(2)-1}(\mu_2) L_{\tau(2)-1}(\mu_1) d\mu_2\right) \left(\prod_{k=3}^N \int_{\mathbb{R}_+^{N-2}} e^{-\mu_k} L_{\sigma(k)-1}(\mu_k) L_{\tau(k)-1}(\mu_k) d\mu_k\right),
\end{aligned}
$$

where S_N is the permutation group on $\{1, 2, \cdots, N\}$.

Denote $I_{kl}^{(q)} := \int_0^\infty L_k(x) L_l(x) e^{-x} x^q dx$, where $q > -1$. It holds that [42]

$$
I_{kl}^{(q)} = \sum_{r=0}^{\min(k,l)} (-1)^{k+l} \binom{q}{k-r} \binom{q}{l-r} \frac{\Gamma(q+r+1)}{r!}, \quad q > -1. \quad (23)
$$

Note that

$$
\int_0^\infty \sqrt{\mu_i} e^{-\mu_i} L_{\sigma(i)-1}(\mu_i) L_{\tau(i)-1}(\mu_i) d\mu_i = I^{(\frac{1}{2})}_{\sigma(i)-1,\tau(i)-1}, \quad i = 1, 2
$$

and

$$
\int_0^\infty \sqrt{\mu_i} e^{-\mu_i} L_{\sigma(1)-1}(\mu_i) L_{\sigma(2)-1}(\mu_i) d\mu_i = I^{(\frac{1}{2})}_{\sigma(1)-1,\sigma(2)-1}, \quad i = 1, 2.
$$

We calculate the integral $\int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right) |\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j$ by considering the following two cases.

Case I: $\sigma = \tau$. Denote $I = \sum_{k=0}^{N-1} I_{kk}^{(\frac{1}{2})}$, we have

$$\begin{aligned}
&\sum_{\sigma,\tau\in S_N,\sigma=\tau} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\left(\int_0^\infty \sqrt{\mu_1}e^{-\mu_1}L_{\sigma(1)-1}(\mu_1)L_{\tau(1)-1}(\mu_1)d\mu_1\right)\\
&\left(\int_0^\infty \sqrt{\mu_2}e^{-\mu_2}L_{\sigma(2)-1}(\mu_2)L_{\tau(2)-1}(\mu_1)d\mu_2\right)\left(\prod_{k=3}^N\int_{\mathbb{R}_+^{N-2}} e^{-\mu_k}L_{\sigma(k)-1}(\mu_k)L_{\tau(k)-1}(\mu_k)d\mu_k\right)\\
&= \sum_{\sigma\in S_N} I_{\sigma(1)-1,\sigma(1)-1}^{(\frac{1}{2})}I_{\sigma(2)-1,\sigma(2)-1}^{(\frac{1}{2})} = (N-2)!\sum_{k\neq l} I_{kk}^{(\frac{1}{2})}I_{ll}^{(\frac{1}{2})}\\
&= (N-2)!\left[\left(\sum_{k=0}^{N-1} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k=0}^{N-1}\left(I_{kk}^{(\frac{1}{2})}\right)^2\right]. \qquad (24)
\end{aligned}$$

Case II: $\sigma \neq \tau$. First, note that if there exists $k_0 \in \{3, 4, \cdots, N\}$ such that $\sigma(k_0) \neq \tau(k_0)$, then by Eq. (31) we have

$$\left(\prod_{k=3}^N\int_{\mathbb{R}_+^{N-2}} e^{-\mu_k}L_{\sigma(k)-1}(\mu_k)L_{\tau(k)-1}(\mu_k)d\mu_k\right) = 0.$$

Thus $\int_{\mathbb{R}_+^N}\sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N\mu_j\right)|\Delta(\mu)|^2\prod_{j=1}^N d\mu_j = 0$. Otherwise, $\sigma(i) = \tau(i)$ for $i = 3, \cdots, N$, which implies that $\sigma(1) = \tau(2)$ and $\sigma(2) = \tau(1)$, i.e., $\tau = \sigma(12)$. Then we have

$$\begin{aligned}
&\sum_{\sigma,\tau\in S_N,\sigma\neq\tau} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\left(\int_0^\infty \sqrt{\mu_1}e^{-\mu_1}L_{\sigma(1)-1}(\mu_1)L_{\tau(1)-1}(\mu_1)d\mu_1\right)\\
&\left(\int_0^\infty \sqrt{\mu_2}e^{-\mu_2}L_{\sigma(2)-1}(\mu_2)L_{\tau(2)-1}(\mu_1)d\mu_2\right)\left(\prod_{k=3}^N\int_{\mathbb{R}_+^{N-2}} e^{-\mu_k}L_{\sigma(k)-1}(\mu_k)L_{\tau(k)-1}(\mu_k)d\mu_k\right)\\
&= \sum_{\sigma\in S_N}(-1)I_{\sigma(1)-1,\sigma(2)-1}^{(\frac{1}{2})}I_{\sigma(2)-1,\sigma(1)-1}^{(\frac{1}{2})} = -(N-2)!\sum_{k\neq l}(I_{kl}^{(\frac{1}{2})})^2. \qquad (25)
\end{aligned}$$

Combining Eqs. (24) and (25), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^N}\sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N\mu_j\right)|\Delta(\mu)|^2\prod_{j=1}^N d\mu_j\\
&= \prod_{k=0}^{N-1}(k!)^2\left[(N-2)!\left(\left(\sum_{k=0}^{N-1} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k=0}^{N-1}\left(I_{kk}^{(\frac{1}{2})}\right)^2\right) - (N-2)!\sum_{k\neq l}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\\
&= (N-2)!\prod_{j=1}^N\Gamma(j)^2\left[\left(\sum_{k=0}^{N-1} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=0}^{N-1}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right], \qquad (26)
\end{aligned}$$

where

$$I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)}(-1)^{k+l}\binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r}\frac{\Gamma(\frac{3}{2}+r)}{r!}.$$

□

Appendix B: Proof of Theorem 4

Proof of Theorem 4. Since $\mathrm{d}\mu_{\mathrm{HS}}$ is a normalized Hilbert-Schmidt measure, by the definition of $C_I(\rho)$, we have

$$\begin{aligned}\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})} C_I(\rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho) &= \int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})}\left[1-\sum_{k=1}^{N}\langle k|\sqrt{\rho}|k\rangle^2\right]\mathrm{d}\mu_{\mathrm{HS}}(\rho)\\ &= 1-\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})}\sum_{k=1}^{N}\langle k^{\otimes 2}|\sqrt{\rho}^{\otimes 2}|k^{\otimes 2}\rangle\mathrm{d}\mu_{\mathrm{HS}}(\rho)\\ &= 1-\sum_{k=1}^{N}\left\langle k^{\otimes 2}\left|\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})}\sqrt{\rho}^{\otimes 2}\mathrm{d}\mu_{\mathrm{HS}}(\rho)\right|k^{\otimes 2}\right\rangle. \qquad (27)\end{aligned}$$

It suffices to compute the integral $\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})}\sqrt{\rho}^{\otimes 2}\mathrm{d}\mu_{\mathrm{HS}}(\rho)$. In fact, by the factorization in Eq. (6), it follows that

$$\begin{aligned}&\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})}\sqrt{\rho}^{\otimes 2}\mathrm{d}\mu_{\mathrm{HS}}(\rho)\\ &= \int \mathrm{d}\nu(\Lambda)\int_{\mathrm{U(N)}}\left[(U\otimes U)(\sqrt{\Lambda}\otimes\sqrt{\Lambda})(U\otimes U)^\dagger\mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U})\right]. \qquad (28)\end{aligned}$$

Using the following formula for integral over unitary groups [65]:

$$\begin{aligned}&\int_{\mathrm{U(N)}}(U\otimes U)A(U\otimes U)^\dagger\mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U})\\ &= \left(\frac{\mathrm{Tr}(A)}{N^2-1}-\frac{\mathrm{Tr}(AF)}{N(N^2-1)}\right)\mathbf{1}_{N^2}-\left(\frac{\mathrm{Tr}(A)}{N(N^2-1)}-\frac{\mathrm{Tr}(AF)}{N^2-1}\right)F, \qquad (29)\end{aligned}$$

where $A\in\mathrm{M}_{\mathrm{N}^2}(\mathbb{C})$ and F is the swap operator defined by $F|ij\rangle=|ji\rangle$ for all $i,j=1,2,\cdots,N$, we have

$$\int_{\mathrm{U(N)}}(U\otimes U)(\sqrt{\Lambda}\otimes\sqrt{\Lambda})(U\otimes U)^\dagger\mathrm{d}\mu_{\mathrm{Haar}}(\mathrm{U}) = \frac{N(\mathrm{Tr}\sqrt{\Lambda})^2-1}{N(N^2-1)}\mathbf{1}_{N^2}+\frac{N-(\mathrm{Tr}\sqrt{\Lambda})^2}{N(N^2-1)}F. \qquad (30)$$

Noting that

$$\begin{aligned}\int(\mathrm{Tr}\sqrt{\Lambda})^2\mathrm{d}\nu(\Lambda) &= \int\mathrm{d}\nu(\Lambda)+2\int\sum_{1\leq i<j\leq N}\sqrt{\lambda_i\lambda_j}\mathrm{d}\nu(\Lambda)\\ &= 1+2\int\sum_{1\leq i<j\leq N}\sqrt{\lambda_i\lambda_j}\mathrm{d}\nu(\Lambda)\\ &= 1+2C^N_{\mathrm{HS}}\int_{\mathbb{R}^N_+}\sum_{1\leq i<j\leq N}\sqrt{\lambda_i\lambda_j}\delta\left(1-\sum_{j=1}^N\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^N d\lambda_j\\ &= 1+2C^N_{\mathrm{HS}}\binom{N}{2}\int_{\mathbb{R}^N_+}\sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^N\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^N d\lambda_j, \qquad (31)\end{aligned}$$

where C_{HS}^N is given in Eq. (8), we only need to calculate

$$\int_{\mathbb{R}_+^N} \sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^N \lambda_j\right)|\Delta(\lambda)|^2 \prod_{j=1}^N d\lambda_j.$$

Denote

$$F(t) = \int_{\mathbb{R}_+^N} \sqrt{\lambda_1\lambda_2}\delta\left(t-\sum_{j=1}^N \lambda_j\right)|\Delta(\lambda)|^2 \prod_{j=1}^N d\lambda_j.$$

By performing Laplace transform ($t \to s$) of $F(t)$, and letting $\mu_j = s\lambda_j, j = 1, 2$, we get

$$\begin{aligned} \tilde{F}(s) &= \int_{\mathbb{R}_+^N} \sqrt{\lambda_1\lambda_2}\exp\left(-s\sum_{j=1}^N \lambda_j\right)|\Delta(\lambda)|^2 \prod_{j=1}^N d\lambda_j \\ &= s^{-(N^2+1)} \int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right)|\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j. \end{aligned} \tag{32}$$

Utilizing the inverse Laplace transform $(s \to t) : \mathscr{L}^{-1}(s^\alpha) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$, we obtain

$$F(t) = \frac{t^{N^2}}{\Gamma(N^2+1)} \int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right)|\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j. \tag{33}$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}_+^N} \sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^N \lambda_j\right)|\Delta(\lambda)|^2 \prod_{j=1}^N d\lambda_j \\ &= \frac{1}{\Gamma(N^2+1)} \int_{\mathbb{R}_+^N} \sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^N \mu_j\right)|\Delta(\mu)|^2 \prod_{j=1}^N d\mu_j. \end{aligned} \tag{34}$$

Substituting Eq. (17) into Eq. (34) yields

$$\begin{aligned} &\int_{\mathbb{R}_+^N} \sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^N \lambda_j\right)|\Delta(\lambda)|^2 \prod_{j=1}^N d\lambda_j \\ &= \frac{(N-2)!\prod_{j=1}^N \Gamma(j)^2}{\Gamma(N^2+1)} \left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right], \end{aligned} \tag{35}$$

which by Eqs. (8) and (31) gives rise to

$$\int (\mathrm{Tr}\sqrt{\Lambda})^2 \mathrm{d}\nu(\Lambda) = 1 + \frac{1}{N^2}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]. \tag{36}$$

Combining Eqs. (28), (30) and (36), we obtain

$$\begin{aligned}
&\int_{\mathrm{D}(\mathbb{C}^{\mathrm{N}})} \sqrt{\rho}^{\otimes 2} \mathrm{d}\mu_{\mathrm{HS}}(\rho) \\
&= \int \left[\frac{N(\mathrm{Tr}\sqrt{\Lambda})^2 - 1}{N(N^2-1)} \mathbf{1}_{N^2} + \frac{N - (\mathrm{Tr}\sqrt{\Lambda})^2}{N(N^2-1)} F \right] \mathrm{d}\nu(\Lambda) \\
&= \frac{N\mathbf{1}_{N^2} - F}{N(N^2-1)} \int (\mathrm{Tr}\sqrt{\Lambda})^2 \mathrm{d}\nu(\Lambda) + \frac{NF - \mathbf{1}_{N^2}}{N(N^2-1)} \int \mathrm{d}\nu(\Lambda) \\
&= \frac{N\mathbf{1}_{N^2} - F}{N(N^2-1)} \left(1 + \frac{1}{N^2} \left[\left(\sum_{k=1}^{N} I_{kk}^{(\frac{1}{2})} \right)^2 - \sum_{k,l=1}^{N} \left(I_{kl}^{(\frac{1}{2})} \right)^2 \right] \right) + \frac{NF - \mathbf{1}_{N^2}}{N(N^2-1)}.
\end{aligned}$$

Finally, by using the fact that $\sum_{k=1}^{N} \langle k^{\otimes 2} | F | k^{\otimes 2} \rangle = N$, we have

$$\sum_{k=1}^{N} \langle k^{\otimes 2} | N\mathbf{1}_{N^2} - F | k^{\otimes 2} \rangle = \sum_{k=1}^{N} \langle k^{\otimes 2} | NF - \mathbf{1}_{N^2} | k^{\otimes 2} \rangle = \frac{N^2 - N}{N(N^2-1)} = \frac{1}{N+1}.$$

From Eq. (27) we get (18). □

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