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on modified generalized skew
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by

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Coherence and complementarity based on modified generalized skew information

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Abstract   We introduce modified generalized Wigner-Yanase-Dyson (MGWYD) skew information and modified weighted generalized Wigner-Yanase-Dyson (MWGWYD) skew information. By revisiting state-channel interaction based on MGWYD skew information, a family of coherence measures with respect to quantum channels is proposed. Furthermore, explicit analytical expressions of these coherence measures of qubit states are derived with respect to different quantum channels. Moreover, complementarity relations based on MGWYD skew information and MWGWYD skew information are also presented. Specifically, the conservation relations are investigated, while two interpretations of them including symmetry-asymmetry complementarity and wave-particle duality have been proposed.

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1. Introduction

Originating from the superposition principle, quantum coherence is a characteristic feature of quantum mechanics. Despite its wide applications in superconductivity, quantum thermodynamics and biological processes, the quantification of quantum coherence from a resource-theoretic perspective was initiated only recently [1]. Since then the study on quantum coherence has attracted much attention in recent years. Various kinds of coherence measures such as relative entropy of coherence, l_1 norm of coherence, intrinsic randomness of measurement, robustness of coherence, averaged coherence, max-relative entropy of coherence, modified trace distance, skew information, Hellinger distance, affin-

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ity distance and generalized α-z-relative Rényi entropy have been proposed to quantify quantum coherence [2–15]. Coherence-generating power of quantum channels has also been investigated [16–18].

On the other hand, the relationships between quantum coherence and other quantum resources such as quantum entanglement, quantum discord and asymmetry [19–26] have been extensively studied. It has been shown that quantum coherence and asymmetry cannot be broadcasted [27, 28]. Quantum coherence is also tightly related to the optimization of quantum observables [29]. Recently, a regime of defining coherence measures by making use of POVMs has been put forward [30]. Utilizing the concept of resource destroying maps, the authors in [31] have established a framework of coherence theory on the level of quantum operations.

Complementarity is another important quantum feature which has been extensively studied since the advent of quantum mechanics. It is well known that the Bohr's complementarity principle plays an indispensable role in the basic theory of quantum mechanics in the early days [32]. This was manifested in wave-particle duality and uncertainty relations by many authors [33–42].

Recently, by decomposing the state-channel interaction into a symmetric and an asymmetric part, the authors in [43] formalized a quantitative symmetry-asymmetry complementarity relation. The asymmetric part is given based on a modified version of Wigner-Yanase skew information, in which a Hermitian operator is replaced by a bounded linear operator (not-necessarily-Hermitian), which can be interpreted as a measure of coherence with respect to a quantum channel. However, as a desired property of a quantum coherence measure, the strong monotonicity of this quantity is not proved in [43]. In [44], Li provided an alternative proof of the monotonicity based on the modified skew information via operator algebra approach, and derived the strong monotonicity.

The generalized Wigner-Yanase-Dyson skew information with parameters α and β has been introduced in [45]. It would be interesting if such generalized Wigner-Yanase-Dyson skew information could be utilized to characterize quantum coherence, general state-channel interactions and complementarity relations among the coherence measures. In this work, we define correspondingly the modified generalized Wigner-Yanase-Dyson (MGWYD) skew information and the modified weighted generalized Wigner-Yanase-Dyson (MWGWYD) skew information. Based on the MGWYD skew information and MWGWYD skew information, we study the state-channel interactions. It is shown that the asymmetric part of the generalized state-channel couplings can be regarded as a family of coherence of states with respect to a channel. Complementarity relations based on MGWYD skew information and MWGWYD skew information are also derived with physical interpretations. Some concluding remarks are given in Section 6. Our results shed new light on the study of coherence, and give rise to a basic framework for quantitatively addressing symmetry-asymmetry complementarity.

2. Preliminaries

Let $\mathcal{H}$ be a d-dimensional Hilbert space, and $\mathcal{B}(\mathcal{H})$, $\mathcal{S}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ the set of all bounded linear operators, Hermitian operators and density operators on $\mathcal{H}$, respectively. Usually, a state and a channel are mathematically described by a density operator (positive operator of trace 1) and a completely positive trace preserving (CPTP) map, respectively [46]. Nevertheless, in this paper when we discuss state-channel interactions, a channel is assumed to be a completely positive trace nonincreasing map, while a quantum operation is assumed to be a completely positive trace preserving (CPTP) map, in accord with Ref. [43].

Fix an orthonormal basis $\{|i\rangle\}_{i=1}^d$ of a d-dimensional Hilbert space $\mathcal{H}$. The density operators which are diagonal in this basis are called incoherent states and the set of all incoherent states is denoted by $\mathcal{I}$, i.e.,

$$\mathcal{I} = \{\delta \in \mathcal{D}(\mathcal{H}) | \delta = \sum_i p_i |i\rangle\langle i|, \ p_i \geq 0, \ \sum_i p_i = 1\}.$$

Let Λ be a completely positive trace preserving (CPTP) map

$$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger,$$

where K_n are Kraus operators satisfying $\sum_n K_n^\dagger K_n = I_d$ with I_d the identity operator. K_n are called incoherent Kraus operators if $K_n^\dagger \mathcal{I} K_n \in \mathcal{I}$ for all n, and the corresponding Λ is called an incoherent operation.

In [1], the authors proposed the conditions that a well-defined coherence measure C should satisfy:

(C_1) (Faithfulness) $C(\rho) \geq 0$ and $C(\rho) = 0$ iff ρ is incoherent.

(C_2) (Monotonicity) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ.

(C_3) (Convexity) $C(\cdot)$ is a convex function of ρ, i.e.,

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n),$$

where $p_n \geq 0, \ \sum_n p_n = 1$.

(C_4) (Strong monotonicity) $C(\cdot)$ does not increase on average under selective incoherent operations, i.e.,

$$C(\rho) \geq \sum_n p_n C(\varrho_n),$$

where $p_n = \text{Tr}(K_n \rho K_n^\dagger)$ are probabilities and $\varrho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ are the post-measurement states, K_n are incoherent Kraus operators.

Now, we recall the concepts of different kinds of skew information. For a density operator $\rho \in \mathcal{D}(\mathcal{H})$ and an observable $A \in \mathcal{S}(\mathcal{H})$, the *Wigner-Yanase* (WY) skew infor-

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mation [47] is defined by

$$I_\rho(A) = -\frac{1}{2}\text{Tr}([\rho^{\frac{1}{2}}, A]^2), \tag{1}$$

where $[X, Y] := XY - YX$ is the commutator of X and Y. A more general quantity was suggested by Dyson,

$$I_\rho^\alpha(A) = -\frac{1}{2}\text{Tr}([\rho^\alpha, A][\rho^{1-\alpha}, A]), \quad 0 \le \alpha \le 1, \tag{2}$$

which is now called the *Wigner-Yanase-Dyson* (WYD) skew information. The quantity in Eq. (2) was further generalized to [45]

$$I_\rho^{\alpha,\beta}(A) = -\frac{1}{2}\text{Tr}([\rho^\alpha, A][\rho^\beta, A]\rho^{1-\alpha-\beta}), \quad \alpha, \beta \ge 0, \ \alpha + \beta \le 1, \tag{3}$$

which is termed as *generalized Wigner-Yanase-Dyson* (GWYD) skew information. It is easy to see that when $\alpha + \beta = 1$, Eq. (3) reduces to Eq. (2), and Eq. (2) reduces to Eq. (1) when $\alpha = \frac{1}{2}$.

Another generalization of WYD skew information was given in [48]

$$K_\rho^\alpha(A) = -\frac{1}{2}\text{Tr}\left(\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]^2\right), \quad 0 \le \alpha \le 1, \tag{4}$$

where $A_0 = A - \text{Tr}(\rho A)I$. We call $K_\rho^\alpha(A)$ the *weighted Wigner-Yanase-Dyson skew information* in the following. Noting that $I_\rho(A) = I_\rho(A_0)$ when $\alpha = \frac{1}{2}$, one sees that Eq. (4) also reduces to Eq. (1) in this case.

Remarkable properties of these quantities and their applications in quantum information theory have been revealed and explored during the past few years [49–58]. Nevertheless, quantum gates [46], generalized quantum gates [59], the Kraus operators of a quantum channel [46] and many other operators are not necessarily Hermitian. Therefore, it is natural to consider the corresponding definitions of the different types of the skew information mentioned above for pseudo-Hermitian and/or PT-symmetric quantum mechanics [60–65].

For a density operator $\rho \in \mathcal{D}(\mathcal{H})$ and an operator $A \in \mathcal{B}(\mathcal{H})$ (not necessarily Hermitian), a generalization of the quantity in Eq. (1) is defined by [66]

$$|I_\rho|(A) = -\frac{1}{2}\text{Tr}([\rho^{\frac{1}{2}}, A^\dagger][\rho^{\frac{1}{2}}, A]), \tag{5}$$

which we refer to *modified Wigner-Yanase* (MWY) skew information. Similarly, a generalization of the quantity in Eq. (2) is defined by [67]

$$|I_\rho^\alpha|(A) = -\frac{1}{2}\text{Tr}([\rho^\alpha, A^\dagger][\rho^{1-\alpha}, A]), \quad 0 \le \alpha \le 1, \tag{6}$$

for any $A \in \mathcal{B}(\mathcal{H})$ and $\rho \in \mathcal{D}(\mathcal{H})$, which we call *modified Wigner-Yanase-Dyson* (MWYD) skew information. A generalization of the quantity in Eq. (4) is given by [68]

$$|K_\rho^\alpha|(A) = -\frac{1}{2}\text{Tr}\left(\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0^\dagger\right]\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]\right), \quad 0 \le \alpha \le 1, \tag{7}$$

for any $A \in L^2(H)$ and $\rho \in D(H)$, which we call *modified weighted Wigner-Yanase-Dyson* (MWWYD) skew information. The related quantity $|L_\rho^\alpha|(A)$ is defined by replacing the commutators in Eq. (7) by anti-commutators.

In addition, a *Schatten p-norm* [69] is defined as

$$\|A\|_p = \left[\sum_{j=1}^{n}(s_j(A))^p\right]^{1/p},$$

where $s_j(A)$ denotes the singular value of A. When $p = 2$, it is called a *Hilbert-Schmidt norm* [69]. Note that the class of Schatten p-norms is a special type of unitarily invariant norms [69] satisfying $|||UAV||| = |||A|||$ for all $A \in M(n)$ and $U, V \in U(n)$, where $M(n)$ denotes the set of $n \times n$ matrices and $U(n)$ the unitary group in $M(n)$.

Recently, Luo et al. defined the following quantity for any operator $K \in \mathcal{B}(\mathcal{H})$ and state $\rho \in \mathcal{D}(\mathcal{H})$ [43],

$$I(\rho, K) = \text{Tr}([\rho^{\frac{1}{2}}, K]^\dagger [\rho^{\frac{1}{2}}, K]) = \|[\rho^{\frac{1}{2}}, K]\|_2^2, \tag{8}$$

where $\|X\|_2^2 = \text{Tr}(X^\dagger X)$ is the Hilbert-Schmidt norm, $[X, Y] = \frac{1}{2}(XY - YX)$ is the commutator. This quantity is in fact the one defined in [66] (up to a constant factor). For the sake of convenience, we call it *modified Wigner-Yanase (MWY) skew information* in the following. Similarly, the following quantity was defined as a measure to quantify the symmetry between ρ and K,

$$J(\rho, K) = \text{Tr}(\{\rho^{\frac{1}{2}}, K\}\{\rho^{\frac{1}{2}}, K\}^\dagger) = \|\{\rho^{\frac{1}{2}}, K\}\|_2^2, \tag{9}$$

where $\{X, Y\} = \frac{1}{2}(XY + YX)$ is the anti-commutator.

Any quantum channel (completely positive trace nonincreasing map) has the following Kraus representation

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger, \tag{10}$$

while the dual channel can be written as

$$\Phi^\dagger(X) = \sum_i K_i^\dagger X K_i, \tag{11}$$

where X is any non-negative operator.

With respect to Eqs. (8) and (9), for a quantum channel in the form of Eq. (10), it is defined that

$$I(\rho, \Phi) = \sum_i I(\rho, K_i) \tag{12}$$

and

$$J(\rho, \Phi) = \sum_i J(\rho, K_i). \tag{13}$$

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For any operator $K \in \mathcal{B}(\mathcal{H})$ and state $\rho \in \mathcal{D}(\mathcal{H})$, the following quantity [44] has been derived from the unified entropy

$$I^\alpha(\rho, K) = \text{Tr}([\rho^\alpha, K]^\dagger[\rho^{1-\alpha}, K]), \quad 0 \leq \alpha \leq 1. \tag{14}$$

Note that this quantity is in fact the one defined in [67] (up to a constant factor). We shall call it *modified Wigner-Yanase-Dyson (MWYD) skew information* in the following. $J^\alpha(\rho, K)$ was defined by using anti-commutator in Eq. (12). Also, $I^\alpha(\rho, \Phi)$ and $J^\alpha(\rho, \Phi)$ have been proposed in a similar manner with the Kraus representation of a quantum channel.

3. State-channel interactions based on modified generalized skew information

In this section, we will study state-channel interaction based on modified generalized skew information. We first define the *modified generalized Wigner-Yanase-Dyson (MGWYD) skew information* for any operator $K \in \mathcal{B}(\mathcal{H})$ and state $\rho \in \mathcal{D}(\mathcal{H})$,

$$I^{\alpha,\beta}(\rho, K) = \text{Tr}([\rho^\alpha, K]^\dagger[\rho^\beta, K]\rho^{1-\alpha-\beta}), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1, \tag{15}$$

and the related quantity $J^{\alpha,\beta}(\rho, K)$,

$$J^{\alpha,\beta}(\rho, K) = \text{Tr}(\{\rho^\alpha, K\}^\dagger\{\rho^\beta, K\}\rho^{1-\alpha-\beta}), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{16}$$

Note that when $\alpha + \beta = 1$, $I^{\alpha,\beta}(\rho, K)$ and $J^{\alpha,\beta}(\rho, K)$ reduce to $I^\alpha(\rho, K)$ and $J^\alpha(\rho, K)$, respectively. Furthermore, corresponding to the map $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$, we define

$$I^{\alpha,\beta}(\rho, \Phi) = \sum_i I^{\alpha,\beta}(\rho, K_i), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1 \tag{17}$$

and

$$J^{\alpha,\beta}(\rho, \Phi) = \sum_i J^{\alpha,\beta}(\rho, K_i), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{18}$$

On the other hand, we define the *modified weighted generalized Wigner-Yanase-Dyson (MWGWYD) skew information* for any operator $K \in \mathcal{B}(\mathcal{H})$ and state $\rho \in \mathcal{D}(\mathcal{H})$,

$$V^{\alpha,\beta}(\rho, K) = \text{Tr}\left(\left[\frac{\rho^\alpha + \rho^\beta}{2}, K\right]^\dagger \left[\frac{\rho^\alpha + \rho^\beta}{2}, K\right]\rho^{1-\alpha-\beta}\right), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1, \tag{19}$$

with the related quantity

$$W^{\alpha,\beta}(\rho, K) = \text{Tr}\left(\left\{\frac{\rho^\alpha + \rho^\beta}{2}, K\right\}^\dagger \left\{\frac{\rho^\alpha + \rho^\beta}{2}, K\right\}\rho^{1-\alpha-\beta}\right), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{20}$$

Obviously, when $\alpha + \beta = 1$, $V^{\alpha,\beta}(\rho, K)$ and $W^{\alpha,\beta}(\rho, K)$ reduce to $|K_\rho^\alpha|(A)$ and $|L_\rho^\alpha|(A)$ defined in [68] (up to a constant factor), respectively. With respect to the map $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$, we define

$$V^{\alpha,\beta}(\rho, \Phi) = \sum_i V^{\alpha,\beta}(\rho, K_i), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1 \tag{21}$$

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and

$$W^{\alpha,\beta}(\rho, \Phi) = \sum_i W^{\alpha,\beta}(\rho, K_i), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{22}$$

Remark 1 As noted in [43] and [44], we can also prove that the quantities $I^{\alpha,\beta}(\rho, \Phi)$, $J^{\alpha,\beta}(\rho, \Phi)$, $V^{\alpha,\beta}(\rho, \Phi)$ and $W^{\alpha,\beta}(\rho, \Phi)$ are independent of the choice of the Kraus operators of Φ, which guarantees that the quantities given by Eqs. (17), (18), (21) and (22) are all well-defined.

From the above definitions, we have the following results of $I^{\alpha,\beta}(\rho, K)$ (the $W^{\alpha,\beta}(\rho, K)$ admits similar properties).

Proposition 1 For $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $K \in \mathcal{B}(\mathcal{H})$ and $\rho \in \mathcal{D}(\mathcal{H})$, it holds that

(i) $I^{\alpha,\beta}(\rho, K) = I^{\beta,\alpha}(\rho, K)$;

(ii) If $\alpha, \beta \in [0, 1]$, $\alpha + 2\beta \leq 1$ and $2\alpha + \beta \leq 1$, then $I^{\alpha,\beta}(\rho, K)$ is convex in ρ. In particular, $I^{\alpha}(\rho, K)$ is convex in ρ for $0 \leq \alpha \leq 1$.

Proof. (i) follows immediately from the definition. It follows from [11] that (ii) holds when $K \in \mathcal{S}(\mathcal{H})$. It can be seen that (ii) also holds for any $K \in \mathcal{B}(\mathcal{H})$ from the proof in [53].

4. A family of coherence measures of a state with respect to a channel

In this section, we demonstrate that $I^{\alpha,\beta}(\rho, \Phi)$ could be viewed as a bona fide family of coherence measures of ρ with respect to a channel Φ under certain conditions. We first prove the following properties of $I^{\alpha,\beta}(\rho, \Phi)$.

Theorem 1 For channel $\Phi(\rho) = \sum_j K_j \rho K_j^\dagger$, the quantity $I^{\alpha,\beta}(\rho, \Phi)$ defined in Eq. (17) for $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ has the following properties:

(i) $I^{\alpha,\beta}(\rho, \Phi) \geq 0$, with the equality holding if and only if $\Phi^\dagger(\rho^\alpha) = \rho^\alpha$, $\Phi^\dagger(\rho^\beta) = \rho^\beta$ and $\Phi^\dagger(\rho^{\alpha+\beta}) = \rho^{\alpha+\beta}$.

(ii) If $\alpha, \beta \in [0, 1]$, $\alpha + 2\beta \leq 1$ and $2\alpha + \beta \leq 1$, then $I^{\alpha,\beta}(\rho, \Phi)$ is convex in ρ.

(iii) (Ancillary independence) $I^{\alpha,\beta}(\rho^a \otimes \rho^b, \Phi^a \otimes \mathbf{1}^b) = I^{\alpha,\beta}(\rho^a, \Phi^a)$, where ρ^a and ρ^b are any states of systems a and b, respectively, and $\mathbf{1}^b$ is the identity channel on system b.

(iv) (Monotonicity) If a channel $\mathcal{E}$ admits the representation $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$, then

$$I^{\alpha,\beta}(\mathcal{E}(\rho), \Phi) \geq I^{\alpha,\beta}(\rho, \Phi),$$

provided that one of the following two conditions is satisfied for all i:

(1) $\mathcal{E}^\dagger(K_i) = K_i$, $\mathcal{E}^\dagger(K_i^\dagger K_i) = K_i^\dagger K_i$, $\mathcal{E}^\dagger(K_i K_i^\dagger) = K_i K_i^\dagger$ and $[\rho^{1-\alpha-\beta}, K_i] = 0$;

(2) $\mathcal{E}^\dagger(K_i) = K_i$, $\mathcal{E}^\dagger(K_i^\dagger K_i) = K_i^\dagger K_i$ and $[\rho^{\alpha+\beta}, K_i] = 0$.

(v) (Strong monotonicity) If a channel $\mathcal{E}$ admits the representation $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$,

then
$$\sum_i p_i I^{\alpha,\beta}(\rho_i, \Phi) \geq I^{\alpha,\beta}(\rho, \Phi),$$

where $p_i = \mathrm{Tr} E_i \rho E_i^\dagger$ and $\rho_i = E_i \rho E_i^\dagger / p_i$, provided that condition (1) or (2) in item (iv) is satisfied for all i and $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$.

Proof. (i) and (ii) can be verified easily from the definitions and the property (ii) in Proposition 1. Direct calculation shows that

$$
\begin{aligned}
&4I^{\alpha,\beta}(\rho^a \otimes \rho^b, \Phi^a \otimes \mathbf{1}^b) \\
=\ & 1 - \mathrm{Tr}[(\rho^a \otimes \rho^b)^{1-\alpha}(\Phi^{a\dagger} \otimes \mathbf{1}^b)((\rho^a \otimes \rho^b)^\alpha)] - \mathrm{Tr}[(\rho^a \otimes \rho^b)^{1-\beta}(\Phi^{a\dagger} \otimes \mathbf{1}^b)((\rho^a \otimes \rho^b)^\beta)] \\
& -\mathrm{Tr}[(\rho^a \otimes \rho^b)^{1-\alpha-\beta}(\Phi^{a\dagger} \otimes \mathbf{1}^b)((\rho^a \otimes \rho^b)^{\alpha+\beta})] \\
=\ & 1 - \mathrm{Tr}[(\rho^a)^{1-\alpha} \otimes (\rho^b)^{1-\alpha}(\Phi^{a\dagger}((\rho^a)^\alpha) \otimes (\rho^b)^\alpha)] - \mathrm{Tr}[(\rho^a)^{1-\beta} \otimes (\rho^b)^{1-\beta}(\Phi^{a\dagger}((\rho^a)^\beta) \otimes (\rho^b)^\beta)] \\
& +\mathrm{Tr}[(\rho^a)^{1-\alpha-\beta} \otimes (\rho^b)^{1-\alpha-\beta}(\Phi^{a\dagger}((\rho^a)^{\alpha+\beta}) \otimes (\rho^b)^{\alpha+\beta})] \\
=\ & 1 - \mathrm{Tr}[(\rho^a)^{1-\alpha}(\Phi^{a\dagger}((\rho^a)^\alpha))] - \mathrm{Tr}[(\rho^a)^{1-\beta}(\Phi^{a\dagger}((\rho^a)^\beta))] + \mathrm{Tr}[(\rho^a)^{1-\alpha-\beta}(\Phi^{a\dagger}((\rho^a)^{\alpha+\beta}))] \\
=\ & 4I^{\alpha,\beta}(\rho^a, \Phi^a).
\end{aligned}
$$

Hence item (iii) holds.

It follows from Eq. (15) that

$$I^{\alpha,\beta}(\rho, K_i) = \frac{1}{4}[\mathrm{Tr}(\rho^{1-\alpha-\beta}K_i^\dagger \rho^{\alpha+\beta}K_i) + \mathrm{Tr}(\rho K_i^\dagger K_i) - \mathrm{Tr}(\rho^{1-\alpha}K_i^\dagger \rho^\alpha K_i) - \mathrm{Tr}(\rho^{1-\beta}K_i^\dagger \rho^\beta K_i)].$$

If $[\rho^{1-\alpha-\beta}, K_i] = 0$, the above equation becomes

$$I^{\alpha,\beta}(\rho, K_i) = \frac{1}{4}[\mathrm{Tr}(\rho K_i K_i^\dagger) + \mathrm{Tr}(\rho K_i^\dagger K_i) - \mathrm{Tr}(\rho^{1-\alpha}K_i^\dagger \rho^\alpha K_i) - \mathrm{Tr}(\rho^{1-\beta}K_i^\dagger \rho^\beta K_i)].$$

If $[\rho^{\alpha+\beta}, K_i] = 0$, the above equation becomes

$$I^{\alpha,\beta}(\rho, K_i) = \frac{1}{4}[2\mathrm{Tr}(\rho K_i^\dagger K_i) - \mathrm{Tr}(\rho^{1-\alpha}K_i^\dagger \rho^\alpha K_i) - \mathrm{Tr}(\rho^{1-\beta}K_i^\dagger \rho^\beta K_i)].$$

For $p \in (0,1)$, $f(x) = x^p$ is operator monotone as well as operator concave [69]. So in either case, one can prove the conclusion (iv) by following the same line of the proof of Theorem 4 in [44] under the assumptions of item (iv).

At last, since (iii) holds, imitating the proof of Theorem 4 in [44], one proves that (v) is also true. $\square$

Remark 2 By Lemma 3 in [44], it is observed that $\mathcal{E}^\dagger(K_i) = K_i$, $\mathcal{E}^\dagger(K_i^\dagger K_i) = K_i^\dagger K_i$ and $\mathcal{E}^\dagger(K_i K_i^\dagger) = K_i K_i^\dagger$ if and only if $[E_i, K_j] = 0$ and $[E_i, K_j^\dagger] = 0$ for all i and j. The later one could be seen as the condition that the channel $\mathcal{E}$ does not disturb channel Φ, which was proposed as the assumption in proving the monotonicity of $I(\rho, \Phi)$ in [43]. Taking this fact into consideration, it follows from Theorem 1 that $I^{\alpha,\beta}(\rho, \Phi)$ $(\alpha, \beta \geq 0,\ \alpha + \beta \leq 1)$ could be viewed as a family of coherence measures with respect to a channel Φ under the restrictive conditions.

Remark 3 Direct computation shows that property (iv) (ancillary independence) in Theorem 1 does not hold for $V^{\alpha,\beta}(\rho, \Phi)$ ($\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$) unless $\alpha = \beta$. But the proof of strong monotonicity in Theorem 1 relies heavily on the property of the ancillary independence. Hence the method used in [44] fails, and we do not know whether $V^{\alpha,\beta}(\rho, \Phi)$ can be regarded as a family of coherence measures with respect to a channel in general. Note that $V^{\alpha,\alpha}(\rho, \Phi) = I^{\alpha,\alpha}(\rho, \Phi)$. $V^{\alpha,\alpha}(\rho, \Phi)$ ($0 \leq \alpha \leq 1$) is just a special class of $I^{\alpha,\beta}(\rho, \Phi)$ ($\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$).

Now we calculate $I^{\alpha,\beta}(\rho, \Phi)$ and $V^{\alpha,\beta}(\rho, \Phi)$ for qubit states with respect to different kinds of quantum channels. Consider the Pauli channel defined by

$$\Phi(\rho) = \sum_{j=0}^{3} p_j \sigma_j \rho \sigma_j, \quad p_j \geq 0, \quad \sum_{j=0}^{3} p_j = 1, \tag{23}$$

where $\sigma_0 = I$, and σ_i, $i = 1, 2, 3$, are Pauli matrices. For a qubit state $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \sigma)$, where $\mathbf{r} = \{r_1, r_2, r_3\}$ and $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, its eigenvalues are $\lambda_{1,2} = (1 \mp |\mathbf{r}|)/2$, and

$$\rho^\alpha = \begin{pmatrix} \frac{\lambda_1^\alpha + \lambda_2^\alpha}{2} + \frac{r_3(\lambda_2^\alpha - \lambda_1^\alpha)}{2|\mathbf{r}|} & \frac{(-r_1 + ir_2)(\lambda_1^\alpha - \lambda_2^\alpha)}{2|\mathbf{r}|} \\ \frac{(-r_1 - ir_2)(\lambda_1^\alpha - \lambda_2^\alpha)}{2|\mathbf{r}|} & \frac{\lambda_1^\alpha + \lambda_2^\alpha}{2} - \frac{r_3(\lambda_2^\alpha - \lambda_1^\alpha)}{2|\mathbf{r}|} \end{pmatrix}.$$

Then we have

$$I^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4} \sum_{j=1}^{3} p_j \frac{|\mathbf{r}|^2 - r_j^2}{|\mathbf{r}|^2} (\lambda_1^\alpha - \lambda_2^\alpha)(\lambda_1^\beta - \lambda_2^\beta)(\lambda_1^{1-\alpha-\beta} + \lambda_2^{1-\alpha-\beta}). \tag{24}$$

In particular, when $p_1 = p_2 = p_3 = p$, the Pauli channel defined in Eq. (23) becomes the depolarizing channel, and we have

$$I^{\alpha,\beta}(\rho, \Phi) = \frac{1}{2} p (\lambda_1^\alpha - \lambda_2^\alpha)(\lambda_1^\beta - \lambda_2^\beta)(\lambda_1^{1-\alpha-\beta} + \lambda_2^{1-\alpha-\beta}), \tag{25}$$

which is an increasing function of p. Taking $p_1 = p$ and $p_2 = p_3 = 0$ in Eq. (23), we get the bit-flipping channel, and have

$$I^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4} \cdot \frac{p(r_2^2 + r_3^2)}{|\mathbf{r}|^2} (\lambda_1^\alpha - \lambda_2^\alpha)(\lambda_1^\beta - \lambda_2^\beta)(\lambda_1^{1-\alpha-\beta} + \lambda_2^{1-\alpha-\beta}). \tag{26}$$

Setting $p_1 = p_2 = 0$ and $p_3 = p$ in Eq. (23), we obtain the phase-flipping channel, and get

$$I^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4} \cdot \frac{p(r_1^2 + r_2^2)}{|\mathbf{r}|^2} (\lambda_1^\alpha - \lambda_2^\alpha)(\lambda_1^\beta - \lambda_2^\beta)(\lambda_1^{1-\alpha-\beta} + \lambda_2^{1-\alpha-\beta}). \tag{27}$$

For the (unital) amplitude damping channel $\Phi(\rho) = \sum_{j=1}^{2} K_j \rho K_j^\dagger$ with

$$K_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \quad K_2 = \sqrt{q}|1\rangle\langle 1|, \quad 0 \leq q \leq 1, \tag{28}$$

we have

$$I^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4} \cdot \frac{(1 - \sqrt{1-q})(r_1^2 + r_2^2)}{2|\mathbf{r}|^2} (\lambda_1^\alpha - \lambda_2^\alpha)(\lambda_1^\beta - \lambda_2^\beta)(\lambda_1^{1-\alpha-\beta} + \lambda_2^{1-\alpha-\beta}). \tag{29}$$

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And for the (nonunital) amplitude damping channel $\Phi(\rho) = \sum_{j=1}^{2} K_j \rho K_j^{\dagger}$ with

$$K_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \quad K_2 = \sqrt{q}|0\rangle\langle 1|, \quad 0 \le q \le 1, \tag{30}$$

we have

$$
\begin{aligned}
I^{\alpha,\beta}(\rho, \Phi) &= \frac{1}{4}\left[\frac{(1-\sqrt{1-q})(r_1^2+r_2^2)+qr_3^2}{2|\mathbf{r}|^2}(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta})\right.\\
&\quad\left.+\frac{qr_3}{2|\mathbf{r}|}(\lambda_1^{1-\alpha-\beta}-\lambda_2^{1-\alpha-\beta})\right](\lambda_1^{\alpha}-\lambda_2^{\alpha})(\lambda_1^{\beta}-\lambda_2^{\beta}).
\end{aligned} \tag{31}
$$

In a similar way, we can also compute $V^{\alpha,\beta}(\rho, \Phi)$ for a qubit state $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$ with respect to different channels. For the Pauli channel Φ defined in Eq. (23), we have

$$V^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4}\sum_{j=1}^{3}p_j\frac{|\mathbf{r}|^2-r_j^2}{4|\mathbf{r}|^2}(\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta}). \tag{32}$$

For the depolarizing channel Φ (a special case when $p_1 = p_2 = p_3 = p$), we have

$$V^{\alpha,\beta}(\rho, \Phi) = \frac{1}{8}p(\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta}), \tag{33}$$

which is also an increasing function of p. For the bit-flipping channel Φ (a special case when $p_1 = p$ and $p_2 = p_3 = 0$), we have

$$V^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4}\cdot\frac{p(r_2^2+r_3^2)}{4|\mathbf{r}|^2}(\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta}). \tag{34}$$

For the phase-flipping channel Φ (a special case when $p_1 = p_2 = 0$ and $p_3 = p$), we have

$$V^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4}\cdot\frac{p(r_1^2+r_2^2)}{4|\mathbf{r}|^2}(\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta}). \tag{35}$$

For the (unital) amplitude damping channel (28), we have

$$V^{\alpha,\beta}(\rho, \Phi) = \frac{1}{4}\cdot\frac{(1-\sqrt{1-q})(r_1^2+r_2^2)}{8|\mathbf{r}|^2}(\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta}). \tag{36}$$

And for the (nonunital) amplitude damping channel (30), we have

$$
\begin{aligned}
V^{\alpha,\beta}(\rho, \Phi) &= \frac{1}{4}\left[\frac{(1-\sqrt{1-q})(r_1^2+r_2^2)+qr_3^2}{8|\mathbf{r}|^2}(\lambda_1^{1-\alpha-\beta}+\lambda_2^{1-\alpha-\beta})\right.\\
&\quad\left.+\frac{qr_3}{8|\mathbf{r}|}(\lambda_1^{1-\alpha-\beta}-\lambda_2^{1-\alpha-\beta})\right](\lambda_1^{\alpha}-\lambda_2^{\alpha}+\lambda_1^{\beta}-\lambda_2^{\beta})^2.
\end{aligned} \tag{37}
$$

We can see that when $\alpha = \beta = \frac{1}{2}$, the above analytical expressions for a qubit state with respect to certain channels reduce to the corresponding ones in Ref. [43].

5. Complementarity relations and information conservation

In Section 3, we have defined $I^{\alpha,\beta}(\rho,\Phi)$ and $J^{\alpha,\beta}(\rho,\Phi)$ using commutator and anti-commutator, which is in a dual fashion in some sense. Another kind of generalization of the quantities $V^{\alpha,\beta}(\rho,\Phi)$ and $W^{\alpha,\beta}(\rho,\Phi)$ have also been introduced. In this section, we discuss the relationships between these two sets of quantities.

Set

$$C^{\alpha,\beta}(\rho,K) = \frac{1}{2}[\operatorname{Tr}(\rho^{1-\alpha}K^{\dagger}\rho^{\alpha}K) + \operatorname{Tr}(\rho^{1-\beta}K^{\dagger}\rho^{\beta}K)], \quad \alpha,\beta \geq 0, \ \alpha+\beta \leq 1.$$

It follows from Eqs. (15) and (16) that

$$I^{\alpha,\beta}(\rho,K) = \frac{1}{4}[\operatorname{Tr}(\rho^{1-\alpha-\beta}K^{\dagger}\rho^{\alpha+\beta}K + K\rho K^{\dagger}) - 2C^{\alpha,\beta}(\rho,K)]$$

and

$$J^{\alpha,\beta}(\rho,K) = \frac{1}{4}[\operatorname{Tr}(\rho^{1-\alpha-\beta}K^{\dagger}\rho^{\alpha+\beta}K + K\rho K^{\dagger}) + 2C^{\alpha,\beta}(\rho,K)].$$

Noting that

$$C^{\alpha,\beta}(\rho,K) = \frac{1}{2}(\|\rho^{\frac{\alpha}{2}}K\rho^{\frac{1-\alpha}{2}}\|_2^2 + \|\rho^{\frac{\beta}{2}}K\rho^{\frac{1-\beta}{2}}\|_2^2) \geq 0,$$

we have

$$I^{\alpha,\beta}(\rho,K) \leq J^{\alpha,\beta}(\rho,K),$$

and thus

$$I^{\alpha,\beta}(\rho,\Phi) \leq J^{\alpha,\beta}(\rho,\Phi).$$

For any channel Φ with Kraus representation in the form of Eq. (10), the quantities $I^{\alpha,\beta}(\rho,\Phi)$ and $J^{\alpha,\beta}(\rho,\Phi)$ can be rewritten as

$$I^{\alpha,\beta}(\rho,\Phi) = \frac{1}{4}\operatorname{Tr}[\rho^{1-\alpha-\beta}\Phi^{\dagger}(\rho^{\alpha+\beta}) + \Phi(\rho) - \rho^{1-\alpha}\Phi^{\dagger}(\rho^{\alpha}) - \rho^{1-\beta}\Phi^{\dagger}(\rho^{\beta})]$$

and

$$J^{\alpha,\beta}(\rho,\Phi) = \frac{1}{4}\operatorname{Tr}[\rho^{1-\alpha-\beta}\Phi^{\dagger}(\rho^{\alpha+\beta}) + \Phi(\rho) + \rho^{1-\alpha}\Phi^{\dagger}(\rho^{\alpha}) + \rho^{1-\beta}\Phi^{\dagger}(\rho^{\beta})],$$

which implies that for $\alpha,\beta \geq 0, \ \alpha+\beta \leq 1$,

$$I^{\alpha,\beta}(\rho,\Phi) + J^{\alpha,\beta}(\rho,\Phi) = \frac{1}{2}\operatorname{Tr}(\rho^{1-\alpha-\beta}\Phi^{\dagger}(\rho^{\alpha+\beta}) + \Phi(\rho)). \tag{38}$$

When $\alpha+\beta=1$, Eq. (38) reduces to the following one:

$$I^{\alpha}(\rho,\Phi) + J^{\alpha}(\rho,\Phi) = \frac{1}{2}\operatorname{Tr}[\Phi^{\dagger}(\rho) + \Phi(\rho)], \quad 0 \leq \alpha \leq 1. \tag{39}$$

In particular, when Φ is a unital quantum operation, i.e., $\sum_i K_i^{\dagger}K_i = I$ and $\sum_i K_iK_i^{\dagger} = I$, we get

$$I^{\alpha}(\rho,\Phi) + J^{\alpha}(\rho,\Phi) = 1, \quad 0 \leq \alpha \leq 1. \tag{40}$$

which could be viewed as a class of conservation relations.

In a similar manner, we can investigate the relationship between $V^{\alpha,\beta}(\rho,\Phi)$ and $W^{\alpha,\beta}(\rho,\Phi)$. Define the following quantity

$$D^{\alpha,\beta}(\rho,K) = \text{Tr}\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}K^\dagger\frac{\rho^\alpha + \rho^\beta}{2}K\right), \quad \alpha,\beta \geq 0, \ \alpha + \beta \leq 1.$$

It follows from Eqs. (19) and (20) that

$$V^{\alpha,\beta}(\rho,K) = \frac{1}{4}\left[\text{Tr}\left(\rho^{1-\alpha-\beta}K^\dagger\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)^2 K + K\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)K^\dagger\right) - 2D^{\alpha,\beta}(\rho,K)\right]$$

and

$$W^{\alpha,\beta}(\rho,K) = \frac{1}{4}\left[\text{Tr}\left(\rho^{1-\alpha-\beta}K^\dagger\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)^2 K + K\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)K^\dagger\right) + 2D^{\alpha,\beta}(\rho,K)\right].$$

Noting that

$$D^{\alpha,\beta}(\rho,K) = \frac{1}{4}(\|\rho^{\frac{\alpha}{2}}K\rho^{\frac{1-\alpha}{2}}\|_2^2 + \|\rho^{\frac{\beta}{2}}K\rho^{\frac{1-\beta}{2}}\|_2^2 + \rho^{\frac{\alpha}{2}}K\rho^{\frac{1-\beta}{2}}\|_2^2 + \rho^{\frac{\beta}{2}}K\rho^{\frac{1-\alpha}{2}}\|_2^2) \geq 0,$$

we have

$$V^{\alpha,\beta}(\rho,K) \leq W^{\alpha,\beta}(\rho,K),$$

and thus

$$V^{\alpha,\beta}(\rho,\Phi) \leq W^{\alpha,\beta}(\rho,\Phi).$$

For any channel Φ with Kraus representation in the form of Eq. (10), the quantities $V^{\alpha,\beta}(\rho,\Phi)$ and $W^{\alpha,\beta}(\rho,\Phi)$ can be rewritten as

$$V^{\alpha,\beta}(\rho,\Phi) = \frac{1}{4}\text{Tr}\left[\rho^{1-\alpha-\beta}\Phi^\dagger\left(\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)^2\right) + \Phi\left(\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)\right) - 2\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\Phi^\dagger(\frac{\rho^\alpha + \rho^\beta}{2})\right]$$

and

$$W^{\alpha,\beta}(\rho,\Phi) = \frac{1}{4}\text{Tr}\left[\rho^{1-\alpha-\beta}\Phi^\dagger\left(\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)^2\right) + \Phi\left(\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)\right) + 2\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\Phi^\dagger\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)\right],$$

which implies that for $\alpha,\beta \geq 0$, $\alpha + \beta \leq 1$,

$$V^{\alpha,\beta}(\rho,\Phi) + W^{\alpha,\beta}(\rho,\Phi)$$
$$= \frac{1}{4}\text{Tr}\left[\rho^{1-\alpha-\beta}\Phi^\dagger\left(\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)^2\right) + \Phi\left(\left(\frac{\rho^{1-\alpha} + \rho^{1-\beta}}{2}\right)\left(\frac{\rho^\alpha + \rho^\beta}{2}\right)\right)\right]. \quad (41)$$

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Furthermore, when $\alpha = \beta = \frac{1}{2}$, Eq. (41) reduces to

$$V^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) + W^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) = \frac{1}{2}\text{Tr}[\Phi^\dagger(\rho) + \Phi(\rho)]. \tag{42}$$

In particular, when Φ is a unital quantum operation, i.e., $\sum_i K_i^\dagger K_i = I$ and $\sum_i K_i K_i^\dagger = I$, we obtain

$$V^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) + W^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) = 1, \tag{43}$$

which could also be regarded as a conservation relation. Note that from the definitions, $V^{\frac{1}{2},\frac{1}{2}}(\rho, K) = I(\rho, K)$ and $W^{\frac{1}{2},\frac{1}{2}}(\rho, K) = J(\rho, K)$, and thus $V^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) = I(\rho, \Phi)$ and $W^{\frac{1}{2},\frac{1}{2}}(\rho, \Phi) = J(\rho, \Phi)$, Eqs. (42) and (43) hold naturally since they are in accord with Eqs. (32) and (33) in Ref. [43], respectively.

Inspired by Ref. [43], we now give two interpretations of the conservation relations Eq. (40) derived above. First of all, for any group representation $\{U_g : g \in G\}$ on a finite group G, we can define a quantum channel

$$\Phi(\rho) = \frac{1}{|G|} \sum_{g \in G} U_g \rho U_g^\dagger, \tag{44}$$

where $|G|$ denotes the number of elements in G. For a compact Lie group G, it can be also defined that

$$\Phi(\rho) = \int \text{d}\mu U_g \rho U_g^\dagger, \tag{45}$$

where U_g is the unitary representation of $g \in G$ and $\text{d}\mu$ is the Haar measure with respect to G. In this regard, it is natural to interpret $J^\alpha(\rho, \Phi)$ and $I^\alpha(\rho, \Phi)$ as symmetry and asymmetry of ρ with respect to the group G, respectively. Therefore, Eq. (40) could be viewed as a family of symmetry-asymmetry complementarity relations.

Secondly, we illustrate Eq. (40) with wave-particle duality in the Mach-Zehnder interferometry with a path detector [70]. The beam-splitter, mirror and phase shift are realized by the following unitary matrices

$$U_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix},$$

respectively, while the path detector is realized by the unitary matrix V. The input state is a bipartite state $\rho^{ab} = \rho \otimes \tau$, where $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \sigma)$ is the initial (external) system state and τ is the initial (internal) state. Here $\mathbf{r} = \{r_1, r_2, r_3\}$ is the Bloch vector with $|\mathbf{r}|^2 \leq 1$ and $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ with σ_i $(i = 1, 2, 3)$ the Pauli matrices. The interferometry channel is

$$\Phi(\rho) = \text{Tr}_b(U(\rho \otimes \tau)U^\dagger), \tag{46}$$

where $U = U_B^{ab} U_M^{ab} V^{ab} U_B^{ab}$, $U_B^{ab} = U_B \otimes \mathbf{1}^b$, $U_M^{ab} = U_M \otimes \mathbf{1}^b$ and $V^{ab} = e^{i\theta}|0\rangle\langle 0| \otimes \mathbf{1}^b +$

$|1\rangle\langle 1| \otimes V$. Direct calculation shows that

$$
\begin{aligned}
I^\alpha(\rho, \Phi) &= \frac{1}{4}\left[2 - (\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})(\lambda_1^\alpha + \lambda_2^\alpha) + \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2} r_1^2 \right. \\
&\quad \left. - \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2}(1 - r_1^2)|\mathrm{Tr}(V\tau)|\cos(\theta - \nu - \gamma) \right],
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
J^\alpha(\rho, \Phi) &= \frac{1}{4}\left[2 + (\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})(\lambda_1^\alpha + \lambda_2^\alpha) - \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2} r_1^2 \right. \\
&\quad \left. + \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2}(1 - r_1^2)|\mathrm{Tr}(V\tau)|\cos(\theta - \nu - \gamma) \right],
\end{aligned}
\tag{48}
$$

where $\nu = \arg(\mathrm{Tr}(V\tau))$, $\gamma = \arctan\frac{2r_2 r_3}{r_2^2 - r_3^2}$ and $\lambda_{1,2} = (1 \mp |\mathbf{r}|)/2$ are the eigenvalues of the qubit state $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \sigma)$.

Since $I^\alpha(\rho, \Phi)$ is a manifestation of the asymmetry, by minimizing over the phase shift, the quantity

$$
\begin{aligned}
\tilde{P}^\alpha(\rho, \Phi) &= \min_\theta I^\alpha(\rho, \Phi) \\
&= \frac{1}{4}\left[2 - (\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})(\lambda_1^\alpha + \lambda_2^\alpha) + \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2} r_1^2 \right. \\
&\quad \left. - \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2}(1 - r_1^2)|\mathrm{Tr}(V\tau)| \right]
\end{aligned}
\tag{49}
$$

gives a quantification for the which-path information. In contrast, since $J^\alpha(\rho, \Phi)$ represents the symmetry, by maximizing over the phase shift, the quantity

$$
\begin{aligned}
\tilde{W}^\alpha(\rho, \Phi) &= \max_\theta J^\alpha(\rho, \Phi) \\
&= \frac{1}{4}\left[2 + (\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})(\lambda_1^\alpha + \lambda_2^\alpha) - \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2} r_1^2 \right. \\
&\quad \left. + \frac{(\lambda_1^{1-\alpha} - \lambda_2^{1-\alpha})(\lambda_1^\alpha - \lambda_2^\alpha)}{|r|^2}(1 - r_1^2)|\mathrm{Tr}(V\tau)| \right]
\end{aligned}
\tag{50}
$$

could be exploited as a measure of the interference visibility. Utilizing these quantities with parameter α, we thus obtain a family of complementarity relations signifying the wave-particle duality:

$$
\tilde{P}^\alpha(\rho, \Phi) + \tilde{W}^\alpha(\rho, \Phi) = 1, \quad 0 \leq \alpha \leq 1,
\tag{51}
$$

which is more general than Eq. (43) in Ref. [43].

6. Conclusions and discussions

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We have showed that the state-channel interactions based on modified generalized Wigner-Yanase-Dyson skew information could be exploited as a family of coherence measures with respect to a quantum channel under certain assumptions. We have also provided explicit analytical expressions of this family of coherence measures for qubit states with respect to various kinds of quantum channels. Moreover, complementarity relations based on MGWYD skew information and MWGWYD skew information have been derived. Inspired by Ref. [43], we have also presented two interpretations of the conservation relations: the interplay between symmetry and asymmetry with respect to a group, and the wave-particle duality.

Coherence and complementarity are both fundamental issues in quantum mechanics and quantum information theory. As is shown in Ref. [43], the quantities $I(\rho, \Phi)$ and $J(\rho, \Phi)$ not only present a basic and an alternative framework for addressing complementarity, but also put forward the study of coherence in a broad context involving quantum channels. The quantity $I(\rho, \Phi)$ has several interpretations including asymmetry, coherence, incompatibility, quantumness and quantum uncertainty with state-channel interactions, while $J(\rho, \Phi)$ possesses the corresponding interpretations in a dual fashion. We have revisited the work of Refs. [43] and [44] based on the interplay of $I^{\alpha,\beta}(\rho, \Phi)$ and $J^{\alpha,\beta}(\rho, \Phi)$ as well as $V^{\alpha,\beta}(\rho, \Phi)$ and $W^{\alpha,\beta}(\rho, \Phi)$ induced by MGWYD skew information and MWGWYD skew information, and investigated the coherence and complementarity based on measures with parameters α and β. Therefore, our results are valid for a large family of quantities, which are more general than the ones induced by MWY skew information and MWYD skew information proposed in Ref. [43] and Ref. [44], respectively.

The quantities in Eqs. (8) and (9) are defined in terms of the Hilbert-Schmidt norm. A natural question arises: could $I(\rho, \Phi) = \sum_i \|[\rho^{\frac{1}{2}}, K_i]\|_p^2$ be exploited as a measure of coherence with respect to a quantum channel? Or more generally, could $I(\rho, \Phi) = \sum_i |||[\rho^{\frac{1}{2}}, K_i]|||^2$ be utilized as a measure of coherence with respect to a quantum channel for other unitarily invariant norms other than Schatten p-norms? These questions deserve further investigations.

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