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Local cohomology on a subexceptional series of representations

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LOCAL COHOMOLOGY ON A SUBEXCEPTIONAL SERIES OF REPRESENTATIONS

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Abstract. We consider a series of four subexceptional representations coming from the third line of the Freudenthal-Tits magic square; using Bourbaki notation, these are fundamental representations (G', X) corresponding to (C_3, ω_3), (A_5, ω_3), (D_6, ω_5) and (E_7, ω_6). In each of these four cases, the group $G = G' \times \mathbb{C}^*$ acts on X with five orbits, and many invariants display a uniform behavior, *e.g.* dimension of orbits, their defining ideals and the character of their coordinate rings as G-modules. In this paper, we determine some more subtle invariants and analyze their uniformity within the series. We describe the category of G-equivariant coherent $\mathcal{D}_X$-modules as the category of representations of a quiver with relations. We construct explicitly the simple G-equivariant $\mathcal{D}_X$-modules and compute the characters of their underlying G-structures. We determine the local cohomology groups with supports given by orbit closures, determining their precise $\mathcal{D}_X$-module structure. As a consequence, we calculate the intersection cohomology groups and Lyubeznik numbers of the orbit closures. While our results for the cases (A_5, ω_3), (D_6, ω_5) and (E_7, ω_6) are still completely uniform, the case (C_3, ω_3) displays a surprisingly different behavior. We give two explanations for this phenomenon: one topological, as the middle orbit of (C_3, ω_3) is not simply-connected; one geometric, as the closure of the orbit is not Gorenstein.

1. Introduction

The subexceptional series coming the third line of the Freudenthal–Tits magic square corresponds to the following four Dynkin formats: C_3, A_5, D_6, E_7 [Fre64, Page 168]. This line stands for the 5-dimensional symplectic geometries as explained in [Fre64]. The third line of the extended magic square has the six Dynkin formats [LM04, Section 6]:

$$A_1, \quad A_1 \times A_1 \times A_1, \quad C_3, \quad A_5, \quad D_6, \quad E_7. \tag{1.1}$$

There are respective parameters

$$m = -2/3, 0, 1, 2, 4, 8.$$

For each of the corresponding Lie groups G', there is a preferred irreducible representation X [LM04, Section 6] that displays some uniform behavior within the series. In all (but the first) cases, $G = G' \times \mathbb{C}^*$ acts on X with five orbits, and $\dim X = 6m + 8$.

The first, second, and fourth representations from the series (1.1) correspond to the space of binary cubic forms, of $2 \times 2 \times 2$ hypermatrices and of alternating senary 3-tensors, respectively. The equivariant $\mathcal{D}$-modules and local cohomology modules for these representations are studied in detail in the articles [LRW19], [Per18] and [LP18], respectively. In this paper we complete the analogous study for the rest of the representations within the series (1.1), emphasizing the uniformity of the methods and results.

For the Dynkin diagrams C_3, D_6, E_7, we use the following conventions on the ordering of nodes:

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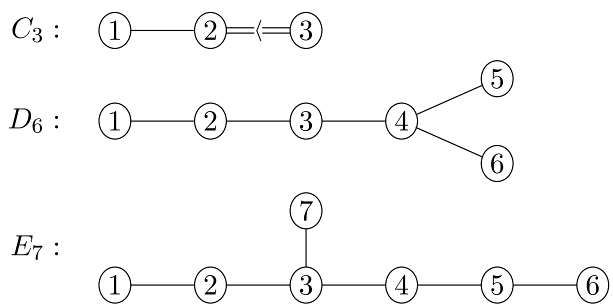


With the exception of Section 5.2, throughout the article the representation (G', X) always denotes either (C_3, ω_3), (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6), which come from the third line of the Freudenthal–Tits magic square. Here (D_6, ω_5) corresponds to the (even) half-spin representation. To display the uniformity of results better, we include the case (A_5, ω_3), albeit the results in this case are worked out completely in [LP18].

The representations in the subexceptional series are representations with finitely many orbits that are not spherical varieties. For the irreducible representations of the latter kind, the categories of equivariant $\mathcal{D}$-modules have been described in [LW19]. Nevertheless, for our four cases the algebra of covariants $\mathbb{C}[X]^U$ is a polynomial ring [Bri83], where U denotes a maximal unipotent subgroup of G.

Throughout $S = \mathbb{C}[X]$ is the polynomial ring on X, and $\mathcal{D} = \mathcal{D}_X$ is the Weyl algebra of differential operators on X with polynomial coefficients. Let $\text{mod}_G(\mathcal{D}_X)$ denote the category of G-equivariant coherent $\mathcal{D}$-modules on X (which are regular and holonomic in our situation). According to the Riemann–Hilbert correspondence, equivariant $\mathcal{D}$-modules correspond to equivariant perverse sheaves, and the simple equivariant $\mathcal{D}$-modules correspond to irreducible equivariant local systems on the orbits. Nevertheless, their explicit realization is in general a difficult problem (see Open Problem 3 in [MV86, Section 6]).

We give explicit constructions for all the simple equivariant $\mathcal{D}$-modules and determine the characters of their underlying G-module structures. The formulas are written as uniformly as possible. Similar formulas for characters were obtained in various other equivariant situations [Rai16, Rai17, LRW19, LP18].

In the case (C_3, ω_3) there is an extra simple due to the fact that its middle orbit is not simply-connected. Moreover, the fact that there are no (semi)-invariant sections for the simple $\mathcal{D}$-module corresponding to the trivial local system on the middle orbit is reflected by the fact that this orbit closure is the only one that is not Gorenstein. More generally, we link the roots of the Bernstein–Sato polynomials of semi-invariant polynomials to the Castelnuovo–Mumford regularity of Gorenstein orbit closures.

As the group is acting with finitely many orbits, the category $\text{mod}_G(\mathcal{D}_X)$ of equivariant coherent $\mathcal{D}$-modules is equivalent to the category of finite-dimensional representations of a quiver with relations (see [Vil94] and [LW19]). We determine the quiver structure of the category of $\text{mod}_G(\mathcal{D}_X)$ (see Section 3.3). The quivers appear also in [LW19] and [LP18], and have finitely many indecomposable representations that are described explicitly [LW19, Theorem 2.11]. Again, only the case (C_3, ω_3) displays exceptional behavior as the equivariant $\mathcal{D}$-module corresponding to the trivial local system on the middle orbit is disconnected from the rest.

For any G-stable closed subset Z in X, the local cohomology modules $H^i_Z(S)$ are G-equivariant coherent $\mathcal{D}$-modules, for all $i \geq 0$. While being objects of great interest, explicit computations of local cohomology modules are in general difficult. Several results been obtained for representations with finitely many orbits (see [RW14, RWW14, RW16, LRW19, LR18, Per18]). In spirit of these results, we determine the explicit $\mathcal{D}$-module structures of local cohomology modules for our series. The results are uniform with respect to the parameter m, with some discrepancy in the case (C_3, ω_3) again.

The article is organized as follows. In Section 2, we introduce the basic terminology regarding representations of reductive algebraic groups (Section 2.1), equivariant $\mathcal{D}$-modules (Section 2.2), representation theory of quivers (Section 2.3) and the Borel–Weil–Bott theorem (Section 2.4). In Section 3 we describe explicitly the category equivariant $\mathcal{D}$-modules. We compute the $\mathcal{D}$-module (Section 3.2) and G-module (Section 3.4 and Theorem 4.9) structure of the simple objects explicitly, and we determine the quivers corresponding to the categories (Section 3.3). In Section 4 we determine all the local cohomology modules with supports given by orbit closures. As an application of our main results, we calculate the intersection cohomology groups and Lyubeznik numbers of orbit closures (Section 5.1). In Section 5.2, we establish a connection between Castelnuovo–Mumford regularity of Gorenstein orbit closures and the roots of the Bernstein–Sato polynomials of semi-invariant polynomials.

2. Preliminaries

Throughout we use the Bourbaki notation for irreducible highest weight representations. The pair (G', X) denotes either (C_3, ω_3), (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6), and we fix the parameter $m = 1, 2, 4, 8$ respectively.

2.1. **Representations and Characters.** Let Λ be the set of isomorphism classes of finite dimensional irreducible representations of the simple, simply-connected algebraic group G'. We identify Λ with the set of dominant integral weights. We put ω_i for the ith fundamental representation of G', and write $(a_1, a_2, \ldots, a_k)$ for the highest weight of the irreducible G'-module $a_1\omega_1 + a_2\omega_2 + \cdots + a_k\omega_k$, where $a_i \in \mathbb{Z}_{\geq 0}$ and k is the rank of G'.

The Cartan product of two irreducible G'-modules is defined by $V_\lambda \cdot V_\mu := V_{\lambda+\mu}$, where $\lambda, \mu \in \Lambda$.

We deal mostly with $\mathbb{Z}$-graded G'-modules, that is, representations of the group $G = G' \times \mathbb{C}^*$. We keep track of the grading using a parameter t. A rational G-representation (possibly infinite-dimensional) M is admissible if each G-representation appears (up to isomorphism) with finite multiplicity in M. Equivalently, any graded piece M_d $(d \in \mathbb{Z})$ decomposes as a direct sum of G'-representations

$$M_d = \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus m_\lambda^d(M)},$$

with $m_\lambda^d(M) \in \mathbb{Z}_{\geq 0}$. Also, we will use the notation

$$[M] = \bigoplus_{d \geq \mathbb{Z}} t^d M_d.$$

For $\lambda \in \Lambda$ we can make sense of admissible G-representations (where 1 denotes the trivial G'-module)

$$\frac{1}{1 - tV_\lambda} = 1 + tV_\lambda + t^2 V_{2\lambda} + \cdots .$$

2.2. **Equivariant $\mathcal{D}$-modules.** A $\mathcal{D}_X$-module M is (strongly) equivariant if we have a $\mathcal{D}_{G\times X}$-isomorphism $\tau : p^*M \to m^*M$, where $p : G \times X \to X$ denotes the projection and $m : G \times X \to X$ the map defining the action, with τ satisfying the usual compatibility conditions (see [HTT08, Definition 11.5.2]).

Another characterization of equivariant $\mathcal{D}$-modules is as follows. Let $\mathfrak{g}$ denote the Lie algebra of G. Differentiating the G-action on X induces a map from $\mathfrak{g}$ to space of vector fields on X, hence a map $\mathfrak{g} \to \mathcal{D}_X$. Then the $\mathcal{D}$-module M is equivariant if and only if it is endowed with an algebraic G-action, such that differentiating this action we recover the $\mathfrak{g}$-action induced from the map $\mathfrak{g} \to \mathcal{D}_X$.

The category $\mathrm{mod}_G(\mathcal{D}_X)$ of equivariant $\mathcal{D}$-modules is a full subcategory of the category $\mathrm{mod}(\mathcal{D}_X)$ of all coherent $\mathcal{D}$-modules, and it is is closed under taking subquotients. If Z is a G-stable closed subset of X, we denote by $\mathrm{mod}_G^Z(\mathcal{D}_X)$ the full subcategory of $\mathrm{mod}_G(\mathcal{D}_X)$ consisting of equivariant $\mathcal{D}$-modules that have support contained in Z.

In all our cases G acts on X with finitely many orbits. This implies that every module in $\mathrm{mod}_G(\mathcal{D}_X)$ is regular and holonomic [HTT08, Theorem 11.6.1]. The category $\mathrm{mod}_G(\mathcal{D}_X)$ is equivalent to the category of finite-dimensional representations of a quiver with relations (see [Vil94, Theorem 4.3] or [LW19, Theorem 3.4]; for quivers see Section 2.3). For more details on categories of equivariant $\mathcal{D}$-modules, cf. [LW19].

Given an equivariant map between two G-varieties, (derived) pushforward and pullback of $\mathcal{D}$-modules preserves equivariance. In particular, so do local cohomology functors $H^i_Z(\bullet)$, for Z an orbit closure in X. Namely, for each $i \geq 0$ and each $M \in \mathrm{mod}_G(\mathcal{D}_X)$, the i-th local cohomology module $H^i_Z(M)$ of M with support in Z is an element of $\mathrm{mod}_G^Z(\mathcal{D}_X)$.

Since G is reductive, another construction of objects in $\mathrm{mod}_G(\mathcal{D}_X)$ comes from considering the (twisted) Fourier transform [LW19, Section 4.3]. This functor gives a self-equivalence

$$\mathcal{F} : \mathrm{mod}_G(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{mod}_G(\mathcal{D}_X).$$

For $M \in \mathrm{mod}_G(\mathcal{D}_X)$ we have as G-modules

$$\mathcal{F}(M) \cong M^* \cdot \det X^*. \tag{2.1}$$

Throughout we work with the convention that polynomials live in non-negative degrees, and note that $X \cong X^*$ as G'-representations. The action of G on X extends to an action on S and the character of S is given by (see [Bri83] or [LM04, Section 6]):

$$[S] = \frac{1}{(1-tX)(1-t^2\mathfrak{g}')(1-t^3X)(1-t^4)(1-t^4X_4)}, \tag{2.2}$$

where $\mathfrak{g}' = 2\omega_1,\, \omega_1+\omega_5,\, \omega_2,\, \omega_1$ is the adjoint representation and $X_4 = 2\omega_2,\, \omega_2+\omega_4,\, \omega_4,\, \omega_5$ for $(G', X) = (C_3, \omega_3)$, (A_5, ω_3), (D_6, ω_5), (E_7, ω_6), respectively. For the simple $\mathcal{D}$-module $E = \mathcal{F}(S)$ with support equal to the origin, we have by (2.1)):

$$[E] = \frac{t^{-6m-8}}{(1-t^{-1}X)(1-t^{-2}\mathfrak{g}')(1-t^{-3}X)(1-t^{-4})(1-t^{-4}X_4)}. \tag{2.3}$$

2.3. **Quivers.** We briefly introduce some basic notions on the representation theory of quivers, following [ASS06]. A quiver Q is an oriented graph, *i.e.* a pair $Q = (Q_0, Q_1)$ formed by a finite set of vertices Q_0 and a finite set of arrows Q_1. An arrow $a \in Q_1$ has a head ha and a tail ta which are elements of Q_0:

$$ta \xrightarrow{a} ha$$

A relation in Q is a linear combination of paths of length at least two that have the same source and target. We define a quiver (with relations) (Q, I) to be a quiver Q together with a finite collection of relations I.

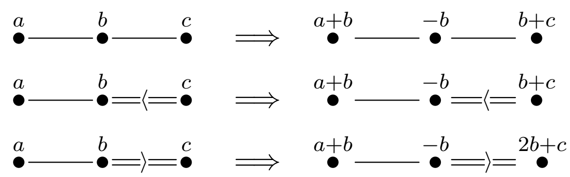
A representation M of a quiver (Q, I) is a family of (finite-dimensional) vector spaces $\{M_x \,|\, x \in Q_0\}$ together with linear maps $\{M(a) : M_{ta} \to M_{ha} \,|\, a \in Q_1\}$ that satisfy the relations induced by I. A morphism $\phi : M \to N$ of two representations M, N of (Q, I) is a set of linear maps $\phi = \{\phi(x) : M_x \to N_x \,|\, x \in Q_0\}$, such that for each $a \in Q_1$ we have $\phi(ha) \circ M(a) = N(a) \circ \phi(ta)$. The category $\mathrm{rep}(Q, I)$ of finite-dimensional representations of (Q, I) is Artinian, has enough projectives and injectives, and contains only finitely many simple objects,that are in bijection with the vertices. For the projective cover (resp. injective envelope) of the simple corresponding to a vertex $x \in Q_0$, the dimension of its space at $y \in Q_0$ is given by the number of paths from x to y (resp. from y to x), considered up to the relations in I (see [ASS06, Section III.2]).

2.4. **Borel–Weil–Bott theorem.** In this section, we present some special cases of the Borel–Weil–Bott theorem that we use in Section 4.1. For more details, see [Wey03], and especially [BE89, Chapters 4,5], as we will use these conventions.

We denote the integral weights of a simple group G with the corresponding root system (which in this paper will be C_3, D_5 or E_7), by labeling the Dynkin diagram by integers. Dominant weights correspond to labelings by nonnegative integers. The weight ρ (the half of sum of positive roots) corresponds to labeling all nodes by 1. The action of the Weyl group on the integral weights is described in [BE89, Recipe 4.1.3] which states the following.

Proposition 2.1. *The simple reflection σ_α acts as follows. To compute $\sigma_\alpha(\lambda)$, let c be the coefficient of the node of X_n associated to α. Add c to the adjacent coefficients, with multiplicity if there is a multiple edge directed towards the adjacent node, and then replace c by $-c$.*

Example 2.1. *Here we reproduce* [BE89, Example 4.1.4]. *In each case we reflect at the middle node:*



Next we define the affine action of the Weyl group on weights via

$$w\cdot\lambda := w(\lambda+\rho)-\rho.$$

Recall that the weight λ is called singular if there exists a nontrivial $w \in W$ such that $w\cdot\lambda = \lambda$. For a non-singular weight λ there exists a unique $w \in W$ such that $w\cdot\lambda$ is dominant.

Now we are ready to state the Borel–Weil–Bott theorem. Recall that parabolic subgroups P of G (up to conjugation) correspond to subsets of positive roots. Let us fix G and P. The weight λ is dominant with respect to P if it is dominant when restricted to the Levi factor $L(P)$. This means that all the labelings of the simple roots that are in $L(P)$ are nonnegative.

The homogeneous vector bundles on G/P correspond to rational P-modules. To a P-dominant λ we associate the homogeneous bundle $\mathcal{V}(\lambda)$ which is ireducible, i.e. the unipotent radical of P acts on it trivially and with the action of $L(P)$ it is the highest weight $L(P)$-module coresponding to the restriction of λ. The Borel–Weil–Bott theorem calculates the sheaf cohomology of bundles $\mathcal{V}(\lambda)$ for P-dominant weights λ:

Theorem 2.2. *Let G be a simply connected complex semisimple Lie group, and $P \subset G$ a parabolic subgroup. Suppose λ is an integral weight for G and dominant with respect to P. Consider the homogeneous bundle $\mathcal{V}(\lambda)$ on G/P. Then*

(1) *If λ is singular for the affine Weyl group action, then*

$$H^r(G/P, \mathcal{V}(\lambda)) = 0$$

for all r.

(2) *If λ is nonsingular for the affine Weyl group action, then as a representation of G,*

$$H^{l(w)}(G/P, \mathcal{V}(\lambda)) = V_{w\cdot\lambda},$$

where $w \in W$ is the unique element for which $w\cdot\lambda$ is dominant. All other cohomology vanishes.

We finish this section with some examples.

Example 2.2. *Let us consider the root system of type C_3. Calculate cohomology of the bundle corresponding to the weight $3\omega_1 - 3\omega_3$ This corresponds to the sequence $(3, 0, -3)$, After adding ρ we get the weight $(4, 1, -2)$. Applying the reflection at the 3-rd vertex we get $(4, -3, 2)$. Next we apply the reflection on the second vertex to get $(1, 3, -1)$ and then after applying the reflection at the third vertex, we get $(1, 1, 1)$. Subtracting ρ we get $(0, 0, 0)$. This means that the third cohomology of the corresponding bundle is a trivial representation with all other cohomology modules equal to zero.*

Example 2.3. *Let us consider the root system of type D_6. Calculate cohomology of the bundle corresponding to the dual of the 9-th symmetric power of the universal quotient bundle Q_1 on the corresponding isotropic Grassmannian. This corresponds to the weight $(-9,0,0,0,0,0)$. Adding ρ we get $(-8,1,1,1,1,1)$. Applying corresponding reflections we get in turn:* $(8,-7,1,1,1,1)$, $(1,7,-6,1,1,1)$, $(1,1,6,-5,1,1)$, $(1,1,1,5,-4,-4)$, $(1,1,1,1,4,-4)$, $(1,1,1,-3,4,4)$, $(1,1,-2,3,1,1)$, $(1,-1,2,1,1,1)$, $(0,1,1,1,1,1)$. *This shows that the weight* $(-9,0,0,0,0,0)$ *is singular, as a zero appears in the algorithm (the corresponding weight is fixed under reflection at the zero node). This shows that all cohomology groups of our bundle are zero.*

Example 2.4. *Let us consider the system of type E_7. Consider the bundle which is the a-th multiple of the dual of the 6-th fundamental representation, with $a \geq 10$. Our weight is $(0,0,0,0,0,-a,0)$ and the above algorithm shows that the 14-th cohomology of our bundle is the representation with highest weight $(a-10)\omega_6+(a-10)\omega_7$. All other cohomology groups are zero.*

3. The category of equivariant coherent $\mathcal{D}$-modules

As usual, the pair (G', X) denotes either (C_3, ω_3), (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6), and recall that $m = 1, 2, 4, 8$, respectively. Whenever possible, we discuss the properties of the action of $G = G' \times \mathbb{C}^*$ on X in a uniform matter within these four cases.

We have $\dim X = 6m + 8$, and G acts on X with five orbits. Accordingly, we write $X = \bigcup_{i=0}^{4} O_i$ with $\overline{O}_{i-1} \subset \overline{O}_i$ (where $1 \leq i \leq 4$), where $O_0 = \{0\}$ and $\overline{O}_4 = X$. The hypersurface $\overline{O}_3$ defined by the vanishing of a G'-invariant polynomial f of degree 4 (unique, up to scalar). The codimensions of $\overline{O}_2$ and $\overline{O}_1$ are $m+3$ and $3m+4$, and their defining ideals are generated by X (in degree 3) and $\mathfrak{g}'$ (in degree 2) from (2.2), respectively. More details can be found in [SK77], [Igu73, Section 7] and [Wey03, Exercise 7.17].

For an orbit $O \subset X$, we denote by $O^\vee \subset X$ its projective (Pyasetskii) dual orbit (see [LW19, Section 4.3]). By [KM87], we have

$$O_i^\vee = O_{4-i}, \text{ for } i = 0, \dots, 4. \tag{3.1}$$

3.1. **Fundamental groups of orbits.** Given a G-orbit $O \cong G/H$ of X, we call the finite group H/H^0 the component group of O (here H^0 stands for the connected component of H containing the identity). If O is simply-connected, then its component group is trivial. The orbits O_1, O_2, O_3 are also G'-orbits. Note that since G' is simply-connected, the fundamental groups of O_1, O_2, O_3 are isomorphic to their component groups under the action of G'. We proceed by determining the fundamental and component groups of all orbits.

Lemma 3.1. *The orbit O_1 is simply-connected.*

Proof. Since O_1 is the orbit of the highest weight vector, this follows from [LW19, Lemma 4.13]. The stabilizer is computed explicitly in [Igu73, Lemma 4.16] □

The following is the first indication that our results for the case (C_3, ω_3) are going to be somewhat different from the rest of the cases.

Lemma 3.2. *The orbit O_2 is simply-connected, except for $(G', X) = (C_3, \omega_3)$ when the component group equals $\pi_1(O_2) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. The claim about the fundamental groups follows from [Igu73, Lemma 17]. Hence, the component group (under the action of G) can be either trivial or $\mathbb{Z}/2\mathbb{Z}$. To show that the latter holds, one can either follow through the computations in [Igu73, Lemma 17], or use the fact that we have two non-isomorphic equivariant $\mathcal{D}$-modules with support $\overline{O}_2$, namely $H^4_{\overline{O}_2}(S)$ and $\mathcal{D}f^{-2}/\mathcal{D}f^{-1}$ (see Theorem 3.7 (b) and Section 4). □

Lemma 3.3. *The orbit O_3 is simply-connected.*

Proof. This follows from [Igu73, Lemma 15]. □

Lemma 3.4. *The variety $\overline{O}_3$ is normal with rational singularities.*

Proof. This follows by [Sai93, Theorem 0.4], since the polynomial $b_f(s)/(s+1)$ has no roots ≥ -1, where $b_f(s)$ stands for the b-function of the invariant f (see Section 3.2). □

Lemma 3.5. *The orbit O_4 has component group $\mathbb{Z}/4\mathbb{Z}$, and $\pi_1(O_4) \cong \mathbb{Z}$.*

Proof. This follows by [LW19, Lemma 4.11 and Remark 4.12] and Lemma 3.4. □

3.2. **Simple equivariant $\mathcal{D}$-modules.** We determine the filtrations of the equivariant $\mathcal{D}$-modules S_f and $S_f \cdot \sqrt{f}$ using the b-function $b_f(s)$ of f. From this we obtain the explicit construction of almost all simple equivariant $\mathcal{D}$-modules.

We list the simple equivariant $\mathcal{D}$-modules on X, which according to [HTT08, 11.6.2] and Section 3.1 are the following.

Notation. For each O_i (with $1 \leq i \leq 3$), we denote the simple $\mathcal{D}$-module corresponding to the trivial local system on O_i by D_i. The $\mathcal{D}$-modules S (the coordinate ring) and its Fourier transform E (the injective envelope of the residue field) correspond to the trivial local system on O_4 and O_0, respectively. Let D_4' and D_2' be the equivariant simple $\mathcal{D}$-modules corresponding to the non-trivial self-dual local systems on O_4 and O_2, respectively (the latter only for $(G', X) = (C_3, \omega_3)$). The remaining simple equivariant $\mathcal{D}$-modules with full support will be denoted by D_{41} and D_{43}.

The roots of $b_f(s)$ are (see [Kim82, Section 12]):

$$-1,\ r_1 := -\frac{m+3}{2},\ r_2 := -\frac{2m+3}{2},\ r_3 := -\frac{3m+4}{2}. \tag{3.2}$$

The following is the corresponding the holonomy diagram (see [Kim82] for more details):

$$O_4 \overset{s+1}{\text{——}} O_3 \overset{s-r_1}{\text{——}} O_2 \overset{s-r_2}{\text{——}} O_1 \overset{s-r_3}{\text{——}} O_0 \tag{3.3}$$

Lemma 3.6. *Let $x_{\mathfrak{g}'}$ denote (up to a non-zero constant) the degree 2 highest weight vector of $\mathfrak{g}'$ in S (see (2.2)), and $\partial_{\mathfrak{g}'}$ the constant differential operator of degree -2 of the same highest weight. Then we have*

$$\partial_{\mathfrak{g}'} \cdot f^{s+1} = (s+1)(s-r_1) \cdot x_{\mathfrak{g}'} f^s,$$

hence the local b-function of f at a point in O_2 is $(s+1)(s-r_1)$.

Proof. This follows by [Lőr19, Theorem 2.5], see also [LP18, Remark 3.6]. The statement about the local b-function also follows from the holonomy diagram (3.3). □

Given a $\mathcal{D}$-module M, we write $\operatorname{charC}(M)$ for its characteristic cycle (see [Kas03]), which is a formal linear combination of the irreducible components of its characteristic variety counted with multiplicities – these for us are always closures of conormal bundles to orbits. Clearly, $\operatorname{charC}(S) = [\overline{T^*_{O_0}X}]$ and $\operatorname{charC}(E) = [\overline{T^*_{O_4}X}]$. The diagram 3.3 has an edge between two orbits O, O' if $\overline{T^*_O X}$ and $\overline{T^*_{O'} X}$ intersect in codimension 1.

Theorem 3.7. *We have the following explicit construction of simple equivariant $\mathcal{D}$-modules:*

- (a) *When (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6):*
  - (0) $E \cong \mathcal{D} f^{r_3} / \mathcal{D} f^{-1}$.
  - (1) $D_1 \cong \mathcal{D} f^{r_2} / \mathcal{D} f^{r_1}$ *with* $\operatorname{charC}(D_1) = [\overline{T^*_{O_1}X}] + [\overline{T^*_{O_0}X}]$.
  - (2) $D_2 \cong \mathcal{D} f^{r_1} / \mathcal{D} f^{r_1+1}$ *with* $\operatorname{charC}(D_2) = [\overline{T^*_{O_2}X}]$.
  - (3) $D_3 \cong \mathcal{D} f^{-1} / S$ *with* $\operatorname{charC}(D_3) = [\overline{T^*_{O_3}X}] + [\overline{T^*_{O_2}X}] + [\overline{T^*_{O_1}X}]$.

(4) $D'_4 \cong \mathcal{D}f^{r_1+1}$ *with* $\mathrm{charC}(D'_4) = [\overline{T^*_{O_4}X}] + [\overline{T^*_{O_3}X}]$, *and* $D_{4i} \cong \mathcal{D}f^{i/4}$ *(where* $i = 1,3$*) with* $\mathrm{charC}(D_{4i}) = [\overline{T^*_{O_4}X}] + [\overline{T^*_{O_3}X}] + [\overline{T^*_{O_2}X}] + [\overline{T^*_{O_1}X}] + [\overline{T^*_{O_0}X}]$.

(b) *When* (G', X) *is* (C_3, ω_3)*:*

(0) $E \cong \mathcal{D}f^{r_3}/\mathcal{D}f^{r_2}$.

(1) $D_1 \cong \mathcal{D}f^{r_2}/\mathcal{D}f^{r_2+1}$ *with* $\mathrm{charC}(D_1) = [\overline{T^*_{O_1}X}]$.

(2) $D'_2 \cong \mathcal{D}f^{r_1}/\mathcal{D}f^{-1}$ *with* $\mathrm{charC}(D'_2) = [\overline{T^*_{O_2}X}] + [\overline{T^*_{O_1}X}] + [\overline{T^*_{O_0}X}]$, *and* $D_2 \cong H^4_{\overline{O_2}}(S)$ *with* $\mathrm{charC}(D_2) = [\overline{T^*_{O_2}X}]$.

(3) $D_3 \cong \mathcal{D}f^{-1}/S$ *with* $\mathrm{charC}(D_3) = [\overline{T^*_{O_3}X}]$.

(4) $D'_4 \cong \mathcal{D}f^{r_2+1}$ *with* $\mathrm{charC}(D'_4) = [\overline{T^*_{O_4}X}] + [\overline{T^*_{O_3}X}]$, *and* $D_{4i} \cong S_f \cdot f^{i/4}$ *(where* $i = 1,3$*) with* $\mathrm{charC}(D_{4i}) = [\overline{T^*_{O_4}X}] + [\overline{T^*_{O_3}X}] + [\overline{T^*_{O_2}X}] + [\overline{T^*_{O_1}X}] + [\overline{T^*_{O_0}X}]$.

Proof. Since part (a) follows as [LP18, Theorem 3.5], we give a proof for the case $(G', X) = (C_3, \omega_3)$ only, when $r_1 = -2$, $r_2 = -5/2$ and $r_3 = -7/2$.

First, by [Kas03, Corollary 6.25] the equivariant $\mathcal{D}$-modules of full support $\mathcal{D}f^{-3/2}$ and $S_f \cdot f^{i/4}$ $(i = 1,3)$ are simple. Viewed as $\mathcal{D}$-modules on $X \setminus \overline{O}_3$, tensoring $S_f \cdot f^{i/4}$ by itself four times yields S_f, hence $D_{4i} \cong S_f \cdot f^{i/4}$ and $D'_4 \cong \mathcal{D}f^{-3/2}$.

By [LW19, Proposition 4.9], we have the following filtrations in S_f and $S_f \cdot \sqrt{f}$.

$$0 \subsetneq S \subsetneq \mathcal{D}f^{-1} \subsetneq \mathcal{D}f^{-2}, \quad 0 \subsetneq \mathcal{D}f^{-3/2} \subsetneq \mathcal{D}f^{-5/2} \subsetneq \mathcal{D}f^{-7/2}.$$

Each of the successive quotients of the filtration has a unique simple $\mathcal{D}$-module quotient, hence we get six non-isomorphic equivariant simple $\mathcal{D}$-modules $S, L^{-1}, L^{-2}, \mathcal{D}f^{-3/2}, L^{-5/2}, L^{-7/2}$ (see [LW19, Proposition 4.9] and notation therein) respectively, all having G'-invariant sections. On the other hand, by Lemma 4.7 we see that $H^4_{\overline{O_2}}(S)$ (and hence D_2, which is always a submodule) has no G'-invariant sections. By Section 3.1 this yields all simple equivariant $\mathcal{D}$-modules, and shows also that $D_2 \cong H^4_{\overline{O_2}}$ must be simple (note that the argument also implies that the component group of O_2 must be indeed $\mathbb{Z}/2\mathbb{Z}$, as discussed in Lemma 3.2).

The local cohomology module $H^1_{\overline{O}_3}(S) = S_f/S$ contains a unique simple $\mathcal{D}$-module, which is D_3. The module $\mathcal{D}f^{-1}/S$ is a submodule of $H^1_{\overline{O}_3}(S)$, hence has unique simple sub- and quotient modules D_3 and L^{-1}, respectively. By looking at G'-invariant sections of the module $\mathcal{D}f^{-1}/S$, we see that the only other simple $\mathcal{D}$-module besides L^{-1} that could appear as its composition factor is D_2. Hence, $D_3 \cong \mathcal{D}f^{-1}/S \cong L^{-1}$.

Since $E = \mathcal{F}(S)$ has a G'-invariant section of degree -14, we must have $L^{-7/2} \cong E$ by (2.3). Similarly, $\mathcal{F}(D_3) \cong L^{-5/2}$, $\mathcal{F}(D_2) \cong D_2$, $\mathcal{F}(L^{-2}) \cong D'_4$ and $\mathcal{F}(D_{4i}) \cong D_{4i}$ (with $i = 1, 3$). In particular, $\overline{T^*_{O_0}X}$ is an irreducible component of the characteristic variety of L^{-2}, D_{41} and D_{43} (see [LW19, Section 4.3]).

By [Kim82], the variety $\overline{T^*_{O_i}X}$ has a dense G-orbit, for each $i = 0, \dots, 4$. Hence, for any equivariant $\mathcal{D}$-module that is generated by a G'-invariant section, its characteristic cycle is multiplicity-free, as can be seen using [LW19, Lemma 3.12] and by proof of [LW19, Proposition 3.14] (for comparison, see end of proof of [LP18, Theorem 3.5]). Moreover, for any equivariant indecomposable $\mathcal{D}$-module, its characteristic variety should have irreducible components connected via the holonomy diagram 3.3 (see [MV86, Theorem 6.7]). We saw that both $\overline{T^*_{O_0}X}$ and $\overline{T^*_{O_4}X}$ are components of $\mathrm{charC}(S_f \cdot f^{i/4})$, for $i = 0, 1, 2, 3$. In conclusion, we have that $\mathrm{charC}(S_f \cdot f^{i/4}) = [\overline{T^*_{O_4}X}] + [\overline{T^*_{O_3}X}] + [\overline{T^*_{O_2}X}] + [\overline{T^*_{O_1}X}] + [\overline{T^*_{O_0}X}]$, for all $i = 0, 1, 2, 3$.

Since $\overline{T^*_{O_0}X}$ is a component of the characteristic variety of L^{-2} and its support cannot be O_0, $\overline{T^*_{O_1}X}$ must also be a component. Since $\overline{T^*_{O_3}X}$ is in $\mathrm{charC}(D_3)$, by (3.1) $\overline{T^*_{O_1}X}$ is a component of the characteristic variety of $\mathcal{F}(D_3) \cong L^{-5/2}$. From the $\mathcal{D}$-modules $L^{-2}, L^{-5/2}$, one must be D_1 and the other D'_2: if $L^{-2} \cong D'_2$, then $\overline{T^*_{O_2}X}$ is a component in the characteristic varieties of L^{-2} and $\mathcal{F}(L^{-2}) \cong D'_4$; on the other hand, if $L^{-5/2} \cong D'_2$, then

$\overline{T^*_{O_2}X}$ is a component in the characteristic varieties of $L^{-5/2}$ and $\mathcal{F}(L^{-5/2}) \cong D_3$. Either way, since $[\overline{T^*_{O_2}X}]$ appears in $\operatorname{charC}(S_f \cdot f^{i/2})$ with multiplicity one (for $i = 0, 1$), this shows that D_2 cannot be a composition factor of $S_f \cdot f^{i/2}$. Hence, S, D_3, L^{-2} (resp. $D'_4, L^{-5/2}, E$) are all the simples appearing as composition factors in $S_f \cdot f$ (resp. $S_f \cdot \sqrt{f}$), and all with multiplicity one. Therefore, we have $L^{-2} \cong \mathcal{D}f^{-2}/\mathcal{D}f^{-1}$ and $L^{-5/2} \cong \mathcal{D}f^{-5/2}/\mathcal{D}f^{-3/2}$.

We are left to show that we have in fact $D_1 \cong \mathcal{D}f^{-5/2}/\mathcal{D}f^{-3/2}$ (and hence $D'_2 \cong \mathcal{D}f^{-2}/\mathcal{D}f^{-1}$), for which it is enough to see that the support of $\mathcal{D}f^{-5/2}/\mathcal{D}f^{-3/2}$ is contained in $\overline{O}_1$. Since $\mathfrak{g}'$ (in degree 2) from (2.2) generates the defining ideal of $\overline{O}_1$, we conclude by Lemma 3.6. □

3.3. **The quiver of** $\operatorname{mod}_G(\mathcal{D}_X)$**.** Here we describe the quivers of $\operatorname{mod}_G(\mathcal{D}_X)$ as discussed in Section 2. The vertices of the quivers are labeled with the simple equivariant $\mathcal{D}$-modules that they correspond to. We state the result without proof, as it is similar to the proof at the end of [LP18, Section 3].

Theorem 3.8. *There is an equivalence of categories*

$$\operatorname{mod}_G(\mathcal{D}_X) \cong \operatorname{rep}(Q, I),$$

where $\operatorname{rep}(Q, I)$ *is the category of finite-dimensional representations of a quiver* Q *with relations* I*. The vertices* D_{4i} *(with* $i = 1, 3$*) are isolated, while the rest of quiver* Q *is given as follows (with the relations* I *given by all* 2*-cycles):*

(a) *When* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6)*:*

$$S \rightleftarrows D_3 \rightleftarrows E \qquad\qquad D'_4 \rightleftarrows D_2 \rightleftarrows D_1$$

(b) *When* (G', X) *is* (C_3, ω_3)*:*

$$S \rightleftarrows D_3 \rightleftarrows D'_2 \qquad\qquad D'_4 \rightleftarrows D_1 \rightleftarrows E$$

and the vertex D_2 *is isolated.*

While it fixes all isolated vertices, the Fourier transform also behaves differently in the two cases above, as seen in the proof of Theorem 3.7: in (a) it reflects each individual component of the quiver, while in (b) it reflects one component into the other.

Note that (each connected component of) this quiver appears also in [LP18, LW19], and (Q, I) has finitely many (isomorphism classes of) indecomposable representations that can be described explicitly [LW19, Theorem 2.11].

3.4. **Characters of equivariant $\mathcal{D}$-modules.** Since G acts on X with finitely many orbits, any equivariant coherent $\mathcal{D}$-module is admissible as a G-representation by [LW19, Proposition 3.14]. In this section, we describe explicitly the G-module structure of all the simple equivariant $\mathcal{D}$-modules. The techniques we use are based on the techniques in [LP18, Section 4].

The characters of S and E are given by (2.2) and (2.3), respectively. The character of S_f is given by

$$[S_f] = \lim_{n\to\infty} [f^{-n} \cdot S] = \frac{1}{(1-tX)(1-t^2\mathfrak{g}')(1-t^3X)(1-t^4X_4)} \cdot t^{4\mathbb{Z}}, \tag{3.4}$$

where $t^{4\mathbb{Z}} = \sum_{i\in\mathbb{Z}} t^{4i}$. Clearly, $[S_f \cdot f^{i/4}] = [S_f] \cdot t^i$, for $i = 0, 1, 2, 3$. In particular, we get formulas for $[D_{4i}]$, for $i = 1, 3$.

In this section we compute the character $[\mathcal{D}f^{r_1+1}]$ in a uniform matter, and explain how this can be used to compute the characters of all the other G-equivariant simple $\mathcal{D}$-modules, with the exception of D'_2 in the case $(G', X) = (C_3, \omega_3)$ which will be considered separately in Section 4.

First, consider the case when (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6). We readily obtain the character of D_3 from $[D_3] = [S_f] - [S] - [E]$. If we know $[\mathcal{D}f^{r_1+1}] = [D_4']$, from $\mathcal{F}(D_1) \cong D_4'$ we have by (2.1) the relation $[D_1] = [D_4']^* \cdot t^{-6m-8}$, and also get $[D_2] = [S_f \cdot \sqrt{f}] - [D_4'] - [D_1]$.

Similarly when $(G', X) = (C_3, \omega_3)$, given $[\mathcal{D}f^{r_1+1}] = [\mathcal{D}f^{-1}]$ yields $[D_3] = [\mathcal{D}f^{-1}] - [S]$ and $[D_2'] = [S_f] - [\mathcal{D}f^{-1}]$. By Fourier transform, we get using (2.1) also $[D_1]$ and $[D_4']$ from $\mathcal{F}(D_3) \cong D_1$ and $\mathcal{F}(D_2') \cong D_4'$.

Recall the notation in (2.2). We have the following formula obtained analogously to [LP18, Theorem 4.1].

Theorem 3.9. *The G-character of Df^{r_1+1} is given by*

$$[Df^{r_1+1}] = \frac{t^{4(r_1+1)}}{(1-t^{-1}X)(1-X_4)(1-tX)(1-t^2\mathfrak{g}')(1-t^4)}.$$

We are left to determine the character of the $\mathcal{D}$-module $D_2 \cong H^4_{\overline{O_4}}(S)$ in the case when $(G', X) = (C_3, \omega_3)$. The formula (see Theorem 4.9) is postponed until Section 4.2 as we calculate it using a different approach.

4. Local cohomology

In this section, we determine the all the local cohomology modules of the coordinate ring supported in the orbit closures. The strategies used are similar to the ones in [LP18, Section 5].

4.1. **Local cohomology of S.**

The goal in this section is to prove the following theorem.

Theorem 4.1. *The following are all the non-zero local cohomology modules of S with support in an orbit closure:*

(a) *When (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6):*
 (0) $H^{6m+8}_{\overline{O_0}}(S) = E$.
 (1) $H^{3m+4}_{\overline{O_1}}(S) = D_1$, $H^{4m+5}_{\overline{O_1}}(S) = E$, $H^{5m+5}_{\overline{O_1}}(S) = E$.
 (2) $0 \to D_2 \to H^{m+3}_{\overline{O_2}}(S) \to D_1 \to 0$, $H^{2m+3}_{\overline{O_2}}(S) = D_1$, $H^{3m+4}_{\overline{O_2}}(S) = E$.
 (3) $0 \to D_3 \to H^{1}_{\overline{O_3}}(S) \to E \to 0$.

(b) *When (G', X) is (C_3, ω_3):*
 (0) $H^{6m+8}_{\overline{O_0}}(S) = E$.
 (1) $0 \to D_1 \to H^{3m+4}_{\overline{O_1}}(S) \to E \to 0$.
 (2) $H^{m+3}_{\overline{O_2}}(S) = D_2$, $H^{3m+4}_{\overline{O_2}}(S) = E$.
 (3) $0 \to D_3 \to H^{1}_{\overline{O_3}}(S) \to D_2' \to 0$.

Part (3) follows by Theorem 3.7 since $H^1_{\overline{O_3}}(S) = S_f/S$. The non-trivial parts are (1) and (2).

Since $H^{m+3}_{\overline{O_2}}(S)$ (resp. $H^{3m+4}_{\overline{O_1}}(S)$) is the injective envelope of D_2 in $\mathrm{mod}_G^{\overline{O_2}}(\mathcal{D}_X)$ (resp. of D_1 in $\mathrm{mod}_G^{\overline{O_1}}(\mathcal{D}_X)$) by [LW19, Lemma 3.11], the claim about their structures follows by our description of the quiver of $\mathrm{mod}_G(\mathcal{D}_X)$ in Theorem 3.8. In fact, in case (a) (resp. case (b)) we have an isomorphism $H^{m+3}_{\overline{O_2}}(S) \cong S_f\sqrt{f}/\mathcal{D}f^{r_1+1}$ (resp. $H^{3m+4}_{\overline{O_1}}(S) \cong S_f\sqrt{f}/\mathcal{D}f^{r_2+1}$).

We proceed with part (1). Since here we consider local cohomology supported in the cone over a smooth projective variety, there are several results available in this direction [Ogu73],[GLS98],[Swi15],[LSW16],[HP16].

Proposition 4.2. *Apart from $H^{3m+4}_{\overline{O_1}}(S)$, the only non-zero local cohomology modules of S with support in $\overline{O_1}$ are $H^{4m+5}_{\overline{O_1}}(S) = H^{5m+5}_{\overline{O_1}}(S) = E$ when (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6).*

Proof. We only need to determine how many copies of E appear in $H^j_{\overline{O_1}}(S)$ for $j > 3m+4$, as the latter is always supported on the origin. Note that the highest weight orbit $\overline{O}_1$ is the affine cone over some (partial) flag variety G/P. Hence, it is enough to determine the Betti numbers of G/P, according to [Swi15, Main Theorem 1.2] (see also [GLS98, Theorem] and [LSW16, Theorem 3.1]). These numbers are given by the Bruhat decomposition. The Poincaré polynomials encoding them are known, and can be computed case-by-case as explained in [Hum90, Sections 1.11 and 3.15] using factorization methods.

Let us consider first the case (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6), when $m = 2, 4, 8$, respectively. The Betti numbers are given by the coefficients of the powers q in Poincaré polynomial of G/P:

$$P_m(q) = (1+q^{m+2})(1+q^{22-32/m}) \cdot \frac{1-q^{3m+4}}{1-q^2}.$$

When $(G', X) = (C_3, \omega_3)$, the Poincaré polynomial of G/P is:

$$P_1(q) = (1+q^6) \cdot \frac{1-q^8}{1-q^2}. \tag{4.1}$$

□

We are left with proving both parts (2) of Theorem 4.1, which we devote the rest of the section to.

The variety $\overline{O}_3$ is the projective dual of the highest weight orbit, hence given by a discriminant in the sense of [Wey94]. Thus, it has a desingularization as the total space $Z = \mathrm{Tot}(\eta^*)$ of a bundle η of 1-jets on G/P, as described in [Wey94, Section 1]. The space Z is a subbundle of the trivial bundle $G/P \times X$, and we denote the first and the second projection (which yields the desingularization of $\overline{O}_3$) by

$$p_1 : Z \longrightarrow G/P, \qquad p_2 : Z \longrightarrow \overline{O}_3.$$

We denote by ξ the locally free sheaf on G/P corresponding to the quotient bundle obtained from the inclusion $Z \subset G/P \times X$. Hence, we have the following exact sequence of locally free sheaves on G/P:

$$0 \to \xi \to X \otimes \mathcal{O}_{G/P} \to \eta \to 0.$$

We give the following uniform description of the bundles η and μ following [Wey94, Section 1] (see also [Wey03, Section 9.3]). The group P is a maximal parabolic which can be represented by distinguishing the corresponding node in the Dynkin diagram. We have $\eta = \eta' \otimes \mathcal{O}(1)$ (here $\mathcal{O}(1)$ is the twisting sheaf), where η' fits in a sequence

$$0 \to \Omega_{G/P} \to \eta' \to \mathcal{O}_{G/P} \to 0. \tag{4.2}$$

The cotangent bundle $\Omega_{G/P}$ in terms of P-dominant weights can be described by labeling the distinguished node by -2 and the rest of the nodes with the number of edges connecting them to the distinguished node:

- (C_3, ω_3): $\mathcal{V}(0,2,-2) = S_2\mathcal{R}$, where the latter is the second symmetric power of the 3-dimensional isotropic tautological subbundle $\mathcal{R}$ (see [Wey03, Chapter 4, Exercise 9]);
- (A_5, ω_3): $\mathcal{V}(0,1,-2,1,0) = \mathcal{R} \otimes Q^*$, where $\mathcal{R}$ (resp. $\mathcal{Q}$) is the tautological subbundle (resp. quotient bundle) (see [Wey03, Proposition 3.3.5]);
- (D_6, ω_5): $\mathcal{V}(0,0,0,1,-2,0) = \bigwedge^2 \mathcal{R}$, where $\mathcal{R}$ denotes the 6-dimensional tautological subbundle (see [Wey03, Chapter 4, Exercise 10]);
- (E_7, ω_6): $\mathcal{V}(0,0,0,0,1,-2,0)$, which is induced from the 27-dimensional representation V_{ω_1} of E_6.

The next result three results follow as in [LP18, Lemma 5.2, Proposition 5.3 and Proposition 5.4].

Lemma 4.3. *We have $H^0(G/P, \mathrm{Sym}\,\eta) = S/(f)$, and $H^i(G/P, \mathrm{Sym}\,\eta) = 0$ for $i \geq 0$. Moreover, $\eta' = \eta \otimes \mathcal{O}(-1)$ is characterized as the unique nonsplit extension (up to isomorphism) in the sequence (4.2).*

Let $U = p_2^{-1}(O_3)$, which is an open subset in Z. Since p_2 is a G-equivariant birational isomorphism, we have $U \cong O_3$ as G-varieties.

Proposition 4.4. *We have $Z \setminus U = D$, where D is a G-stable divisor on Z. Moreover,, the ideal sheaf of D is $p_1^*(\mathcal{L})$, where $\mathcal{L}$ is the line bundle on G/P:*

- $(C_3, \omega_3):\ \mathcal{L} = \mathcal{V}(0,0,-2) \otimes \mathcal{O}(3) \cong \mathcal{V}(0,0,1)$;
- $(A_5, \omega_3):\ \mathcal{L} = \mathcal{V}(0,0,-2,0,0) \otimes \mathcal{O}(3) \cong \mathcal{V}(0,0,1,0,0)$;
- $(D_6, \omega_5):\ \mathcal{L} = \mathcal{V}(0,0,0,0,-2,0) \otimes \mathcal{O}(3) \cong \mathcal{V}(0,0,0,0,1,0)$;
- $(E_7, \omega_6):\ \mathcal{L} = \mathcal{V}(0,0,0,0,0,-2,0) \otimes \mathcal{O}(3) \cong \mathcal{V}(0,0,0,0,0,1,0)$.

Proposition 4.5. *For each $i \geq 2$, we have an isomorphism of G-modules*

$$H^i_{\overline{O_2}}(S/(f)) \cong \varinjlim_k H^{i-1}(G/P, \mathcal{L}^{-k} \otimes \operatorname{Sym} \eta).$$

The following exact sequence of G-equivariant S-modules

$$0 \to S(-4) \xrightarrow{f} S \longrightarrow S/(f) \to 0.$$

gives a long exact sequence in local cohomology modules

$$\cdots \to H^{i-1}_{\overline{O_2}}(S/(f)) \to H^i_{\overline{O_2}}(S)(-4) \xrightarrow{f} H^i_{\overline{O_2}}(S) \to H^i_{\overline{O_2}}(S/(f)) \to H^{i+1}_{\overline{O_2}}(S)(-4) \to \dots \tag{4.3}$$

We estimate cohomology by working first with the associated graded of η:

$$\operatorname{gr} \eta = (\Omega_{G/P} \oplus \mathcal{O}_{G/P}) \otimes \mathcal{O}(1). \tag{4.4}$$

To finish part (2) of Theorem 4.1, we argue as in [LP18, Section 5] case-by-case. We only give details for the cases (C_3, ω_3) and (D_6, ω_5), as the case (E_7, ω_6) is completely analogous to the latter.

4.2. **Local cohomology supported in $\overline{O_2}$ for** (C_3, ω_3)**.** Throughout this section $(G', X) = (C_3, \omega_3)$.

Lemma 4.6. *For all $k \gg 0$, the space of G'-invariants*

$$H^i(G/P, \mathcal{L}^{-k} \otimes \operatorname{Sym}(\operatorname{gr} \eta))^{G'}$$

is nonzero if and only if $i = 0, 1, 5, 6$. Among these for $i = 5, 6$, the invariant spaces of $\mathbb{C}^*$-degree -6 are zero, and the space of $\mathbb{C}^*$-degree -10 is one-dimensional for $i = 5$ and zero for $i = 6$.

Proof. We start with the decomposition (by [Wey03, Proposition 2.3.8] since $\mathcal{V}(0,2,-2) = S_2\mathcal{R}$)

$$\operatorname{Sym}_d \mathcal{V}(0,2,-2) = \bigoplus_{\substack{a,b \geq 0,\ c \geq a+b \\ 3c-2b-a=d}} \mathcal{V}(2a, 2b, -2c).$$

Hence, in degree $d - 3k$ we have a decomposition

$$\mathcal{L}^{-k} \otimes \operatorname{Sym}_d(\operatorname{gr} \eta) = \bigoplus_{\substack{a,b \geq 0,\ c \geq a+b \\ 3c-2b-a \leq d}} \mathcal{V}(2a, 2b, d-2c-k). \tag{4.5}$$

We now apply Theorem 2.2 to a summand of type $\mathcal{V}(2a, 2b, -x)$ with $a, b, x \geq 0$, to see when it gives the trivial G'-representation $(0,0,0)$. Clearly, $(2a+1, 2b+1, -x+1)$ is $(1,1,1)$ for $a = b = x = 0$, giving the trivial representation for the $i = 0$ cohomology. Otherwise, in order to continue we must have $x > 1$ and reflecting at the third node gives $\mathcal{V}(2a+1, 2b-2x+3, x-1)$. This gives $(1,1,1)$ for $a = 0, b = 1, x = 2$. Otherwise, we must have $2x - 2b + 3 < 0$ and we proceed likewise to $i = 2$. At the next step $i = 3$, we can potentially reflect at either the first or third node. All in all, we can encounter the following weights for $i = 2,3,3,4$ respectively:

$$\begin{aligned}(2a+2b+4-2x, 2x-2b-3, 2b+2-x), \quad (2x-2a-2b-4, 2a+1, 2b+2-x),\\ (2a+2b+4-2x, 2b+1, x-2b-2), \quad (2x-2a-2b-4, 2a+4b+5-2x, x-2b-2).\end{aligned} \tag{4.6}$$

Due to parity reasons, we can not get the trivial representation for $i = 2,3,4$. After $i = 5$ steps, we arrive at the weight (with $x \geq a+2b+3$):

$$(2b+1, 2x-2a-4b-5, 2a+2b+3-x).$$

This gives $(1,1,1)$ for $a=1, b=0, x=4$. Otherwise, we must have $x > 2a+2b+3$ and reflecting at the third node yields $(2b+1, 2a+1, x-2a-2b-3)$. This gives $(1,1,1)$ for $a=0, b=0, x=4$ at $i=6$, and we stop since all entries of the weight vector are non-negative.

In other words, the only possible summands giving the trivial G'-representation $(0,0,0)$ are

$$\mathcal{V}(0,0,0),\ \mathcal{V}(0,2,-2),\ \mathcal{V}(2,0,-4),\ \mathcal{V}(0,0,-4),$$

when $i = 0,1,5,6$, respectively. Now when the degree is $d-3k=-6$, we see by inspection that there are no summands $\mathcal{V}(2,0,-4)$ or $\mathcal{V}(0,0,-4)$ in (4.5). If $d-3k=-10$ then $\mathcal{V}(0,0,-4)$ is not a summand, but $\mathcal{V}(2,0-4)$ is a summand in (4.5) with $a=1, b=0, c=k-3$, for $k \geq 4$. □

Lemma 4.7. *Each G'-module in the decomposition of $H^4_{\overline{O_2}}(S)$ is of the form $(2p+1)\omega_1 + 2q\omega_2 + r\omega_3$, for some $p,q,r \in \mathbb{Z}_{\geq 0}$. In particular, $H^4_{\overline{O_2}}(S)$ has no G'-invariant sections.*

Proof. By Proposition 4.5, any G-representation in $H^3_{\overline{O_2}}(S/(f))$ must appear in $H^2(G/P, \mathcal{L}^{-k} \otimes \mathrm{Sym}(\mathrm{gr}\,\eta))$ for $k \gg 0$. We see from (4.6) with $i=2$ that the only possible G'-representations in $H^2(G/P, \mathcal{L}^{-k} \otimes \mathrm{Sym}(\mathrm{gr}\,\eta))$ are of the form $(2p+1)\omega_1 + 2q\omega_2 + r\omega_3$, for some $p,q,r \in \mathbb{Z}_{\geq 0}$. The long exact sequence (4.3) gives an exact sequence

$$0 \to H^3_{\overline{O_2}}(S/(f)) \to H^4_{\overline{O_2}}(S)(-4) \xrightarrow{f} H^4_{\overline{O_2}}(S).$$

Since any element of $H^4_{\overline{O_2}}(S)$ is annihilated by a power of the G'-invariant f, this proves the claim. □

We now finish the proof of Theorem 4.1(b)(2).

Proposition 4.8. *We have $H^i_{\overline{O_2}}(S) = 0$ for $i > 4, i \neq 7$ and $H^7_{\overline{O_2}}(S) = E$.*

Proof. The modules $H^i_{\overline{O_2}}(S) = 0$ for $i > 4$ have support contained in $\overline{O_1}$. Hence, by the description of the category in Theorem 3.8 (b), they must be direct sums of modules of type D_1, E, N, M, where $N = \mathcal{D}f^{-5/2}/\mathcal{D}f^{-3/2}$ and $M = \mathbb{D}N$ is the holonomic dual to N. There is a nonsplit exact sequence

$$0 \to E \to M \to D_1 \to 0.$$

The modules D_1, N (resp. E, M) have G-semi-invariant elements of degree -10 (resp. -14) that are annihilated by f (see Theorem 3.7). Since E does not have a semi-invariant of degree -6, we see that M also has a semi-invariant element of degree -10 annihilated by f. In particular, any non-trivial local cohomology module $H^i_{\overline{O_2}}(S)$ (for $i > 4$) has a non-zero G'-invariant element annihilated by f.

Since $H^4_{\overline{O_2}}(S)$ and $H^4_{\overline{O_2}}(S/(f)$ have no G'-invariants (Lemma 4.7, Lemma 4.6 and Proposition 4.5), the module $H^5_{\overline{O_2}}(S)$ has no G'-invariant elements annihilated by f by (4.3). Hence, $H^5_{\overline{O_2}}(S) = 0$.

By the long exact sequence (4.3) together with Proposition 4.5 and Lemma 4.6, we see that the first potential non-zero local cohomology (when $i > 4$) is for $i = 7$. Under the inclusion $\mathcal{L}^{-k} \otimes \mathrm{Sym}\,\eta \hookrightarrow \mathcal{L}^{-k-1} \otimes \mathrm{Sym}\,\eta$ the limit maps in Proposition 4.5 map the semi-invariant of degree -10 non-trivially (compare [LP18, Proposition

5.7]). This shows that we must have $H^7_{\overline{O_2}}(S) = E$. Since multiplication by f is surjective on $E^{G'}$ (see Theorem 3.7) we conclude again by the sequence (4.3), Proposition 4.5 and Lemma 4.6 that $H^i_{\overline{O_2}}(S) = 0$ for $i \geq 8$. $\square$

We now proceed determining the G-character of the equivariant $\mathcal{D}$-module D_2.

Theorem 4.9. *When $(G', X) = (C_3, \omega_3)$, the G-character of D_2 is given by*

$$[D_2] = \frac{t^{-7}\omega_1}{(1-t^{-2}\mathfrak{g}')(1-t^{-1}X)(1-X_4)(1-tX)(1-t^2\mathfrak{g}')}.$$

Proof. By Proposition 4.8, the long exact sequence (4.3) gives the exact sequence

$$0 \to H^3_{\overline{O_2}}(S/(f)) \to D_2(-4) \xrightarrow{f} D_2 \to H^4_{\overline{O_2}}(S/(f)) \to 0. \tag{4.7}$$

For any representation λ of G', this gives the formula (we use the notation as in Section 2.1)

$$m_\lambda^{d-4}(D_2) - m_\lambda^d(D_2) = m_\lambda^d(H^3(S/(f))) - m_\lambda^d(H^4(S/(f))), \text{ for any } d \in \mathbb{Z}. \tag{4.8}$$

By Lemma 4.7, we can assume that $\lambda = (2p+1)\omega_1 + 2q\omega_2 + r\omega_3$, for some $p, q, r \in \mathbb{Z}_{\geq 0}$. We now proceed to compute the right-hand side of (4.8).

Consider again the decomposition (4.5). Due to parity reasons, the only cohomologies that yield a representation of the form λ via the Borel-Weil-Bott theorem occur in steps $i = 2, 3, 4$ for weights of the form (4.6). For simplicity, put $N_i = H^i(G/P, \mathcal{L}^{-k} \otimes \mathrm{Sym}(\mathrm{gr}\,\eta))$ $(k \gg 0)$. An elementary calculation shows that the representation $\lambda = (2p+1)\omega_1 + 2q\omega_2 + r\omega_3$ occurs in N_i in $\mathbb{C}^*$-degree $d \in \mathbb{Z}$ if and only if $d + r$ is odd and we are in the following situation:

$$\begin{cases} d \leq 2p + r - 3, & \text{when } i = 2; \\ d \leq 2p - r - 5 \text{ or } d \leq -2p + r - 7, & \text{when } i = 3; \\ d \leq -2p - r - 9, & \text{when } i = 4. \end{cases} \tag{4.9}$$

Moreover, $m_\lambda^d(N_i)$ is equal to the number of the corresponding inequalities satisfied above. Since $H^5(S/(f)) = 0$, by Proposition 4.5 the representations in N_4 must cancel out completely via representations in N_3 in the spectral sequence corresponding to (4.4), otherwise the limit maps from in Proposition 4.5 would map them non-trivially (compare with [LP18, Proposition 5.7]). Hence, we have

$$m_\lambda^d(H^3(S/(f))) - m_\lambda^d(H^4(S/(f))) = m_\lambda^d(N_2) - m_\lambda^d(N_3) + m_\lambda^d(N_4).$$

Together with (4.8) and (4.9), this yields the recursive formula in d (when $d + r$ is odd):

$$m_\lambda^{d-4}(D_2) - m_\lambda^d(D_2) = \begin{cases} 0 & \text{when } d > 2p + r - 3; \\ 1 & \text{when } d \leq 2p + r - 3 \text{ and } d > \max\{2p - r - 5, -2p + r - 7\}; \\ 0 & \text{when } d \leq \max\{2p - r - 5, -2p + r - 7\} \text{ and } d > \min\{2p - r - 5, -2p + r - 7\}; \\ -1 & \text{when } d \leq \min\{2p - r - 5, -2p + r - 7\} \text{ and } d > -2p - r - 9; \\ 0 & \text{when } d \leq -2p - r - 9. \end{cases} \tag{4.10}$$

We now show the initial condition $m_\lambda^d(D_2) = 0$, whenever $d > 2p + r - 7$. Assume by contradiction that this is not the case, hence there exists a non-zero highest weight vector $v \in D_2$ of weight λ with $\deg v > 2p + r - 7$. Since the support of D_2 is $\overline{O_2}$, there exist a minimal integer $l \geq 1$ with $f^l \cdot v = 0$. Then the element $w = f^{l-1} \cdot v \neq 0$ has highest weight λ with $\deg w > 2p + r - 7$. By the sequence (4.7), this gives a non-zero element in $H^3_{\overline{O_2}}(S/(f))$ of highest weight λ with degree $> 2p + r - 3$. By Proposition 4.5, this contradicts (4.9), showing that $m_\lambda^d(D_2) = 0$, whenever $d > 2p + r - 7$. This initial condition together with the recursive formula (4.10) determines the character of D_2, and it is elementary to see that it can be written in the form of the rational function as claimed. $\square$

Remark 4.10. We can give an explicit $\mathcal{D}$-module presentation for D_2 as follows. From Theorem 4.9 we see that for $\lambda = \omega_1$ we have $m_\lambda^{-7}(D_2) = 1$, and from Section 3.4 that $m_\lambda^{-7}(M) = 0$ for any other simple equivariant $\mathcal{D}$-module M. Denote by V the irreducible G-representation corresponding to λ with $\mathbb{C}^*$-degree -7. Then the $\mathcal{D}$-module $P = \mathcal{D}_X \otimes_{U\mathfrak{g}} V$ is the projective cover of D_2 in $\mathrm{mod}_G(\mathcal{D}_X)$ (see [LW19, Lemma 2.1 and Proposition 2.7]). By Theorem 3.8, in fact we have $P \cong D_2$. Now P can be given an explicit presentation as explained in [LW19, Page 435].

4.3. **Local cohomology supported in $\overline{O_2}$ for the other cases.** In this section (G', X) is one of the cases $(A_5, \omega_3), (D_6, \omega_5)$ or (E_7, ω_6). While we adhere to uniformity as much as possible, in the following result we use the Borel-Weil-Bott theorem on a case-by-case basis (for the case (A_5, ω_3), see [LP18, Proposition 5.6]).

Lemma 4.11. *Let (G', X) be either . Then for all $k \gg 0$, the space of G'-invariants*

$$H^i(G/P, \mathcal{L}^{-k} \otimes \mathrm{Sym}(\mathrm{gr}\,\eta))^{G'}$$

is nonzero if and only if $i = 0, 1, m+1, m+2, 2m+1, 2m+2, 3m+2, 3m+3$. Among these, the spaces of $\mathbb{C}^*$-degree $-4m-2$ are one-dimensional for $i = 0, 1, m+1, m+2, 2m+1$ and zero otherwise, and of $\mathbb{C}^*$-degree $-6m-4$ are one-dimensional for $i = 0, 1, m+1, m+2, 2m+1, 2m+2, 3m+2$ and zero for $i = 3m+3$.

Proof. First, consider the case $(G', X) = (D_6, \omega_5)$. We have a decomposition (by [Wey03, Proposition 2.3.8] since $\mathcal{V}(0,0,0,1,-2,0) = \bigwedge^2 \mathcal{R}$)

$$\mathrm{Sym}_d\, \mathcal{V}(0,0,0,1,-2,0) = \bigoplus_{\substack{a,b\geq 0,\ c\geq a+b \\ 3c-2b-a=d}} \mathcal{V}(0,a,0,b,-2c,0).$$

Hence, in degree $d - 3k$ we have a decomposition

$$\mathcal{L}^{-k} \otimes \mathrm{Sym}_d(\mathrm{gr}\,\eta) = \bigoplus_{\substack{a,b\geq 0,\ c\geq a+b \\ 3c-2b-a\leq d}} \mathcal{V}(0,a,0,b,d-2c-k,0). \tag{4.11}$$

We apply Theorem 2.2 to a summand of type $\mathcal{V}(0,a,0,b,-x,0)$ with $a,b,x \geq 0$. Computing as in Lemma 4.6 we see that we obtain the trivial G'-representation only for the following types:

$$\mathcal{V}(0,0,0,0,0,0),\ \mathcal{V}(0,0,0,1,-2,0),\ \mathcal{V}(0,2,0,1,-6,0),\ \mathcal{V}(0,3,0,0,-6,0),$$
$$\mathcal{V}(0,0,0,3,-10,0),\ \mathcal{V}(0,1,0,2,-10,0),\ \mathcal{V}(0,1,0,0,-10,0),\ \mathcal{V}(0,0,0,0,-10,0),$$

when $i = 0,1,5,6,9,10,14,15$, respectively. Assuming $k \gg 0$, we see by inspection that the corresponding summands appear in the decomposition (4.11) for degree $d - 3k = -18$ only when $i = 0,1,5,6,9$, and for degree $d - 3k = -28$ all but the last summand $\mathcal{V}(0,0,0,0,-10,0)$ appear.

Next, consider the case $(G', X) = (E_7, \omega_6)$. We have a decomposition (see [Joh80, Section 4])

$$\mathrm{Sym}_d\, \mathcal{V}(0,0,0,0,1,-2,0) = \bigoplus_{\substack{a,b\geq 0,\ c\geq a+b \\ 3c-2b-a=d}} \mathcal{V}(a,0,0,0,b,-2c,0).$$

We apply Theorem 2.2 to a summand of type $\mathcal{V}(a,0,0,0,b,-x,0)$ with $a,b,x \geq 0$. Computing as above we see that we obtain the trivial G'-representation only for the following types:

$$\mathcal{V}(0,0,0,0,0,0,0),\ \mathcal{V}(0,0,0,0,1,-2,0),\ \mathcal{V}(4,0,0,0,1,-10,0),\ \mathcal{V}(5,0,0,0,0,-10,0),$$
$$\mathcal{V}(0,0,0,0,5,-18,0),\ \mathcal{V}(1,0,0,0,4,-18,0),\ \mathcal{V}(1,0,0,0,0,-18,0),\ \mathcal{V}(0,0,0,0,0,-18,0),$$

when $i = 0,1,9,10,17,18,26,27$, respectively. The rest of the proof follows similarly to the previous case. $\square$

The following completes the proof of Theorem 4.1(a)(2). The reasoning is analogous to [LP18], hence we only give a sketch of the argument.

Proposition 4.12. *The only non-zero* $H^i_{\overline{O_2}}(S)$ *for* $i > m+3$ *are* $H^{2m+3}_{\overline{O_2}}(S) = D_1$, $H^{3m+4}_{\overline{O_2}}(S) = E$.

Proof. The modules $H^i_{\overline{O_2}}(S) = 0$ for $i > m+3$ have support contained in $\overline{O_1}$. Hence, by the description of the category in Theorem 3.8 (a), they must be direct sums of modules of type D_1 and E.

The modules D_1 (resp. E) have G-semi-invariant elements of degree $-4m-6$ (resp. $-6m-8$) that are annihilated by f (see Theorem 3.7). Moreover, multiplication by f is surjective both on $D_1^{G'}$ and on $E^{G'}$ (see Theorem 3.7) and also on the G'-invariant space of $H^{m+3}_{\overline{O_2}}(S) \cong S_f\sqrt{f}/\mathcal{D}f^{r_1+1}$. By the long exact sequence (4.3) together with Proposition 4.5 and Lemma 4.11, we conclude that D_1 (resp. E) can only appear in the local cohomology modules $H^i_{\overline{O_2}}(S)$ for $i = m+4, 2m+3$ (resp. $i = m+4, 2m+3, 2m+4, 3m+4$). Since there are no cancelations possible for $i = 2m+3$ (resp. $i = 3m+4$) in the spectral sequence when passing from the associated graded $\operatorname{gr}\eta$ to η (4.4), we see that $H^{2m+3}_{\overline{O_2}}(S) = D_1$ (resp. $H^{3m+4}_{\overline{O_2}}(S) = E$). We will now show that the rest of the terms must cancel out in the spectral sequence.

We observe that $H^{m+3}_{\overline{O_2}}(S) = S_f\sqrt{f}/\mathcal{D}f^{r_1+1}$ has no G-semi-invariant of degree $-4m-2$ (resp. $-6m-4$) that is annihilated by f. Together with the sequence (4.3) and Proposition 4.5, this implies that for $i = m+1, m+2$ the semi-invariants in degree $-4m-2$ (resp. $-6m-4$) from Lemma 4.11 must cancel each other out in the spectral sequence (again, otherwise the limit maps would be non-trivial in Proposition 4.5).

We are left to show that the semi-invariants in Lemma 4.11 in degree $-6m-4$ between $i = 2m+1, 2m+2$ cancel each other out in the spectral sequence. This can be seen as in [LP18, Proposition 5.9], by comparing the connecting homomorphism between them to another connecting homomorphism corresponding to semi-invariants in a different degree that are known to cancel each other out. □

5. Other invariants

5.1. **Lyubeznik numbers and intersection cohomology groups of orbit closures.** In this section, we determine some local cohomology groups of equivariant $\mathcal{D}$-modules with support in the origin. We then use these computations to determine the Lyubeznik numbers (for the (A_5, ω_3) case see [LP18, Section 5]) and the (middle perversity) intersection cohomology groups of the orbit closures $\overline{O_1}, \overline{O_2}, \overline{O_3}$.

We start with the following observation relating the intersection cohomology groups to the local cohomology groups.

Proposition 5.1. *Let* $p \in \{1,2,3\}$ *and* $c_p = \operatorname{codim}_X O_p$. *For all* $i \in \mathbb{Z}$, *we have*

$$H^{i+c_p}_{\{0\}}(\mathcal{F}(D_p)) = E^{\oplus \dim IH^i(\overline{O}_p)}.$$

Proof. By the Riemann–Hilbert correspondence (especially [HTT08, Theorem 7.1.1]), the intersection cohomology groups $IH^i(\overline{O}_p)$ can be computed as the (derived) pushforward of the module D_p to a point. The latter is equivalent to the restriction of the Fourier transform $\mathcal{F}(D_p)$ to a point [HTT08, Proposition 3.2.6]. By [HTT08, Proposition 1.7.1], this can be computed as local cohomology supported at the origin. □

Lemma 5.2. *The module* $H^i_{\{0\}}(D_1)$ *is non-zero (in which case it is isomorphic to* E*) if and only if:*

(a) *When* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6) : $i = m+2, 2m+2, 3m+4$;
(b) *When* $(G', X) = (C_3, \omega_3)$: $i = 1,\ 3m+4$.

Proof. This follows readily by considering the spectral sequence $H^i_{\{0\}}(H^j_{\overline{O_1}}(S)) \Rightarrow H^{i+j}_{\{0\}}(S)$ together with Theorem 4.1. For case (b) this spectral sequence degenerates, and the claim follows by using additionally the long the exact sequence associated to $0 \to D_1 \to H^{3m+4}_{\overline{O_1}}(S) \to E \to 0$. □

Lemma 5.3. *The module* $H^i_{\{0\}}(D_2)$ *is non-zero (in which case it is isomorphic to* E*) if and only if:*

(a) *When* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6)*:* $\quad i = m+3, 2m+3, 3m+3, 3m+5, 4m+5, 5m+5$*;*
(b) *When* $(G', X) = (C_3, \omega_3)$*:* $\quad i = m+3,\ 5m+5$.

Proof. We consider the spectral sequence $H^i_{\{0\}}(H^j_{\overline{O_2}}(S)) \Rightarrow H^{i+j}_{\{0\}}(S)$ together with Theorem 4.1. Part (b) follows easily. For part (a) assuming that $H^i_{\{0\}}(H^{m+3}_{\overline{O_2}}(S)) = 0$ for $i < 3m+3$, it follows from the spectral sequence (using Lemma 5.2) that

$$H^i_{\{0\}}(H^{m+3}_{\overline{O_2}}(S)) = E, \text{ for } i = 3m+3, 4m+5, 5m+5,$$

and it is zero for all other i. Using the exact sequence $0 \to D_2 \to H^{m+3}_{\overline{O_2}}(S) \to D_1 \to 0$ proves the claim.

We are left to show the vanishing of $H^i_{\{0\}}(H^{m+3}_{\overline{O_2}}(S))$ for $i < 3m+3$. Note that $H^{m+3}_{\overline{O_2}}(S)) \cong H^1_{\overline{O_3}}(D'_4)$, and also $H^i_{\{0\}}(H^1_{\overline{O_3}}(D'_4)) = H^{i+1}_{\{0\}}(D'_4)$, since the corresponding spectral sequence degenerates. Hence, we are left to show that $H^i_{\{0\}}(D'_4) = 0$, for $i < 3m+4$. Since $\mathcal{F}(D'_4) = D_1$, by Proposition 5.1 the latter follows since $\operatorname{codim}_X O_p = 3m+4$. □

Lemma 5.4. *The module* $H^i_{\{0\}}(D_3)$ *is non-zero (in which case it is isomorphic to* E*) if and only if:*

(a) *When* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6)*:* $\quad i = 1, 6m+7$*;*
(b) *When* $(G', X) = (C_3, \omega_3)$*:* $\quad i = 3m+4,\ 6m+7$.

Proof. For part (a) we have $H^i_{\{0\}}(H^1_{\overline{O_3}}(S)) = H^{i+1}_{\{0\}}(S)$, since the corresponding spectral sequence degenerates. Hence, the claim follows readily by Theorem 4.1.

For part (b), since $\mathcal{F}(D_3) = D_1$ it is enough to compute $IH^i(\overline{O}_1)$ by Proposition 5.1. Because $\overline{O}_1$ is the affine cone over G/P, the latter can be computed as explained in [Bea08, Section I.5]. Using the Poincaré polynomial (4.1) we get that $\dim IH^i(\overline{O}_1) = 1$ for $i = 0, 6$ and it is 0 otherwise. □

Our results above determine also the Lyubeznik numbers $\lambda_{i,j}(R_p)$ (see [Lyu93]) of the orbit closures $\overline{O_p}$ for $p = 1, 2, 3$, where $R_p := \mathbb{C}[\overline{O_p}]_{\mathfrak{m}}$ is the localization of the coordinate ring of $\overline{O_p}$ at the maximal homogeneous ideal $\mathfrak{m}$. Since $\lambda_{i,j}(R_p)$ equals the multiplicity of E in $H^i_{\{0\}}(H^{\dim X - j}_{\overline{O_p}}(S))$, we obtain the following by our previous calculations in this section together with Theorem 4.1.

Corollary 5.5. *For* $p \in \{1, 2, 3\}$ *the following are the only non-zero Lyubeznik numbers* $\lambda_{i,j}(R_p)$ *(in which case they are equal to 1):*

(a) *If* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6)*:*
  (1) *When* $p = 1:$ $(i, j) = (0, m+3)$, $(0, 2m+3)$, $(m+2, 3m+4)$, $(2m+2, 3m+4)$, $(3m+4, 3m+4)$;
  (2) *When* $p = 2:$ $(i, j) = (0, 3m+4)$, $(m+2, 4m+5)$, $(2m+2, 4m+5)$, $(3m+4, 4m+5)$, $(3m+3, 5m+5)$, $(4m+5,\ 5m+5)$, $(5m+5, 5m+5)$;
  (3) *When* $p = 3:$ $(i, j) = (6m+7, 6m+7)$.
(b) *If* (G', X) *is* (C_3, ω_3)*:*
  (1) *When* $p = 1:$ $(i, j) = (3m+4, 3m+4)$;
  (2) *When* $p = 2:$ $(i, j) = (0, 3m+4)$, $(2m+2, 5m+5)$, $(5m+5, 5m+5)$;
  (3) *When* $p = 3:$ $(i, j) = (6m+7, 6m+7)$.

Finally, we give a list of the intersection cohomology groups of the orbit closures. These follow by our previous calculations together with Proposition 5.1.

Corollary 5.6. *For* $p \in \{1, 2, 3\}$ *the following are the only non-zero intersection cohomology groups* $IH^i(\overline{O}_p)$ *(in which case they are 1-dimensional):*

(a) *If* (G', X) *is* (A_5, ω_3), (D_6, ω_5) *or* (E_7, ω_6)*:*
   (1) *When* $p = 1$*, then* $i = 0,\ m+2,\ 2m+2$;
   (2) *When* $p = 2$*, then* $i = 0,\ m,\ 2m,\ 2m+2,\ 3m+2,\ 4m+2$;
   (3) *When* $p = 3$*, then* $i = 0,\ 6m+6$.
(b) *If* (G', X) *is* (C_3, ω_3)*:*
   - *When* $p = 1, 2, 3$*, then* $i = 0,\ 4m+2$.

5.2. **Gorenstein property and Castelnuovo–Mumford regularity.** In the last section, we establish some results for Gorenstein varieties based on observations that we extracted from the previous sections.

Throughout this section $G = G' \times \mathbb{C}^*$ denotes a linearly reductive complex connected algebraic group, and X is a finite-dimensional rational representation of G. Here the factor $\mathbb{C}^*$ of G acts on X by the usual scaling.

We start with a result describing the elements of a local cohomology module that are annihilated by its supporting ideal, which is relevant even in the case when $G' = \{1\}$.

Lemma 5.7. *Let Z be a Gorenstein G-stable closed subvariety of X with $c = \operatorname{codim}_X Z > 0$, and I the defining ideal of Z. Then there is a G-equivariant isomorphism of S-modules*

$$\operatorname{Hom}_S(S/I,\, H^c_Z(S)) \;\cong\; S/I \otimes \chi\,,$$

for some character $\chi : G \to \mathbb{C}^*$. In particular, the module $H^c_Z(S)$ has a unique a G-semi-invariant section h of degree $-\operatorname{reg}(I) - c$ with $\operatorname{Ann}_S(h) = I$ (here $\operatorname{reg}(I)$ denotes the Castelnuovo–Mumford regularity of I).

Proof. First, note that $\operatorname{Hom}_S(S/I,\, M) \cong \operatorname{Hom}_S(S/I,\, H^0_Z(M))$, for any S-module M. Hence, we have a spectral sequence

$$\operatorname{Ext}^i_S(S/I,\, H^j_Z(S)) \implies \operatorname{Ext}^{i+j}_S(S/I, S).$$

This yields a natural isomorphism

$$\operatorname{Hom}_S(S/I,\, H^c_Z(S)) \cong \operatorname{Ext}^c_S(S/I, S).$$

Since Z is Gorenstein, we have a G-equivariant isomorphism $\operatorname{Ext}^c_S(S/I, S) \cong S/I \otimes \chi$ for a character χ as required. ☐

The result above provides an interesting technique for proving that a variety is not Gorenstein. For example, by Lemma 4.7 we obtain that for $(G', X) = (C_3, \omega_3)$ the variety $\overline{O}_2$ is not Gorenstein. Similarly, [LRW19, Lemma 3.4] implies the (well-known) result that the affine cone over the twisted cubic curve is not Gorenstein.

Since the $\mathcal{D}_X$-module $H^c_Z(S)$ has a unique simple submodule L (corresponding to the intersection cohomology sheaf of the trivial local system on Z_{reg}), it is interesting to see when the element h as above lies in L. While this happens frequently (e.g., for our subexceptional series), it is not always the case as can be seen already when Z is a hypersurface (e.g., the discriminant of cubics [LRW19]).

In [Lev09, Conjecture 5.17], a conjecture has been made for the existence of (semi)-invariant sections for some $\mathcal{D}$-modules. The conjecture has been disproved precisely for orbit closures that are not Gorenstein (see [Rai17] and [LW19, Proposition 5.8]). On the other hand, from [KW12], [KW13] and [KW], we see that indeed all orbit closures in our exceptional series (G', X) are Gorenstein, with the only exception for $\overline{O}_2$ when $(G', X) = (C_3, \omega_3)$ due to the reason mentioned above (nevertheless, its regularity is obtained in [KW12, Page 38]). In conclusion, when the group G is large enough, the existence of a semi-invariant section as in Lemma 5.7 gives strong evidence for the Gorenstein property of a G-stable subvariety Z.

Proposition 5.8. *Consider Z as in Lemma 5.7. Assume $\mathbb{C}[X]^{G'} = \mathbb{C}[f]$, and that there is a surjective map of $\mathcal{D}$-modules $\pi : \mathcal{D}f^\alpha \twoheadrightarrow H^c_Z(S)$, for some $\alpha \in \mathbb{Q}$. Let $r \in \alpha + \mathbb{N}$ be maximal with the property $\pi(f^r) \neq 0$.*

Then r is a root of the Bernstein-Sato polynomial of f and $\operatorname{reg}(I) = -r \cdot \deg f - c$.

Proof. The existence of the map π implies that we have $f \in I$. In particular, this shows the existence of $r \in \alpha + \mathbb{N}$. By [LW19, Proposition 4.9], r is a root of the Bernstein-Sato polynomial of f.

Since $\mathbb{C}[X]^{G'} = \mathbb{C}[f]$, the only semi-invariants (up to constant) in $\mathcal{D}f^{\alpha}$ are powers of f. Hence, the only semi-invariant in $H^c_Z(S)$ that is annihilated by f is $\pi(f^r)$. By Lemma 5.7, $H^c_Z(S)$ has a semi-invariant section h annihilated by I. Since h is annihilated by f, this shows that $h = \pi(f^r)$ (up to non-zero constant). □

Remark 5.9. The existence of a map π can often be seen directly from the quiver of $\operatorname{mod}_G(\mathcal{D}_X)$. Namely, the module $S_f \cdot f^{\alpha}$ (for α such that this is G-equivariant) is an injective object in $\operatorname{mod}_G(\mathcal{D}_X)$, and the module $H^c_O(S)$ is an injective object in $\operatorname{mod}^{\overline{O}}_G(\mathcal{D}_X)$ for an orbit O (see [LRW19, Lemma 2.4], [LW19, Lemma 3.11]).

We established a fundamental link between the roots of the Bernstein–Sato polynomial of f and Castelnuovo–Mumford regularity of Gorenstein varieties that appear in the localizations at powers of f as above. The assumption for the existence of π as above is satisfied frequently for prehomogeneous vector spaces with semi-invariants. It is satisfied for our subexceptional series for all (Gorenstein) orbit closures. Using Remark 5.9, it is not difficult to see that in all the cases encountered in [LRW19], [Per18] and [LW19, Section 5], the map π exists for Gorenstein orbit closures with only one exception from [LW19, Section 5.5]:

Example 5.1. *Consider $G' = \operatorname{Spin}(9)$ and let X be its 16-dimensional spin representation. The group G acts on X with 4 orbits O_0, O_1, O_2, O_3 of codimensions* 16, 5, 1, 0*, respectively. Then $\mathbb{C}[X]^{G'} = \mathbb{C}[f]$, with* $\deg f = 2$*. The roots of the Bernstein-Sato polynomial of f are $-1, -8$. It follows from* [KW12, Section 5.1] *that $\overline{O}_1$ is Gorenstein with $\operatorname{reg}(\overline{O}_1) = 3$. Since $H^5_{\overline{O}_1}(S)$ is a simple $\mathcal{D}$-module that corresponds to an isolated vertex of the quiver of $\operatorname{mod}_G(\mathcal{D}_X)$ (see* [LW19, Section 5.5]*), it is not a composition factor of $S_f \cdot f^{\alpha}$, for any $\alpha \in \mathbb{Q}$. Of course, for the orbits O_0, O_2 the map π exists, and we have $1 = \operatorname{reg}(\overline{O}_2) = -(-1)\cdot 2 - 1$ and $0 = \operatorname{reg}(O_0) = -(-8)\cdot 2 - 16$.*

The exceptional behavior of $\overline{O}_1$ can be also seen from the fact that it is a self-dual highest weight orbit closure [KM87]*. Moreover, both of the simple equivariant $\mathcal{D}$-modules corresponding to O_2 and O_1 (by Lemma 5.7) have G'-invariant elements in degree -8. This demonstrates the sharpness of* [LW19, Corollary 3.23]*, as X is a spherical G-variety that is not of Capelli type.*

Finally, we apply the results above to our subexceptional series again.

Corollary 5.10. *For $p \in \{1,2\}$ the Castelnuovo–Mumford regularity $\operatorname{reg}(I_p)$ of the defining ideal I_p of $\overline{O}_p$ is:*

(a) *If (G', X) is (A_5, ω_3), (D_6, ω_5) or (E_7, ω_6) :* $\operatorname{reg}(I_1) = m + 2$, $\operatorname{reg}(I_2) = m + 3$;
(b) *If (G', X) is (C_3, ω_3) :* $\operatorname{reg}(I_1) = \operatorname{reg}(I_2) = m + 2$.

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