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Risk Preferences in Time Lotteries*

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An important but understudied question in economics is how people choose when facing uncertainty in the timing of events. Here we study preferences over time lotteries, in which the payment amount is certain but the payment time is uncertain. Expected discounted utility theory (EDUT) predicts decision makers to be risk-seeking over time lotteries. We explore a normative model of growth-optimality, in which decision makers maximise the long-term growth rate of their wealth. Revisiting experimental evidence on time lotteries, we find that growth-optimality accords better with the evidence than EDUT. We outline future experiments to scrutinise further the plausibility of growth-optimality.

Keywords Ergodicity Economics, Growth-Optimal Preferences, Time Lotteries

JEL Classification C61 · D01 · D81 · D9

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1 Introduction

Real-world economic decisions are usually a combination of at least two types of uncertainties. First, uncertainty about the exact payment *amount*, and second, uncertainty about the exact payment *time*. Examples for the latter include investing in an R&D venture with unknown completion date, exercising an American option, deciding on an optimal payment schedule of a life assurance with a specified death benefit, or choosing a delivery at a random or a guaranteed date. Uncertainty about the exact timing of events also abounds in many decisions that involve expenditures or losses like climate catastrophes with hard-to-predict timing but more or less known costs. While uncertainty about payment time is crucial for many real-world problems, both theoretical and empirical literatures on decision-making are mostly focused on random payment amounts.

A recent paper by DeJarnette et al. (2020) coined the term *time lotteries*. These are lotteries in which the payment amount is known with certainty, but the payment time is random. Expected Discounted Utility theory (EDUT) predicts decision makers to be risk-seeking over time lotteries – they would choose the lottery that corresponds to higher uncertainty in the payment time (for the same expected payment time).[1]

Yet, the experimental evidence on decision makers' risk preferences in time lotteries contrasts this prediction. Chesson and Viscusi (2003), Onay and Önçüler (2007), Ebert (2018) and DeJarnette et al. (2020) all find heterogeneous results that contradict the prediction of EDUT. At most, decision makers in experiments generally preferred lotteries with lower uncertainty in payment time.

This paper studies risk preferences in time lotteries by exploring a normative decision model that differs from EDUT in its optimand: the growth rate of the decision maker's wealth. More specifically, we study time lotteries under growth-optimality, *i.e.* when decision makers maximise the long-term growth rate of their wealth. The paper further aims to reconcile the experimental evidence with a theory of choice.

In time lotteries, the growth rate of wealth is a random variable. To make a growth-optimal choice between time lotteries it is thus necessary to collapse the random growth rate into a scalar. This can be done by taking different averages, leading to two approaches for calculating the growth rate associated with a time lottery. In the first approach, the scalar

[1]Like DeJarnette et al. (2020), "we interpret outcomes [...] as representing goods that are consumed on date t rather th[a]n prizes received in a certain date that need not coincide with the time of actual consumption." We thus follow the Money Earlier or Later framework (Cohen et al. 2020), keeping in mind the existing controversy in the literature regarding its relevance to decisions made in the context of consumption rather than money payments.

growth rate associated with a time lottery is the *time-average* growth rate, experienced by the decision maker if the lottery is indefinitely repeated over time. We call this the time approach. Maximisation of this growth rate predicts risk-neutral behaviour in time lotteries. The time approach is the implementation of "ergodicity economics" (PETERS 2019) to derive an optimal decision behaviour from the decision maker's experience over time. Studying time lotteries with the maximisation of the time-average growth rate as a choice criterion is the focus of this paper.

In the second approach, the scalar growth rate is derived as the expected value of the random growth rate or synonymously the *ensemble-average* growth rate. A decision maker experiences this growth rate as if the lottery is simultaneously realised infinitely many times, which might be impossible under most circumstances. We call this approach the ensemble approach. Maximising the ensemble-average growth rate is equivalent to maximising the expected change in discounted utility, *i.e.* to EDUT. It thus predicts risk-seeking behaviour in time lotteries.

We then use the experimental results of ONAY and ÖNÇÜLER (2007) and DEJARNETTE et al. (2020) to reconcile the predictions of the two approaches. For the experiment conducted by DEJARNETTE et al. (2020) we find that when the difference between the ensemble-average and time-average growth rates was small, decision makers were neither significantly risk-seeking nor risk-averse. Since the time approach predicts decision makers will be risk-neutral in time lotteries, it accords better with the experimental evidence than the ensemble approach.

As observed in the literature, decision makers' choices substantially deviated from the prediction of the ensemble approach. Yet, the difference between the ensemble-average and time-average growth rates may still be informative on these choices. As the ensemble-average growth rate got larger with respect to the time-average growth rate, decision makers were more likely to prefer less risky lotteries in experiments. Thus, the higher the ensemble-average growth rate was relative to the growth rate associated with a sure payment, the less attractive it became. The reanalysis of the experimental data suggests that decision makers may not consider the ensemble-average growth rate, equivalent to EDUT, as a relevant criterion for their choices. This result is found for both experiments done by ONAY and ÖNÇÜLER (2007) and DEJARNETTE et al. (2020).

Our main contribution is the rationalisation of risk-seeking, risk-neutral and risk-averse behaviour in time lotteries in a normative model with a single choice criterion. We emphasise that growth-optimality satisfies standard axioms of choice (VON NEUMANN and MORGENSTERN 1944). It assumes neither behavioural bias nor dynamic inconsistency in the decision maker's behaviour. At all times she prefers the option with the highest growth rate.

This paper also provides a framework for future experiments. It is possible to design choice problems between time lotteries for which one lottery is preferred in the ensemble approach and the other in the time approach. Such experiments, based on the model provided in this paper, are able to confirm or falsify the two approaches, and are planned.

In addition, this paper contributes to a growing body of work in ergodicity economics. Our analysis reveals in a novel way a conceptual problem in the implicit assumption of ergodicity in decision theory (Peters 2019), which manifests in using expected values as optimands. Boltzmann called the ergodic hypothesis and the interchangeable use of time averages and ensemble averages a "Kunstgriff" (trick) because only under ergodicity are the two types of averages identical, such that an expected value captures the behaviour over time (Boltzmann 1872, p. 345). This trick fails in the context of time lotteries. It is impossible to resurrect any decision criterion based on expected values, like in EDUT, if the uncertainty appears in the time domain. Our findings thus reinforce the conceptual insight of ergodicity economics to derive an optimal decision behaviour from the decision maker's experience over time.

The paper is organised as follows. Section 2 sets out the choice problem and the theoretical framework for analysing time lotteries. In Section 3 we describe how growth-optimality is used to evaluate time lotteries and determine the risk preferences predicted by the ensemble and time approaches. Section 4 reconciles growth-optimality with existing experimental evidence on risk preferences in time lotteries and presents distinguishing experimental setups able to falsify the predictions of the time approach. We conclude in Section 5 with a discussion on the significance of the results.

1.1 Related Literature

Economics Research on time lotteries appears under different aliases in the economic literature, such as lottery timing, time ambiguity, uncertain delay (Chesson and Viscusi 2003), timing risk (Onay and Öncüler 2007), delay risk (Ebert 2018), time risk (Ebert 2020) and money earlier or later tasks (Cohen et al. 2020). We adopt the term *Time Lottery* coined recently in DeJarnette et al. (2020).

As mentioned above, EDUT predicts risk-seeking behaviour in time lotteries (RSTL). The experimental studies thus emphasise any finding of deviant risk-averse behaviour in time lotteries (RATL). Indeed, in all of these experiments (Chesson and Viscusi 2003; Onay and Öncüler 2007; Ebert 2018; DeJarnette et al. 2020) a large fraction of subjects were risk-averse. Yet, there were many risk-seeking subjects across all studies. We show how growth-optimality offers an informative analysis of the experimental evidence in Section 4.

In an unincentivised survey sent to 373 business owners and managers, Chesson and Viscusi (2003) explain the deviations from the EDUT prediction by ambiguity aversion. Onay and Önçüler (2007) performed incentivised, yet inconsequential, experiments with 55 economics undergraduates. They find a large fraction of RATL decision makers, though not in all the conducted experiments, and especially not when timing risk is large. They suggest such preferences might be explained by probability weighting. Ebert (2018) analysed the choices of 80 students and found substantial heterogeneity of risk preferences over time lotteries. DeJarnette et al. (2020) performed incentivised experiments with 196 students and recruited 156 participants on Amazon Mechanical Turk. In line with the previous literature, they report consistent violations of the EDUT prediction, and find a significant fraction of RATL decision makers. To rationalise their findings they propose a generalisation of EDUT, adding additional curvature by wrapping the discount function in another function to absorb the excess risk aversion.

In summary, the remedies offered so far in the literature to the surprising experimental results are incompatible with EDUT. They either suggest behavioural explanations or add free parameters. Overall, however, both theoretical and experimental research on human behaviour in time lotteries is scant, especially when compared to related work on discounting or on uncertainty in payment amount.

Behavioural Ecology Contrary to this, in the context of optimal foraging theory in behavioural ecology, animal behaviour in time lotteries has been extensively studied over many species. Models in optimal foraging theory involve different optimands, referred to as currencies. These optimands are usually determined experimentally and differ across species and environmental conditions. In the classical model of optimal foraging the optimand is the long-term average rate of energy gain (Stephens and Krebs 1986, p. 7). Adding constraints to the model renders competing models and optimands plausible, such as maximisation of the optimal use of time, total food uptake, opportunity cost, mean time to reach satiation or specific nutrient maximisation (Gross 1986). Ultimately, the optimand strictly determines a suitable fitness criterion.

In Section 3 we study a model of growth rate maximisation. We present two approaches to find a suitable scalar growth rate. One is equivalent to EDUT, and the other is based on the maximisation of the time-average growth rate. It turns out that both approaches are used to explain risk preferences in behavioural ecology and have well-understood currency analogues:

- The time approach of growth rate maximisation (see Subsec. 3.1) corresponds to an optimand known as the ratio of expectations (RoE), *i.e.* the ratio of the expected gain over the expected delay (Bateson and Kacelnik 1995, p. 314, eq. 1).
- The standard model in economics, EDUT, follows the ensemble approach, see Subsec. 3.2. This growth rate corresponds to an optimand known as expectation of ratios (EoR) of gains over delay (Bateson and Kacelnik 1995, p. 314, eq. 2) or per patch rate maximisers (Stephens and Krebs 1986, Box 2.1).

The difference between the two approaches matters greatly for at least two reasons. First, they lead to different values and thus to different predicted behaviours in time lotteries. Second, they highlight the inability of arbitrary growth rates to correspond to real-world experiences. This correspondence is clearly desirable if a model aims to capture not only choices or behaviours, but also mechanisms and logic.

The importance of real-world relevance of any proposed mathematical behavioural model is well understood in behavioural ecology. For example, Stephens and Krebs (1986, p. 16, see esp. Box 2.1) mention that there is no clear mechanism generating the associated time interval of the ensemble-average growth rate. This is in line with Subsec. 3.2, where we show that it is possible to construct a growth rate which is equivalent to EDUT preferences, however there is no realistic mechanism that could generate this particular growth rate. Furthermore, it is known that the difference between the EoR and the RoE is a consequence of Jensen's inequality (Jensen 1906; Bateson and Kacelnik 1996). Jensen's inequality reappears in our proof of Proposition 3 which establishes the equivalence of EDUT and the ensemble approach. Paradoxical statements result from the erroneous treatment of Jensen's *inequality* as an *equality* and have led to the *Fallacy of Averages* (Templeton and Lawlor 1981; Smallwood 1996).

Studies on non-human animals report an extremely stable pattern of exclusively RSTL preferences for a wide range of species (see the surveys of Bateson and Kacelnik 1995; Kacelnik and Bateson 1996). The contrasting behaviour of humans (varying shares of RATL) and non-human animals (all RSTL) is interesting in its own right, yet it exceeds the scope of this paper.

Ergodicity Economics Our work also contributes to the growing field of ergodicity economics, exploring decision-making under the postulate that decision makers maximise the long-time growth rate of resources (Peters and Gell-Mann 2016; Peters and Adamou 2018a; Berman et al. 2020; Peters 2019). The conceptual focus of ergodicity economics is the embedding of uncertainty within time and not in the ensemble. Despite conceptual differences, ergodicity economics is capable of mechanistically resurrecting EDUT as a special case if the appropriate utility function happens to generate an ergodic observable in the expectation operator, see PETERS and ADAMOU (2018b) and ADAMOU et al. (2021). In the case of time lotteries, when uncertainty appears in the time domain it enters the growth rate calculation in the denominator. This makes it impossible to find a corresponding mapping using the expected value, see Subsec. 3.3. Thus from a theoretical perspective, our treatment of timing uncertainty complements the ergodicity economics literature.

From an experimental perspective, our reanalysis of data in Section 4 joins recent experimental evidence of strong dependence on wealth dynamics in human decisions under uncertainty (MEDER et al. 2019) and may be used to design similar experiments outlined in Subsec. 4.1, which are planned.

2 Theoretical Framework

We begin by formalising a choice problem between *time lotteries*. A decision maker has to choose between two possible gambles, or time lotteries, which involve fixed payments at either certain or uncertain timing. We follow the formalism introduced in DEJARNETTE et al. (2020) with small modifications to notation.

Definition 1 (Timed Payment)
A Timed Money Payment $\mathcal{M}$ is an ordered pair $(t, \Delta x)$, which denotes receiving an amount of money Δx at time t.

The payment amount and the payment time can be negative in general. However, for concreteness we confine our attention to positive payments ($\Delta x > 0$) and only to future payments, which are the most commonly considered dilemmas. Timed payments can now be used to define a time lottery.

Definition 2 (Time Lottery)
Consider two timed payments $\mathcal{M}_1$ and $\mathcal{M}_2$, which correspond to the same payment Δx at times t_1 and t_2 and a probability $0 \leq p \leq 1$ to receive the payment at the earlier time $t_1 (\leq t_2)$. Then the tuple $(t_1, t_2, p, \Delta x)$ defines a time lottery L. A time lottery provides a decision maker with the option to receive a certain payment Δx either at t_1 (with probability p) or the same payment at t_2 (with probability $1-p$).

For every time lottery L there exists a unique corresponding degenerate time lottery $\mathcal{L}$ for which the payment Δx is received with certainty at the expected payment time, $\langle t \rangle = pt_1 + (1-p)\,t_2$. Thus any degenerate time lottery $\mathcal{L}$ is a timed payment.

The setup assumed in the definitions above is presented in Fig. 1, which illustrates the choice problem between a risky time lottery L and its corresponding riskless degenerate time lottery $\mathcal{L}$.

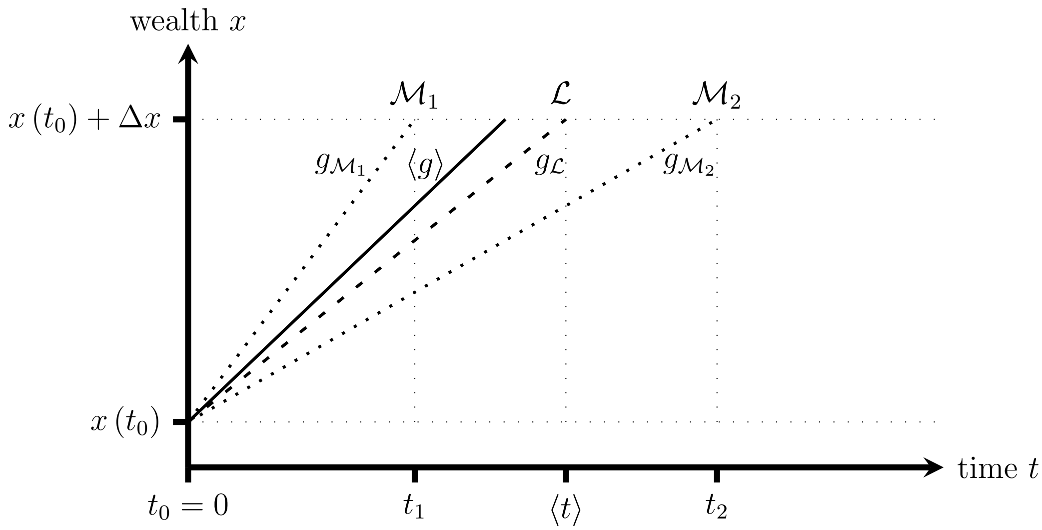

Figure 1: The time lottery L and its corresponding degenerate time lottery $\mathcal{L}$. In a time lottery L a certain payment Δx is received either at an earlier time t_1 with probability p or at a later time t_2 with probability $1-p$. The depicted slopes correspond to the growth rates of the corresponding timed payments. Receiving the payment at the earlier (later) time t_1 (t_2) leads to the growth rate $g_{\mathcal{M}_1}$ ($g_{\mathcal{M}_2}$). The ensemble average of these two growth rates associated with the time lottery leads to the ensemble-average growth rate $\langle g \rangle$. In the degenerate time lottery, $\mathcal{L}$, the payment Δx is received with certainty at $\langle t \rangle = pt_1 + (1-p)\,t_2$, which yields the growth rate $g_{\mathcal{L}}$.

Our aim is to provide a theory of how decision makers choose between time lotteries, and we define formal criteria for such choices in the next section. Therefore, we continue to describe the problem setup and the necessary definitions for evaluating time lotteries. An important property of preference relations over time lotteries is whether they are risk-averse or risk-seeking. Following DeJarnette et al. (2020), we define risk-averse behaviour over time lotteries.

Definition 3 (Risk-Averse over Time Lotteries)
The relation $\succsim$ is Risk-Averse over Time Lotteries (RATL) if for all payments Δx, for all

times t_1*,* t_2*, where* $t_1 \leq t_2$*, and for all probabilities* $0 \leq p \leq 1$ *then*

$$\mathcal{L} \succsim L \ . \tag{2.1}$$

Similarly, a relation $\succsim$ would be "Risk Seeking over Time Lotteries (RSTL) or Risk Neutral over Time Lotteries (RNTL) if the above holds with $\precsim$ or $\sim$, respectively" (DeJarnette et al. 2020, p. 4). Put simply, a decision maker is RATL if her preferences are best described by a RATL preference relation, *i.e.* if she would always prefer a certain timed payment over receiving the same amount at an uncertain time, with the same expected payment time as the certain timed payment. A RSTL decision maker would always prefer the lottery with the uncertain time in such a case.

It is also possible to compare two time lotteries with the same payment amount and the same expected payment time, but different payment time risk, *i.e.* variance of payment time. In such a case, we could consider the time lottery with the larger variance of payment time as more risky. Thus, preference to lotteries with smaller variance of payment time would be a natural extension to the definition of RATL. Growth-optimality will provide a decision criterion general enough to predict such behaviours as well.

3 Evaluating Time Lotteries

A formal decision theory has to provide a criterion for choosing between two time lotteries. Here we explore what happens if that criterion is maximisation of the growth rate of wealth, or *growth-optimality*. Under this criterion, a time lottery L_1 is chosen when it corresponds to a higher growth rate of the decision maker's wealth than another time lottery, L_2, and *vice versa*. This criterion requires defining properly the growth rate that corresponds to a time lottery. In this section we study two possible definitions for such a growth rate and their implications on risk preferences in time lotteries.

A growth rate, g, is defined as the scale parameter of time for an underlying dynamic of wealth.[2] In the case of time lotteries, as defined above, it is implicit in the literature that

[2] Assuming wealth follows some dynamic $x(t)$, it is possible to define a growth rate if there is a transformation $v(x)$ such that for any time t and time interval Δt

$$g = \frac{\Delta v(x)}{\Delta t} \tag{3.1}$$

is constant, *i.e.* it depends neither on t nor on Δt (where $\Delta v(x)$ is the change in v between t and $t + \Delta t$). This is explained in detail in Peters and Gell-Mann (2016) and Peters and Adamou (2018b). We note that this assumption is inconsequential to our mathematical analysis.

the dynamics are additive. Under additive dynamics the change of wealth Δx over a time interval, between t and $t+\Delta t$, is independent of the wealth level at time t. In such a case, the wealth, x, follows

$$x\left(t+\Delta t\right)=x\left(t\right)+\Delta x \ , \tag{3.2}$$

and the growth rate of wealth is defined as the rate at which wealth changes over the time interval, or

$$g=\frac{x\left(t+\Delta t\right)-x\left(t\right)}{\Delta t}=\frac{\Delta x}{\Delta t} \ . \tag{3.3}$$

It follows that a timed payment, $\mathcal{M}$, defined by the payment Δx and the payment time $t=t_0+\Delta t$, where t_0 is present time and Δt is a time interval, corresponds to a unique growth rate $\Delta x/\Delta t$. For simplicity we assume $t_0=0$, so a timed payment $\mathcal{M}$ corresponds to a growth rate

$$g_{\mathcal{M}}=\frac{\Delta x}{t} \ . \tag{3.4}$$

A time lottery, L, is composed of two timed payments, $\mathcal{M}_1$ and $\mathcal{M}_2$, corresponding to growth rates $g_{\mathcal{M}_1}={}^{\Delta x}/_{t_1}$ and $g_{\mathcal{M}_2}={}^{\Delta x}/_{t_2}$, respectively. L also defines its unique degenerate time lottery, $\mathcal{L}$, which is a timed payment that corresponds to the growth rate $g_{\mathcal{L}}={}^{\Delta x}/_{\langle t\rangle}$. These growth rates are illustrated in Fig. 1: the slope of the line that connects $(t_0, x\left(t_0\right))$ and $(t_1, x\left(t_0\right)+\Delta x)$ is $g_{\mathcal{M}_1}$; the slope of the line that connects $(t_0, x\left(t_0\right))$ and $(t_2, x\left(t_0\right)+\Delta x)$ is $g_{\mathcal{M}_2}$; the slope of the line that connects $(t_0, x\left(t_0\right))$ and $(\langle t\rangle, x\left(t_0\right)+\Delta x)$ is $g_{\mathcal{L}}$.

To make a growth-optimal choice between time lotteries we would like to define a single growth rate that describes a time lottery. However, the growth rate of wealth in time lotteries is a random variable

$$g=\begin{cases} g_{\mathcal{M}_1}=\frac{\Delta x}{t_1} & \text{with probability } p \\ g_{\mathcal{M}_2}=\frac{\Delta x}{t_2} & \text{with probability } 1-p \ , \end{cases} \tag{3.5}$$

so it is necessary to collapse the growth rate into a scalar. The scalar growth rates are then used as the criteria to choose between time lotteries.

In general, there are many ways to define a scalar growth rate for a time lottery. We study two possible approaches, the time approach and the ensemble approach. The rationale is to arrive at a growth rate that corresponds to the decision maker's experience of a time lottery. As it turns out, only the time approach truly leads to such a growth rate. Nevertheless, it is still possible that the ensemble approach is in better accordance with decision makers' choices. In Section 4 we put that to the test using experiments conducted by Onay and Öncüler (2007) and DeJarnette et al. (2020).

3.1 Time Approach

One approach for collapsing the random growth rate of a time lottery into a scalar is the time approach. In this approach the growth rate is defined as the *time-average* growth rate. Technically, this is achieved by taking the long-time limit of the time average of the growth rate. This quantity corresponds to the growth rate experienced by the decision maker if the lottery is indefinitely repeated over time.

Practically, we assume the lottery is sequentially repeated T times and evaluate the growth rate in the limit $T \to \infty$. We assume that in n_1 of the times the early payment was realised, and in $n_2 = T - n_1$ of the times it was the later payment.

We denote the time-average growth rate of a time lottery L by $\bar{g}$, given by

$$\bar{g} \equiv \lim_{T\to\infty} \frac{\text{total payment after } T \text{ rounds}}{\text{total time elapsed after } T \text{ rounds}} \tag{3.6}$$

$$= \lim_{T\to\infty} \frac{T\Delta x}{n_1 t_1 + n_2 t_2} = \lim_{T\to\infty} \frac{\Delta x}{{}^{n_1}\!/_T \cdot t_1 + {}^{n_2}\!/_T \cdot t_2} \tag{3.7}$$

$$= \frac{\Delta x}{p t_1 + (1-p)\, t_2} = \frac{\Delta x}{\langle t \rangle} \,. \tag{3.8}$$

The time-average growth rate, $\bar{g}$, is illustrated by the slope of the dashed line in Fig. 1 and denoted by $g_{\mathcal{L}}$, the notation will become clear shortly.

3.2 Ensemble Approach

Another approach for collapsing the random growth rate into a scalar is the ensemble approach. Here, the scalar growth rate is defined as the *ensemble-average* growth rate or, synonymously, as the expected value of the growth rate. Technically, this is achieved by taking the large sample limit of the ensemble average of the growth rate, which simply gives the arithmetic mean of the random growth rate.

Practically, we assume the lottery is simultaneously realised N times and denote by g_i the i-th realised growth rate. As before, we assume that in n_1 realisations the early payment was realised and in $n_2 = N - n_1$ realisations it was the later payment. We then evaluate the mean of these growth rates in the limit $N \to \infty$.

We denote the ensemble-average growth rate of a time lottery L by $\langle g \rangle$, given by

$$\langle g \rangle \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} g_i = \lim_{N\to\infty} \frac{1}{N} \left(n_1 \cdot g_{\mathcal{M}_1} + n_2 \cdot g_{\mathcal{M}_2} \right) \tag{3.9}$$

$$= \lim_{N\to\infty} \left(\frac{n_1}{N} \cdot \frac{\Delta x}{t_1} + \frac{n_2}{N} \cdot \frac{\Delta x}{t_2} \right) \tag{3.10}$$

$$= p \frac{\Delta x}{t_1} + (1-p) \frac{\Delta x}{t_2} = \left\langle \frac{\Delta x}{t} \right\rangle \ . \tag{3.11}$$

The ensemble-average growth rate, $\langle g \rangle$, is illustrated by the slope of the solid line in Fig. 1. This quantity describes the decision maker's experience as if $N \to \infty$ simultaneous lotteries were realised, with their outcomes pooled together and divided by N.

We note that a decision maker may never experience the ensemble-average growth rate. This is also illustrated in Fig. 1 by the difference of the slopes of $\langle g \rangle$ and $g_{\mathcal{L}}$. The ensemble-average and time-average growth rates differ in the way the decision maker's experience in time lotteries is quantified. It is therefore no surprise that the ensemble-average growth rate of a time lottery, $\langle g \rangle$, does not coincide with the time-average growth rate, $\bar{g}$, where the former is generally higher. They will only coincide if $t_1 = t_2$ or if p is either 0 or 1, *i.e.* if the time lottery comprises only a single timed payment, *i.e.* if it is, in fact, a degenerate time lottery. Put differently, there exists no real-world mechanism under the time lottery that would result in the time interval that corresponds to $\langle g \rangle$.[3]

3.3 The Failure of the "Kunstgriff"

The inequality between the scalar growth rates

$$\bar{g} \neq \langle g \rangle \tag{3.12}$$

implies that the growth rate of wealth is non-ergodic in time lotteries. In the context of statistical mechanics, BOLTZMANN referred to ergodicity, *i.e.* the equality of the time average and the ensemble average of an observable (in our case $\bar{g} = \langle g \rangle$) and hence their substitutability, as a mathematical "Kunstgriff" (trick). This trick turns out to be helpful in simplifying many calculations (BOLTZMANN 1872, p. 345), and was labelled later the ergodic hypothesis.

[3]Given a time lottery that pays Δx at either t_1 with probability p or at t_2 with probability $(1-p)$, this effective time interval is the time at which the solid line in Fig. 1 crosses the horizontal line $x(t_0) + \Delta x$. It equals $\frac{t_1 t_2}{t_1 + p(t_2 - t_1)}$. This effective time does not correspond to an actual payment time or an average payment time at which the decision maker receives Δx.

Every decision theory that relies on an expected value in its optimand is paired with the ergodicity hypothesis explicitly or implicitly. Under certain conditions, transforming a non-ergodic observable (such as the wealth of a decision maker) into an ergodic observable yields equivalence between expected utility theory (EUT) and the maximisation of the time-average growth rate (Peters and Gell-Mann 2016; Peters and Adamou 2018b). Under the appropriate transformation of wealth, both theories would give identical predictions of decision makers' choices. Furthermore, this equivalence produces predictions of decision makers' utility functions, which can be experimentally tested (Peters and Gell-Mann 2016; Meder et al. 2019).

In the case of time lotteries, such transformation of wealth does not exist. Since the dynamics are additive the appropriate transformation or utility function is the identity, $u(x) = x$ (Peters and Gell-Mann 2016; Peters and Adamou 2018b). Although the appropriate mapping is applied in Eq. (3.11), the ensemble-average and time-average growth rates are not equal. This is a result of introducing the uncertainty in the payment time and not in the payment amount, where the ergodicity transformation could counteract the non-ergodicity. Thus the uncertainty enters Eq. (3.8) in the denominator, which makes it impossible to construct an ergodic growth rate. Put differently, there exists no function u of wealth x, which induces an equality between the ensemble average on the left-hand side and the time average on the right-hand side of the following equation:

$$\left\langle \frac{\Delta u(x)}{t} \right\rangle \stackrel{?}{=} \frac{\Delta x}{\langle t \rangle} \tag{3.13}$$

$$p\frac{\Delta u(x)}{t_1} + (1-p)\,\frac{\Delta u(x)}{t_2} \stackrel{?}{=} \frac{\Delta x}{pt_1 + (1-p)\,t_2} \tag{3.14}$$

$$\Delta u(x) \stackrel{?}{=} \frac{t_1 t_2}{(pt_1 + (1-p)\,t_2)(pt_2 + (1-p)\,t_1)}\Delta x\ . \tag{3.15}$$

For the equality in Eq. (3.13) to be obtained it is therefore necessary for $u(x)$ to also depend on the payment times, *i.e.* on the problem setup, rather than on wealth x and its dynamics.

3.4 Using Growth Rates to Evaluate Time Lotteries

3.4.1 Growth-Optimality

We will now use the definitions above as choice criteria between time lotteries. Choosing between two time lotteries, L_a and L_b, requires evaluating the growth rate associated with each, *i.e.* g_a and g_b. We define a preference relation of a decision maker who chooses growth-optimally between time lotteries.

Definition 4 (Growth-Optimal over Time Lotteries)
The relation $\succsim$ *is Growth-Optimal over Time Lotteries (GOTL) if given two time lotteries,* L_a *and* L_b*, with growth rates* g_a *and* g_b*, respectively,*

1. $L_a \succ L_b$ *[‘L_a is preferred to L_b’] if and only if* $g_a > g_b$
2. $L_a \sim L_b$ *[‘indifference between L_a and L_b’] if and only if* $g_a = g_b$
3. $L_a \prec L_b$ *[‘L_b is preferred to L_a’] if and only if* $g_a < g_b$.

In words, a decision maker is GOTL if she prefers lottery L_a if her wealth grows faster under this choice than under choice L_b, and vice versa. While intuitive and straightforward, such preferences are not commonly studied in economics, but they adhere to the standard rationality definitions.

Proposition 1 (Optimization of Growth satisfies von Neumann-Morgenstern axioms)
If a relation $\succsim$ *is GOTL, it satisfies the axioms by* von Neumann-Morgenstern.

Proof. See Appendix A. ■

We can now turn to test what growth-optimal preferences mean for decision makers’ choices in time lotteries, based on the two approaches described above.

3.4.2 Risk-Neutrality over Time Lotteries in the Time Approach

Proposition 2 (Time Approach Predicts RNTL)
In the time approach, if a relation $\succsim$ *is GOTL, then* $\succsim$ *is RNTL (risk-neutral over time lotteries).*

Proof. We would like to show that in the time approach, if a relation $\succsim$ is growth-optimal, then $\succsim$ is RNTL (risk-neutral over time lotteries).

To show that $\succsim$ is RNTL, a decision maker must be indifferent between the time lottery and its degenerate time lottery, $L \sim \mathcal{L}$. In the time approach the growth rate of the time lottery L is given by Eq. (3.6):

$$\bar{g} = \frac{\Delta x}{pt_1 + (1-p)\,t_2} = \frac{\Delta x}{\langle t \rangle} \,, \tag{3.16}$$

The growth rate of the degenerate time lottery $\mathcal{L}$ is $g_{\mathcal{L}} = {}^{\Delta x}/_{\langle t \rangle}$. Hence, the time-average growth rate of the risky time lottery coincides with the growth rate of its riskless degenerate time lottery. Because $\succsim$ is growth-optimal, $L \sim \mathcal{L}$.

■

Thus, maximising the time-average growth rate predicts decision makers would be indifferent between risky time lotteries and their corresponding riskless degenerate time lotteries. The ensemble approach yields a different prediction of growth-optimal behaviour.

3.4.3 Risk-Seeking Behaviour over Time Lotteries in the Ensemble Approach

Proposition 3 (Ensemble Approach Predicts RSTL)
In the ensemble approach, if a relation $\succsim$ is GOTL, then $\succsim$ is RSTL (risk-seeking over time lotteries).

Proof. We would like to show that in the ensemble approach, if a relation $\succsim$ is growth-optimal, then $\succsim$ is RSTL (risk-seeking over time lotteries).

For that purpose we look at a general time lottery L, which is composed of two timed payments, $\mathcal{M}_1$ and $\mathcal{M}_2$, corresponding to growth rates $g_{\mathcal{M}_1} = {}^{\Delta x}/_{t_1}$ and $g_{\mathcal{M}_2} = {}^{\Delta x}/_{t_2}$, respectively. The time lottery L also defines its unique degenerate time lottery, $\mathcal{L}$, which is a timed payment that corresponds to the growth rate $g_{\mathcal{L}} = {}^{\Delta x}/_{\langle t \rangle}$. We also assume that the time lottery L is not just a timed payment, *i.e.* that $t_1 \neq t_2$ and p is neither 0 nor 1. To show that $\succsim$ is RSTL, L must be preferred to $\mathcal{L}$.

In the ensemble approach the growth rate of L is given by Eq. (3.11):

$$\langle g \rangle = \left\langle \frac{\Delta x}{t} \right\rangle = p \frac{\Delta x}{t_1} + (1-p) \frac{\Delta x}{t_2} \; . \tag{3.17}$$

We define $f(z) = {}^1/_z$, thus f is a convex function and from Jensen's inequality follows

$$\begin{aligned}
f\big(\langle t \rangle\big) &< \big\langle f(t) \big\rangle && (3.18)\\
\Delta x \frac{1}{\langle t \rangle} &< \Delta x \left\langle \frac{1}{t} \right\rangle && (3.19)\\
\frac{\Delta x}{\langle t \rangle} &< p \frac{\Delta x}{t_1} + (1-p) \frac{\Delta x}{t_2} && (3.20)\\
g_{\mathcal{L}} &< \langle g \rangle \; . && (3.21)
\end{aligned}$$

Because $\succsim$ is growth-optimal, the time lottery is preferred over the corresponding degenerate time lottery, $L \succ \mathcal{L}$.

■

Thus, maximising the ensemble-average growth rate means that decision makers would prefer risky time lotteries over their corresponding riskless degenerate time lotteries.[4] This is mathematically equivalent to maximising expected discounted utility, $\mathrm{E}\left[D(t)\Delta u(x)\right]$, for linear utility ($u(x) = x$) and hyperbolic discounting ($D(t) = {}^{1}\!/_{t}$). As described in Peters and Adamou (2018b) and Adamou et al. (2021), these are indeed the appropriate functions for the specified problem and dynamics. Thus, this coincides with the prediction of EDUT that expected discounted utility maximisers must be RSTL for any utility function and any convex discount function.

It is worth repeating the main argument from Subsec. 3.3 about the failure of the ergodicity trick here. There exists no concave utility function u which if inserted in the right-hand side of Eq. (3.19) induces an equality.

The ensemble and time approaches not only result in different growth rates but also lead to different risk preferences in time lotteries. We can now put the two decision criteria – the ensemble-average growth rate and the time-average growth rate – to the test by revisiting experiments conducted by Onay and Önçüler (2007) and DeJarnette et al. (2020). We would like to find out which of the two, if any, is in accordance with the experimental evidence.

4 Revisiting Experimental Evidence on Time Lotteries

The growth-optimality criterion makes two predictions on risk preferences in time lotteries, depending on how growth rates are computed. We have just established that in the ensemble approach decision makers must be risk-seeking over time lotteries. Practically, when facing a choice between a time lottery and its corresponding degenerate time lottery, the ensemble approach predicts a preference for the risky lottery. In the case of a choice between two time lotteries with the same payment amount and the same expected payment time, but a different payment time risk, decision makers must prefer the lottery with the higher variance in payment time.

The time approach predicts decision makers are risk-neutral over time lotteries. This means

[4] This is unless $g_a = g_b$. In the ensemble approach this happens only if $t_1 = t_2$, or if $p \in \{0, 1\}$, *i.e.* the time lottery is a timed payment.

decision makers must be indifferent between a time lottery and its corresponding degenerate time lottery, and between lotteries with the same payment amount and the same expected payment time. If decision makers are forced to choose, the time approach does not provide a prediction for this choice. It may be hypothesised that indifference means that subjects are neither significantly risk-seeking nor risk-averse. Yet, it is possible that there are other higher-order effects that would systematically affect subjects' choices and are not captured by our model of choice alone.

We put these predictions to the test using evidence from two experiments (Onay and Önçüler 2007; DeJarnette et al. 2020). For each experiment we calculate the corresponding growth rate of every time lottery under the ensemble and time approaches, using Eq. (3.11) in the ensemble approach and Eq. (3.8) in the time approach.

Table 1 presents the results for Part I of the experiment in DeJarnette et al. (2020). We reproduce Table 2 of their Appendix D adding the corresponding growth rates. In this experiment a total number of 196 subjects had to choose between two time lotteries in ten different settings.[5] In six settings subjects had to choose between a time lottery and its corresponding degenerate time lottery (questions 1–3). In the other four settings the choice was between two time lotteries with the same payment amount and the same expected payment time, but a different payment time risk. In Tab. 1 option I refers to the less risky option (or the degenerate time lottery) and option II to the riskier option.

Table 2 shows similar results for Study 1 in Onay and Önçüler (2007). In this study 55 subjects had to choose between time lotteries and their corresponding degenerate time lottery in six different settings. Unlike the experiments in DeJarnette et al. (2020), this experiment was not consequential, and participants only received a flat rate payment for their participation.

In each table and for every choice problem we include the growth rate associated with each option using the ensemble approach ($\langle g \rangle$) and the time approach ($\bar{g}$). We also include the difference between these growth rates and the fraction of subjects who were RATL, thus violating the EDUT prediction. We note that the growth rate magnitudes are not comparable between the experiments. The growth rates in the tables are given in the relevant units for each of the experiments, *i.e.* US Dollars per week ($\$/wk$) in Tab. 1 and New Turkish Lira per month (NTL/mth) in Tab. 2.

These results demonstrate that the ensemble-average growth rate is always higher than the

[5]Each subject had to make only five choices in total and received payment for only one of the five choices, selected randomly. For more details on the experiment please refer to Appendix D in DeJarnette et al. (2020).

Table 1: Growth rates for the experimental results in DeJarnette et al. (2020) (in $\$/wk$).

Question	$\langle g\rangle^{\mathrm{I}}$	$\langle g\rangle^{\mathrm{II}}$	$\bar{g}^{\mathrm{I}}$ $(=\bar{g}^{\mathrm{II}})$	$\langle g\rangle^{\mathrm{II}}-\bar{g}^{\mathrm{II}}$	Fraction of RATL subjects (%)
Question 1, long treatment	10.0	16.0	10.0	6.0	65.7
Question 1, short treatment	10.0	16.0	10.0	6.0	56.0
Question 2, long treatment	5.0	6.9	5.0	1.9	50.5
Question 2, short treatment	5.0	9.0	5.0	4.0	55.0
Question 3, long treatment	5.0	6.7	5.0	1.7	48.6
Question 3, short treatment	5.0	6.7	5.0	1.7	37.4
Question 4, long treatment	8.3	12.5	8.0	4.5	64.8
Question 4, short treatment	8.3	8.8	8.0	0.8	38.5
Question 5, long treatment	5.3	11.6	4.3	7.3	73.3
Question 5, short treatment	3.5	3.0	2.9	0.1	52.8

Notes: In each question subjects had to choose between two options, labelled I and II. $\langle g\rangle^{\mathrm{I}}$ ($\langle g\rangle^{\mathrm{II}}$) refers to the ensemble-average growth rate associated with option I (II). Similarly $\bar{g}^{\mathrm{I}}$ ($\bar{g}^{\mathrm{II}}$) refers to the time-average growth rate associated with option I (II). The payment amount and the expected payment time were similar for the two options, so $\bar{g}^{\mathrm{I}}=\bar{g}^{\mathrm{II}}$. In questions 1–3 option I was the degenerate time lottery corresponding to option II in the respective question. In these cases $\langle g\rangle^{\mathrm{I}}=\bar{g}^{\mathrm{I}}$, since option I was a timed payment. In questions 4 and 5, options I and II were two non-degenerate time lotteries with the same payment amount and the same expected payment time, while in option I the payment time had lower variance than in option II. Thus, in all questions option I was the less risky time lottery and option II the more risky. The fraction of RATL subjects is the fraction of subjects who preferred option II to option I in each question. The long and short treatments refer to two different experiments with a similar structure but with different parameters, *i.e.* different payment amounts, payment times and probabilities. Questions 1 and 3 were similar in both treatments. For more details on the experiments please refer to Appendix D in DeJarnette et al. (2020).

Table 2: Growth rates for the experimental results in Onay and Öncüler (2007, Tables 2 & 5) (in NTL/mth).

Case	$\langle t\rangle$ $(mths)$	Δx (NTL)	$\langle g\rangle$	$\bar{g}$	$\langle g\rangle-\bar{g}$	Fraction of RATL subjects (%)
1	9	160	43.6	27.8	25.9	22
2	9	140	38.2	15.6	22.6	9
3	6	160	87.3	26.7	60.6	62
4	6	140	76.4	23.3	53.0	40
5	2	160	145.5	80.0	65.5	75
6	2	140	127.3	70.0	57.3	93

Notes: In all cases subjects had to choose between a time lottery and its corresponding degenerate time lottery. In cases 2, 4 and 6, the question presented to the subjects was framed in terms of a loss of 140 NTL rather than a gain. For comparability with the other experiments we consider absolute amounts. The fraction of RATL subjects in cases 2, 4 and 6 is therefore found by subtracting the "proportion of subjects that behaved as predicted by DEU" in Onay and Öncüler (2007, Tab. 2) from 100%. For more details on the experiments please refer to Section 2 in Onay and Öncüler (2007).

time-average growth rate, $\langle g\rangle>\bar{g}$, a result of Jensen's inequality (see also the proof of Proposition 3). It is also illustrated in Fig. 1, where $\langle g\rangle$ corresponds to the slope of the solid line, which is higher than the time-average growth rate $\bar{g}$, the slope of the line that connects the points $(t_0, x(t_0))$ and $(\langle t\rangle, x(t_0)+\Delta x)$. This creates the illusion of faster growth, while in practice, $\langle g\rangle$ is inaccessible as has been clearly identified in the behavioural ecology Stephens and Krebs (see 1986, p. 16, Box 2.1). For degenerate time lotteries there is no difference between the two growth rates.

The experimental results are not straightforward to interpret, and they do not provide a clear-cut answer as to whether decision makers are generally risk-seeking, risk-neutral or risk-averse in time lotteries. In both experiments, many subjects were risk-seeking and many were risk-averse. Nevertheless, the difference between the ensemble-average and time-average growth rates reveals a striking regularity. In both experiments, as the ensemble-average growth rate gets larger with respect to the time-average growth rate, decision makers are more likely to prefer less risky lotteries, *i.e.* the bigger the difference is, the more subjects were RATL. This regularity is seemingly counterintuitive. Under growth-optimality, when the growth rate associated with a particular choice is very high, one would expect it to be attractive to decision makers. But the higher the ensemble-average growth rate is, relative to the sure payment, the less attractive it becomes. This is illustrated in Fig. 2.

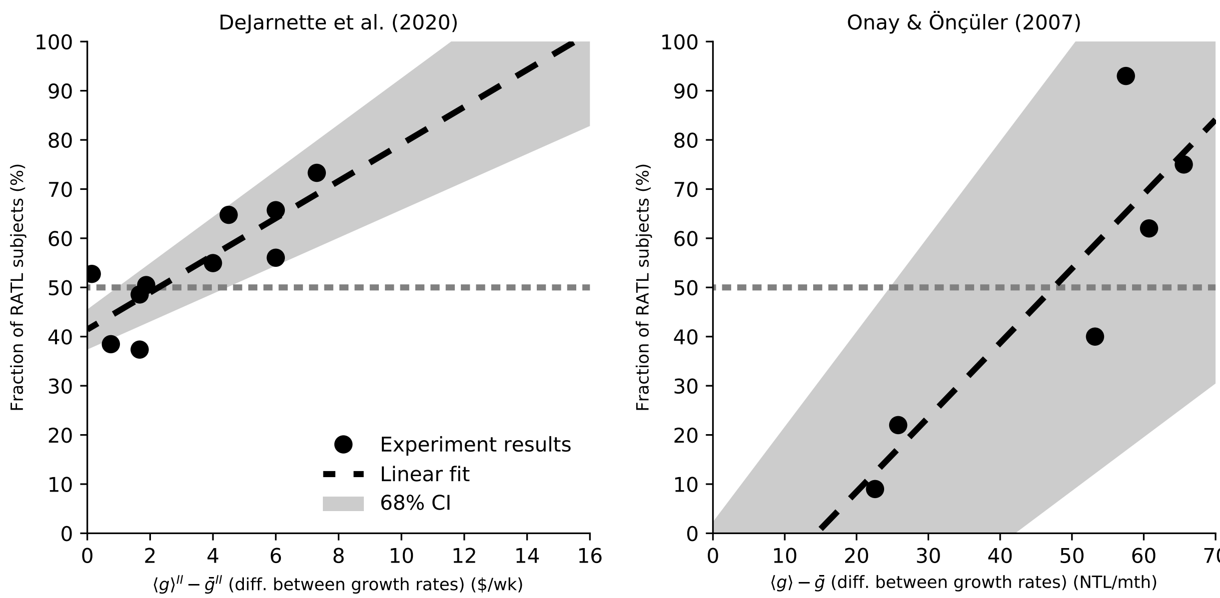

Figure 2: The relationship between the fraction of RATL subjects and the difference between the ensemble-average and time-average growth rates ($\langle g\rangle - \bar{g}$) in incentivised experiments (see Tab. 1 and Tab. 2). Left) Results for Part I of the experiment in DeJarnette et al. (2020). Right) Results for Study 1 in Onay and Önçüler (2007). The black lines are OLS linear fits for the fraction of RATL subjects as a function of $\langle g\rangle - \bar{g}$ ($R^2 = 0.67$ in DeJarnette et al. (2020), $R^2 = 0.76$ in Onay and Önçüler (2007)). The grey areas represent 68% (1σ) confidence intervals based on the linear fits.

Thus, the experimental results falsify the ensemble approach, or EDUT. Strictly speaking, EDUT predicts that all subjects must be risk-seeking. More realistically, it would predict an inverse relationship between the fraction of RATL subjects and the difference $\langle g\rangle - \bar{g}$, since the riskier option becomes more attractive if evaluated using its ensemble-average growth rate. However, the relationship found in the experiments is significantly positive.

In the experiment conducted by DeJarnette et al. (2020) we find that if the difference between the ensemble-average and time-average growth rates is small, decision makers are neither significantly risk-seeking nor risk-averse. Thus, in such cases, the prediction of the time approach, that decision makers will be risk-neutral in time lotteries, is in line with the experimental evidence. This indicates that the time approach may provide a better model of decision-making, also echoing the experimental results in Meder et al. (2019).

The risk-neutrality predicted in the time approach may practically lead to arbitrary fractions of RATL. Thus, the time approach predicts neither an upward nor a downward sloping relationship for the fraction of RATL subjects and the difference $\langle g \rangle - \bar{g}$, and the experimental results do not support or falsify it. Other effects, possibly systematic, may affect decision makers' choices. It is also possible that if decision makers are forced to choose, there are higher-order decision criteria that are not captured by growth-optimality alone. Other experiments, with lotteries for which the time approach does not predict indifference, are therefore necessary to confirm or falsify it.

4.1 Distinguishing Experimental Design

In our basic setup, decision makers had to choose between a risky time lottery and its corresponding riskless degenerate time lottery. This is a choice between two options with the same payment amount and the same expected payment time. In such a case, maximising the time-average growth rate predicts risk-neutrality, *i.e.* indifference between the risky time lottery and the riskless degenerate time lottery. To design experiments which may falsify or confirm the time approach prediction the choice has to be between lotteries that differ either in the payment amount or in the expected payment time.

Setup 1 – adjusting times A simple setup of such a choice problem would be between a risky time lottery, L, and a riskless timed payment, $\mathcal{M}$, with the same payment amount, Δx and is depicted in Fig. 3. The expected payment time of the time lottery, $\langle t \rangle$, would now differ from the payment time of the timed payment, $t_{\mathcal{M}}$. If $\langle t \rangle > t_{\mathcal{M}}$ the time approach predicts $\mathcal{M} \succ L$, since

$$\bar{g}_{\mathcal{M}} = {}^{\Delta x}/_{t_{\mathcal{M}}} > {}^{\Delta x}/_{\langle t \rangle} = \bar{g}_L \ , \tag{4.1}$$

and $\mathcal{M} \prec L$ if $\langle t \rangle < t_{\mathcal{M}}$. The advantage of this setup is the invariance to subjective perception of money.

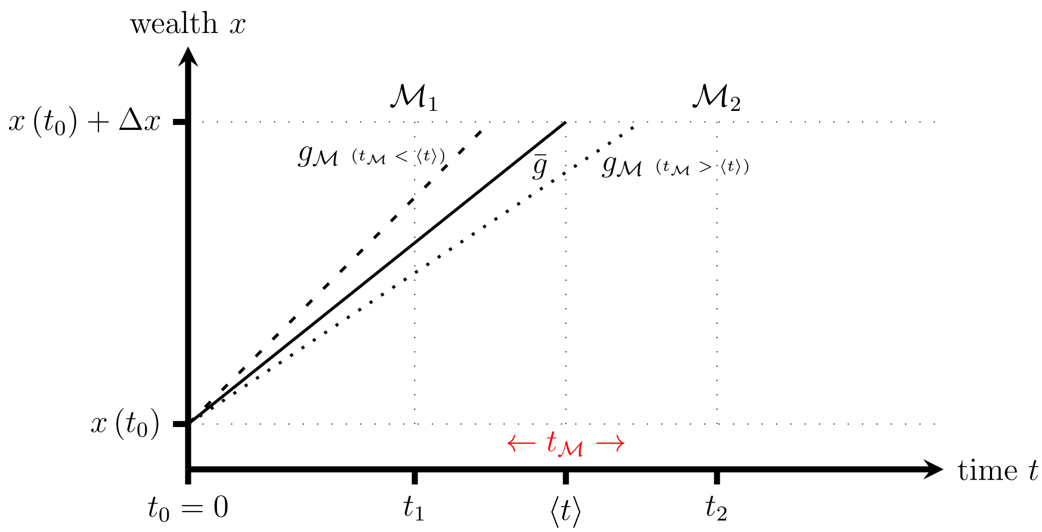

Figure 3: Setup 1 – adjusting the payment times. If the payment time $t_{\mathcal{M}}$ of the riskless timed payment is earlier (later) than the expected payment time, then $t_{\mathcal{M}} < \langle t \rangle$ ($t_{\mathcal{M}} > \langle t \rangle$) and the corresponding growth rate is higher (lower) than $\bar{g}_L$. The time approach predicts in this case that decision makers are RATL (RSTL).

Setup 2 – adjusting amounts A different distinguishing setup would be a choice problem between a risky time lottery, L, and a riskless timed payment, $\mathcal{M}$, with payment amounts Δx_L and $\Delta x_{\mathcal{M}}$, respectively, depicted in Fig. 4. The expected payment time of the time lottery is the same as the payment time of the timed payment, $\langle t \rangle = t_{\mathcal{M}}$. If $\Delta x_{\mathcal{M}} > \Delta x_L$ the prediction is $\mathcal{M} \succ L$, since

$$\bar{g}_{\mathcal{M}} = {}^{\Delta x_{\mathcal{M}}}/_{t_{\mathcal{M}}} > {}^{\Delta x_L}/_{\langle t \rangle} = \bar{g}_L \ , \tag{4.2}$$

and $\mathcal{M} \prec L$ if $\Delta x_{\mathcal{M}} < \Delta x_L$.

We note that in EDUT there is no clear-cut prediction of choices in these setups as EDUT is underdetermined. In the first setup the prediction would require specifying a subjective discount function. In the second setup it would require specifying a subjective utility function. If we use the ensemble-average growth rate as the equivalent choice criterion of EDUT instead, we can achieve a clear-cut prediction. It is possible to design setups in which using the time-average growth rate and the ensemble-average growth rate as choice criteria will lead to opposite predictions. For example, in the first setup, if the possible payment times in L are t_1 and t_2, with probability p to receive the payment at t_1 such that

$$\langle t \rangle > t_{\mathcal{M}} > \left({}^{p}/_{t_1} + {}^{(1-p)}/_{t_2} \right)^{-1} \tag{4.3}$$

then

$$\bar{g}_L < \bar{g}_{\mathcal{M}} = \langle g \rangle_{\mathcal{M}} < \langle g \rangle_L \ , \tag{4.4}$$

so maximising the time-average growth rate predicts preference for the riskless timed payment whereas EDUT predicts preference for the risky time lottery.

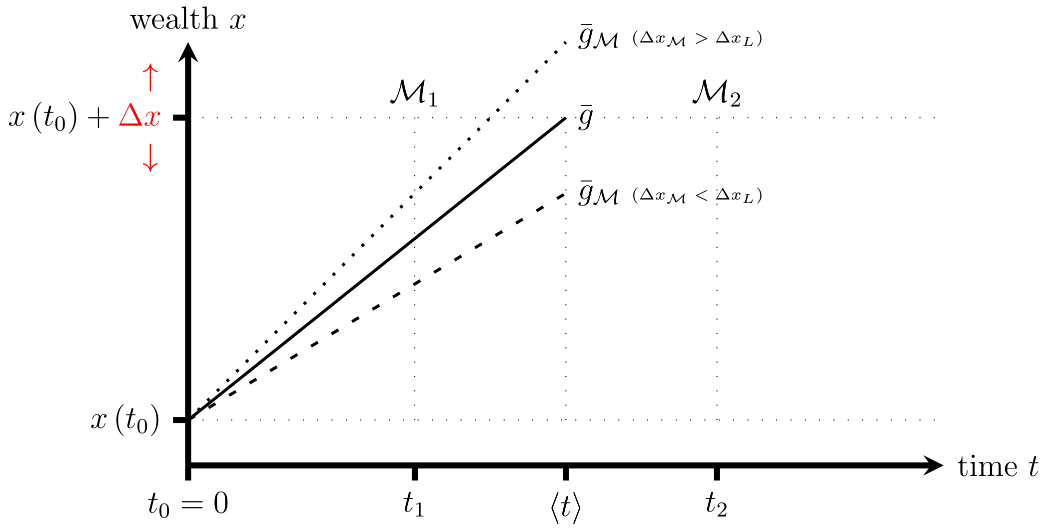

Figure 4: Setup 2 – adjusting the payment amounts. If the payment amount $\Delta x_{\mathcal{M}}$ of the riskless timed payment is higher (lower) than the amount of the risky time lottery, then the corresponding growth rate is higher (lower) than $\bar{g}_L$, and the time approach predicts that decision makers are RATL (RSTL).

5 Discussion

This paper explores time lotteries under the postulate that decision makers maximise the growth rate of their wealth. We consider a basic choice problem between a certain payment at a certain time, to the same payment but with an uncertain payment time, assuming the expected payment time is equal to the certain one.

In time lotteries, the growth rate of wealth is a random variable and to make a growth-optimal choice between two lotteries requires collapsing this random variable into a scalar. We present two approaches to compute this scalar. In the first approach, the time approach, the growth rate associated with a time lottery is the growth rate experienced by the decision maker if the lottery is indefinitely repeated over time. Maximisation of the time-average growth rate predicts risk-neutral behaviour in time lotteries.

In the second approach, the ensemble approach, the growth rate associated with a time lottery is the mean or ensemble average of the growth rates, as if the lottery was simultaneously realised infinitely many times. This growth rate does not correspond to any real-world experience of a decision maker. Maximising the ensemble-average growth rate is equivalent to maximising the expected change in discounted utility, and predicts risk-seeking behaviour in time lotteries. Both approaches are consistent with standard axioms of choice and assume neither behavioural bias nor dynamic inconsistency in the decision maker's behaviour. At all times she prefers the option with the highest growth rate.

Of the two approaches, only the time-average growth rate is a physically meaningful decision criterion, as it corresponds to a decision maker's experience if the time lottery is indefinitely repeated. Together with the empirical evidence, our analysis thus invalidates the decision criterion put forth by the ensemble approach, or EDUT. The time approach predicts risk-neutrality in time lotteries. Empirically testing a theory where its prediction is risk-neutrality is challenging, as systematic effects not captured by the choice model may influence decisions. This complicates the validation of the time approach. It is not a weakness of this particular approach, but merely a statement that the existing experiments were not designed to test the time approach. Therefore we propose two setups for distinguishing experiments. Applying the core idea of ergodicity economics, using the decision maker's experience over time as a decision criterion, still helps reconciling decision-theoretical reasoning with the empirical evidence.

The main contribution of this paper is the rationalisation of risk-seeking, risk-neutral and risk-averse behaviour in time lotteries in a normative model with a single choice criterion. This allows revisiting existing experimental evidence on risk preferences in time lotteries in light of growth-optimality. We find that the time approach accords better with the experimental evidence than the ensemble approach. We also find, seemingly surprisingly in the context of EDUT, that the higher the ensemble-average growth rate is, relative to the sure payment, the less attractive it becomes. It demonstrates that decision makers may not consider the ensemble-average growth rate as a relevant criterion for their choices. This may be due to the lack of realism in this growth rate, *i.e.* it does not correspond to the real-life experience of decision makers in time lotteries.

Moreover, this paper provides a framework for future experiments. It is possible to design choice problems between time lotteries for which one lottery will be preferred in the ensemble approach and the other in the time approach. Such experiments are able to confirm or falsify the two approaches, and are planned.

We end with a conceptual remark. Whether or not decision makers' choices maximise the growth rate of their wealth is an open question. Yet, if they do, there is also the question of which growth rate they have in mind when making their choices? Here we present two options. The ensemble approach uses the expected value, which effectively averages over an ensemble of "copies" of the decision maker that cannot be accessed. Moreover, the growth rates that characterise the inaccessible copies cannot be pooled to yield a single realistic growth rate. The ensemble-average growth rate is thus a quantity that never captures the decision maker's experience. The time approach uses the long time limit, which may also be incompatible with the decision maker's choices if their time horizon is realistically short. We let data decide which approach better describes how decision makers choose. In the case of time lotteries the time approach seems to be better aligned with the experimental evidence than the ensemble approach. This may reflect the view of Kacelnik (1997, p. 60) that the same "process used for one-off events seems to obey a law that evolved as an adaptation to cope with repetitive events."

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A Proof of Proposition 1

We would like to show that if a relation $\succsim$ is growth-optimal over time lotteries, then it satisfies the VON NEUMANN-MORGENSTERN axioms: completeness, transitivity, continuity and independence. For completeness and transitivity there is no difference between the ensemble and time approaches, and the proof is similar. For continuity and independence we separate the proofs for each of the approaches. The ensemble approach is fully consistent with the independence axiom. In the time approach, growth-optimal preferences satisfy independence for time lotteries with the same payment (as commonly described). If time lotteries with different payments are allowed to be compared, this axiom may be violated, and we describe a counterexample.

Completeness

Any two time lotteries L_a and L_b correspond to growth rates g_a and g_b, respectively. It follows that

$$L_a \succ L_b \iff g_a > g_b \; , \tag{A.1}$$

$$L_a \sim L_b \iff g_a = g_b \; , \tag{A.2}$$

$$L_a \prec L_b \iff g_a < g_b \; , \tag{A.3}$$

so either L_a is preferred, L_b is preferred, or the decision maker is indifferent.

■

Transitivity

For any three lotteries L_a, L_b and L_c, corresponding to growth rates g_a, g_b and g_c, such that

$$L_a \prec L_b \iff g_a < g_b \; , \tag{A.4}$$

$$L_b \prec L_c \iff g_b < g_c \; , \tag{A.5}$$

if follows that $g_a < g_c$, so $L_a \prec L_c$.

Similarly, if

$$L_a \sim L_b \iff g_a = g_b \ , \tag{A.6}$$

$$L_b \sim L_c \iff g_b = g_c \ , \tag{A.7}$$

then also $g_a = g_c$ and $L_a \sim L_c$.

■

Continuity

Given three time lotteries L_a, L_b and L_c, such that $L_a \prec L_b$ and $L_b \prec L_c$, we wish to show that there exists a weight $\alpha \in [0,1]$ such that $\alpha L_a + (1-\alpha) L_c \sim L_b$.

To show that we first need to define the "natural" operation (as described in VON NEUMANN and MORGENSTERN 1944) between the lotteries, *i.e.*, a combination of two time lotteries with a weight α. We assume L_a corresponds to the tuple $(t_{a,1}, t_{a,2}, p_a, \Delta x_a)$ (see the definition of a time lottery in Section 2), L_b to the tuple $(t_{b,1}, t_{b,2}, p_b, \Delta x_b)$ and L_c to the tuple $(t_{c,1}, t_{c,2}, p_c, \Delta x_c)$. The combination $\alpha L_a + (1-\alpha) L_c$ is defined as a lottery that pays:

$$\Delta x_a \text{ at time } t_{a,1}, \text{ with probability } \alpha \cdot p_a \ , \tag{A.8}$$

$$\Delta x_a \text{ at time } t_{a,2}, \text{ with probability } \alpha \cdot (1-p_a) \ , \tag{A.9}$$

$$\Delta x_c \text{ at time } t_{c,1}, \text{ with probability } (1-\alpha) \cdot p_c \ , \tag{A.10}$$

$$\Delta x_c \text{ at time } t_{c,2}, \text{ with probability } (1-\alpha) \cdot (1-p_c) \ . \tag{A.11}$$

Continuity in the ensemble approach We begin with the ensemble approach. In the ensemble approach the time lotteries L_a, L_b and L_c correspond to the following respective growth rates

$$\langle g_a \rangle = p_a \frac{\Delta x_a}{t_{a,1}} + (1-p_a) \frac{\Delta x_a}{t_{a,2}} \ , \tag{A.12}$$

$$\langle g_b \rangle = p_b \frac{\Delta x_b}{t_{b,1}} + (1-p_b) \frac{\Delta x_b}{t_{b,2}} \ , \tag{A.13}$$

$$\langle g_c \rangle = p_c \frac{\Delta x_c}{t_{c,1}} + (1-p_c) \frac{\Delta x_c}{t_{c,2}} \ . \tag{A.14}$$

The combined lottery $\alpha L_a + (1-\alpha) L_c$ has the growth rate

$$\alpha \cdot p_a \frac{\Delta x_a}{t_{a,1}} + \alpha \cdot (1-p_a) \frac{\Delta x_a}{t_{a,2}} + (1-\alpha) \cdot p_c \frac{\Delta x_c}{t_{c,1}} + (1-\alpha) \cdot (1-p_c) \frac{\Delta x_c}{t_{c,2}} = \alpha \langle g_a \rangle + (1-\alpha) \langle g_c \rangle \ . \tag{A.15}$$

To show that growth-optimality is continuous, we look for a weight $\alpha \in [0,1]$ for which the growth rate of the combined lottery, $\alpha \langle g_a \rangle + (1-\alpha) \langle g_c \rangle$, is exactly the same as $\langle g_b \rangle$. Solving

$$\langle g_b \rangle = \alpha \langle g_a \rangle + (1-\alpha) \langle g_c \rangle \tag{A.16}$$

we get

$$\alpha = \frac{\langle g_c \rangle - \langle g_b \rangle}{\langle g_c \rangle - \langle g_a \rangle} \ , \tag{A.17}$$

so there is indifference between L_b and the combination $\alpha L_a + (1-\alpha) L_c$.

Since $L_a \prec L_b$ and $L_b \prec L_c$, $\langle g_a \rangle < \langle g_b \rangle$ and $\langle g_b \rangle < \langle g_c \rangle$. It follows that $\alpha \in [0,1]$.

Continuity in the time approach We follow a similar proof for the time approach. In the time approach the time lotteries L_a, L_b and L_c correspond to the following respective growth rates

$$\bar{g}_a = \frac{\Delta x_a}{p_a t_{a,1} + (1-p_a) t_{a,2}} = \frac{\Delta x_a}{\langle t_a \rangle} \ , \tag{A.18}$$

$$\bar{g}_b = \frac{\Delta x_b}{p_b t_{b,1} + (1-p_b) t_{b,2}} = \frac{\Delta x_b}{\langle t_b \rangle} \ , \tag{A.19}$$

$$\bar{g}_c = \frac{\Delta x_c}{p_c t_{c,1} + (1-p_c) t_{c,2}} = \frac{\Delta x_c}{\langle t_c \rangle} \ . \tag{A.20}$$

The combined lottery $\alpha L_a + (1-\alpha) L_c$ is the same as in the ensemble approach. However, in the time approach it corresponds to a different growth rate. For calculating this growth rate we assume the lottery is sequentially repeated N times and then evaluate the growth rate after the infinite time limit $N \to \infty$. We denote by n_1 the number of times Δx_a was paid after time $t_{a,1}$, by n_2 the number of times Δx_a was paid after time $t_{a,2}$, by n_3 the number of times Δx_c was paid after time $t_{c,1}$, by n_4 the number of times Δx_c was paid after time $t_{c,2}$.

If follows that the growth rate of the combined lottery is

$$\lim_{N\to\infty} \frac{\text{Total payment after } N \text{ rounds}}{\text{Total time elapsed after } N \text{ rounds}} \tag{A.21}$$

$$= \lim_{N\to\infty} \frac{(n_1+n_2)\,\Delta x_a + (n_3+n_4)\,\Delta x_c}{n_1 t_{a,1} + n_2 t_{a,2} + n_3 t_{c,1} + n_4 t_{c,2}} \tag{A.22}$$

$$= \lim_{N\to\infty} \frac{{}^{n_1+n_2}/_N \cdot \Delta x_a + {}^{n_3+n_4}/_N \cdot \Delta x_c}{{}^{n_1}/_N \cdot t_{a,1} + {}^{n_2}/_N \cdot t_{a,2} + {}^{n_3}/_N \cdot t_{c,1} + {}^{n_4}/_N \cdot t_{c,2}} \tag{A.23}$$

$$= \frac{\alpha \Delta x_a + (1-\alpha)\,\Delta x_c}{\alpha p_a t_{a,1} + \alpha\,(1-p_a)\,t_{b,1} + (1-\alpha)\,p_c t_{c,1} + (1-\alpha)\,(1-p_c)\,t_{c,2}} \tag{A.24}$$

$$= \frac{\alpha \Delta x_a + (1-\alpha)\,\Delta x_c}{\alpha \langle t_a \rangle + (1-\alpha)\,\langle t_c \rangle} \,. \tag{A.25}$$

To show that growth-optimality is continuous, we look for a weight $\alpha \in [0,1]$ for which the growth rate of the combined lottery, $\frac{\alpha \Delta x_a + (1-\alpha)\Delta x_c}{\alpha\langle t_a\rangle + (1-\alpha)\langle t_c\rangle}$, is exactly the same as $\bar{g}_b = \frac{\Delta x_b}{\langle t_b \rangle}$. Solving

$$\frac{\alpha \Delta x_a + (1-\alpha)\,\Delta x_c}{\alpha \langle t_a \rangle + (1-\alpha)\,\langle t_c \rangle} = \frac{\Delta x_b}{\langle t_b \rangle} \tag{A.26}$$

for α yields

$$\alpha = \frac{\Delta x_c \langle t_b \rangle - \langle t_c \rangle \Delta x_b}{\langle t_b \rangle\,(\Delta x_c - \Delta x_a) + \Delta x_b\,(\langle t_a \rangle - \langle t_c \rangle)} \,, \tag{A.27}$$

so there is indifference between L_b and the combination $\alpha L_a + (1-\alpha)\,L_c$.

We now need to show that the value found for α is within the interval $[0,1]$. First, since $L_b \prec L_c$, $\bar{g}_b < \bar{g}_c$, so $\frac{\Delta x_b}{\langle t_b \rangle} < \frac{\Delta x_c}{\langle t_c \rangle}$. It follows that the numerator in Eq. (A.27) is positive.

Comparing the numerator and the denominator in Eq. (A.27) we get:

$$\Delta x_c \langle t_b \rangle - \langle t_c \rangle \Delta x_b < \langle t_b \rangle\,(\Delta x_c - \Delta x_a) + \Delta x_b\,(\langle t_a \rangle - \langle t_c \rangle) \tag{A.28}$$

$$\iff 0 < \Delta x_b \langle t_a \rangle - \Delta x_a \langle t_b \rangle \tag{A.29}$$

$$\iff \frac{\Delta x_a}{\langle t_a \rangle} < \frac{\Delta x_b}{\langle t_b \rangle} \tag{A.30}$$

$$\iff L_a \prec L_b \,, \tag{A.31}$$

so the denominator is larger than the positive numerator, which renders $\alpha \in [0,1]$.

■

Independence

Given three time lotteries L_a, L_b and L_c, such that $L_a \prec L_b$ and a weight $\alpha \in [0,1]$, we would like to show that a growth-optimal decision maker would prefer the combined lottery $\mathcal{B} = \alpha L_b + (1-\alpha)\, L_c$ over the combined lottery $\mathcal{A} = \alpha L_a + (1-\alpha)\, L_c$.

We define the combined lotteries similarly to the proof of the continuity axiom above. $\mathcal{A}$ is a lottery that pays

$$\Delta x_a \text{ at time } t_{a,1}, \text{ with probability } \alpha \cdot p_a \ , \tag{A.32}$$

$$\Delta x_a \text{ at time } t_{a,2}, \text{ with probability } \alpha \cdot (1-p_a) \ , \tag{A.33}$$

$$\Delta x_c \text{ at time } t_{c,1}, \text{ with probability } (1-\alpha) \cdot p_c \ , \tag{A.34}$$

$$\Delta x_c \text{ at time } t_{c,2}, \text{ with probability } (1-\alpha) \cdot (1-p_c) \ , \tag{A.35}$$

and $\mathcal{B}$ is a lottery that pays

$$\Delta x_b \text{ at time } t_{b,1}, \text{ with probability } \alpha \cdot p_b \ , \tag{A.36}$$

$$\Delta x_b \text{ at time } t_{b,2}, \text{ with probability } \alpha \cdot (1-p_b) \ , \tag{A.37}$$

$$\Delta x_c \text{ at time } t_{c,1}, \text{ with probability } (1-\alpha) \cdot p_c \ , \tag{A.38}$$

$$\Delta x_c \text{ at time } t_{c,2}, \text{ with probability } (1-\alpha) \cdot (1-p_c) \ . \tag{A.39}$$

Independence in the ensemble approach We begin with the ensemble approach. In the ensemble approach the combined lotteries, $\mathcal{A}$ and $\mathcal{B}$, correspond to the respective growth rates

$$\langle g_{\mathcal{A}} \rangle = \alpha \langle g_a \rangle + (1-\alpha) \langle g_c \rangle \ , \tag{A.40}$$

$$\langle g_{\mathcal{B}} \rangle = \alpha \langle g_b \rangle + (1-\alpha) \langle g_c \rangle \ . \tag{A.41}$$

It follows that

$$\langle g_{\mathcal{A}} \rangle < \langle g_{\mathcal{B}} \rangle \iff \alpha \langle g_a \rangle + (1-\alpha) \langle g_c \rangle < \alpha \langle g_b \rangle + (1-\alpha) \langle g_c \rangle \iff \langle g_a \rangle < \langle g_b \rangle \ , \tag{A.42}$$

and since $L_a \prec L_b$, $\mathcal{B}$ is indeed preferred to $\mathcal{A}$.

■

Independence in the time approach In the time approach the combined lotteries, $\mathcal{A}$ and $\mathcal{B}$, correspond to the following respective growth rates

$$\bar{g}_{\mathcal{A}} = \frac{\alpha \Delta x_a + (1-\alpha)\,\Delta x_c}{\alpha\langle t_a\rangle + (1-\alpha)\,\langle t_c\rangle}\ , \tag{A.43}$$

$$\bar{g}_{\mathcal{B}} = \frac{\alpha \Delta x_b + (1-\alpha)\,\Delta x_c}{\alpha\langle t_b\rangle + (1-\alpha)\,\langle t_c\rangle}\ . \tag{A.44}$$

Assuming $0 < \alpha \leq 1$ (as otherwise the combined lottery is not a combined lottery but simply L_c), it follows that

$$\bar{g}_{\mathcal{A}} < \bar{g}_{\mathcal{B}} \tag{A.45}$$

$$\Longleftrightarrow$$

$$\alpha^2 \left[\Delta x_a \left(\langle t_b\rangle - \langle t_c\rangle\right) + \Delta x_b \left(\langle t_c\rangle - \langle t_a\rangle\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right)\right] + \tag{A.46}$$

$$\alpha \left[\langle t_c\rangle \left(\Delta x_a - \Delta x_b\right) + \Delta x_c \left(\langle t_b\rangle - \langle t_a\rangle\right)\right] < 0 \tag{A.47}$$

$$\Longleftrightarrow$$

$$\alpha \left[\Delta x_a \left(\langle t_b\rangle - \langle t_c\rangle\right) + \Delta x_b \left(\langle t_c\rangle - \langle t_a\rangle\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right)\right] + \tag{A.48}$$

$$\left[\langle t_c\rangle \left(\Delta x_a - \Delta x_b\right) + \Delta x_c \left(\langle t_b\rangle - \langle t_a\rangle\right)\right] < 0 \tag{A.49}$$

$$\Longleftrightarrow$$

$$\alpha < -\frac{\langle t_c\rangle \left(\Delta x_a - \Delta x_b\right) + \Delta x_c \left(\langle t_b\rangle - \langle t_a\rangle\right)}{\Delta x_a \left(\langle t_b\rangle - \langle t_c\rangle\right) + \Delta x_b \left(\langle t_c\rangle - \langle t_a\rangle\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right)} \tag{A.50}$$

$$\Longleftrightarrow$$

$$\frac{\langle t_c\rangle \left(\Delta x_b - \Delta x_a\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right)}{\langle t_c\rangle \left(\Delta x_b - \Delta x_a\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right) + \left(\Delta x_a\langle t_b\rangle - \Delta x_b\langle t_a\rangle\right)} > 1\ . \tag{A.51}$$

To last line is of the form $\frac{a}{a+b}$, where $a = \langle t_c\rangle \left(\Delta x_b - \Delta x_a\right) + \Delta x_c \left(\langle t_a\rangle - \langle t_b\rangle\right)$, and $b = \Delta x_a\langle t_b\rangle - \Delta x_b\langle t_a\rangle$. To show that the independence condition $\bar{g}_{\mathcal{A}} < \bar{g}_{\mathcal{B}}$ in Eq. (A.45) is fulfilled, it is therefore required that $a > 0$, $b < 0$ and $a + b > 0$. While it is guaranteed that $b < 0$ from $L_a \prec L_b$, it is possible that $a < 0$. In such a case, there might be weights α and time lotteries L_c such that $L_a \prec L_b$ but $\alpha L_b + (1-\alpha)\,L_c \prec \alpha L_a + (1-\alpha)\,L_c$. In such cases, the independence axiom is violated.

For example, if we consider the time lotteries detailed in Tab. 3 we get that $L_a \prec L_b$, since $\bar{g}_a = 6.67$ \$/sec and $\bar{g}_b = 8.42$ \$/sec. However, for $\alpha = 0.1$, we get that the combined lottery $\mathcal{A} = \alpha L_a + (1-\alpha)\,L_c$ corresponds to a growth rate of $\bar{g}_{\mathcal{A}} = 0.87$ \$/sec, and $\mathcal{B} = \alpha L_b + (1-\alpha)\,L_c$ to $\bar{g}_{\mathcal{B}} = 0.82$ \$/sec.

Yet, as commonly described in the literature, the standard case is a comparison between

Table 3: A counterexample for the independence criterion in the time approach.

Time lottery	t_1 earlier payment time (sec)	t_2 later payment time (sec)	p earlier payment probability	Δx payment ($)
L_a	1	2	0.5	10
L_b	0.5	2	0.7	8
L_c	2	4	0.3	2

time lotteries with the same payment. In the context of the time lotteries described herein, this means $\Delta x = \Delta x_a = \Delta x_b = \Delta x_c$. It follows that

$$\bar{g}_{\mathcal{A}} < \bar{g}_{\mathcal{B}} \iff \alpha \Delta x \left(\langle t_b \rangle - \langle t_a \rangle \right) < 0 \ . \tag{A.52}$$

Under the assumption of equal payments $L_a \prec L_b$ implies ${}^{\Delta x}/_{\langle t_a \rangle} < {}^{\Delta x}/_{\langle t_b \rangle}$ and $\langle t_a \rangle > \langle t_b \rangle$. Thus, indeed, $\bar{g}_{\mathcal{A}} < \bar{g}_{\mathcal{B}}$ and independence is satisfied.

■