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We study the trade-off relations given by the l_1-norm coherence of general multipartite states. Explicit trade-off inequalities are derived with lower bounds given by the coherence of either bipartite or multipartite reduced density matrices. In particular, for pure three-qubit states, it is explicitly shown that the trade-off inequality is lower bounded by the three tangle of quantum entanglement.

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INTRODUCTION

As one of the central concepts in quantum mechanics which distinguish quantum from classical physics, coherence plays a significant role in many quantum phenomena such as the phase space distributions in quantum optics [1] and higher order correlation functions [2]. It is highly desirable to precisely quantify the usefulness of coherence as a resource. In the classical work of Baumgratz, Cramer and Plenio [3], this was achieved by defining the key concepts such as incoherent states, maximally coherent states and incoherent operations. Rapid developments have been made since then on the fundamental theory of quantum coherence and its applications [4, 5].

A successful and secure quantum network relies on quantum correlations distributed among the subsystems [6]. The so-called monogamy relation of the distribution of quantum resources characterizes such correlation distributions. Based on the entanglement measure concurrence, one has that the concurrences of the reduced states ρ_{AB} and ρ_{AC} of an arbitrary three-qubit state ρ_{ABC} satisfy the Coffman-Kundu-Wootters relation [7]. Monogamy relations have been investigated for quantum entanglement [8–11], quantum discord [12, 13], quantum streering [14], Bell nonlocality [15–19], indistinguishability [20], other nonclassical correlations [21, 22] and quantum coherence [23–25].

Distributions of different quantum resources have been also studied, such as the fundamental monogamy relation between contextually and nonlocality [26], Bell nonlocality and three tangle for three-qubit states [27], the internal entanglement and external correlations [28, 29]. More recently, the trade-off relations for Bell inequality violations in qubit networks [30], for quantum steering in distributed scenario [31], for state-dependent error-disturbance in sequential measurements [32], and for the entanglement cost and classical communication complexity of causal order structure of a protocol in distributed quantum information processing [33] have been investigated.

The distribution of quantum coherence in multipartite systems based on relative entropy is given in [23–25] with nice geometrical intuition, although the relative entropy is not easily calculated. The N-partite monogamy of coherence is given by defining $M = \sum_{n=2}^{N} C_{1:n} - C_{1:2\cdots N}$, where $C_{1:n}$ is the intrinsic coherence between the partitions 1 and n [23]. For $M \leq 0$, i.e, $C_{1:2\cdots N} \geq \sum_{n=2}^{N} C_{1:n}$, one obtains a monogamy relation (e.g. for the GHZ states). If $M > 0$, i.e, $C_{1:2\cdots N} < \sum_{n=2}^{N} C_{1:n}$, one has a polygamous relation (e.g. for the W states).

In [3] two different measures of coherence, the relative entropy of coherence C_r and the l_1 norm of coherence C_{l_1}, have been proposed. C_r is an entropic measure, while C_{l_1} is a geometric (distance) measure. Both C_r and C_{l_1} satisfy the strong monotonicity for all states, and the corresponding quantum resources theories have been rigorously established [3]. Some relations between C_r and C_{l_1} have been also studied in [34].

For any d-dimensional quantum state ρ, one has $C_r(\rho) \leq log(d)$, where the upper bound is attained for maximally coherent states, $|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$ [3, 35]. For bipartite states ρ_{AB}, its correlated coherence $C_{cc}(\rho_{AB})$ is defined by $C_{cc}(\rho_{AB}) = C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_A) - C_{l_1}(\rho_B)$, with ρ_A and ρ_B the reduced density matrices of the subsystems. $C_{cc}(\rho_{AB})$ is always nonnegative [36]. Namely, $C_r(\rho_{AB}) \geq C_r(\rho_A)+C_r(\rho_B)$, which gives a kind of trade-off relations among the bipartite coherence and the coherence of the subsystems [35]. $C_f(\rho_{AB}) \geq C_f(\rho_A)+C_f(\rho_B)$ is given in [37], $C_f(\rho)$ is a convex roof coherence measure, and defined as $C_f(\rho) = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_f(|\varphi_i\rangle)$ with $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$. For tripartite states ρ_{ABC}, [38] has been discussed whether a similar trade-off relation like $C_r(\rho_{ABC}) \geq C_r(\rho_{AB}) + C_r(\rho_{AC})$ holds. Unfortunately, this conjecture is invalid. An interesting and important question one would ask is then what trade-off relations hold among the tripartite or multipartite coherence and the coherence of the reduced subsystems.

In this paper, we investigate the distribution of quantum coherence in multi-qubit systems by using the easily calculated l_1-norm of quantum coherence [3]. We derive explicit trade-off inequalities lower bounded by the coherence of either bipartite or multipartite reduced density matrices. For pure three-qubit states, we show an trade-off relation between the coherence distribution and the three tangle of quantum entanglement.

TRADE-OFF RELATIONS OF MULTI-QUBIT COHERENCE

The l_1 norm quantum coherence of any quantum state $\rho = \sum \rho_{ij}|i\rangle\langle j|$ is given by the non-diagonal entries of ρ [3],

$$C_{l_1}(\rho) = \sum_{i,j,i\neq j} |\rho_{ij}|. \tag{1}$$

In the following, we denote for a tripartite state ρ_{ABC}, $C_{123} = C_{l_1}(\rho_{ABC})$, $C_{12} = C_{l_1}(\rho_{AB})$, $C_{13} = C_{l_1}(\rho_{AC})$, $C_{23} = C_{l_1}(\rho_{BC})$, where $\rho_{AB} = Tr_C(\rho_{ABC})$, $\rho_{AC} = Tr_B(\rho_{ABC})$ and $\rho_{BC} = Tr_A(\rho_{ABC})$ are the reduced density matrices.

According to the definition of l_1 norm quantum coherence, for any d-dimensional quantum state ρ, one has $C_{l_1}(\rho) \leq d-1$, where the upper bound is attained for maximally coherent states. It is obvious that the coherence of each subsystem is less than or equal to the coherence of whole system, for example, $C_{123} \geq C_{12}$, $C_{123} \geq C_{13}$ and $C_{123} \geq C_1$. In order to study the trade-off relation between quantum states and their subsystems, we first consider the three-qubit case.

Theorem 1. *For any three-qubit quantum state* $\rho_{ABC} = \sum_{i,j,k,i',j',k'=0}^{1} \rho_{ijk}^{i'j'k'} |ijk\rangle\langle i'j'k'|$, *we have*

$$C_{123} \geq \frac{C_{12}+C_{13}+C_{23}}{2}. \tag{2}$$

Proof: From

$$C_{123} = \sum_{i,j,k=0}^{1} \sum_{i',j',k'=0_{i\neq i' or j\neq j' or k\neq k'}}^{1} |\rho_{ijk}^{i'j'k'}|,$$

by using the triangle inequality $|a|+|b| \geq |a+b|$, we have

$$\begin{aligned} &2C_{123} \\ \geq\ & \sum_{i,j=0}^{1} \sum_{i',j'=0_{i\neq i' or j\neq j'}}^{1} |\sum_{k=0}^{1} \rho_{ijk}^{i'j'k}| \\ &+\sum_{i,k=0}^{1} \sum_{i',k'=0_{i\neq i' or k\neq k'}}^{1} |\sum_{j=0}^{1} \rho_{ijk}^{i'jk'}| \\ &+\sum_{j,k=0}^{1} \sum_{j',k'=0_{j\neq j' or k\neq k'}}^{1} |\sum_{i=0}^{1} \rho_{ijk}^{ij'k'}| \\ &+D \\ =\ & C_{12}+C_{13}+C_{23}+D, \end{aligned}$$

where $D = |\rho_{000}^{011}|+|\rho_{000}^{101}|+|\rho_{000}^{110}|+|\rho_{001}^{010}|+|\rho_{001}^{100}|+|\rho_{001}^{111}|+|\rho_{010}^{100}|+|\rho_{010}^{111}|+|\rho_{011}^{101}|+|\rho_{011}^{110}|+|\rho_{100}^{111}|+|\rho_{101}^{110}|+|\rho_{011}^{000}|+|\rho_{101}^{000}|+|\rho_{110}^{000}|+|\rho_{010}^{001}|+|\rho_{100}^{001}|+|\rho_{111}^{001}|+|\rho_{100}^{010}|+|\rho_{111}^{010}|+|\rho_{101}^{011}|+|\rho_{110}^{011}|+|\rho_{111}^{100}|+|\rho_{110}^{101}|+2(|\rho_{000}^{111}|+|\rho_{001}^{110}|+|\rho_{010}^{101}|+|\rho_{011}^{100}|+|\rho_{100}^{011}|+|\rho_{101}^{010}|+|\rho_{110}^{001}|+|\rho_{111}^{000}|)$, which gives rise to (2). ■

For example, for a pure incoherent product state $|\psi\rangle = a_{ijk}|ijk\rangle$, one has trivially $C_{123} = C_{12} = C_{13} = C_{23} = 0$. For coherent state of the form, $|\psi\rangle = a_{000}|000\rangle + a_{100}|100\rangle$, we get $C_{123} = C_{12} = C_{13} = |a_{000}a_{100}^*| + |a_{100}a_{000}^*|$ and $C_{23} = 0$. The equality holds in this case, $C_{123} = (C_{12}+C_{13}+C_{23})/2$, which gives rise to $C_{123} \leq C_{12}+C_{13}$ as $C_{23} = 0$. Therefore, the conjecture in [38], $C_{123} \geq C_{12}+C_{13}$ is invalid in this case.

In [38], the authors discussed that the trade-off relation $C_r(\rho_{ABC}) \geq C_r(\rho_{AB}) + C_r(\rho_{AC})$ does not hold by the relative entropy. Here, we also give a class of quantum states that violate the trade-off relation $C_{123} \geq C_{12}+C_{13}$.

Due to that correlated coherence $C_{cc}(\rho_{AB})$ is always nonnegative [36], we have $C_{l_1}(\rho_{AB}) \geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)$, $C_{l_1}(\rho_{AC}) \geq C_{l_1}(\rho_A)+C_{l_1}(\rho_C)$ and $C_{l_1}(\rho_{BC}) \geq C_{l_1}(\rho_B)+C_{l_1}(\rho_C)$, namely, $C_{12} \geq C_1+C_2$, $C_{13} \geq C_1+C_3$ and $C_{23} \geq C_2+C_3$.

$$\begin{aligned} C_{123} &\geq \frac{C_{12}+C_{13}+C_{23}}{2} \\ &\geq C_1+C_2+C_3. \end{aligned} \tag{3}$$

For the trade-off relation $C_{123} \geq C_{12}+C_{13}$, we have

$$\begin{aligned} C_{123} &\geq C_{12}+C_{13} \\ &\geq 2C_1+C_2+C_3. \end{aligned} \tag{4}$$

When $2C_1 \geq C_{123}$, the inequality (4) does not hold. Similarly, when $2C_r(\rho_A) \geq C_r(\rho_{ABC})$, the trade-off relation $C_r(\rho_{ABC}) \geq C_r(\rho_{AB}) + C_r(\rho_{AC})$ is invalid. This is why we give the trade-off relation between the tripartite coherence and the bipartite coherence in Theorem 1. In addition, one can get the trade-off relation by using the triangle inequality as follows,

$$C_{123} \geq C_1 + C_{23}, \tag{5}$$

where $C_1 = \sum_{i=0}^{1}\sum_{i'=0 i\neq i'}^{1}|\sum_{j=0}^{1}\sum_{k=0}^{1}\rho_{ijk}^{i'jk}|$. Similarly, one has $C_{123} \geq C_2 + C_{13}$ and $C_{123} \geq C_3 + C_{12}$.

Generalizing Theorem 1 to n-qubit case, we have, see proof in Appendix,

Theorem 2. *For any n-qubit quantum state* $\rho = \sum \rho_{i_1 i_2\cdots i_n}^{j_1 j_2\cdots j_n}|i_1 i_2\cdots i_n\rangle\langle j_1 j_2\cdots j_n|$, *we have*

$$C_{123\cdots n} \geq \frac{C_{123\cdots(n-1)} + C_{123\cdots(n-2)n} + \cdots + C_{234\cdots(n-1)n}}{n-1}. \quad (6)$$

The lower bound of (6) can be further expressed as the coherence of all m-partite reduced states of the n-qubit state. Let $\Gamma(m,n) = \{a_1 a_2\cdots a_m | 1 \leq a_1 < a_2 < \cdots < a_m \leq n\}$ denote the set of m different elements from n. For example, $\Gamma(2,4) = \{12,13,14,23,24,34\}$. Then for any given m, we have, see proof in Appendix,

Corollary 1. *For any n-qubit quantum state* $\rho = \sum \rho_{i_1 i_2\cdots i_n}^{j_1 j_2\cdots j_n}|i_1 i_2\cdots i_n\rangle\langle j_1 j_2\cdots j_n|$, *we have*

$$C_{123\cdots n} \geq \frac{\sum_{a\in\Gamma(m,n)} C_a}{C_{n-1}^{m-1}}, \quad (7)$$

where the combination C_{n-1}^{m-1} *represents the maximum number of occurrences of the element* $\rho_{i_1 i_2\cdots i_n}^{j_1 j_2\cdots j_n}$ *on the right side of the inequality.*

In particular, taking $a \in \Gamma(2,3)$ or $a \in \Gamma(n-1,n)$, one gets the Theorem 1 or Theorem 2, respectively.

The results in Corollary 1 can be straightforwardly generalized to multi-qudit case.

Corollary 2. *For any n-qudit* $\rho = \sum_{i_1,i_2,\cdots,i_n,j_1,j_2,\cdots,j_n=0}^{d-1} \rho_{i_1 i_2\cdots i_n}^{j_1 j_2\cdots j_n}|i_1 i_2\cdots i_n\rangle\langle j_1 j_2\cdots j_n|$, *we have*

$$C_{123\cdots n} \geq \frac{\sum_{a\in\Gamma(m,n)} C_a}{C_{n-1}^{m-1}}. \quad (8)$$

(8) can be proved by similar derivations to the proof of Theorem 2 and Corollary 1. In fact, it is valid for any multipartite states with different individual dimensions.

Above results valid for any mixed quantum states. Next, we consider the relationship between the coherence and the entanglement for the 3-qubit pure states.

Theorem 3. *For any three-qubit pure state* $|\psi\rangle_{ABC} = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$, *we have*

$$C_{123} \geq \frac{C_{12} + C_{13} + C_{23}}{2} + \tau_{123}, \quad (9)$$

where $\tau_{123} = 4|d_1 - 2d_2 + 4d_3|$ *is entanglement tangle [7],* $d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2$, $d_2 = a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}$ *and* $d_3 = a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}$.

In [23], the authors proposed a trade-off upper bound for tripartite systems, $C_{123} \leq C_1 + C_2 + C_3 + C_{1:2:3}$, where $C_{1:2:3}$ is an intrinsic coherence, defined by minimizing over the set of separable tripartite states. For the case of three-qubit pure states, we have a lower bound inequality given by the entanglement τ_{123} in Theorem 3. According to the inequality (3), we have

$$C_{123} \geq C_1 + C_2 + C_3 + \tau_{123}. \quad (10)$$

As examples, let us consider the GHZ state and the W state. For the GHZ state $|GHZ\rangle = \cos\phi|000\rangle + \sin\phi|111\rangle$, $\phi \in [0, 2\pi)$, we have $C_{123} = 2|\sin\phi\cos\phi|$, $C_{12} = C_{13} = C_{23} = 0$, $\tau_{123} = 4|\cos^2\phi\sin^2\phi|$. As $2|\sin\phi\cos\phi| \geq 4|\cos^2\phi\sin^2\phi|$, one gets the inequality (9). For the W state $|W\rangle = \sin\theta\cos\phi|100\rangle + \sin\theta\sin\phi|010\rangle + \cos\theta|001\rangle$ with $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$, we get $C_{123} = 2(|\sin^2\theta\sin\phi\cos\phi| + |\sin\theta\cos\theta\cos\phi| + |\sin\theta\cos\theta\sin\phi|)$, $C_{12} = 2|\sin^2\theta\sin\phi\cos\phi|$, $C_{13} = 2|\sin\theta\cos\theta\cos\phi|$, $C_{23} = 2|\sin\theta\cos\theta\sin\phi|$ and $\tau_{123} = 0$, which satisfy the inequality (9). This example shows that both GHZ and W states obey the same inequality (9), which is different from the case in [23], where the GHZ and W states satisfy different inequalities under the relative entropy of coherence.

CONCLUSION AND DISCUSSION

We have studied the trade-off relations under the l_1-norm of quantum coherence. The general trade-off relations satisfied by the coherence of multipartite quantum states have been derived. For pure three-qubit case, it has been explicitly shown that the trade-off relation is lower bounded by the three tangle of quantum entanglement. These results may highlight further studies on coherence and correlation distributions in multipartite quantum systems.

We have proved that the trade-off relation $C(\rho_{ABC}) \geq C(\rho_{AB}) + C(\rho_{AC})$ is invalid for relative entropy coherence and l_1 norm coherence. Here, We have used the l_1 norm coherence in defining the correlated coherence. Obviously one may also define the correlated coherence in terms of the entopic coherence measure or the convex roof coherence measure. One may expect that the inequality like (2) hold for correlated coherence defined by other quantum coherence measures.

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APPENDIX

Proof of Theorem 2

Proof: From the definitions

$$C_{123\cdots n} = \sum_{\substack{i_1,i_2,\cdots,i_n=0 \\ i_1\neq j_1\ or\ i_2\neq j_2\ or\cdots or\ i_n\neq j_n}}^{1} \sum_{j_1,j_2,\cdots,j_n=0}^{1} |\rho_{i_1i_2\cdots i_n}^{j_1j_2\cdots j_n}|,$$

$$C_{123\cdots(n-1)} = \sum_{\substack{i_1,i_2,\cdots,i_{n-1}=0 \\ i_1\neq j_1\ or\ i_2\neq j_2\ or\cdots or\ i_{n-1}\neq j_{n-1}}}^{1} \sum_{j_1,j_2,\cdots,j_{n-1}=0}^{1} |\sum_{i_n=0}^{1} \rho_{i_1i_2\cdots i_{n-1}i_n}^{j_1j_2\cdots j_{n-1}i_n}|,$$

$$C_{123\cdots(n-2)n} = \sum_{\substack{i_1,\cdots,i_{n-2},i_n=0 \\ i_1\neq j_1 or\cdots or}}^{1} \sum_{\substack{j_1,\cdots,j_{n-2},j_n=0 \\ i_{n-2}\neq j_{n-2} or i_n\neq j_n}}^{1} |\sum_{i_{n-1}=0}^{1} \rho_{i_1\cdots i_{n-2}i_{n-1}i_n}^{j_1\cdots j_{n-2}i_{n-1}j_n}|,$$

$$\vdots$$

$$C_{234\cdots n} = \sum_{\substack{i_2,i_3,\cdots,i_n=0 \\ i_2\neq j_2\ or\ i_3\neq j_3\ or\cdots or\ i_n\neq j_n}}^{1} \sum_{j_2,j_3,\cdots,j_n=0}^{1} |\sum_{i_1=0}^{1} \rho_{i_1i_2\cdots i_n}^{i_1j_2\cdots j_n}|,$$

similar to the proof of Theorem 1, by using triangular inequalities and taking into account the number of times of the same element appearing on both sides of the inequalities, we obtain (3). ∎

Proof of Corollary 1

Proof: According to Theorem 2, we get

$$\begin{aligned} & C_{123\cdots n} \\ \geq\ & \frac{C_{123\cdots(n-1)} + C_{123\cdots(n-2)n} + \cdots + C_{234\cdots(n-1)n}}{n-1} \\ =\ & \frac{\sum_{a\in\Gamma(n-1,n)} C_a}{C_{n-1}^{n-2}}. \end{aligned}$$

Hence

$$\begin{aligned} & C_{123\cdots n} \\ \geq\ & \frac{C_{123\cdots(n-1)}+C_{123\cdots(n-2)n}+\cdots+C_{234\cdots(n-1)n}}{n-1}. \\ \geq\ & \frac{C_{123\cdots(n-2)}+C_{12\cdots(n-3)(n-1)}+\cdots+C_{234\cdots(n-1)}}{(n-2)(n-1)} \\ & +\frac{C_{123\cdots(n-2)}+C_{12\cdots(n-3)n}+\cdots+C_{234\cdots(n-2)n}}{(n-2)(n-1)} \\ & +\cdots \\ & +\frac{C_{234\cdots(n-1)}+C_{23\cdots(n-2)n}+\cdots+C_{345\cdots n}}{(n-2)(n-1)} \\ =\ & \frac{C_{123\cdots(n-2)}+C_{12\cdots(n-3)(n-1)}+\cdots+C_{345\cdots n}}{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

According to the characteristics of the combination, each element appears twice on the right side of the second inequality. One has

$$C_{123\cdots n} \geq \frac{\sum_{a\in\Gamma(n-2,n)} C_a}{C_{n-1}^{n-3}}.$$

Similarly, we obtain

$$\begin{aligned} C_{123\cdots n} &\geq \frac{\sum_{a\in\Gamma(n-3,n)} C_a}{C_{n-1}^{n-4}}. \\ C_{123\cdots n} &\geq \frac{\sum_{a\in\Gamma(n-4,n)} C_a}{C_{n-1}^{n-5}}. \\ &\vdots \\ C_{123\cdots n} &\geq \frac{\sum_{a\in\Gamma(1,n)} C_a}{C_{n-1}^{0}}, \end{aligned}$$

which give rise to (7). ■

Proof of Theorem 3

Proof: The two-qubit reduced density matrices of $\rho_{ABC}=|\psi\rangle_{ABC}\langle\psi|$ are given by

$$\rho_{AB}=\sum_{i,j=0}^{1}\sum_{i',j'=0}^{1}\sum_{k=0}^{1} a_{ijk}a_{i'j'k}^{*}|ij\rangle\langle i'j'|,$$

$$\rho_{AC}=\sum_{i,k=0}^{1}\sum_{i',k'=0}^{1}\sum_{j=0}^{1} a_{ijk}a_{i'jk'}^{*}|ik\rangle\langle i'k'|,$$

$$\rho_{BC}=\sum_{j,k=0}^{1}\sum_{j',k'=0}^{1}\sum_{i=0}^{1} a_{ijk}a_{ij'k'}^{*}|jk\rangle\langle j'k'|.$$

According to the proof of Theorem 1 and the fact that $|xy^*|=|xy|$ for any complex numbers x and y, we have

$$C_{123} \geq \frac{C_{12}+C_{13}+C_{23}}{2}+\frac{D'}{2},$$

where

$$\begin{aligned} \frac{D'}{2} =\ & |a_{000}a_{011}|+|a_{000}a_{101}|+|a_{000}a_{110}|+|a_{001}a_{010}| \\ & +|a_{001}a_{100}|+|a_{001}a_{111}|+|a_{010}a_{100}|+|a_{010}a_{111}| \\ & +|a_{011}a_{110}|+|a_{011}a_{101}|+|a_{100}a_{111}|+|a_{101}a_{110}| \\ & +2(|a_{000}a_{111}|+|a_{001}a_{110}|+|a_{010}a_{101}|+|a_{011}a_{100}|). \end{aligned}$$

From the inequality $a^2+b^2\geq 2ab$ for $a\geq 0$ and $b\geq 0$, we have

$$\begin{aligned} 1 &= \sum_{i,j,k=0}^{1}|a_{ijk}|^2 \\ &\geq 2(|a_{000}a_{111}|+|a_{001}a_{110}|+|a_{010}a_{101}|+|a_{100}a_{011}|) \\ &\geq 0. \end{aligned}$$

Hence

$$\begin{aligned} & 2(|a_{000}a_{111}|+|a_{001}a_{110}|+|a_{010}a_{101}|+|a_{100}a_{011}|) \\ \geq\ & [2(|a_{000}a_{111}|+|a_{001}a_{110}|+|a_{010}a_{101}|+|a_{100}a_{011}|)]^2 \\ =\ & 4[|a_{000}|^2|a_{111}|^2+|a_{001}|^2|a_{110}|^2 \\ & +|a_{010}|^2|a_{101}|^2+|a_{100}|^2|a_{011}|^2 \\ & +2(|a_{000}a_{111}a_{011}a_{100}|+|a_{000}a_{111}a_{101}a_{010}| \\ & +|a_{000}a_{111}a_{110}a_{001}|+|a_{011}a_{100}a_{101}a_{010}| \\ & +|a_{011}a_{100}a_{110}a_{001}|+|a_{101}a_{010}a_{110}a_{001}|)] \\ \geq\ & 4|d_1-2d_2|. \end{aligned}$$

Similarly

$$\begin{aligned} 1 &\geq |a_{000}|^2+|a_{011}|^2+|a_{101}|^2+|a_{110}|^2 \\ &\geq 2(|a_{000}a_{011}|+|a_{101}a_{110}|) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & |a_{000}a_{011}|+|a_{101}a_{110}| \\ \geq\ & 2(|a_{000}a_{011}|+|a_{101}a_{110}|)^2 \\ =\ & 2(|a_{000}a_{011}|^2+|a_{101}a_{110}|^2+2|a_{000}a_{110}a_{101}a_{011}|) \\ \geq\ & 8|a_{000}a_{110}a_{101}a_{011}|. \end{aligned}$$

And similarly,

$$\begin{aligned} |a_{000}a_{101}|+|a_{011}a_{110}| &\geq 8|a_{000}a_{110}a_{101}a_{011}|, \\ |a_{000}a_{110}|+|a_{011}a_{101}| &\geq 8|a_{000}a_{110}a_{101}a_{011}|, \\ |a_{001}a_{010}|+|a_{100}a_{111}| &\geq 8|a_{111}a_{001}a_{010}a_{100}|, \\ |a_{001}a_{100}|+|a_{010}a_{111}| &\geq 8|a_{111}a_{001}a_{010}a_{100}|, \\ |a_{001}a_{111}|+|a_{010}a_{100}| &\geq 8|a_{111}a_{001}a_{010}a_{100}|. \end{aligned}$$

Thus we have

$$\begin{aligned}
& |a_{000}a_{011}| + |a_{000}a_{101}| + |a_{000}a_{110}| + |a_{001}a_{010}| \\
& +|a_{001}a_{100}| + |a_{001}a_{111}| + |a_{010}a_{100}| + |a_{010}a_{111}| \\
& +|a_{011}a_{110}| + |a_{011}a_{101}| + |a_{100}a_{111}| + |a_{101}a_{110}| \\
\geq \; & 16(|a_{000}a_{110}a_{101}a_{011}| + |a_{111}a_{001}a_{010}a_{100}|) \\
\geq \; & 4|4d_3|,
\end{aligned}$$

Therefore, we obtain

$$\frac{D'}{2} \geq 4|d_1 - 2d_2 + 4d_3| = \tau_{123},$$

and then (9). ■