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Detecting EPR steering via two classes of local measurements

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Abstract We study the Einstein-Podolsky-Rosen (EPR) steering and present steerability criteria for arbitrary qubit-qudit (qudit-qubit) systems based on mutually unbiased measurements (MUMs) and general symmetric informationally complete measurements (general SIC-POVMs). Avoiding the usual complicated steering inequalities, these criteria can be more operational than some existing criteria, and implemented experimentally. Detailed examples are given to illustrate the efficiency of the criteria in both computation and experimental implementation.

Keywords EPR steering · Steerability criterion · MUM · General SIC-POVM

1 Introduction

As a distinctive and key feature in quantum world, the nonlocality challenges our intuition and comprehension about the nature. In the heart of nonlocality is the concept "EPR paradox" raised by Einstein, Podolsky, and Rosen in their seminal paper [1], which indicates that there were some conflicts between quantum mechanics and local realism. They proposed the possible existence of "local hidden variable" (LHV) models. With respect to the EPR paradox Schrödinger introduced the concept "steering" [2, 3] to characterize the Alice's ability of remotely steering Bob's

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state by local measurements. These counterintuitive nonlocal effects, or "spooky action at a distance", were collectively dubbed "entanglement". In 1964, Bell introduced his famous inequality for local hidden variable theories, which crucially brought the nonlocality debate to an experimentally testable form [4]. Thus, three distinct types of nonlocal correlations: entanglement, Schrödingers steering and Bell nonlocality, were intuitively elaborated, which have opened an epoch of unrelenting exploration of quantum correlations.

The nonlocality and quantum entanglement play important roles in our fundamental understandings of physical world as well as in various novel quantum informational tasks. A bipartite quantum state admits no LHV models if it violates some Bell inequalities such that the local measurement outcomes can not be modeled by classical random distributions over probability spaces, termed as Bell nonlocal [4–6]. A quantum state without entanglement must admit LHV models. However, not all the entangled quantum states are of nonlocality.

Entanglement and Bell nonlocality have attained flourishing developments. The concept of EPR steering was only introduced in 2007 [10]. The task of quantum steering is that a referee has to determine, by using the measurement outcomes communicated classically from the two parties to the referee, whether two spatially separated parties share entanglement, when one of the two parties is untrusted. The notion of EPR steering was introduced as the inability to construct a local hidden state (LHS) model to explain the joint probabilities of measurement outcomes. It has been shown that EPR steering is an intermediate between entanglement and Bell nonlocality. According to the hierarchy of nonlocality, the set of steerable states is a strict subset of entangled states and a strict superset of Bell nonlocal states [11]. Moreover, the EPR steering is inherently asymmetric with respect to the observers, unlike quantum nonlocality and entanglement [12]. There exist entangled states which are one-way steerable, demonstrating steerability from one observer to another spatially separated observer, but not vice-versa [12–14].

EPR steering not only has foundational significance of describing the nonlocality, but also has a vast range of information theoretic applications, ranging from one-sided device-independent quantum key distribution [15], advantages in sub-channel discrimination [16], secure quantum teleportation [17], quantum communication [18], detecting bound entanglement [19], one-sided device-independent randomness generation [20], to one-sided device-independent self-testing of pure maximally as well as non-maximally entangled states [21].

Against the above backdrop, from a fundamental viewpoint as well as an information-theoretic perspective, it is important to detect whether a quantum state is steerable or not. A number of criteria have been proposed till date [22–41]. Recently, in [42, 43], the authors focused on detecting arbitrary qubit-qudit state ρ_{AB} and gave a criterion by detecting the entanglement of a new constructed state, $\mu\rho_{AB}+(1-\mu)\frac{\mathbb{I}}{2}\otimes\rho_B$, without using any steering inequality. Following the positive partial transposition criterion [44,45], the authors in [43] present a brief idea on how their result can be implemented in experiments for two-qubit states. Although such result consumes some resources, it provides a way of detecting EPR steering by avoiding steering inequalities.

In [48] the authors formulated an effective tool called mutually unbiased measurements (MUMs) to study the problem of quantum entanglement. Besides, there is another useful tool called general symmetric informationally complete measurements (general SIC-POVMs) [49,50]. Both the MUMs and general SIC-POVMs can be used to detect quantum entanglement [46, 51–55]. These entanglement criteria are shown to be powerful and can be implemented experimentally. Due to the relationship between the entanglement and the EPR steering, we present steering criteria in terms of MUMs and general SIC-POVMs.

2 Detection of EPR Steering

The EPR steering is usually formulated by considering a quantum information task [10,11]. Suppose two spatially separated observers, say Alice and Bob, want to share entanglement between each other. Alice prepares a bipartite quantum state ρ_{AB} and sends one partite to Bob. Bob trusts his own but not Alice's apparatus. He will be convinced that they share an entangled state only if there exists evidence that Alice can "steer" Bob's state by performing measurements on their respective subsystems. If Alice (Bob) performs projective measurement A (B) with measurement outcomes a (b) on her (his) system, the joint probability of obtaining the outcomes a and b is given by

$$P(a, b|A, B; \rho_{AB}) = \mathrm{Tr}[(\Pi_a^A \otimes \Pi_b^B)\rho_{AB}], \tag{1}$$

where Π_a^A and Π_a^A are the corresponding projective operators for Alice and Bob, respectively.

The only way that the dishonest Alice pretends to steer Bobs state, is to send some local hidden states (LHS) with ensemble $\{p_\lambda \rho_\lambda\}$, where λ is the hidden variable, ρ_λ is the state that Alice sends with probability p_λ ($\sum_\lambda p_\lambda = 1$). She announces an outcome according to her knowledge about the sent states. In this case the correlation will be of the form

$$P(a, b|A, B; \rho_{AB}) = \sum_\lambda p_\lambda \, P(a|A, \lambda) \, P_Q(b|B, \rho_\lambda), \tag{2}$$

where $P(a|A, \lambda)$ can be any possible probability distribution that Alice designed, $P_Q(b|B, \rho_\lambda) = \mathrm{Tr}[\Pi_b^B \rho_\lambda]$ denotes the quantum probability of outcome b given by measuring B on the local hidden state ρ_λ. If Bob finds that any LHS models fail to satisfy such correlation Eq. (2), he has to admit that Alice can steer his system and the corresponding bipartite state is entangled. In short, the bipartite state ρ_{AB} is unsteerable by Alice to Bob if and only if the joint probability distributions satisfy the relation (2) for all measurements A and B.

2.1 Detecting EPR Steering via Mutually Unbiased Measurements

Two orthonormal bases $\mathcal{B}_1 = \{|i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|j\rangle\}_{j=1}^d$ of $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle i|j\rangle| = \frac{1}{\sqrt{d}}, \qquad \text{for all} \;\; i, j = 1, 2, \cdots, d. \tag{3}$$

A set of orthonormal bases $\{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_m\}$ in $\mathbb{C}^d$ is called a set of mutually unbiased bases (MUBs) if every pair of bases in the set is mutually unbiased. In a d dimensional Hilbert space, there are at most $d+1$ pairwise unbiased bases. This set is called a complete set of MUBs. It is still an open problem whether complete set of MUBs exists for arbitrary d.

In Ref. [48], the authors introduced the concept of MUMs. Two POVM measurements on $\mathbb{C}^d$, $\mathcal{P}^{(b)} = \{P_n^{(b)}\}_{n=1}^d$, $b = 1, 2$, are said to be mutually unbiased measurements if

$$\begin{aligned} &\mathrm{Tr}[P_n^{(b)}] = 1, \\ &\mathrm{Tr}[P_n^{(b)} P_{n'}^{(b')}] = \frac{1}{d}, \quad b \neq b' \\ &\mathrm{Tr}[P_n^{(b)} P_{n'}^{(b)}] = \delta_{nn'} \kappa + (1 - \delta_{nn'}) \frac{1-\kappa}{d-1}, \end{aligned} \tag{4}$$

where $1/d < \kappa \leqslant 1$, and $\kappa = 1$ if and only if $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ reduce to projective measurements with respect to two MUBs.

A general construction of $d+1$ MUMs has been presented in [48]. Let $\{F_{n,b} : n = 1, 2, \cdots, d-1, b = 1, 2, \cdots, d+1\}$ be a set of $d^2 - 1$ Hermitian and traceless operators acting on $\mathbb{C}^d$, satisfying $\mathrm{Tr}(F_{n,b}F_{n',b'}) = \delta_{nn'}\delta_{bb'}$. Define $d(d+1)$ operators

$$F_n^{(b)} = \begin{cases} F^{(b)} - (d + \sqrt{d})F_{n,b}, & n = 1, 2, \cdots, d-1, \\ (1 + \sqrt{d})F^{(b)}, & n = d, \end{cases} \tag{5}$$

where $F^{(b)} = \sum\limits_{n=1}^{d-1} F_{n,b}$, $b = 1, 2, \cdots, d+1$. Then the $d+1$ MUMs are given by

$$P_n^{(b)} = \frac{1}{d}\mathbb{I} + tF_n^{(b)}, \tag{6}$$

with $b = 1, 2, \cdots, d+1$, $n = 1, 2, \cdots, d$, and t is so chosen such that $P_n^{(b)} \geqslant 0$. $d+1$ MUMs can be expressed in such form for any dimension d.

Now we study the steering criteria for qudit-qubit and qubit-qudit quantum systems based on MUMs.

Theorem 1 *Let ρ_{AB} be a qudit-qubit state in $\mathbb{C}^d \otimes \mathbb{C}^2$ shared by Alice and Bob, and $\{\mathcal{P}^{(b)}\}_{b=1}^{d+1}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{3}$ be any two complete MUMs on $\mathbb{C}^d$ and $\mathbb{C}^2$ with the parameter κ_1 and κ_2, respectively, where $\mathcal{P}^{(b)} = \{P_n^{(b)}\}_{n=1}^{d}$ and $\mathcal{Q}^{(b)} = \{Q_n^{(b)}\}_{n=1}^{2}$. Set $R_n^{(b)} = Q_n^{(b)}$ for $1 \leqslant b \leqslant 3$, $1 \leqslant n \leqslant 2$, and $R_n^{(b)} = \frac{\mathbb{I}}{2}$ for $3 < b \leqslant d+1$ or $3 \leqslant n \leqslant d$. Define $J(\rho) = \sum\limits_{b=1}^{d+1}\sum\limits_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})\rho]$. If*

$$J(\rho_{AB}) > \frac{\sqrt{\kappa_1 + 1}\sqrt{4\kappa_2 + 4 + (d+3)(d-2)}}{2\mu} - \frac{(d+1)(1-\mu)}{2\mu}, \tag{7}$$

then ρ_{AB} is steerable from Bob to Alice, where $\mu \in (0, \frac{1}{\sqrt{3}}]$.

Proof Denote $\tau_{AB} = \mu\rho_{AB} + (1-\mu)\rho_A \otimes \frac{\mathbb{I}}{2}$, where $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ is the reduced state at Alice's side. we have

$$\begin{aligned}
J(\tau_{AB}) &= \sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})\tau_{AB}] \\
&= \sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})(\mu\rho_{AB} + (1-\mu)\rho_A \otimes \frac{\mathbb{I}}{2})] \\
&= \mu\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})(\rho_{AB})] + (1-\mu)\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})(\rho_A \otimes \frac{\mathbb{I}}{2})] \\
&= \mu\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})(\rho_{AB})] + (1-\mu)\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[P_n^{(b)}\rho_A \otimes R_n^{(b)}\frac{\mathbb{I}}{2}] \\
&= \mu\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(P_n^{(b)} \otimes R_n^{(b)})(\rho_{AB})] + (1-\mu)\sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[P_n^{(b)}\rho_A]\mathrm{Tr}[R_n^{(b)}\frac{\mathbb{I}}{2}] \\
&= \mu J(\rho_{AB}) + (1-\mu)\sum_{b=1}^{3}\{\sum_{n=1}^{2} \mathrm{Tr}[P_n^{(b)}\rho_A]\mathrm{Tr}[Q_n^{(b)}\frac{\mathbb{I}}{2}] + \sum_{n=3}^{d}\mathrm{Tr}[P_n^{(b)}\rho_A]\mathrm{Tr}[\frac{\mathbb{I}^2}{4}]\} \\
&\quad + (1-\mu)\sum_{b=4}^{d+1}\sum_{n=1}^{d}\mathrm{Tr}[P_n^{(b)}\rho_A]\mathrm{Tr}[\frac{\mathbb{I}^2}{4}] \\
&= \mu J(\rho_{AB}) + \frac{(d+1)(1-\mu)}{2} \\
&> \sqrt{\kappa_1+1}\sqrt{\kappa_2+1+\frac{(d+3)(d-2)}{4}}.
\end{aligned}$$

The last inequality follows from (7).

In [55], the authors presented a separability criterion: if a bipartite state τ_{AB} in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ ($d_1 \geq d_2$) is separable, one has $J(\tau) \leqslant \sqrt{\kappa_1+1}\sqrt{\kappa_2+1+\frac{(d_1-d_2)(d_1+d_2+1)}{4}}$. In particular, for the case $d_1 = d$ and $d_2 = 2$, one has $J(\tau) \leqslant \sqrt{\kappa_1+1}\sqrt{\kappa_2+1+\frac{(d+3)(d-2)}{4}}$ for all separable states τ in $\mathbb{C}^d \otimes \mathbb{C}^2$. From this criterion we have that τ_{AB} must be entangled. In addition, from that any qudit-qubit state ρ_{AB} is EPR steering from Bob to Alice if the state $\tau_{AB} = \mu\rho_{AB} + (1-\mu)\rho_A \otimes \frac{\mathbb{I}}{2}$ is entangled [42, 43], we complete the proof.

□

On the other hand, for a qubit-qudit state ρ_{AB} in $\mathbb{C}^2 \otimes \mathbb{C}^d$ shared by Alice and Bob, we can detect the EPR steering from Alice to Bob through the following theorem.

Theorem 2 *Let $\{\mathcal{P}^{(b)}\}_{b=1}^{3}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{d+1}$ be any two complete MUMs on $\mathbb{C}^2$ and $\mathbb{C}^d$ with the parameter κ_1 and κ_2, respectively, where $\mathcal{P}^{(b)} = \{P_n^{(b)}\}_{n=1}^{2}$, $\mathcal{Q}^{(b)} = \{Q_n^{(b)}\}_{n=1}^{d}$. Set $R_n^{(b)} = P_n^{(b)}$, for $1 \leqslant b \leqslant 3$, $1 \leqslant n \leqslant 2$, and $R_n^{(b)} = \frac{\mathbb{I}}{2}$ for $3 < b \leqslant d+1$ or $3 \leqslant n \leqslant d$. For a qubit-qudit state ρ_{AB} in $\mathbb{C}^2 \otimes \mathbb{C}^d$ shared by Alice and Bob, if*

$$J(\rho_{AB}) = \sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}[(R_n^{(b)} \otimes Q_n^{(b)})\rho_{AB}] > \frac{\sqrt{4\kappa_1+4+(d+3)(d-2)}\sqrt{\kappa_2+1}}{2\mu} - \frac{(d+1)(1-\mu)}{2\mu} \quad (8)$$

then ρ_{AB} is steerable from Alice to Bob, where $\mu \in (0, \frac{1}{\sqrt{3}}]$.

The proof is similar to that of Theorem 1, by defining the state $\sigma_{AB} = \mu\rho_{AB} + (1-\mu)\frac{\mathbb{I}}{2} \otimes \rho_B$, where $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ is the reduced state at Bob's side.

As a particular case, let us consider a two-qubit state ρ_{AB} in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Denote σ_1, σ_2 and σ_3 the Pauli matrices. We have the following corollary:

Corollary 1 *Set $H(\rho_{AB}) = \mathrm{Tr}[(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)\rho_{AB}]$. If $H(\rho_{AB}) > \dfrac{1}{\mu}$, $\mu \in (0, \frac{1}{\sqrt{3}}]$, then the qubit-qubit state ρ_{AB} is steerable from Bob to Alice and from Alice to Bob.*

Proof A two-qubit state ρ_{AB} can be written in the following form under local unitary transformation,

$$\rho_{AB} = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \mathbf{a} \cdot \boldsymbol{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbf{b} \cdot \boldsymbol{\sigma} + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i), \tag{9}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ are the Bloch vectors, $a_i = \mathrm{Tr}[(\sigma_i \otimes \mathbb{I})\rho_{AB}]$, $b_i = \mathrm{Tr}[(\mathbb{I} \otimes \sigma_i)\rho_{AB}]$, $c_i = \mathrm{Tr}[(\sigma_i \otimes \sigma_i)\rho_{AB}]$, $i = 1, 2, 3$.

Let $\{P_n^{(b)}\}_{n=1}^2$, $b = 1, 2, 3$, be the three MUMs with the parameter κ constructed from the generalized Gell-Mann operators [48], and $\bar{P}_n^{(b)}$ the conjugation of $P_n^{(b)}$. It is obvious that $\{\bar{P}_n^{(b)}\}_{n=1}^2$, $b = 1, 2, 3$ are the three MUMs with the same parameter κ. We get

$$\begin{aligned} J(\rho_{AB}) &= \sum_{b=1}^{3}\sum_{n=1}^{2} \mathrm{Tr}[(P_n^{(b)} \otimes \bar{P}_n^{(b)})\rho_{AB}] \\ &= \frac{3 + (2\kappa - 1)(\mathrm{Tr}[(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)\rho_{AB}])}{2} \\ &= \frac{3 + (2\kappa - 1)H(\rho_{AB})}{2}. \end{aligned}$$

According to Theorem 1 (Theorem 2), we have ρ_{AB} is steerable from Bob to Alice and from Alice to Bob, if $J(\rho_{AB}) > \frac{3\mu+2\kappa-1}{2\mu}$ for $d = 2$ and $\kappa_1 = \kappa_2 = \kappa$, which follows from $H(\rho_{AB}) > \dfrac{1}{\mu}$, $\mu \in (0, \frac{1}{\sqrt{3}}]$.

□

In the following, we detect EPR steering of different families of two-qubit mixed states by using our results. We show by those detailed examples that our criterion based on MUMs is more convenient and operational, and more powerful than some criteria using steering inequality.

Example 1. We consider the Werner derivative states [56], which are a class of non-maximally entangled mixed states and can be obtained by applying a nonlocal unitary operator on the Werner state,

$$\rho_{\mathrm{wd}} = p|\psi_\theta\rangle\langle\psi_\theta| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}, \tag{10}$$

where $|\psi_\theta\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $0 \leqslant \theta \leqslant \pi/4$, $0 \leqslant p \leqslant 1$. From Corollary 1, we have $H(\rho_{\mathrm{wd}}) = p(1 + 2\sin(2\theta))$. Therefore, ρ_{wd} is steerable (from Alice to Bob and from Bob to Alice) for $1/\sqrt{3} \leqslant p < 1$ and $\arcsin[\frac{1}{2}(\sqrt{3} - 1)]/2 < \theta \leqslant \pi/4$, see Fig. 1. It should be noted that this steering criterion can be also derived from the result of Ref. [57]. However, our criteria from Theorem 1 and 2 do not need to know the detailed state. The detection of the steerability of a state can be done by direct measurements on the state.

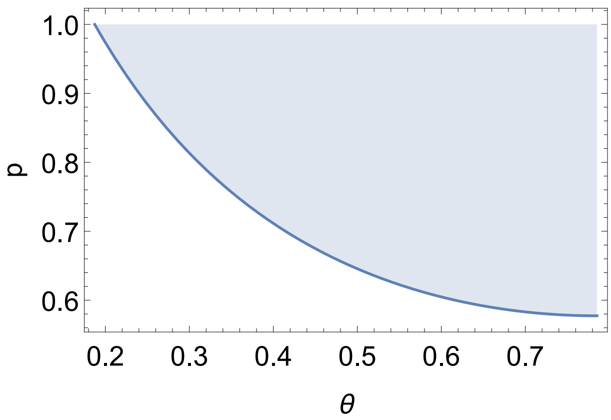

Fig. 1 The gray area represents the range of steerability that can be detected experimentally.

Example 2. Consider the following class of maximally steerable mixed states (the states that violate the most the steering inequality for a given mixedness) proposed in [58],

$$\rho^{\tau} = \begin{pmatrix} (1-\tau)/4 & 0 & 0 & (1-\tau)/4 \\ 0 & (1+\tau)/4 & (1+\tau)/4 & 0 \\ 0 & (1+\tau)/4 & (1+\tau)/4 & 0 \\ (1-\tau)/4 & 0 & 0 & (1-\tau)/4 \end{pmatrix}, \tag{11}$$

where $-1 \leqslant \tau \leqslant 1$. By straightforward computation, we have that $H(\rho^{\tau}) = 1-2\tau > 1/\mu$, namely, $-1 \leqslant \tau \leqslant (1-\sqrt{3})/2$. Thus our criterion can detect the both-way steerability of the state ρ_{τ} for $-1 \leqslant \tau \leqslant (1-\sqrt{3})/2$. Here the upper bound $(1-\sqrt{3})/2$ is approximately -0.366 given in [43].

Example 3. Consider maximally entangled mixed states presented in [59],

$$\rho_{\mathrm{Munro}} = \begin{pmatrix} h(C) & 0 & 0 & C/2 \\ 0 & 1-2h(C) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C/2 & 0 & 0 & h(C) \end{pmatrix}, \tag{12}$$

where C is the concurrence [60] of ρ_{Munro}, $h(C) = 1/3$ for $C < 2/3$ and $h(C) = C/2$ for $C \geqslant 2/3$. Here, the concurrence of a pure state $|\psi\rangle$ is defined by $C(|\psi\rangle) = \sqrt{2(1-\mathrm{Tr}\rho_A^2)}$ with $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ the reduced density matrix. The concurrence of a mixed state ρ is defined by the convex roof extension: $C(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$ with $p_i > 0$, $\sum_i p_i = 1$, and the minimization goes over all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. It is straightforward to obtain $H(\rho_{\mathrm{Munro}}) = 4C - 1$. Thus, the both-way EPR steerability of ρ_{Munro} is detected for $C > (1+\sqrt{3})/4 \approx 0.683$.

It has been shown that the state (12) demonstrates both-way steerability for $C > 0.707$ by using the two-setting linear steering inequality [25](see Appendix). Therefore, in the region $0.683 < C \leqslant 0.707$, the steerability of state ρ_{Munro} can be detected by our criterion, but not by the two-setting linear steering inequality.

2.2 Detecting EPR Steering via general SIC-POVMs

A POVM $\{P_j\}$ with d^2 rank-1 operators acting on $\mathbb{C}^d$ is called symmetric informationally (SIC) complete, if

$$P_j = \frac{1}{d}|\phi_j\rangle\langle\phi_j|, \qquad \sum_{j=1}^{d^2} P_j = \mathbb{I}, \tag{13}$$

where $j = 1, 2, \cdots, d^2$, the vectors $|\phi_j\rangle$ satisfy $|\langle\phi_j|\phi_k\rangle|^2 = 1/(d+1)$, $j \neq k$.

The general SIC measurements were introduced in Refs. [49,50]. A set of d^2 positive semidefinite operators $\{P_\alpha\}_{\alpha=1}^{d^2}$ on $\mathbb{C}^d$ is said to be a general SIC measurements if

$$\begin{aligned} &\sum_{\alpha=1}^{d^2} P_\alpha = \mathbb{I}, \quad \mathrm{Tr}[P_\alpha^2] = a, \\ &\mathrm{Tr}[P_\alpha P_\beta] = \frac{1-da}{d(d^2-1)}, \end{aligned} \tag{14}$$

where $\alpha, \beta \in \{1, 2, \cdots, d^2\}$, $\alpha \neq \beta$, the parameter a satisfies $1/d^3 < a \leqslant 1/d^2$. $a = 1/d^2$ if and only if all P_α are rank one, which gives rise to a SIC-POVM. It can be shown that $\mathrm{Tr}(P_\alpha) = 1/d$ for all α, and general SIC-POVM can be explicitly constructed [50]. Let $\{F_\alpha\}_{\alpha=1}^{d^2-1}$ be a set of d^2-1 Hermitian, traceless operators acting on $\mathbb{C}^d$, satisfying $\mathrm{Tr}(F_\alpha F_\beta) = \delta_{\alpha,\beta}$. Set $F = \sum_{\alpha=1}^{d^2-1} F_\alpha$. The d^2 operators

$$\begin{aligned} P_\alpha &= \frac{1}{d^2}\mathbb{I} + t[F - d(d+1)F_\alpha], \quad \alpha = 1, 2, \cdots, d^2-1, \\ P_{d^2} &= \frac{1}{d^2}\mathbb{I} + t(d+1)F \end{aligned} \tag{15}$$

form a general SIC measurement. Here t should be chosen such that $P_\alpha \geqslant 0$ and the parameter a is given by

$$a = \frac{1}{d^3} + t^2(d-1)(d+1)^3. \tag{16}$$

Instead of the MUMs used in Theorem 1 (Theorem 2), now we consider the general SIC-POVMs. We have the following EPR steering criteria for qubit-qudit and qudit-qubit states. The proofs of the following theorems are similar to the case of MUMs.

Theorem 3 *Let ρ_{AB} be a qudit-qubit state in $\mathbb{C}^d \otimes \mathbb{C}^2$ shared by Alice and Bob. Denote $\mathcal{P} = \{P_j\}_{j=1}^{d^2}$ and $\mathcal{Q} = \{Q_j\}_{j=1}^4$ two sets of general SIC-POVMs on $\mathbb{C}^d$ and $\mathbb{C}^2$ with the efficiency parameters a_1 and a_2, respectively. Set $R_j = Q_j$ for $j = 1, 2, 3, 4$, and $R_j = \mathbb{I}/4$ for $j = 5, 6, \cdots, d^2$. Define $J(\rho) = \sum_{j=1}^{d^2} \mathrm{Tr}[(P_j \otimes R_j)\rho]$. Then, the ρ_{AB} is steerable from Bob to Alice if*

$$J(\rho_{AB}) > \frac{\sqrt{\frac{a_1 d^2+1}{d(d+1)}}\sqrt{\frac{4a_2+1}{6} + \frac{d^2-4}{16}}}{\mu} - \frac{1-\mu}{4\mu}, \tag{17}$$

where $\mu \in (0, \frac{1}{\sqrt{3}}]$.

Theorem 4 *Let ρ_{AB} be a qubit-qudit state in $\mathbb{C}^2 \otimes \mathbb{C}^d$ shared by Alice and Bob, $\mathcal{P} = \{P_j\}_{j=1}^4$ and $\mathcal{Q} = \{Q_j\}_{j=1}^{d^2}$ be two sets of general SIC-POVMs on $\mathbb{C}^2$ and $\mathbb{C}^d$ with the efficiency parameters a_1 and a_2, respectively. Denote $R_j = P_j$ for $j = 1, 2, 3, 4$, $R_j = \mathbb{I}/4$ for $j = 5, 6, \cdots, d^2$. If*

$$J(\rho_{AB}) > \frac{\sqrt{\frac{4a_1+1}{6} + \frac{d^2-4}{16}}\sqrt{\frac{a_2 d^2+1}{d(d+1)}}}{\mu} - \frac{1-\mu}{4\mu}, \tag{18}$$

then ρ_{AB} is steerable from Alice to Bob, where $\mu \in (0, \frac{1}{\sqrt{3}}]$.

As a direct application of Theorems 3 and 4, for two-qubit states we can get the same results as the ones from corollary 1. Namely, the EPR steerable criteria based on MUMs works as well as the criteria based on general SIC-POVMs for two-qubit systems.

3 Conclusion

We have presented criteria for detecting EPR steering of arbitrary qubit-qudit states and qudit-qubit states through MUMs and general SIC-POVMs. These criteria can be more convenient and efficient, and can be implemented experimentally. The novelty of the results is that it allows one to detect EPR steering without using the usual complicated steering inequalities. From experimental point of view, our results enable one to test EPR steering of an arbitrary qudit-qubit and qubit-qudit state indirectly through two classes of measurements. Our approach may be helpful to avoid the locality loophole in EPR steering test, as the degree of correlation required for entanglement testing via MUMs and general SIC-POVMs is smaller than that for violation of a steering inequality. By detailed examples it has been shown that our criteria based on MUMs and general SIC-POVMs are more convenient and operational than some existing criteria in both computation and experimental implementation.

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APPENDIX

We presened some seminal standard steering inequalities used experimentally in the following.

Experimentally useful criterion for the EPR paradox were only proposed in 1989 by Reid [22] using conditional variances and the Heisenberg uncertainty relation.

The two experimenters, Alice and Bob, can measure the conditional probabilities of Bob finding outcome x_B in a measurement of $\hat{x}_B$ given that Alice finds outcome x_A in a measurement of $\hat{x}_A$, i.e., $P(x_B|x_A)$. Based on a result x_A, Alice can make an estimate of the result for Bob's outcome x_B. Denote this estimate $x_B^{\rm est}(x_A)$. The average inference variance of x_B given estimate $x_B^{\rm est}(x_A)$ is defined as

$$\triangle^2_{\rm inf}\, x_B = \langle [x_B - x_B^{\rm est}(x_A)]^2\rangle = \int dx_A dx_B P(x_A, x_B)[x_B - x_B^{\rm est}(x_A)]^2. \tag{19}$$

An inference variance $\triangle^2_{\rm inf} y_B$ is defined similarly.

Reid showed that the violation of the following uncertainty relation is a signature of the EPR steering for any bipartite state:

$$\triangle_{\inf} x_B \triangle_{\inf} y_B \geqslant 1. \tag{20}$$

This is the EPR-Reid criterion, which has been experimentally demonstrated. For a detailed review and further development see [26].

In a seminal paper [25], Cavalcanti, Jones, Wiseman, and Reid (CJWR) developed a general construction of experimental EPR-steering inequalities based on the assumption of existence of local hidden state (LHS) model, where Reid's criterion was shown to emerge as a special case.

In particular, CJWR gave the following series of linear steering inequalities usefully in experiment to check whether a two-qubit state is steerable from Alice to Bob when both the parties are allowed to perform n dichotomic measurements on his or her part:

$$F_n^{\mathrm{CJWR}}(\rho,\mu) = \frac{1}{\sqrt{n}}|\sum_1^n \langle A_i \otimes B_i\rangle| \leqslant 1, \tag{21}$$

where $A_i = \mathbf{u}_i \cdot \boldsymbol{\sigma}, B_i = \mathbf{v}_i \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector composed of the Pauli matrices, $\mathbf{u}_i \in \mathbb{R}^3$ are unit vectors, $\mathbf{v}_i \in \mathbb{R}^3$ are orthonormal vectors, $\mu = \{\mathbf{u}_1, \cdots, \mathbf{u}_n, \mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is the set of measurement directions, $\langle A_i \otimes B_i\rangle = \mathrm{Tr}[(A_i \otimes B_i)\rho]$, and $\rho \in \mathcal{H}_A \times \mathcal{H}_B$ is two-qubit state shared between two spatially separated parties (Alice and Bob)..

These are called n-setting linear steering inequalities. The linear steering inequalities with $n = 2$ and $n = 3$ (which are relevant for spin-$\frac{1}{2}$ observables) are of the form:

$$F_2(\rho,\mu) = \frac{1}{\sqrt{2}}|\sum_1^2 \langle A_i \otimes B_i\rangle| \leqslant 1, \tag{22}$$

and

$$F_3(\rho,\mu) = \frac{1}{\sqrt{3}}|\sum_1^3 \langle A_i \otimes B_i\rangle| \leqslant 1. \tag{23}$$

Cavalcanti et al. [35]considered a scenario in which Alice performs two dichotomic measurements while Bob performs two mutually unbiased qubit measurements. The authors then derived the following CHSH-like steering inequality:

$$F_2^{\mathrm{CHSH}}(\rho,\mu) = \frac{1}{2}[\sqrt{f_+(\rho,\mu)} + \sqrt{f_-(\rho,\mu)}] \leqslant 1, \tag{24}$$

where $f_\pm(\rho,\mu) = \langle (A_1 \pm A_2) \otimes B_1\rangle^2 + \langle (A_1 \pm A_2) \otimes B_2\rangle^2$. In [36], It was shown that the maximal value of the function $2F_2^{\mathrm{CHSH}}(\rho,\mu)$ can reach is $2\sqrt{2}$.

For any bipartite quantum state ρ, violations of the above inequalities imply the EPR steerability.