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VERONESE AND SEGRE MORPHISMS BETWEEN NON-COMMUTATIVE PROJECTIVE SPACES

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Abstract. We study Veronese and Segre morphisms between non-commutative projective spaces. We compute finite, reduced Gröbner bases for their kernels, and we compare them with their analogues in the commutative case.

1. Introduction

In this work, we describe Veronese and Segre morphisms for a class of non-commutative quadratic algebras which have permeated the literature under different names. They appeared as *quantum affine spaces* in [29, Section 1 and Section 4], and more recently as *non-commutative projective spaces* in the work [5] on mirror symmetry, as well as in the study of deformations of toric varieties [11, 12].

Motivated by the interpretation of morphisms between non-commutative algebras as "maps between non-commutative spaces", we consider here non-commutative analogues of the Veronese and Segre embeddings, two fundamental maps that play pivotal roles not only in classical algebraic geometry, but also in applications to other fields of mathematics.

The d-Veronese map is the non-degenerate embedding of the projective space $\mathbb{P}^n$ via the very ample line bundle $\mathcal{O}(d)$. Its image, called the Veronese variety, has a capital importance in algebraic geometry. Just to mention an example, every projective variety is isomorphic to the intersection of a Veronese variety and a linear space (see [24, Exercise 2.9]). The Segre map is the embedding of $\mathbb{P}^m \times \mathbb{P}^n$ via the very ample line bundle $\mathcal{O}(1,1)$. It is used in projective geometry to endow the Cartesian product of two projective spaces with the structure of a projective variety. In quantum mechanics and quantum information theory, it is a natural mapping for describing non-entangled states (see [7, Section 4.3]). Both are studied for the theory of tensor decomposition [27, Section 4.3], as the image of the Segre morphism is the locus of rank 1 tensors, while the image of the Veronese morphism plays a similar role for symmetric tensors. Moreover, these constructions are central in the field of algebraic statistics: the variety of moments of a Gaussian random variable is a Veronese variety (see [1, Section 6]), while independence models are encoded by Segre varieties (see [14]).

The natural problem of finding non-commutative counterparts of those fundamental constructions has been addressed from different perspectives, for instance in [37] and [36].

In this work, we study the properties of these maps and of the corresponding algebras from the point of view of the theory of Gröbner bases. In classical algebraic geometry, a variety V is completely determined by its defining ideal. When V is the image of a variety morphism f, the ideal of V is the kernel of the algebra morphism corresponding to f. Computing a Gröbner basis for the defining ideal can provide valuable information about the properties of V. With this motivation in mind, we are interested in computing Gröbner bases for the kernels of the non-commutative Veronese and Segre morphisms.

The paper is structured as follows. In Section 2 we recall some basics of the theory of Gröbner bases for ideals in the free associative algebra. Our Lemma 2.7 gives a criterion for quadratic Gröbner bases, which is crucial for the proof of our main results, Theorems 5.5 and 6.10. In Section 3 we present

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the quadratic algebras $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$, called *quantum spaces*, or *non-commutative projective spaces*, and we recall some of their basic properties. In Section 4 we analyse their d-Veronese subalgebras. The main result of the section is Theorem 4.5, which gives a presentation of the d-Veronese subalgebra in terms of generators and quadratic relations. In Section 5 we introduce and study non-commutative analogues of the Veronese maps for non-commutative projective spaces. We present a modification of the theory of Gröbner bases for ideals in a quantum space and find explicitly a Gröbner basis for the kernel of the Veronese map in Theorem 5.5. Using a similar approach and methods, in Section 6 we introduce and study non-commutative analogues of Segre maps and Segre products. Theorem 6.10 describes the reduced Gröbner basis for the kernel of the Segre map. Finally, in Section 7 we present various examples that illustrate our results.

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2. Preliminaries

We start with notation, conventions, and facts which will be used throughout the paper, and recall some basics on Gröbner bases for ideals in the free associative algebra. Lemma 2.7 gives a criterion for quadratic Gröbner bases which is particularly useful in our settings.

2.1. **Basic notations and conventions.** Throughout the paper $X_n = \{x_0, \dots, x_n\}$ denotes a non-empty set of indeterminates. To simplify notation, we shall often write X instead of X_n. We denote by $\mathbb{C}\langle x_0, \dots, x_n\rangle$ the complex free associative algebra with unit generated by X_n, while $\mathbb{C}[X_n]$ denotes the commutative polynomial ring in the variables $x_0, \dots, x_n$. $\langle X_n\rangle$ is the free monoid generated by X_n, where the unit is the empty word, denoted by 1.

We fix the degree-lexicographic order $<$ on $\langle X_n\rangle$, where we set $x_0 < x_1 < \cdots < x_n$. As usual, $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{N}_0$ is the set of all non-negative integers. Given a non-empty set $F \subset \mathbb{C}\langle X_n\rangle$, we write (F) for the two-sided ideal of $\mathbb{C}\langle X_n\rangle$ generated by F.

In more general settings, we shall also consider associative algebras over a field $\mathbf{k}$. Suppose $A = \bigoplus_{m\in\mathbb{N}_0} A_m$ is a graded $\mathbf{k}$-algebra such that $A_0 = \mathbf{k}$, and such that A is finitely generated by elements of positive degree. Recall that its Hilbert function is $h_A(m) = \dim A_m$ and its Hilbert series is the formal series $H_A(t) = \sum_{m\in\mathbb{N}_0} h_A(m)t^m$. In particular, the algebra $\mathbb{C}[X_n]$ of commutative polynomials satisfies

$$h_{\mathbb{C}[X_n]}(d) = \binom{n+d}{n} \quad \text{and} \quad H_{\mathbb{C}[X_n]} = \frac{1}{(1-t)^{n+1}}. \tag{2.1}$$

We shall use two well-known gradings on the free associative algebra $\mathbb{C}\langle X_n\rangle$: the *natural grading by length* and the $\mathbb{N}_0^{n+1}$-*grading*.

Let X^m be the set of all words of length m in $\langle X\rangle$. Then

$$\langle X\rangle = \bigsqcup_{m\in\mathbb{N}_0} X^m,\ X^0 = \{1\},\ \text{and}\ \ X^k X^m \subseteq X^{k+m},$$

so the free monoid $\langle X\rangle$ is naturally *graded by length*.

Similarly, the free associative algebra $\mathbb{C}\langle X\rangle$ is also graded by length:

$$\mathbb{C}\langle X\rangle = \bigoplus_{m\in\mathbb{N}_0} \mathbb{C}\langle X\rangle_m, \quad \text{where}\ \ \mathbb{C}\langle X\rangle_m = \mathbb{C}X^m.$$

A polynomial $f \in \mathbb{C}\langle X\rangle$ is *homogeneous of degree* m if $f \in \mathbb{C}X^m$. We denote by

$$\mathcal{T}^n = \mathcal{T}(X_n) := \{x_0^{\alpha_0}\cdots x_n^{\alpha_n} \in \langle X_n\rangle \mid \alpha_i \in \mathbb{N}_0, i \in \{0,\dots,n\}\}$$

the set of ordered monomials (terms) in $\langle X_n\rangle$ and by

$$\mathcal{T}^n_d = \mathcal{T}_d(X_n) := \left\{ x_0^{\alpha_0}\cdots x_n^{\alpha_n} \in \mathcal{T}^n \mid \sum_{i=0}^n \alpha_i = d \right\}$$

the set of ordered monomials of length d. It is well known that the cardinality $|\mathcal{T}_d(X_n)|$ is given by the Hilbert function (Hilbert polynomial) $h_{\mathbb{C}[X_n]}(d)$ of the polynomial ring in the variables X_n:

$$|\mathcal{T}_d(X_n)| = \binom{n+d}{n} = h_{\mathbb{C}[X_n]}(d). \tag{2.2}$$

Definition 2.1. A monomial $w \in \langle X \rangle$ has *multi-degree* $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}_0^{n+1}$, if w, considered as a commutative term, can be written as $w = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In this case we write $\deg(w) = \alpha$. Clearly, w has length $|w| = \alpha_0 + \cdots + \alpha_n$. In particular, the unit $1 \in \langle X \rangle$ has multi-degree $\mathbf{0} = (0, \dots, 0)$, and $\deg(x_0) = (1, 0, \dots, 0), \dots, \deg(x_n) = (0, 0, \dots, 1)$. For each $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^{n+1}$ we define

$$T_\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T}(X_n) \quad \text{and} \quad X_\alpha := \{w \in \langle X \rangle \mid \deg(w) = \alpha\}. \tag{2.3}$$

The free monoid $\langle X_n \rangle$ is naturally $\mathbb{N}_0^{n+1}$-graded:

$$\langle X_n \rangle = \bigsqcup_{\alpha \in \mathbb{N}_0^{n+1}} X_\alpha, \text{ where } X_{\mathbf{0}} = \{1\}, \text{ and } X_\alpha X_\beta \subseteq X_{\alpha+\beta}.$$

In a similar way, the free associative algebra $\mathbb{C}\langle X_n \rangle$ is also canonically $\mathbb{N}_0^{n+1}$ -graded:

$$\mathbb{C}\langle X_n \rangle = \bigoplus_{\alpha \in \mathbb{N}_0^{n+1}} \mathbb{C}\langle X_n \rangle_\alpha, \text{ where } \mathbb{C}\langle X_n \rangle_\alpha = \mathbb{C}X_\alpha.$$

It follows straightforwardly from (2.3) that $X_\alpha \cap \mathcal{T}(X_n) = \{T_\alpha\}$, for every $\alpha \in \mathbb{N}_0^{n+1}$. Moreover, every $u \in X_\alpha \setminus \{T_\alpha\}$ satisfies $u > T_\alpha$, i.e., T_α is the *minimal element* of X_α with respect to the ordering $<$.

2.2. **Gröbner bases for ideals in the free associative algebra.** In this subsection $\mathbf{k}$ is an arbitrary field and $X = X_n = \{x_0, \dots, x_n\}$. Suppose $f \in \mathbf{k}\langle X \rangle$ is a nonzero polynomial. Its leading monomial with respect to $<$ will be denoted by $\mathbf{LM}(f)$. One has $\mathbf{LM}(f) = u$ if $f = cu + \sum_{1 \le i \le m} c_i u_i$, where $c, c_i \in \mathbf{k}$, $c \neq 0$ and $u > u_i \in \langle X \rangle$, for every $i \in \{1, \dots, m\}$.

Given a set $F \subseteq \mathbf{k}\langle X \rangle$ of non-commutative polynomials, $\mathbf{LM}(F)$ denotes the set

$$\mathbf{LM}(F) = \{\mathbf{LM}(f) \mid f \in F\}.$$

A monomial $u \in \langle X \rangle$ is *normal modulo* F if it does not contain any of the monomials $\mathbf{LM}(f)$ as a subword. The set of all normal monomials modulo F is denoted by $N(F)$.

Let I be a two sided graded ideal in $K\langle X \rangle$ and let $I_m = I \cap \mathbf{k}X^m$. We shall consider graded algebras with a minimal presentation. Without loss of generality, we may assume that I *is generated by homogeneous polynomials of degree* ≥ 2 and $I = \bigoplus_{m \geq 2} I_m$. Then the quotient algebra $A = \mathbf{k}\langle X \rangle / I$ is finitely generated and inherits its grading $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ from $\mathbf{k}\langle X_n \rangle$. We shall work with the so-called *normal* $\mathbf{k}$*-basis of* A.

We say that a monomial $u \in \langle X_n \rangle$ is *normal modulo* I if it is normal modulo $\mathbf{LM}(I)$. We set $N(I) := N(\mathbf{LM}(I))$. In particular, the free monoid $\langle X \rangle$ splits as a disjoint union

$$\langle X \rangle = N(I) \sqcup \mathbf{LM}(I). \tag{2.4}$$

The free associative algebra $\mathbf{k}\langle X \rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces $\mathbf{k}\langle X \rangle \simeq \mathrm{Span}_{\mathbf{k}} N(I) \oplus I$, and there is an isomorphism of vector spaces

$$A \simeq \mathrm{Span}_{\mathbf{k}} N(I). \tag{2.5}$$

We define

$$N(I)_m = \{u \in N(I) \mid u \text{ has length } m\}.$$

Then $A_m \simeq \mathrm{Span}_{\mathbf{k}} N(I)_m$ for every $m \in \mathbb{N}_0$.

Definition 2.2. Let $I \subset \mathbf{k}\langle X_n \rangle$ be a two-sided ideal.

(1) A subset $G \subseteq I$ of monic polynomials is a *Gröbner basis* of I (with respect to the ordering $<$) if
   (a) G generates I as a two-sided ideal, and
   (b) for every $f \in I$ there exists $g \in G$ such that $\mathbf{LM}(g)$ is a subword of $\mathbf{LM}(f)$, that is $\mathbf{LM}(f) = a\mathbf{LM}(g)b$, for some $a, b \in \langle X \rangle$.

(2) A Gröbner basis G is *minimal* if the set $G \setminus \{f\}$ is not a Gröbner basis of I, whenever $f \in G$.

(3) A minimal Gröbner basis G of I is *reduced* if each $f \in G$ is a linear combination of normal monomials modulo $G \setminus \{f\}$. In this case we say that f is *reduced modulo* $G \setminus \{f\}$.
(4) If I has a finite Gröbner basis G, then the algebra $A = \mathbf{k}\langle X\rangle/(G)$ is called a *standard finitely presented algebra*, or shortly an *s.f.p. algebra*.

It is well-known that every ideal I of $\mathbf{k}\langle X\rangle$ has a unique reduced Gröbner basis $G_0 = G_0(I)$ with respect to $<$. However, G_0 may be infinite. For more details, we refer the reader to [28, 31].

Definition 2.3. Let $h_1, \dots, h_s \in \mathbf{k}\langle X\rangle$ ($h_i = 0$ is also possible). For every $i \in \{1, \dots, s\}$, let $w_i \in \langle X\rangle$ be a monomial of degree at least 2, such that $w_i > \mathbf{LM}(h_i)$, whenever $h_i \neq 0$, and let $g_i = w_i - h_i$. Each g_i is a monic polynomial with $\mathbf{LM}(g_i) = w_i$. Let $G = \{g_1 \dots, g_s\} \subset \mathbf{k}\langle X\rangle$ and let $I = (G)$ be the two-sided ideal of $\mathbf{k}\langle X\rangle$ generated by G. For $u, v \in \langle X\rangle$ and for $i \in \{1, \dots, s\}$, we consider the $\mathbf{k}$-linear operators $r_{uiv} : \mathbf{k}\langle X_n\rangle \to \mathbf{k}\langle X_n\rangle$ called *reductions*, defined on the basis elements $c \in \langle X_n\rangle$ by

$$r_{uiv}(c) = \begin{cases} uh_iv & \text{if } c = uw_iv \\ c & \text{otherwise.} \end{cases}$$

Then the following conditions hold:

(1) $c - r_{uiv}(\omega) \in I$.
(2) $\mathbf{LM}(r_{uiv}(c)) \leq c$.
(3) More precisely, $\mathbf{LM}(r_{uiv}(c)) < c$ if and only if $c = uw_iv$.

More generally, for $f \in \mathbf{k}\langle X\rangle$ and for any finite sequence of reductions $r = r_{u_1 i_1 v_1} \circ \cdots \circ r_{u_t i_t v_t}$ one has

$$f \equiv r(f)(\mathrm{mod} I) \text{ and } \mathbf{LM}(f) \geq \mathbf{LM}(r(f)).$$

A polynomial $f \in \mathbf{k}\langle X_n\rangle$ is in *normal form (mod G)* if none of its monomials contains as a subword any of the w_i's. (In particular, the 0 element is in normal form.)

The degree-lexicographic ordering $<$ on $\langle X_n\rangle$ satisfies the decreasing chain condition, and therefore for every $f \in \mathbf{k}\langle X\rangle$ one can find a normal form of f by means of a finite sequence of reductions defined via G. In general, f may have more than one normal forms (mod G). It follows from Bergman's Diamond Lemma (see [8, Theorem 1.2]) that G is a Gröbner basis of I if and only if every $f \in \mathbf{k}\langle X\rangle$ has a unique normal form (mod G), which will be denoted by $\mathrm{Nor}(f)$. In this case $f \in I$ if and only if f can be reduced to 0 via a finite sequence of reductions.

Definition 2.4. Let $G = \{g_i = w_i - h_i \mid i \in \{1, \dots, s\}\} \subset \mathbf{k}\langle X_n\rangle$ as in Definition 2.3 and let $I = (G)$. Let $u = w_i$ and $v = w_j$ for some $i, j \in \{1, \dots, s\}$ and let $a, b, t \in \langle X\rangle \setminus \{1\}$.

(1) Suppose that $u = ab$, $v = bt$ and let $\omega = abt = ut = av$. The difference

$$(u, v)_\omega = g_it - ag_j = ah_j - h_it$$

is called *a composition of overlap*. Note that $(u, v)_\omega \in I$ and $\mathbf{LM}(g_it) = \omega = \mathbf{LM}(ag_j)$, so

$$\mathbf{LM}((u, v)_\omega) = \mathbf{LM}(sh_j - h_it) < \omega.$$

The composition of overlap $(u, v)_\omega$ is *solvable* if it can be reduced to 0 by means of a finite sequence of reductions defined via G.

(2) Suppose that $\omega = w_j = aw_ib$. The *composition of inclusion* corresponding to the pair (u, ω) is

$$(u, \omega)_\omega := (ag_ib) - g_j = h_j - ah_ib.$$

One has $(u, \omega)_\omega \in I$ and $\mathbf{LM}(u, \omega)_\omega = \mathbf{LM}(h_j - ah_ib) < \omega$. The composition of inclusion $(u, \omega)_\omega$ is *solvable* if it can be reduced to 0 by means of a finite sequence of reductions defined via G.

The lemma below is a modification of the Diamond Lemma and follows easily from Bergman's result [8, Theorem 1.2].

Lemma 2.5. *Let $G = \{w_i - h_i \mid i \in \{1, \dots, s\}\} \subset \boldsymbol{k}\langle X_n\rangle$ as in Definition 2.3. Let $I = (G)$ and let $A = \boldsymbol{k}\langle X_n\rangle/I$. Then the following conditions are equivalent.*

(1) The set G is a Gröbner basis of I.
(2) All compositions of overlap and all compositions of inclusion are solvable.

(3) *Every element $f \in \boldsymbol{k}\langle X_n\rangle$ has a unique normal form modulo G, denoted by* $\mathrm{Nor}_G(f)$.

(4) *There is an equality $N(G) = N(I)$, so there is an isomorphism of vector spaces*

$$\boldsymbol{k}\langle X_n\rangle \simeq I \oplus \boldsymbol{k}N(G).$$

(5) *The image of $N(G)$ in A is a $\boldsymbol{k}$-basis of A. In this case A can be identified with the $\boldsymbol{k}$-vector space $\boldsymbol{k}N(G)$, made a $\boldsymbol{k}$-algebra by the multiplication $a \bullet b := \mathrm{Nor}(ab)$.*

Suppose furthermore that G consists of homogeneous polynomials. Then A is graded by length and each of the above conditions is equivalent to

(6) $\dim A_m = \dim(\boldsymbol{k}N(G)_m) = |N(G)_m|$ *for every $m \in \mathbb{N}_0$.*

Corollary 2.6. *Let $G = \{w_i - h_i \mid i \in \{1,\dots,s\}\} \subset \boldsymbol{k}\langle X_n\rangle$ as above and let $I = (G)$. Let $N(G)$ and $N(I)$ be the corresponding sets of normal monomials in $\boldsymbol{k}\langle X_n\rangle$. Then $N(G) \supseteq N(I)$, where an equality holds if and only if G is a Gröbner basis of I.*

It is shown in [25, Corollary 6.3] that there exist ideals in the free associative algebra $\mathbf{k}\langle x_0,\dots,x_n\rangle$ for which the existence of a finite Gröbner basis is an undecidable problem.

In this paper, we focus on a class of quadratic standard finitely presented algebras $\mathcal{A}$ known as *non-commutative projective spaces* or *quantum spaces*. Each such algebra $\mathcal{A}$ is *strictly ordered* in the sense of [15, Definition 1.9], so there is a well-defined notion of Gröbner basis of a two-sided ideal in $\mathcal{A}$ (cf. [15, Definition 1.2]). Moreover, every two-sided ideal in $\mathcal{A}$ has a finite reduced Gröbner basis.

2.3. **Quadratic algebras and quadratic Gröbner bases.** As usual, let $X = X_n = \{x_0,\dots,x_n\}$. Let M be a non-empty proper subset of $\{0,\dots,n\}^2$. For every $(j,i) \in M$, let $h_{ji} \in \mathbf{k}\langle X\rangle$ be either 0 or a homogeneous polynomial of degree 2 with $\mathbf{LM}(h_{ji}) < x_jx_i$. Let

$$\mathcal{R} = \{f_{ji} = x_jx_i - h_{ji} \mid (j,i) \in M\} \subset \mathbf{k}\langle X\rangle. \tag{2.6}$$

Define $I = (\mathcal{R})$ and consider the quadratic algebra $A = \mathbf{k}\langle X_n\rangle/I$. As in Subsection 2.2, let $N(I)_m = N(I) \cap (X_n)^m$ and $N(\mathcal{R})_m = N(\mathcal{R}) \cap (X_n)^m$ be the corresponding subsets of normal words of length m. By construction, $\mathcal{R}$ is a $\mathbf{k}$-basis for I_2, so

$$\dim I_2 = |\mathcal{R}| = |M| \quad \text{and} \quad N(I)_2 = N(\mathcal{R})_2 = X_n^2 \setminus \mathbf{LM}(\mathcal{R}).$$

As vector spaces,

$$\mathbf{k}\langle X\rangle = I \oplus \mathbf{k}N(I) \text{ and } A \cong \mathbf{k}N(I).$$

Moreover, for the canonical grading by length one has

$$(\mathbf{k}\langle X\rangle)_m = (I)_m \oplus \mathbf{k}N(I)_m \text{ and } A_m \cong \mathbf{k}N(I)_m,$$

for every $m \in \mathbb{N}$.

The following Lemma is crucial for the proofs of several results in the paper.

Lemma 2.7. *Let $\mathcal{R}$ be defined as in* (2.6). *The following conditions are equivalent.*

(1) *The set $\mathcal{R}$ is a (quadratic) Gröbner basis of the ideal $I = (\mathcal{R})$.*

(2) $\dim A_3 = |N(\mathcal{R})_3|$.

(3) *All ambiguities of overlap determined by $\mathbf{LM}(\mathcal{R}) = \{x_jx_i \mid (j,i) \in M\}$ are $\mathcal{R}$-solvable.*

In this case A is a PBW *algebra in the sense of* [34, Section 5].

Proof. First note that there are no compositions of inclusions. By Corollary 2.6,

$$N(I)_m \subseteq N(\mathcal{R})_m \text{ and } \dim A_m = |N(I)_m| \le |N(\mathcal{R})_m|$$

for every $m \ge 2$. The implications (1) $\iff$ (3) and (1) $\Rightarrow$ (2) follow from Lemma 2.5.

(2) $\Rightarrow$ (3) A composition of overlap is either 0, or it produces only homogeneous polynomials of degree three. Suppose $\omega = x_kx_jx_i$, where $(k,j),(j,i) \in M$, so $f_{kj} = x_kx_j - h_{kj} \in \mathcal{R}$ and $f_{ji} = x_jx_i - h_{ji} \in \mathcal{R}$. Then the corresponding composition of overlap is

$$(x_kx_j, x_jx_i)_\omega = (f_{kj})x_i - x_k(f_{ji}) = -h_{kj}x_i + x_kh_{ji} \in I.$$

By Definition 2.4, a composition is solvable if and only if it can be reduced to 0. Assume by contradiction that the composition $(x_kx_j, x_jx_i)_\omega$ is not solvable. Then $(x_kx_j, x_jx_i)_\omega \neq 0$ and we can reduce it by means of a finite sequence of reductions to a (not necessarily unique) normal form

$$F := \mathrm{Nor}((x_kx_j, x_jx_i)_\omega) = cu + \sum_{s=1}^{t} c_s u_s \in \mathbf{k}N(\mathcal{R}),$$

where $u > u_s$ and $c \neq 0$. In particular, $x_kx_jx_i > \mathbf{LM}(F) = u \in N(\mathcal{R})$. However, the polynomial F is in the ideal I, hence $\mathbf{LM}(F) \in \mathbf{LM}(I_3)$ and $\mathbf{LM}(F)$ is not in $N(I)_3$. Therefore

$$N(I)_3 \subsetneqq N(\mathcal{R})_3.$$

Note that we have an isomorphism of vector spaces

$$A_3 \cong \mathbf{k}N(I)_3,$$

hence $\dim A_3 = |N(I)_3| < |N(\mathcal{R})_3|$, a contradiction. □

Remark 2.8. Lemma 2.7 is very useful for the case when we want to show that an algebra A with explicitly given quadratic defining relations $\mathcal{R} \subset \mathbf{k}\langle X_n\rangle$ is *PBW* (that is $\mathcal{R}$ is a Gröbner basis of the ideal $I = (\mathcal{R})$) and we have precise information about the dimension $\dim A_3 = d_3$. In this case, instead of following the standard procedure (algorithm) of checking whether all compositions are solvable, we suggest a new simpler procedure:

(1) find the set $N(\mathcal{R})_3$ and its order $|N(\mathcal{R})_3|$, and
(2) compare the order $|N(\mathcal{R})_3|$ with $\dim A_3 = d_3$.

One has $|N(\mathcal{R})_3| \geq \dim A_3$ and an equality holds if and only if $\mathcal{R}$ is a Gröbner basis of the ideal $I = (\mathcal{R})$. This method is particularly useful when we work in general settings–general n and general quadratic relations $\mathcal{R}$. It implies a similar procedure for ideals in the quantum space $\mathcal{A}^N_{\mathbf{g}}$.

We use this result in Section 5, see the proof of Theorem 5.2. In Subsection 3.2 we give some basics on Gröbner bases for ideals in a quantum space $\mathcal{A}^N_{\mathbf{g}}$. Lemma 3.14 is an important analogue of Lemma 2.7 designed for quadratic Gröbner bases of ideals in a quantum space.

3. Quantum spaces

In this section we introduce a class of quadratic algebras which are central for the paper. We shall refer to them as *quantum spaces*. They form a special case of the non-commutative deformation of projective spaces defined by Auroux, Katzarkov, and Orlov in the context of mirror symmetry [5]. These algebras are a particular case of the skew-polynomial rings with binomial relations studied in [17, 18]. We point out that these objects appear with different names in the literature: they are sometimes referred to as *non-commutative projective spaces* and *quantum affine spaces*. We shall now recall their definition and main properties.

3.1. Basic definitions and results.

Definition 3.1. A square matrix $\mathbf{q} = \|q_{ij}\|$ over the complex numbers is *multiplicatively anti-symmetric* if $q_{ij} \in \mathbb{C}^\times$, $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$ for all i, j. We shall sometimes refer to $\mathbf{q}$ as *a deformation matrix*.

Definition 3.2. Let $\mathbf{q}$ be an $(n+1)\times(n+1)$ multiplicatively anti-symmetric matrix. We denote by $\mathcal{A}^n_{\mathbf{q}}$ the complex quadratic algebra with $n+1$ generators $x_0, \ldots, x_n$ subject to the $\binom{n+1}{2}$ quadratic binomial relations

$$\mathcal{R} = \mathcal{R}_{\mathbf{q}} := \{x_jx_i - q_{ji}x_ix_j \mid 0 \leq i < j \leq n\}. \tag{3.1}$$

In other words $\mathcal{A}^n_{\mathbf{q}} = \mathbb{C}\langle X_n\rangle/(\mathcal{R})$. We refer to $\mathcal{A}^n_{\mathbf{q}}$ as the *quantum space defined by the multiplicatively anti-symmetric matrix* $\mathbf{q}$.

Clearly, the algebra $\mathcal{A}^n_{\mathbf{q}}$ is commutative if and only if all entries of $\mathbf{q}$ are 1. In this case $\mathcal{A}^n_{\mathbf{q}}$ is isomorphic the algebra of commutative polynomials $\mathbb{C}[x_0, \ldots, x_n]$. Although $\mathcal{A}^n_{\mathbf{q}}$ is non-commutative whenever $\mathbf{q}$ has at least one entry different from 1, it preserves all 'good properties' of the commutative polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, see Facts 3.7.

Example 3.3. For $n = 2$ and

$$\mathbf{q} = \begin{pmatrix} 1 & q^{-2} & 1 \\ q^2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

one obtains the non-commutative variety $\mathbb{P}^2_{q,\hbar=0}$ defined in [26, Section 3.7]. The quantum space $\mathcal{A}^2_{\mathbf{q}}$ is an Artin–Schelter regular algebra of global dimension three, see [2].

Remark 3.4. It is easy to prove that the set $\mathcal{R}$ defined in (3.3) is a reduced Gröbner basis for the ideal $I = (\mathcal{R})$ and this fact is well known, see for example [25, Proposition 5.5]. Therefore

$$N(I) = N(\mathcal{R}) = \mathcal{T}(X_n).$$

In other words the set $\mathcal{T}(X_n)$ of ordered monomials is the normal basis of the $\mathbb{C}$-vector space $\mathcal{A}^n_{\mathbf{q}}$. The free monoid $\langle X_n \rangle$ splits as a disjoint union

$$\langle X_n \rangle = \mathcal{T}(X_n) \sqcup \mathbf{LM}(I), \tag{3.2}$$

and $\mathbb{C}\langle X_n \rangle \simeq \mathrm{Span}_{\mathbb{C}} \mathcal{T}(X_n) \oplus I$.

Remark 3.5.
(1) Every element $f \in \mathbb{C}\langle X_n \rangle \setminus I$ has unique normal form $\mathrm{Nor}(f) = \mathrm{Nor}_{\mathcal{R}}(f) = \mathrm{Nor}_I(f)$, which satisfies

$$\mathrm{Nor}(f) = \sum_{i=1}^{s} c_i T_i \in \mathbb{C}\mathcal{T}(X_n),$$

where $c_i \in \mathbb{C}^{\times}$, $T_1 < T_2 < \ldots < T_s \leq \mathbf{LM}(f)$, and the equality $f = \mathrm{Nor}(f)$ holds in the algebra $\mathcal{A}^n_{\mathbf{q}}$. Moreover, $\mathrm{Nor}(f) = 0$ if and only if $f \in I$.
(2) The normal form $\mathrm{Nor}(f)$ can be found effectively using a finite sequence of reductions defined via $\mathcal{R}$.
(3) There is an equality $\mathrm{Nor}_{\mathcal{R}}(x_j x_i) = q_{ji} x_i x_j$, for every $0 \leq i < j \leq n$.

When the ideal I, or its generating set $\mathcal{R}$ is understood from the context, we shall denote the normal form of f by $\mathrm{Nor}(f)$.

More generally, recall that a quadratic algebra is an associative graded algebra $A = \bigoplus_{i \geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$. The Koszul dual algebra of A, denoted by $A^!$ is the quadratic algebra $T(V^*)/(R^{\perp})$, see [29, 30]. The algebra $A^!$ is also referred to as *the quadratic dual algebra to a quadratic algebra A*, see [33], p.6.

Note that every quantum space $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$ is *a skew-polynomial ring with binomial relations* in the sense of [17, 18], and *a quantum binomial algebra* in the sense of [21]. Thus the next corollary follows straightforwardly from [20, Theorem A], see also [21], Lemma 5.3, and Theorem 1.1.

Corollary 3.6. *Let $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$ be a* quantum space defined by the multiplicatively anti-symmetric matrix $\mathbf{q}$. *Then*

(1) The Koszul dual $\mathcal{A}^!$ has a presentation $\mathcal{A}^! = \mathbb{C}\langle \xi_0, \xi_1, \cdots, \xi_n \rangle / (\mathcal{R}^{\perp})$, where $\mathcal{R}^{\perp}$ consists of $\binom{n+1}{2}$ quadratic binomial relations and $n+1$ monomials

$$\mathcal{R}^{\perp} = \{\xi_j \xi_i - q_{ji}^{-1} \xi_i \xi_j \mid 0 \leq i < j \leq n\} \bigcup \{\xi_j^2 \mid 0 \leq j \leq n\}. \tag{3.3}$$

(2) The set $\mathcal{R}^{\perp}$ is a Gröbner basis of the ideal $(\mathcal{R}^{\perp})$ in $\mathbb{C}\langle \xi_0, \xi_1, \cdots, \xi_n \rangle$, so $\mathcal{A}^!$ is a PBW *algebra with* PBW *generators $\xi_0, \xi_1, \cdots, \xi_n$.*
(3) $\mathcal{A}^!$ is a quantum Grassmann algebra of dimension $n+1$.

The following result can be extracted from [18, 19], and [21, Theorem 1.1]. We use the well-known equality $\binom{n+d}{n} = \binom{n+d}{d}$.

Facts 3.7. Let $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$ be a quantum space.

(1) $\mathcal{A}$ is canonically graded by length, it is generated in degree one, and $\mathcal{A}_0 = \mathbb{C}$.

(2) $\mathcal{A}$ is a *PBW*-algebra in the sense of Priddy [34, Section 5], with a *PBW* basis $\mathcal{T}(X_n)$. For every $d \in \mathbb{N}$ there is an isomorphism of vector spaces $\mathcal{A}_d \simeq \operatorname{Span}_{\mathbb{C}} \mathcal{T}(X_n)_d$, so

$$\dim \mathcal{A}_d = |\mathcal{T}(X_n)_d| = \binom{n+d}{d}. \tag{3.4}$$

(3) $\mathcal{A}$ is Koszul.
(4) $\mathcal{A}$ is a left and a right Noetherian domain.
(5) $\mathcal{A}$ is an Artin–Schelter regular algebra, that is
  (a) $\mathcal{A}$ has polynomial growth of degree $n+1$ (equivalently, $\operatorname{GKdim} \mathcal{A} = n+1$);
  (b) $\mathcal{A}$ has finite global dimension $\operatorname{gl\,dim} \mathcal{A} = n+1$;
  (c) $\mathcal{A}$ is Gorenstein.
(6) The Hilbert series of $\mathcal{A}$ is $H_{\mathcal{A}}(t) = 1/(1-t)^{n+1}$.

Remark 3.8. The algebra $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$ is a *quantum projective space* in the sense of [35, Definition 2.1] and it is *solvable* in the sense of Kandri-Rodi and Weispfenning [25, Section 1].

Suppose a monomial $u \in \langle X_n \rangle$ has multi-degree $\deg(u) = \alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ and let $T_\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, as in Definition 2.1. Since all relations in $\mathcal{R}$ are binomials which preserve the multi-grading, there exists a unique $\zeta_u \in \mathbb{C}^\times$ such that

(1) ζ_u is a monomial in the entries of $\mathbf{q}$,
(2) $\operatorname{Nor}_{\mathcal{R}}(u) = \zeta_u T_\alpha$,
(3) $u \equiv \zeta_u T_\alpha$ modulo I, i.e., the equality $u = \zeta_u T_\alpha$ holds in $\mathcal{A}^n_{\mathbf{q}}$.

Convention 3.9. Following [8] (see also our Lemma 2.5), we consider the space $\mathbb{C}\mathcal{T}^n$ endowed with multiplication defined by

$$f \bullet g := \operatorname{Nor}_{\mathcal{R}}(fg),$$

for every $f, g \in \mathbb{C}\mathcal{T}^n$. Then $(\mathbb{C}\mathcal{T}^n, \bullet)$ has a well-defined structure of a graded algebra, and there is an isomorphism of graded algebras

$$\mathcal{A}^n_{\mathbf{q}} \cong (\mathbb{C}\mathcal{T}^n, \bullet). \tag{3.5}$$

By convention we shall identify the algebra $\mathcal{A}^n_{\mathbf{q}}$ with $(\mathbb{C}\mathcal{T}^n, \bullet)$.

3.2. **Some basics of Gröbner bases theory for ideals in quantum spaces.** In Sections 5 and 6 we shall introduce analogues of the Veronese map $v_{n,d}$ and of the Segre map $s_{n,m}$ for quantum spaces. A natural problem in this context is to describe the reduced Gröbner bases of $\ker(v_{n,d})$ and $\ker(s_{n,m})$. Each of the kernels is an ideal of an appropriate quantum space $\mathcal{A}^N_{\mathbf{g}}$, so we need a Gröbner bases theory which is admissible for quantum spaces. Proposition 3.10 shows that each quantum space $\mathcal{A}^N_{\mathbf{g}}$ is a *strictly ordered* algebra in the sense of [15, Definition 1.9], and the Gröbner bases theory for ideals in strictly ordered algebras presented by the third author in [15] and [16] seems natural and convenient for our quantum spaces. Here we follow the approach of these works. Note that the results of [15] and [16] are independent from and agree with [25] and [31].

In the sequel we often work simultaneously with two distinct quantum spaces whose sets of generators $X_n = \{x_0, \dots, x_n\}$ and $Y_N = \{y_0, \dots, y_N\}$ are disjoint and have different cardinalities, $N > n$. To avoid ambiguity we denote by $\prec$ the degree-lexicographic ordering on $\langle Y_N \rangle$ and by $\prec_0$ the restriction $\prec_{|\mathcal{T}(Y_N)}$ of $\prec$ on the set of ordered monomials $\mathcal{T}(Y_N) \subset \langle Y_N \rangle$.

Given an arbitrary multiplicatively anti-symmetric $(N+1) \times (N+1)$ matrix $\mathbf{g} = \|g_{ij}\|$, let $\mathcal{A}^N_{\mathbf{g}} = \mathbb{C}\langle Y_N \rangle / (\mathcal{R}_{\mathbf{g}})$ be the associated quantum space, where

$$\mathcal{R}_{\mathbf{g}} := \{y_j y_i - g_{ji} y_i y_j \mid 0 \le i < j \le N\}.$$

Following Convention 3.9, we identify the two algebras

$$\mathcal{A}^N_{\mathbf{g}} \cong (\mathbb{C}\mathcal{T}(Y_N), \bullet).$$

Let $\mathfrak{J}_{\mathbf{g}} = (\mathcal{R}_{\mathbf{g}})$. We shall write $\operatorname{Nor}(f)$ for the normal form of $f \in \mathbb{C}\langle Y_N \rangle$, keeping the ideal $\mathfrak{J}_{\mathbf{g}}$ fixed. The operation $\bullet$ on $\mathbb{C}\mathcal{T}(Y_N)$ induces also an operation $\star$ on the set $\mathcal{T}(Y_N)$ defined by

$$u \star v := \mathbf{LM}(\operatorname{Nor}(uv)) = \mathbf{LM}(u \bullet v),$$

for every $u, v \in \mathcal{T}(Y_N)$. It is not difficult to see that $(\mathcal{T}(Y_N), \star)$ is a monoid.

Let $u, v \in \mathcal{T}(Y_N)$, and let $\alpha = \deg u + \deg v$. We know that $u \bullet v = \zeta(u,v)T(u,v)$, where $\zeta = \zeta(u,v) \in \mathbb{C}^\times$ and $T(u,v) \in \mathcal{T}(Y_n)$, with $\deg T(u,v) = \alpha$. Similarly, $v \bullet u = \eta(v,u)T(v,u)$, where $\eta(v,u) \in \mathbb{C}^\times$ and $\deg T(v,u) = \alpha = \deg T(u,v)$. The unique ordered monomial in $\langle Y_N \rangle$ with multi-degree α is T_α, therefore

$$u \star v = v \star u = T_\alpha.$$

It follows that there is an isomorphism of monoids $(\mathcal{T}(Y_n), \star) \cong [y_0, \dots, y_N]$, the free abelian monoid generated by Y_N. This agrees with [15, Theorem I and Theorem II].

Note that identifying $\mathcal{A}_{\mathbf{g}}^N$ with $\mathbb{C}\mathcal{T}(Y_N)$ we also have the degree-lexicographic well-ordering $\prec_0$ on the free abelian monoid $(\mathcal{T}(Y_N), \star)$. For every $f \in \mathbb{C}\mathcal{T}(Y_N)$, its leading monomial with respect to $\prec_0$ is denoted by $\mathbf{LM}(f)_{\prec_0}$. In fact $\mathbf{LM}_{\prec_0}(f) = \mathbf{LM}_{\prec}(f)$ and we shall simply write $\mathbf{LM}(f)$.

The proposition below follows straightforwardly from [15].

Proposition 3.10. *(1) The quantum space $\mathcal{A}_{\mathbf{g}}^N = (\mathbb{C}\mathcal{T}(Y_N), \bullet)$ is a* strictly ordered algebra *in the sense of* [15, Definition 1.9]*, that is, each of the following two equivalent conditions is satisfied:*

- **SO1:** *Let $a, b, c \in \mathcal{T}(Y_N)$. If $a \prec_0 b$, then $a \star c \prec_0 b \star c$ and $c \star a \prec_0 c \star b$;*
- **SO2:** $\mathbf{LM}(f \bullet h) = \mathbf{LM}(\mathbf{LM}(f) \bullet \mathbf{LM}(h))$*, for all $f, h \in \mathcal{A}_{\mathbf{g}}^N$.*

(2) Every two-sided (respectively, one-sided) ideal $\mathfrak{K}$ of $\mathcal{A}_{\mathbf{g}}^N$ has a finite reduced Gröbner basis with respect to the ordering $\prec_0$ on $(\mathcal{T}(Y_N), \star)$, see Definition 3.12.

The properties **SO1** and **SO2** allow to define Gröbner bases for ideals of a quantum space $\mathcal{A}_{\mathbf{g}}^N$ in a natural way, and to use a standard Gröbner bases theory, analogous to the theory of non-commutative Gröbner bases for ideals of the free associative algebra (Diamond Lemma) proposed by Bergman.

Definition 3.11. Let $P \subset \mathcal{A}_{\mathbf{g}}^N$ be an arbitrary subset, and let $\mathbf{LM}(P) = \{\mathbf{LM}(f) \mid f \in P\}$. A monomial $T \in \mathcal{T}(Y_N)$ *is normal modulo* P if it does not contain as a subword any $u \in \mathbf{LM}(P)$. We denote

$$N_{\prec_0}(P) = \{T \in \mathcal{T}(Y_N) \mid T \text{ is normal mod P}\}. \tag{3.6}$$

Definition 3.12. Suppose $\mathfrak{K}$ is an ideal of $\mathcal{A}_{\mathbf{g}}^N = \mathbb{C}\mathcal{T}(Y_N)$. A set $F \subset \mathfrak{K}$ is *a Gröbner basis* of $\mathfrak{K}$ if for any $h \in \mathfrak{K}$ there exists an $f \in F$, and monomials $a, b \in \mathcal{T}(Y_N)$ such that $\mathbf{LM}(h) = a \star \mathbf{LM}(f) \star b$. Due to the commutativity of the operation $\star$ this is equivalent to $\mathbf{LM}(h) = u \star \mathbf{LM}(f)$, for some $u \in \mathcal{T}$.

An interested reader can find various equivalent definitions of a Gröbner basis in [31, 25], and numerous papers which appeared later. Given an ideal $\mathfrak{K}$ generated by a finite set F one can verify algorithmically whether F is a Gröbner basis for the ideal $\mathfrak{K}$, see for example [31].

Lemma 3.13. *Let $\mathfrak{K} = (F)$ be an ideal of $\mathcal{A}_{\mathbf{g}}^N$ generated by the set $F \subset \mathbb{C}\mathcal{T}(Y_N)$. Then F is a Gröbner basis of $\mathfrak{K}$ if and only if $N(F) = N_{\prec_0}(F) = N_{\prec_0}(\mathfrak{K})$. In this case the vector space $\mathcal{A}_{\mathbf{g}}^N$ splits as a direct sum*

$$\mathcal{A}_{\mathbf{g}}^N = \mathbb{C}\mathcal{T}(Y_N) = \mathfrak{K} \oplus \mathbb{C}N_{\prec_0}(F)$$

and the set $N_{\prec_0}(F) \subset \mathcal{T}(Y_N)$ projects to a $\mathbb{C}$-basis of the quotient algebra $\mathcal{A}_{\mathbf{g}}^N/\mathfrak{K}$. Moreover, if F consists of homogeneous polynomials, then

$$(\mathcal{A}_{\mathbf{g}}^N)_j = (\mathbb{C}\mathcal{T}(Y_N))_j = (\mathfrak{K})_j \oplus (\mathbb{C}N_{\prec_0}(F))_j, \tag{3.7}$$

for every $j \geq 2$.

The following is an analogue of Lemma 2.7 for ideals of $\mathcal{A}_{\mathbf{g}}^N$ generated by quadratic polynomials.

Lemma 3.14. *Let $\mathfrak{K} = (F)$ be an ideal of $\mathcal{A}_{\mathbf{g}}^N$ generated by a set of quadratic polynomials $F \subset (\mathbb{C}\mathcal{T}(Y_N))_2$ and let $B = \mathcal{A}_{\mathbf{g}}^N/\mathfrak{K}$. We consider the canonical grading of B induced by the grading of $\mathcal{A}_{\mathbf{g}}^N$. Then F is a Gröbner basis of $\mathfrak{K}$ if and only if*

$$\dim B_3 = |(N_{\prec_0}(F))_3|. \tag{3.8}$$

4. The d-Veronese subalgebra of $\mathcal{A}_{\mathbf{q}}^n$, its generators and relations

In this section we study the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of the quantum space $\mathcal{A} = \mathcal{A}_{\mathbf{q}}^n$. This is an algebraic construction which mirrors the Veronese embedding. First we recall some basic definitions and facts about Veronese subalgebras of general graded algebras. Our main reference is [33, Section 3.2]. The main result of the section is Theorem 4.5 which presents the d-Veronese subalgebra $\mathcal{A}^{(d)}$ in terms of generators and explicit quadratic relations.

Definition 4.1. Let $A = \bigoplus_{k\in\mathbb{N}_0} A_k$ be a graded algebra. For $d \in \mathbb{N}$, the *d-Veronese subalgebra* of A is the graded algebra

$$A^{(d)} = \bigoplus_{k\in\mathbb{N}_0} A_{kd}.$$

Remark 4.2. (1) By definition the algebra $A^{(d)}$ is a subalgebra of A. However, the embedding is not a graded algebra morphism. The Hilbert function of $A^{(d)}$ satisfies

$$h_{A^{(d)}}(t) = \dim(A^{(d)})_t = \dim(A_{td}) = h_A(td).$$

(2) Let $\mathcal{A} = \mathcal{A}_{\mathbf{q}}^n$ be the quadratic algebra with relations $\mathcal{R}$ introduced in Definition 3.2. It follows from [33, Proposition 2.2], and Facts 3.7 that its d-Veronese subalgebra $\mathcal{A}^{(d)}$ is one-generated, quadratic and Koszul. Moreover, $\mathcal{A}^{(d)}$ is left and right Noetherian.

We fix a multiplicatively anti-symmetric matrix $\mathbf{q}$ and set $\mathcal{A} = \mathcal{A}_{\mathbf{q}}^n$. By Convention 3.9, $\mathcal{A}$ is identified with the algebra $(\mathbb{C}\mathcal{T}^n, \bullet)$ and

$$\mathcal{A} = \bigoplus_{k\in\mathbb{N}_0} \mathcal{A}_k \cong \bigoplus_{k\in\mathbb{N}_0} \mathbb{C}(\mathcal{T}^n)_k.$$

Hence its d-Veronese subalgebra satisfies

$$\mathcal{A}^{(d)} = \bigoplus_{k\in\mathbb{N}_0} \mathcal{A}_{kd} \cong \bigoplus_{k\in\mathbb{N}_0} \mathbb{C}(\mathcal{T}^n)_{kd}.$$

The ordered monomials $w \in (\mathcal{T}^n)_d$ of length d are degree one generators of $\mathcal{A}^{(d)}$, hence

$$\dim \mathcal{A}_d = |(\mathcal{T}^n)_d| = \binom{n+d}{d}.$$

We set $N = \binom{n+d}{d} - 1$ and we order the elements of $(\mathcal{T}^n)_d$ lexicographically, so

$$(\mathcal{T}^n)_d = \{w_0 = x_0^d < w_1 = (x_0)^{d-1}x_1 < \cdots < w_N = x_n^d\}. \tag{4.1}$$

The d-Veronese $\mathcal{A}^{(d)}$ is a quadratic algebra (one)-generated by $w_0, w_1, \ldots, w_N$. We shall find a minimal set of its quadratic relations, each of which is a linear combination of products w_iw_j for some $i, j \in \{0, \ldots, N\}$. The following notation will be used throughout the paper.

Notation 4.3. Let $N = \binom{n+d}{d} - 1$. For every integer j, $1 \le j \le N$, we denote by α^j the multi-degree $\deg(w_j)$, thus

$$\alpha^j = (\alpha_{j_0}, \ldots, \alpha_{j_n}) \text{ whenever } w_j = x_0^{\alpha_{j_0}} \ldots x_n^{\alpha_{j_n}}.$$

We define

$$m(j) = \min\{s \in \{0, \ldots, n\} \mid \alpha_{j_s} \ge 1\} \text{ and } M(j) = \max\{s \in \{0, \ldots, n\} \mid \alpha_{j_s} \ge 1\}. \tag{4.2}$$

In other words, if $w_j = x_{j_1}^{\alpha_{j_1}} x_{j_2}^{\alpha_{j_2}} \ldots x_{j_d}^{\alpha_{j_d}}$ for some $0 \le j_1 \le j_2 \le \cdots \le j_d$ and $\alpha_{j_1}, \ldots, \alpha_{j_d} \ge 1$, then $m(j) = j_1$ and $M(j) = j_d$. For example, if $w_j = x_2x_4^3x_7^2$, then $m(j) = 2$ and $M(j) = 7$. We further define

$$\begin{aligned}
\mathrm{P}(n,d) &= \{(i,j) \mid 0 \le i \le j \le N\};\\
\mathrm{C}(n,2,d) &= \{(i,j) \in \mathrm{P}(n,d) \mid M(i) \le m(j)\} = \{(i,j) \in \mathrm{P}(n,d) \mid w_iw_j \in (\mathcal{T}^n)_{2d}\};\\
\mathrm{C}(n,3,d) &= \{(i,j,k) \,|\, 0 \le i \le j \le k \le N,\ (i,j), (j,k) \in \mathrm{C}(n,2,d)\};\\
\mathrm{MV}(n,d) &= \{(i,j) \in \mathrm{P}(n,d) \mid M(i) > m(j)\} = \{(i,j) \in \mathrm{P}(n,d) \mid w_iw_j \notin (\mathcal{T}^n)_{2d}\}.
\end{aligned}$$

Lemma 4.4. *Let $(\mathcal{T}^n)_p = (\mathcal{T}(X_n))_p$ be the set of all ordered monomials $w \in \langle X_n\rangle$ of length $|w| = p$.*

(1) The maps

$$\Phi : \mathrm{C}(n,2,d) \to (\mathcal{T}^n)_{2d} \qquad \text{and} \qquad \Psi : \mathrm{C}(n,3,d) \to (\mathcal{T}^n)_{3d}$$
$$(i,j) \mapsto w_i w_j \qquad\qquad (i,j,k) \mapsto w_i w_j w_k$$

are bijective. Therefore

$$|\mathrm{C}(n,2,d)| = |(\mathcal{T}^n)_{2d}| = \binom{n+2d}{n} \quad \text{and} \quad |\mathrm{C}(n,3,d)| = |(\mathcal{T}^n)_{3d}| = \binom{n+3d}{n}. \tag{4.3}$$

(2) The set $\mathrm{P}(n,d)$ *is a disjoint union* $\mathrm{P}(n,d) = \mathrm{C}(n,2,d) \sqcup \mathrm{MV}(n,d)$*. Moreover*

$$|\mathrm{P}(n,d)| = \binom{N+2}{2} \quad \text{and} \quad |\mathrm{MV}(n,d)| = \binom{N+2}{2} - \binom{n+2d}{n}. \tag{4.4}$$

Proof. (1) Given $w_i, w_j \in (\mathcal{T}^n)_d$, their product $w = w_i w_j$ belongs to $(\mathcal{T}^n)_{2d}$ if and only if $(i,j) \in \mathrm{C}(n,2,d)$, hence Φ is well-defined. Observe that every $w \in (\mathcal{T}^n)_{2d}$ can be written uniquely as

$$w = x_{i_1} \dots x_{i_d} x_{j_1} \dots x_{j_d}, \text{ where } 0 \le i_1 \le \dots \le i_d \le j_1 \le \dots \le j_d. \tag{4.5}$$

It follows that w has a unique presentation $w = w_i w_j$, where

$$w_i = x_{i_1} \dots x_{i_d} \in (\mathcal{T}^n)_d,\ w_j = x_{j_1} \dots x_{j_d} \in (\mathcal{T}^n)_d,$$
$$M(i) = i_d \le m(j) = j_1 \text{ and } (i,j) \in \mathrm{C}(n,2,d).$$

This implies that Φ is a bijection.

Consider now the map Ψ. Given $w_i, w_j, w_k \in (\mathcal{T}^n)_d$, their product $\omega = w_i w_j w_k$ (considered as an element in $\langle X_n \rangle$) belongs to $(\mathcal{T}^n)_{3d}$ if and only if $(i,j,k) \in \mathrm{C}(n,3,d)$, hence Ψ is well-defined. The proof that Ψ is bijective is similar to the case of Φ.

(2) It is clear that

$$|\mathrm{P}(n,d)| = \binom{N+1}{2} + N + 1 = \binom{N+2}{2}.$$

By definition $\mathrm{P}(n,d) = \mathrm{C}(n,2,d) \sqcup \mathrm{MV}(n,d)$ is a disjoint union of sets, hence

$$|\mathrm{MV}(n,d)| = |\mathrm{P}(n,d)| - |\mathrm{C}(n,2,d)| = \binom{N+2}{2} - \binom{n+2d}{n}. \qquad \square$$

The following result describes the d-Veronese subalgebra $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$ of the quantum space $\mathcal{A}^n_{\mathbf{q}}$ in terms of generators and quadratic relations.

Theorem 4.5. *Let* $\mathbf{q}$ *be an* $(n+1) \times (n+1)$ *multiplicatively anti-symmetric matrix and let* $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$*. The d-Veronese subalgebra* $\mathcal{A}^{(d)} \subseteq \mathcal{A}$ *is a quadratic algebra with* $\binom{n+d}{d}$ *generators, namely the elements of* $(\mathcal{T}^n)_d$*, subject to* $(N+1)^2 - \binom{n+2d}{n}$ *independent quadratic relations which split into two disjoint sets* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *given below.*

(1) The set $\mathcal{R}_1$ *contains exactly* $\binom{N+1}{2}$ *relations*

$$\mathcal{R}_1 = \{f_{ji} = w_j w_i - \varphi_{ji} w_{i'} w_{j'} \mid 0 \le i < j \le N,\ (i',j') \in \mathrm{C}(n,2,d),\ \varphi_{ji} \in \mathbb{C}^\times\}, \tag{4.6}$$

where for each pair $j > i$ *the product* $w_j w_i$ *occurs exactly once in* $\mathcal{R}_1$*, and there is unique pair* $(i',j') \in \mathrm{C}(n,2,d)$ *such that* $\mathrm{Nor}(w_j w_i) = \varphi_{ji} w_{i'} w_{j'} = \varphi_{ji} T_\beta$*, with* $\beta = \deg(w_j w_i) = \deg(w_{i'} w_{j'})$*. One has*

$$\mathbf{LM}(f_{ji}) = w_j w_i > w_{i'} w_{j'} = T_\beta \in (\mathcal{T}^n)_{2d}.$$

Moreover, for every pair $(i,j) \in \mathrm{C}(n,2,d)$ *such that* $i < j$*, the product* $w_i w_j = T_\beta \in (\mathcal{T}^n)_{2d}$ *occurs in a relation* $w_j w_i - \varphi_{ji} w_i w_j \in \mathcal{R}_1$*. Each coefficient* φ_{ji} *is a non-zero complex number, uniquely determined by* $\mathbf{q}$*.*

(2) The set $\mathcal{R}_2$ *consists of exactly* $\binom{N+2}{2} - \binom{n+2d}{n}$ *relations*

$$\mathcal{R}_2 = \{f_{ij} = w_i w_j - \varphi_{ij} w_{i'} w_{j'} \mid (i,j) \in \mathrm{MV}(n,d),\ (i',j') \in \mathrm{C}(n,2,d),\ \varphi_{ij} \in \mathbb{C}^\times\}, \tag{4.7}$$

where for each pair $(i,j) \in \mathrm{MV}(n,d)$ the word w_iw_j occurs exactly once in $\mathcal{R}_2$, and determines uniquely a pair $(i',j') \in \mathrm{C}(n,2,d)$ with $i' < j'$, and a nonzero complex number φ_{ij} such that $\mathrm{Nor}(w_iw_j) = \varphi_{ij}w_{i'}w_{j'} = \varphi_{ij}T_\beta$, with $\beta = \deg(w_iw_j) = \deg(w_{i'}w_{j'})$. In particular,

$$\mathbf{LM}(f_{ij}) = w_iw_j > w_{i'}w_{j'} = T_\beta \in (\mathcal{T}^n)_{2d}.$$

(3) The relations $\mathcal{R}_1 \cup \mathcal{R}_2$ imply a set $\mathcal{R}'_1$ of $\binom{N+1}{2}$ additional relations:

$$\mathcal{R}'_1 = \{w_j \cdot w_i - g_{ji}w_i \cdot w_j \mid g_{ji} \in \mathbb{C}^\times, 0 \le i < j \le N\}, \tag{4.8}$$

where for each $i < j$ the coefficient $g_{ji} = \frac{\varphi_{ji}}{\varphi_{ij}}$ is uniquely determined by the matrix $\mathbf{q}$. We set $\varphi_{ij} = 1$ whenever $(i,j) \in \mathrm{C}(n,2,d)$.

(4) Conversely, the relations $\mathcal{R}' = \mathcal{R}'_1 \cup \mathcal{R}_2$ imply the relations $\mathcal{R}_1$. Moreover, $\mathcal{R}'$ is also a complete set of independent relations for the d-Veronese algebra $\mathcal{A}^{(d)}$.

Proof. (1) Suppose that $0 \le i < j \le N$. Then $w_j > w_i$, and it is not difficult to see that $M(j) > m(i)$, so w_jw_i is not in normal form. By Remark 3.5, its normal form has the shape $\mathrm{Nor}(w_jw_i) = \varphi_{ji}T_\beta$, where $\beta = \deg(w_jw_i) = \alpha^i + \alpha^j$, and $\varphi_{ji} \in \mathbb{C}^\times$ is uniquely determined by the entries of $\mathbf{q}$. By Lemma 4.4, $T_\beta = w_{i'}w_{j'}$ for a unique pair $(i',j') \in \mathrm{C}(n,2,d)$ of ordered monomials $w_{i'} \le w_{j'}$ of length d. We claim that $w_{i'} < w_{j'}$.

Assume by contradiction that $w_{i'} = w_{j'} = x_{i_1}x_{i_2}\dots x_{i_d}$, where $x_{i_1} \le x_{i_2} \le \dots \le x_{i_d}$. This implies that $w_{i'}w_{j'} = w_{i'}^2 = x_{i_1}x_{i_2}\dots x_{i_d}x_{i_1}x_{i_2}\dots x_{i_d} \in (\mathcal{T}^n)_{2d}$. But this is possible if and only if $x_{i_k} = x_{i_1}$ for every $k \in \{2,\dots,d\}$, that is $w_{i'} = w_{j'} = (x_p)^d$, for some $p \in \{0,\dots,n\}$, so $T_\beta = (x_p)^{2d}$. In other words $\beta = (\beta_0,\dots,\beta_n)$, where $\beta_p = 2d$ and $\beta_i = 0$ for every $i \ne p$. One has $\beta = \deg(w_jw_i) = \deg(w_j) + \deg(w_i) = \alpha^j + \alpha^i$, which together with $|w_i| = |w_j| = d$ imply $\alpha^i = \alpha^j$ and $w_i = w_j = (x_p)^d$, which is impossible, since by assumption $i < j$. Hence $w_{i'} < w_{j'}$ and $i' < j'$. We know that the equality $w_jw_i = \mathrm{Nor}(w_jw_i)$ holds in $\mathcal{A}$, hence it is an equality in $\mathcal{A}^{(d)}$. This implies that the equality $(w_jw_i) = \varphi_{ji}w_{i'}w_{j'}$ holds in $\mathcal{A}^{(d)}$, for all $0 \le i < j \le N$. It follows that $\mathcal{A}^{(d)}$ satisfies the relations $f_{ji} = 0$, for all $f_{ji} \in \mathcal{R}_1$, see (4.6). Moreover, the relations satisfy the properties given in part (1). It is clear that the order of $\mathcal{R}_1$ is exactly $\binom{N+1}{2}$.

(2) Suppose that $(i,j) \in \mathrm{MV}(n,d)$. Then the following are equalities in $\mathcal{A}$:

$$w_iw_j = \mathrm{Nor}(w_iw_j) = \varphi_{ij}T_\beta, \text{ where } T_\beta < w_iw_j, \beta = \alpha^i + \alpha^j,$$

and $\varphi_{ij} \in \mathbb{C}^\times$ is uniquely determined by the entries of $\mathbf{q}$. By Lemma 4.4, $T_\beta = w_{i'}w_{j'}$ for a unique pair $(i',j') \in \mathrm{C}(n,2,d)$. We claim that $w_{i'} < w_{j'}$. As in part (1), assuming that $w_{i'} = w_{j'}$ we obtain that $w_i = w_j = (x_p)^d$, but then $w_iw_j = (x_p^d)(x_p^d) \in (\mathcal{T}^n)$, which contradicts our assumption $(i,j) \in \mathrm{MV}(n,d)$. The equality $w_iw_j = \mathrm{Nor}(w_iw_j)$ holds in $\mathcal{A}$, therefore it is an equality in $\mathcal{A}^{(d)}$. We have shown that for every pair $(i,j) \in \mathrm{MV}(n,d)$ there is unique pair $(i',j') \in \mathrm{C}(n,2,d)$ such that $i' < j'$ and $w_jw_i = \varphi_{ii}w_{i'}w_{j'}$ holds in $\mathcal{A}^{(d)}$. Therefore $\mathcal{A}^{(d)}$ satisfies the relations (4.7) from $\mathcal{R}_2$. It is clear that all properties listed in part (2) hold and $|\mathcal{R}_2| = |\mathrm{MV}(n,d)| = \binom{N+2}{2} - \binom{n+2d}{n}$. Note that

$$\mathbf{LM}(\mathcal{R}_1) = \{w_jw_i \mid w_j > w_i\} \text{ and } \mathbf{LM}(\mathcal{R}_2) = \{w_iw_j \mid w_i \le w_j, (i,j) \in \mathrm{MV}(n,d)\}.$$

It follows that $\mathbf{LM}(\mathcal{R}_1) \cap \mathbf{LM}(\mathcal{R}_2) = \emptyset$ and therefore $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$. Hence the set of relations $\mathcal{R}$ is a disjoint union $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$ and

$$\begin{aligned} |\mathcal{R}| &= |\mathcal{R}_1| + |\mathcal{R}_2| = \tbinom{N+1}{2} + \tbinom{N+2}{2} - \tbinom{n+2d}{n} \\ &= (N+1)^2 - \tbinom{n+2d}{n} \\ &= \tbinom{n+d}{n}^2 - \tbinom{n+2d}{n}. \end{aligned} \tag{4.9}$$

(3) Assume now that $0 \le i < j \le N$. Two cases are possible.

(a) $(i,j) \in \mathrm{C}(n,2,d)$. In this case $(i',j') = (i,j)$ and $w_jw_i = \varphi_{ji}w_iw_j = \varphi_{ji}w_{i'}w_{j'}$, so $g_{ji} = \varphi_{ji}$.

(b) $(i,j) \in \mathrm{MV}(n,d)$. Then the two relations

$$w_jw_i = \varphi_{ji}w_{i'}w_{j'} \text{ and } w_iw_j = \varphi_{ij}w_{i'}w_{j'}$$

imply

$$(\varphi_{ji})^{-1}w_j \cdot w_i = w_{i'}w_{j'} = (\varphi_{ij})^{-1}w_i \cdot w_j,$$

and therefore $w_j \cdot w_i = \frac{\varphi_{ji}}{\varphi_{ij}} w_i \cdot w_j$. It follows that $w_j w_i = g_{ji} w_i w_j$, where the nonzero coefficient $g_{ji} = \frac{\varphi_{ji}}{\varphi_{ij}}$ is uniquely determined by $\mathbf{q}$.

(4) This is analogous to (3). □

Observe that Theorem 4.5 contains important numerical data about the d-Veronese $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$, which will be used in the sequel, and which we summarise below.

Notation 4.6. Let $\mathcal{A}^n_{\mathbf{q}}$ be the quantum space defined via a multiplicatively anti-symmetric $(n+1)\times(n+1)$ matrix $\mathbf{q}$. Let $d \geq 2$ and $N = \binom{n+d}{n} - 1$. We associate to the d-Veronese $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$ a list $D(\mathcal{A}^n_{\mathbf{q}})^{(d)}$ of invariants uniquely determined by $\mathbf{q}$ and d.

Let $\mathfrak{F}_1 = \{\varphi_{ji} \mid 0 \leq i < j \leq N\}$ be the set of coefficients occurring in $\mathcal{R}_1$ (see (4.6)) and let $\mathfrak{F}_2 = \{\varphi_{ij} \mid (i,j) \in \mathrm{MV}(n,d)\}$ be the set of coefficients occurring in $\mathcal{R}_2$ (see (4.7)). Let $\mathbf{g} = \|g_{ij}\|$ be the multiplicatively anti-symmetric $(N+1)\times(N+1)$ matrix whose entries $g_{ij}, 0 \leq i < j \leq N$ are the coefficients occurring in $\mathcal{R}'_1$ see (4.8). We collect this information about $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$ in the following data:

$D(\mathcal{A}^n_{\mathbf{q}})^{(d)}$:
- $\mathbf{q} = \|q_{ij}\|$;
- $\mathfrak{F}_1 = \{\varphi_{ji} \mid 0 \leq i < j \leq N\}$, the set of coefficients occurring in (4.6);
- $\mathfrak{F}_2 = \{\varphi_{ij} \mid i \leq j, (i,j) \in \mathrm{MV}(n,d)\}$, the set of coefficients occurring in (4.7);
- $\mathbf{g} = \|g_{ij}\|$, a multiplicatively anti-symmetric $(N+1)\times(N+1)$ matrix with

$$g_{ji} = \begin{cases} 1 & \text{for } i = j \\ (\varphi_{ji})/(\varphi_{ij}) & \text{for } (i,j) \in \mathrm{MV}(n,d) \text{ and } i < j \\ \varphi_{ji} & \text{for } (i,j) \in \mathrm{C}(n,2,d) \text{ and } i < j. \end{cases}$$

5. Veronese maps

Let $n, d \in \mathbb{N}$ and let $N = \binom{n+d}{d} - 1$. In this section, we introduce and study non-commutative analogues of the Veronese embeddings $V_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$. The main result of the section is Theorem 5.2, which describes explicitly the reduced Gröbner bases for the kernel of the non-commutative Veronese map.

We keep the notation and conventions from the previous sections, so $X_n = \{x_0, \dots, x_n\}$ and $\mathcal{T}^n = \mathcal{T}(X_n) \subset \langle X_n \rangle$ is the set of ordered monomials (terms) in the alphabet X_n. The set $(\mathcal{T}^n)_d$ of all degree d terms is enumerated according the degree-lexicographic order in $\langle X_n \rangle$:

$$(\mathcal{T}^n)_d = \{w_0 = x_0^d < w_1 = (x_0)^{d-1}x_1 < \cdots < w_N = x_n^d\}. \tag{5.1}$$

We introduce a second set of variables $Y_N = \{y_0, \dots, y_N\}$, and given an arbitrary multiplicatively anti-symmetric $(N+1)\times(N+1)$ matrix $\mathbf{g} = \|g_{ij}\|$, we present the corresponding quantum space as $\mathcal{A}^N_{\mathbf{g}} = \mathbb{C}\langle Y_N \rangle/(\mathcal{R}_{\mathbf{g}})$, where

$$\mathcal{R}_{\mathbf{g}} := \{y_j y_i - g_{ij} y_i y_j \mid 0 \leq i < j \leq N\}.$$

5.1. Definitions and first results.

Lemma 5.1. *Let $n, d \in \mathbb{N}$ and let $N = \binom{n+d}{d} - 1$. Let $(\mathcal{T}^n)_d$ and Y_N be as above. For every $(n+1)\times(n+1)$ multiplicatively anti-symmetric matrix $\mathbf{q}$, there exists a unique $(N+1)\times(N+1)$ multiplicatively anti-symmetric matrix $\mathbf{g} = \|g_{ij}\|$ such that the assignment*

$$y_0 \mapsto w_0, y_1 \mapsto w_1, \dots, y_N \mapsto w_N$$

extends to an algebra homomorphism

$$v_{n,d} : \mathcal{A}^N_{\mathbf{g}} \to \mathcal{A}^n_{\mathbf{q}}.$$

The entries of $\mathbf{g}$ are given explicitly in terms of the data $D((\mathcal{A}^n_{\mathbf{q}})^{(d)})$ of the d-Veronese $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$, see (4.6). The image of the map $v_{n,d}$ is the d-Veronese subalgebra $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$.

We call $v_{n,d}$ *the (n,d)-Veronese map.*

Proof. Suppose $\mathbf{q}$ is an $(n+1)\times(n+1)$ multiplicatively anti-symmetric matrix, and let $\mathcal{A}^n_{\mathbf{q}}$ be the corresponding quantum space. Assume that there exists an $(N+1)\times(N+1)$ multiplicatively anti-symmetric matrix $\mathbf{g}$ such that the map $v_{n,d}$ is a homomorphism of $\mathbb{C}$-algebras. Then

$$w_j w_i = v_{n,d}(y_j y_i) = v_{n,d}(g_{ji} y_i y_j) = g_{ji} w_i w_j,$$

for every $0 \le i \le j \le N$. By Theorem 4.5,

$$w_j w_i = \varphi_{ji} T_\beta \text{ and } w_i w_j = \varphi_{ij} T_\beta,$$

for every $0 \le i < j \le N$, where $T_\beta \in (\mathcal{T}^n)_{2d}$ is the unique ordered monomial of multi-degree $\beta = \deg(w_j) + \deg(w_i)$. In the particular cases when $(i,j) \in \mathrm{C}(n,2,d)$, one has $w_i w_j = T_\beta$, so $\varphi_{ij} = 1$. The nonzero coefficients φ_{ji} and φ_{ij} are uniquely determined by the matrix $\mathbf{q}$, see (4.6). It follows that the equalities

$$\varphi_{ji} T_\beta = w_j w_i = g_{ji} w_i w_j = g_{ji}\varphi_{ij} T_\beta$$

hold in $\mathcal{A}^n_{\mathbf{q}}$, so $(g_{ji}\varphi_{ij} - \varphi_{ji})T_\beta = 0$. But T_β is in the $\mathbb{C}$-basis of $\mathcal{A}^n_{\mathbf{q}}$, and therefore

$$g_{ji} = \frac{\varphi_{ji}}{\varphi_{ij}} \in \mathbb{C}^\times, \tag{5.2}$$

for all $0 \le i \le j \le N$, which agrees with (4.6). This determines a unique multiplicatively anti-symmetric matrix $\mathbf{g}$ with the required properties, and therefore the quantum space $\mathcal{A}^N_{\mathbf{g}}$ is also uniquely determined. The image of $v_{n,d}$ is the subalgebra of $\mathcal{A}^n_{\mathbf{q}}$ generated by the ordered monomials $\mathcal{T}_d$, which by Theorem 4.5 is exactly the d-Veronese $(\mathcal{A}^n_{\mathbf{q}})^{(d)}$.

Conversely, if $\mathbf{g} = \|g_{ij}\|$ is an $(N+1)\times(N+1)$ matrix whose entries satisfy (5.2) then $\mathbf{g}$ is a multiplicatively anti-symmetric matrix which determines a quantum space $\mathcal{A}^N_{\mathbf{g}}$ and the Veronese map $v_{n,d} : \mathcal{A}^N_{\mathbf{g}} \to \mathcal{A}^n_{\mathbf{q}}$, $y_i \mapsto w_i, 0 \le i \le N$, is well-defined. □

We fix an $(n+1)\times(n+1)$ multiplicatively anti-symmetric matrix $\mathbf{q}$ defining the quantum space $\mathcal{A}^n_{\mathbf{q}}$. Let $\mathcal{A}^N_{\mathbf{g}}$ be the quantum space defined via the $(N+1)\times(N+1)$ matrix $\mathbf{g}$ from Lemma 5.1. To simplify notation, as in the previous subsection, we shall write $\mathcal{A} = \mathcal{A}^n_{\mathbf{q}}$. We know that there is a standard finite presentation $\mathcal{A}^N_{\mathbf{g}} = \mathbb{C}\langle Y_N\rangle/(\mathcal{R}_{\mathbf{g}})$, where

$$\mathcal{R}_{\mathbf{g}} := \{y_j y_i - g_{ji} y_i y_j \mid 0 \le i < j \le N\} \tag{5.3}$$

is the reduced Gröbner basis of the ideal $J = (\mathcal{R}_{\mathbf{g}}) = \ker\rho$, where ρ is the canonical projection

$$\rho : \mathbb{C}\langle Y_N\rangle \to \mathbb{C}\langle Y_N\rangle/(\mathcal{R}_{\mathbf{g}}) = \mathcal{A}^N_{\mathbf{g}}. \tag{5.4}$$

We can lift the Veronese map $v_{n,d} : \mathcal{A}^N_{\mathbf{g}} \to \mathcal{A}$ to a uniquely determined homomorphism $V : \mathbb{C}\langle Y_N\rangle \to \mathcal{A}^{(d)}$ extending the assignment

$$y_0 \mapsto w_0, y_1 \mapsto w_1, \dots, y_N \mapsto w_N.$$

It is clear that the map V is surjective, since the restriction $V_{|Y_N} : Y_N \to (\mathcal{T}^n)_d$ is bijective, and the set of ordered monomials $(\mathcal{T}^n)_d$ generates $\mathcal{A}^{(d)}$.

Let $K := \ker V \subset \mathbb{C}\langle Y_N\rangle$. We want to find the reduced Gröbner basis $\mathcal{R}_0$ of the ideal K with respect to the degree-lexicographic order $\prec$ on $\langle Y_N\rangle$, where $y_0 \prec \cdots \prec y_n$.

Heuristically, we use the explicit information on the d-Veronese subalgebra $\mathcal{A}^{(d)}$ given in terms of generators and relations in Theorem 4.5, (4.6), and (4.7). In each of these relations we replace w_i with y_i, $0 \le i \le N$, preserving the remaining data (the coefficients and the sets of indices), and obtain a polynomial in $\mathbb{C}\langle Y_N\rangle$. This yields two disjoint sets of linearly independent quadratic binomials $\Re_1$ and $\Re_2$ in $\mathbb{C}\langle Y_N\rangle$:

(1) the set $\Re_1$, corresponding to the set $\mathcal{R}_1$ defined in (4.6), consists of $\binom{N+1}{2}$ quadratic relations:

$$\Re_1 = \{F_{ji} = y_j y_i - \varphi_{ji} y_{i'} y_{j'} \mid 0 \le i < j \le N, i' < j', (i',j') \in \mathrm{C}(n,2,d), y_j y_i \succ y_{i'} y_{j'}, \varphi_{ji} \in \mathbb{C}^\times\}; \tag{5.5}$$

(2) the set $\Re_2$, corresponding to the set $\mathcal{R}_2$ defined in (4.7), has exactly $\binom{N+2}{2} - \binom{n+2d}{n}$ relations:

$$\Re_2 = \{F_{ij} = y_i y_j - \varphi_{ij} y_{i'} y_{j'} \mid (i,j) \in \mathrm{MV}(n,d), i' < j', (i',j') \in \mathrm{C}(n,2,d), y_i y_j \succ y_{i'} y_{j'}, \varphi_{ij} \in \mathbb{C}^\times\}. \tag{5.6}$$

There is one more set which is contained in K: the set $\mathcal{R}_{\mathbf{g}}$ of defining relations for $\mathcal{A}_{\mathbf{g}}^N$. Note that $\mathcal{R}_{\mathbf{g}}$ corresponds exactly to $\mathcal{R}_1'$ from (4.8). We set $\Re = \Re_1 \cup \Re_2$ and $\Re' = \mathcal{R}_{\mathbf{g}} \cup \Re_2$. It is not difficult to see that there are equalities of ideals in $\mathbb{C}\langle Y_N\rangle$:

$$(\Re) = (\Re_1, \Re_2) = (\Re') = (\mathcal{R}_{\mathbf{g}}, \Re_2)$$

and that the set of relations $\Re$ and $\Re'$ are equivalent.

It is clear that the set $\Re = \Re_1 \cup \Re_2$ of quadratic polynomials in $\mathbb{C}\langle Y_N\rangle$ and the set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ of relations of the d-Veronese subalgebra $\mathcal{A}^{(d)}$ from Theorem 4.5 have the same cardinality. In fact

$$|\Re'| = |\Re| = |\mathcal{R}| = (N+1)^2 - \binom{n+2d}{n}, \tag{5.7}$$

as computed in (4.9). We shall prove that the set $\Re = \Re_1 \cup \Re_2$ is the reduced Gröbner basis of K, while $\Re'$ is a minimal Gröbner basis of K.

Theorem 5.2. *With notation as above, let $V : \mathbb{C}\langle Y_N\rangle \to \mathcal{A}^{(d)}$ be the algebra homomorphism extending the assignment*

$$y_0 \mapsto w_0, y_1 \mapsto w_1, \dots, y_N \mapsto w_N,$$

let K be the kernel of V. Let $\Re = \Re_1 \cup \Re_2$ be the set of quadratic polynomials given in (5.5) and (5.6), and let $\Re' = \mathcal{R}_{\mathbf{g}} \cup \Re_2$, where $\mathcal{R}_{\mathbf{g}}$ is given in (5.3). Then

(1) $\Re$ is the reduced Gröbner basis of the ideal K.

(2) $\Re'$ is a minimal Gröbner basis of the ideal K.

Proof. We start with a general observation. The quantum space $\mathcal{A} = \mathcal{A}_{\mathbf{q}}^n$ is a quadratic algebra, therefore its d-Veronese $\mathcal{A}^{(d)} \cong \mathbb{C}\langle Y_N\rangle/K$ is also quadratic, see Remark 4.2. Hence K is generated by quadratic polynomials and it is graded by length.

Remark 5.3. It is clear that the sets of leading monomials and the sets of normal monomials satisfy the following equalities in $\langle Y_N\rangle$:

$$\begin{aligned}
&\mathbf{LM}(\mathcal{R}_{\mathbf{g}}) = \mathbf{LM}(\Re_1) = \{y_j y_i \mid 0 \le i < j \le N\}\\
&\mathbf{LM}(\Re_2) = \{y_i y_j \mid (i,j) \in \mathrm{MV}(n,d)\}\\
&\mathbf{LM}(\Re) = \mathbf{LM}(\Re_1) \cup \mathbf{LM}(\Re_2) = \mathbf{LM}(\Re')\\
&N(\Re) = N(\Re').
\end{aligned} \tag{5.8}$$

Therefore $\Re'$ is a minimal Gröbner basis of the ideal K if and only if $\Re$ is a reduced Gröbner basis of K.

By Theorem 4.5, the quadratic polynomials $F_{ji}(Y_n)$ in (5.5) and $F_{ij}(Y_n)$ in (5.6) satisfy

$$V(F_{ji}(y_0, \dots, y_N)) = f_{ji}(w_0, \dots, w_N) = 0, \text{ for every } 0 \le i < j \le N$$

and

$$V(F_{ij}(y_0, \dots, y_N)) = f_{ij}(w_0, \dots, w_N) = 0, \text{ for every } (i,j) \in \mathrm{MV}(n,d).$$

Thus $\Re \subset K$ and, in a similar way, $\Re' \subset K$. We shall show that $\Re$ is a reduced Gröbner basis of K.

As usual, $N(K) \subset \mathbb{C}\langle Y_N\rangle$ denotes the set of normal monomials modulo K, and $N(\Re) \subset \mathbb{C}\langle Y_N\rangle$ denotes the set of normal words modulo $\Re$. In general,

$$N(K) \subseteq N(\Re),$$

and by Corollary 2.6 equality holds if and only if $\Re$ is a Gröbner basis of K. Recall from Subsection 2.3 that there are isomorphisms of vector spaces

$$\mathbb{C}\langle Y_N\rangle = K \oplus \mathbb{C}N(K), \quad \text{and} \quad \mathbb{C}N(K) \cong \mathbb{C}\langle Y_N\rangle/K \cong \mathcal{A}^{(d)}.$$

The ideal K is graded by length, i.e. $K = \bigoplus_{j\ge 0} K_j$, with $K_0 = K_1 = 0$.

For $j \ge 0$, let $N(K)_j$ be the set of normal words of length j, with the convention that $N(K)_0 = \{1\}$, $N(K)_1 = Y_n$. As vector spaces,

$$(\mathbb{C}\langle Y_N\rangle)_j = K_j \oplus \mathbb{C}N(K)_j, \quad \text{and} \quad \mathbb{C}N(K)_j \cong \mathcal{A}_j^{(d)} = \mathcal{A}_{jd}, \text{ for every } j \ge 2.$$

In particular, $(\mathbb{C}\langle Y_N\rangle)_2 = K_2 \oplus \mathbb{C}N(K)_2$, so

$$\dim(\mathbb{C}\langle Y_N\rangle)_2 = \dim K_2 + \dim(\mathbb{C}N(K)_2) = \dim K_2 + \dim \mathcal{A}_{2d}.$$

We know that
$$\dim \mathcal{A}_{2d} = \binom{n+2d}{n} \quad \text{and} \quad \dim(\mathbb{C}\langle Y_N\rangle)_2 = |(Y_n)^2| = (N+1)^2,$$
where Y_n^2 is the set of all words of length two in $\langle Y_N\rangle$. This, together with (5.7), implies
$$\dim K_2 = (N+1)^2 - \binom{n+2d}{n} = |\Re|.$$
Clearly, the set $\Re$ consists of linearly independent polynomials, therefore $\dim K_2 = \dim \mathbb{C}\Re = |\Re|$. It follows that $\mathbb{C}\Re = K_2$, and since K is generated by quadratic polynomials, one has $K = (\Re)$.

We shall use the following remark.

Remark 5.4. The following are equivalent:

(1) $y_iy_jy_k \in N(\Re)_3$;
(2) $y_iy_j \in N(\Re)_2$ and $y_jy_k \in N(\Re)_2$;
(3) $(i,j,k) \in \mathrm{C}(n,3,d)$.

Moreover, there are equalities
$$|N(\Re)_3| = |\mathrm{C}(n,3,d)| = \binom{n+3d}{n}. \tag{5.9}$$

We know that $\mathcal{A}_3^{(d)} = \mathcal{A}_{3d}$, so $\dim \mathcal{A}_3^{(d)} = \dim \mathcal{A}_{3d} = \binom{n+3d}{n}$, which together with (5.9) imply
$$|N(\Re)_3| = \dim \mathcal{A}_{3d}.$$
It follows from Lemma 2.7 that the set $\Re$ is a Gröbner basis of the ideal K. The set of leading monomials $\mathbf{LM}(\Re)$ is an antichain of monomials, hence $\Re$ is a minimal Gröbner basis. For $j > i$, every $F_{ji} \in \Re$ defined in (5.5) is in normal form modulo $\Re \setminus \{F_{ji}\}$. Similarly, for $(i,j) \in \mathrm{MV}(n,d)$, every $F_{ij} \in \Re$ defined in (5.6) is in normal form modulo $\Re \setminus \{F_{ij}\}$. We have proven that $\Re$ is a reduced Gröbner basis of the ideal K.

It follows from Remark 5.3 that $\Re'$ is a minimal Gröbner basis of K. □

5.2. **The Veronese map $v_{n,d}$ and the reduced Gröbner basis of its kernel.**

Theorem 5.5. *Let $n, d \in \mathbb{N}$ and $N = \binom{n+d}{d} - 1$. Let $\mathcal{A}_{\mathbf{q}}^n$ be a quantum space defined by an $(n+1) \times (n+1)$ deformation matrix $\mathbf{q}$ and let $\mathcal{A}_{\mathbf{g}}^N$ be the quantum space whose multiplicatively anti-symmetric $(N+1) \times (N+1)$ matrix $\mathbf{g}$ is determined by Lemma 5.1. Let*
$$v_{n,d} : \mathcal{A}_{\mathbf{g}}^N \to \mathcal{A}_{\mathbf{q}}^n$$
be the Veronese map extending the assignment
$$y_0 \mapsto w_0,\ y_1 \mapsto w_1,\ \ldots,\ y_N \mapsto w_N.$$

(1) The image of $v_{n,d}$ is the d-Veronese subalgebra $(\mathcal{A}_{\mathbf{q}}^n)^{(d)}$ of $\mathcal{A}_{\mathbf{q}}^n$.
(2) The kernel $\mathfrak{K} := \ker(v_{n,d})$ of the Veronese map has a reduced Gröbner basis consisting of exactly $\binom{N+2}{2} - \binom{n+2d}{n}$ binomials:
$$\mathcal{R}_{\mathbf{q}}^{\mathrm{v}} := \{y_iy_j - \varphi_{ij}y_{i'}y_{j'} \mid (i,j) \in \mathrm{MV}(n,d), (i',j') \in \mathrm{C}(n,2,d), \varphi_{ij} \in \mathbb{C}^\times\}, \tag{5.10}$$
where $\mathrm{Nor}(v_{n,d}(y_iy_j)) = \varphi_{ij}v_{n,d}(y_{i'}y_{j'})$, $y_iy_j \succ y_{i'}y_{j'}$, and $\varphi_{ij} \in \mathbb{C}^\times$ are invariants of $(\mathcal{A}_{\mathbf{q}}^n)^{(d)}$ given in Notation 4.6.

Proof. Part (1) follows from Lemma 5.1. For part (2), we first prove that the set $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ generates $\mathfrak{K}$. The proof is similar to the argument describing the kernel $K = \ker V$ in Theorem 5.2.

Note that $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}} \subset \mathfrak{K}$. Indeed, by direct computation, one shows that $v_{n,d}(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}) = \mathcal{R}_2$, the set of relations of the d-Veronese $(\mathcal{A}_{\mathbf{q}}^n)^{(d)}$ given in (4.7), so $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}} \subset \mathfrak{K}$. Moreover, it follows from (5.10) that for each pair $(i,j) \in \mathrm{MV}(n,d)$ the set $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ contains exactly one element, namely $y_iy_j - \varphi_{ij}y_{i'}y_{j'}$, where $\mathrm{Nor}(v_{n,d}(y_iy_j)) = \varphi_{ij}v_{n,d}(y_{i'}y_{j'})$. Here we consider the normal form $\mathrm{Nor}(v_{n,d}(y_iy_j)) = \mathrm{Nor}(w_iw_j) = \varphi_{ij}w_{i'}w_{j'}$, see Theorem 4.5(2). Hence
$$|\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}| = |\mathrm{MV}(n,d)| = \binom{N+2}{2} - \binom{n+2d}{n}, \tag{5.11}$$

where the last equality follows from Lemma 4.4. By Convention 3.9, we identify $\mathcal{A}_{\mathbf{g}}^N \cong (\mathbb{C}\mathcal{T}(Y_N), \bullet)$. Our goal is to show that the two set of normal words $N(\mathfrak{K})$ and $N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})$ coincide, where

$$N(\mathfrak{K}) = N_{\prec_0}(\mathfrak{K}) \subset \mathbb{C}\mathcal{T}(Y_N), \quad \text{and} \quad N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}) = N_{\prec_0}(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}) \subset \mathbb{C}\mathcal{T}(Y_N),$$

as in Definition 3.11. There are obvious isomorphisms of vector spaces

$$\mathcal{A}_{\mathbf{g}}^N = \mathbb{C}\mathcal{T}(Y_N) = \mathfrak{K} \oplus \mathbb{C}N(\mathfrak{K}).$$

For simplicity of notation, we set $B = \mathcal{A}_{\mathbf{g}}^N/\mathfrak{K}$ and consider the canonical grading of B induced by the grading of $\mathcal{A}_{\mathbf{g}}^N$. Then

$$B = \mathcal{A}_{\mathbf{g}}^N / \ker(v_{n,d}) \cong \mathrm{im}(v_{n,d}) = (\mathcal{A}_{\mathbf{q}}^n)^{(d)},$$

so there are equalities

$$(\mathcal{A}_{\mathbf{g}}^N)_m = (\mathbb{C}\mathcal{T}(Y_N))_m = (\mathfrak{K})_m \oplus (\mathbb{C}N(\mathfrak{K}))_m \text{ and } B_m \cong (\mathcal{A}_{\mathbf{q}}^n)_m^{(d)} = (\mathcal{A}_{\mathbf{q}}^n)_{md}, \tag{5.12}$$

for every $m \geq 2$. In particular, for $m = 2$ one has $B_2 \cong (\mathcal{A}_{\mathbf{q}}^n)_2^{(d)} = (\mathcal{A}_{\mathbf{q}}^n)_{2d}$ and

$$\dim(\mathcal{A}_{\mathbf{g}}^N)_2 = \dim(\mathfrak{K})_2 + \dim(\mathcal{A}_{\mathbf{q}}^n)_{2d}, \text{ hence } \binom{N+2}{2} = \dim(\mathfrak{K})_2 + \binom{n+2d}{2},$$

which implies

$$\dim(\mathfrak{K})_2 = \binom{N+2}{2} - \binom{n+2d}{2} = |\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}|.$$

It is clear that the set $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ is linearly independent, so it is a basis of the graded component $\mathfrak{K}_2$, and $(\mathfrak{K})_2 = \mathbb{C}\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$. But the ideal $\mathfrak{K}$ is generated by homogeneous polynomials of degree 2, therefore

$$\mathfrak{K} = \mathfrak{K}_2 = (\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}), \tag{5.13}$$

so $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ generates the kernel $\mathfrak{K}$.

We are now ready to prove that $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ is a Gröbner basis of $\mathfrak{K}$. We shall provide two proofs.

First proof. Here we use an analogue of Remark 5.4 in the settings of a quantum space.

Remark 5.6. The following are equivalent:

(1) $y_i y_j y_k \in N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})_3$;
(2) $y_i y_j \in N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})_2$ and $y_j y_k \in N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})_2$;
(3) $(i, j, k) \in \mathrm{C}(n, 3, d)$.

Moreover there are equalities

$$|N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})_3| = |\mathrm{C}(n, 3, d)| = \binom{n+3d}{n}. \tag{5.14}$$

By (5.12), $\dim B_3 = \dim \mathcal{A}_{3d} = \binom{n+3d}{n}$, which together with (5.14) implies

$$|N(\mathcal{R}_{\mathbf{q}}^{\mathrm{v}})_3| = \dim B_3.$$

Now Lemma 3.14 implies that $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ is a Gröbner basis of the ideal $\mathfrak{K} = \ker(v_{n,d})$. It is clear that $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ is the reduced Gröbner basis of $\mathfrak{K}$.

Second proof. We shall use Theorem 5.2 and ideas from [31]. By (5.13), we know that the set $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ generates $\mathfrak{K}$. Consider now the ideal $\mathrm{Nor}^{-1}(\mathfrak{K})$ in $\mathbb{C}\langle Y_N \rangle$. It is easy to see that

$$\mathrm{Nor}^{-1}(\mathfrak{K}) = \mathfrak{J}_{\mathbf{g}} + (\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}) = (\mathcal{R}_{\mathbf{g}}) + (\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}) = K,$$

where $K = \ker V$ is the kernel of the epimorphism $V : \mathbb{C}\langle Y_N \rangle \to \mathcal{A}^{(d)}$ from Theorem 5.2. Indeed, the polynomials in $\mathcal{R}_{\mathbf{g}}$ and $\mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$, considered as elements of the free associative algebra $\mathbb{C}\langle Y_N \rangle$, satisfy

$$\mathcal{R}_{\mathbf{g}} = \Re_1' \text{ and } \mathcal{R}_{\mathbf{q}}^{\mathrm{v}} = \Re_2,$$

where $\Re_1'$ and $\Re_2$ are the relations given in Theorem 5.2, see (4.8) and (5.6). Hence by the same theorem, the set $\Re' = \mathcal{R}_{\mathbf{g}} \cup \mathcal{R}_{\mathbf{q}}^{\mathrm{v}}$ is a minimal Gröbner basis of the ideal K. Theorem 5.2 also implies that the disjoint union of quadratic relations $\Re = \Re_1 \cup \Re_2$ is the reduced Gröbner basis of K in $\mathbb{C}\langle Y_N \rangle$. It follows from [31, Proposition 9.3(3)] that the intersection

$$G = \Re \cap \mathbb{C}N(\mathfrak{J}_{\mathbf{g}}) = \Re \cap \mathbb{C}N(\mathcal{R}_{\mathbf{g}})$$

is the reduced Gröbner basis of the ideal $\mathfrak{K} = \ker(v_{n,d})$. Moreover, we have $N(\mathfrak{J}_{\mathbf{g}}) = \mathbb{C}\mathcal{T}(Y_N)$. Then the obvious equalities

$$G = \Re \cap \mathbb{C}N(\mathfrak{J}_{\mathbf{g}}) = (\Re_1 \cup \Re_2) \cap \mathbb{C}\mathcal{T}(Y_N) = \Re_2 = \mathcal{R}^{\mathrm{v}}_{\mathbf{q}}$$

imply that $\mathcal{R}^{\mathrm{v}}_{\mathbf{q}}$ is the reduced Gröbner basis of $\mathfrak{K}$. □

We remark that [31, Proposition 9.3(4)] implies that the set $\mathcal{R}_{\mathbf{g}} \cup G = \Re_1' \cup \Re_2$ is the reduced Gröbner basis of the ideal K. This fact agrees with Part (3) of our Theorem 5.2, proven independently.

Corollary 5.7. *The set of leading monomials for the Gröbner basis $\mathcal{R}^{\mathrm{v}}_{\mathbf{q}}$ does not depend on the deformation matrix $\mathbf{q}$ and equals*

$$\mathbf{LM}(\mathcal{R}^{\mathrm{v}}_{\mathbf{q}}) = \{y_i y_j \mid (i,j) \in \mathrm{MV}(n,d)\}.$$

6. Segre products and Segre maps

In this section we introduce and investigate non-commutative analogues of the Segre embedding $S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$. The main result of the section is Theorem 6.10, which describes explicitly the reduced Gröbner basis for the kernel of the non-commutative Segre map. We first recall the notion of Segre product of graded algebras, following [33, Section 3.2].

Definition 6.1. Let

$$R = \bigoplus_{k\in\mathbb{N}_0} R_k \text{ and } S = \bigoplus_{k\in\mathbb{N}_0} S_k$$

be graded algebras. The *Segre product* of R and S is the graded algebra

$$R \circ S := \bigoplus_{k\in\mathbb{N}_0} R_k \otimes S_k.$$

Clearly, the Segre product $R \circ S$ is a subalgebra of the tensor product algebra $R \otimes S$. Note that the embedding is not a graded algebra morphism, as it doubles grading. The Hilbert function of $R \circ S$ satisfies

$$h_{R\circ S}(t) = \dim(R \circ S)_t = \dim(R_t \otimes S_t) = \dim(R_t) \cdot \dim(S_t) = h_R(t) \cdot h_S(t).$$

Given $n, m \in \mathbb{N}$, let

$$N := (n+1)(m+1) - 1.$$

Let $\mathbf{q}$ and $\mathbf{q}'$ be two multiplicatively anti-symmetric matrices of sizes $(n+1)\times(n+1)$ and $(m+1)\times(m+1)$, respectively, and let $\mathcal{A}^n_{\mathbf{q}}$ and $\mathcal{A}^m_{\mathbf{q}'}$ be the corresponding quantum spaces. We shall construct a quantum space $\mathcal{A}^N_{\mathbf{g}}$ defined via $N+1$ (double indexed) generators

$$Z_{nm} = \{z_{i\alpha} \mid i \in \{0,\dots,n\}, \alpha \in \{0,\dots,m\}\}$$

and an $(N+1)\times(N+1)$ multiplicatively anti-symmetric matrix $\mathbf{g}$ uniquely determined by $\mathbf{q}$ and $\mathbf{q}'$.

Convention 6.2. We order the set Z_{nm} using the lexicographic ordering on the pairs of indices (i,α), $0 \le i \le n$, $0 \le \alpha \le m$, that is, $z_{i\alpha} \prec z_{j\beta}$ if and only if either (a) $i < j$, or (b) $i = j$, and $\alpha < \beta$. Thus

$$Z_{nm} = \{z_{00} \prec z_{01} \prec \cdots \prec z_{0m} \prec z_{10} \prec \cdots \prec z_{nm-1} \prec z_{nm}\}. \tag{6.1}$$

When no confusion arises, we write Z for Z_{nm}. As usual, we consider the free associative algebra $\mathbb{C}\langle Z\rangle$ and fix the degree-lexicographic ordering $\prec$ induced by (6.1) on the free monoid $\langle Z\rangle$.

In this section, we shall work simultaneously with three disjoint sets of variables, $X = X_n$, $Y = Y_m$, and $Z = Z_{nm}$. We shall use notation $\mathcal{T}(X) = \mathcal{T}^n$, $\mathcal{T}(Y) = \mathcal{T}^m$ and $\mathcal{T}(Z)$ for the corresponding sets of ordered terms in variables X, respectively Y, respectively Z. In particular, the set $\mathcal{T}(Z)$ of ordered monomials in Z with respect to the ordering (6.1) is

$$\mathcal{T}(Z) = \{z_{00}^{k_{00}} z_{01}^{k_{01}} \dots z_{10}^{k_{10}} \dots z_{nm}^{k_{nm}} \mid k_{i\alpha} \in \mathbb{N}_0,\ i \in \{0,\dots,n\},\ \alpha \in \{0,\dots,m\}\}.$$

As in Convention 3.9, we identify $\mathcal{A}^n_{\mathbf{q}}$ with $(\mathbb{C}\mathcal{T}(X), \bullet)$ and $\mathcal{A}^m_{\mathbf{q}'}$ with $(\mathbb{C}\mathcal{T}(Y), \bullet)$.

Remark 6.3. Consider the free associative algebra $\mathbb{C}\langle X;Y\rangle = \mathbb{C}\langle x_0,\dots,x_n,y_0,\dots y_m\rangle$, generated by the disjoint union $X_n \sqcup Y_m$, and the free monoid $\langle X;Y\rangle = \langle x_0,\dots,x_n,y_0,\dots y_m\rangle$ with the canonical degree-lexicographic ordering $\prec$ extending $x_0 \prec x_1 \prec \dots x_n \prec y_0 \prec y_1 \prec \dots y_m$. Let

$$\mathcal{R}_0 = \mathcal{R}(\mathcal{A}^n_{\mathbf{q}} \otimes \mathcal{A}^m_{\mathbf{q}'}) = \mathcal{R}_{\mathbf{q}} \cup \mathcal{R}_{\mathbf{q}'} \cup \{y_\alpha x_i - x_i y_\alpha \mid i \in \{0,\dots,n\},\ \alpha \in \{0,\dots,m\}\}.$$

Then $\mathcal{R}_0$ is the reduced Gröbner basis of the two-sided ideal $(\mathcal{R}_0)$ of $\mathbb{C}\langle X;Y\rangle$ and there is an isomorphism of algebras

$$\mathbb{C}\langle X;Y\rangle/(\mathcal{R}_0) \cong \mathcal{A}^n_{\mathbf{q}} \otimes \mathcal{A}^m_{\mathbf{q}'}.$$

Proposition 6.4. *In notation as above, let $\mathcal{A}^n_{\mathbf{q}}$, and $\mathcal{A}^m_{\mathbf{q}'}$ be quantum spaces and let $N := (n+1)(m+1)-1$. Then there exists a unique $(N+1)\times(N+1)$ multiplicatively anti-symmetric matrix $\mathbf{g} = \|g_{i\alpha,j\beta}\|$ such that the assignment*

$$z_{i\alpha} \mapsto x_i \otimes y_\alpha, \quad \textit{for every } i \in \{0,\dots,n\} \textit{ and every } \alpha \in \{0,\dots,m\},$$

extends to a well-defined $\mathbb{C}$-algebra homomorphism

$$s_{n,m} : \mathcal{A}^N_{\mathbf{g}} \to \mathcal{A}^n_{\mathbf{q}} \otimes \mathcal{A}^m_{\mathbf{q}'}. \tag{6.2}$$

Moreover, the following conditions hold

(1) The quantum space $\mathcal{A}^N_{\mathbf{g}}$ is presented as

$$\mathcal{A}^N_{\mathbf{g}} = \mathbb{C}\langle Z\rangle/(\mathcal{R}_{\mathbf{g}}),$$

where

$$\mathcal{R}_{\mathbf{g}} := \{z_{j\beta}z_{i\alpha} - (g_{j\beta,i\alpha})z_{i\alpha}z_{j\beta} \mid z_{j\beta} \succ z_{i\alpha}, z_{j\beta}, z_{i\alpha} \in Z\} \tag{6.3}$$

is a reduced Gröbner basis for the two-sided ideal $(\mathcal{R}_{\mathbf{g}})$ in $\mathbb{C}\langle Z\rangle$.

(2) There is an isomorphism of algebras $\mathcal{A}^N_{\mathbf{g}} \cong (\mathbb{C}\mathcal{T}(Z), \bullet)$, where the multiplication $\bullet$ is defined as $u \bullet v := \mathrm{Nor}_{\mathcal{R}_{\mathbf{g}}}(uv)$.

(3) The image $s_{n,m}(\mathcal{A}^N_{\mathbf{g}})$ is the Segre subalgebra $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$ of $\mathcal{A}^n_{\mathbf{q}} \otimes \mathcal{A}^m_{\mathbf{q}'}$.

We call the homomorphism $s_{n,m}$ the (n,m)-*Segre map*.

Proof. Assume that there exists an $(N+1)\times(N+1)$ multiplicatively anti-symmetric matrix $\mathbf{g}$ such that $s_{n,m}$ is a homomorphism of $\mathbb{C}$-algebras. Let $Z = Z_{nm}$ as above be the set of generators of $\mathcal{A}^N_{\mathbf{g}}$. We compute $s_{n,m}(z_{i\alpha}z_{j\beta})$ in two different ways:

$$\begin{aligned}
s_{n,m}(z_{i\alpha}z_{j\beta}) &= s_{n,m}(z_{i\alpha})s_{n,m}(z_{j\beta}) \\
&= (x_i \otimes y_\alpha)(x_j \otimes y_\beta) = (x_i x_j) \otimes (y_\alpha y_\beta) \\
s_{n,m}(z_{i\alpha}z_{j\beta}) &= s_{n,m}(g_{i\alpha,j\beta}(z_{j\beta}z_{i\alpha})) = g_{i\alpha,j\beta}s_{n,m}(z_{j\beta}z_{i\alpha}) \\
&= g_{i\alpha,j\beta}s_{n,m}(z_{j\beta})s_{n,m}(z_{i\alpha}) = g_{i\alpha,j\beta}(x_j x_i \otimes y_\beta y_\alpha) \\
&= g_{i\alpha,j\beta}q_{ji}q'_{\beta\alpha}(x_i x_j) \otimes (y_\alpha y_\beta).
\end{aligned}$$

Therefore,

$$(x_i x_j) \otimes (y_\alpha y_\beta) = (g_{i\alpha,j\beta}q_{ji}q'_{\beta\alpha})(x_i x_j) \otimes (y_\alpha y_\beta)$$

for every $i,j \in \{0,\dots,n\}$ and every $\alpha,\beta \in \{0,\dots,m\}$. It follows that $\mathbf{g} = \|g_{i\alpha,j\beta}\|$ is a multiplicatively anti-symmetric matrix uniquely determined by the equalities

$$g_{i\alpha,j\beta} = (q_{ji}q'_{\beta\alpha})^{-1} = q_{ij}q'_{\alpha\beta}, \tag{6.4}$$

We remark that the matrix $\mathbf{g}$ is equal to the the Kronecker product $\mathbf{q} \otimes \mathbf{q}'$ of the matrices $\mathbf{q}$ and $\mathbf{q}'$.

Conversely, if $\mathbf{g}$ is the multiplicatively anti-symmetric matrix defined via (6.4), then the Segre map (6.2) is a well-defined algebra homomorphism. Conditions (1) and (2) follow straightforwardly from the discussion in Section 3, see Remarks 3.4 and Convention 3.9. The Segre subalgebra $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$ is generated by the elements $x_i \otimes y_\alpha$ for $i \in \{0,\dots,n\}$ and $\alpha \in \{0,\dots,m\}$. By construction $s_{n,m}(z_{i\alpha}) = x_i \otimes y_\alpha$, hence the image $s_{n,m}(\mathcal{A}^N_{\mathbf{g}})$ is the Segre subalgebra $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$, which proves (3). ☐

As usual, we identify the quantum space $\mathcal{A}^N_{\mathbf{g}}$ with $(\mathbb{C}\mathcal{T}(Z), \bullet)$, see Convention 3.9.

Remark 6.5. Being a Segre product, the algebra $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'} = s_{n,m}(\mathcal{A}^N_{\mathbf{g}})$ inherits various properties from the two algebras $\mathcal{A}^n_{\mathbf{q}}$ and $\mathcal{A}^m_{\mathbf{q}'}$. In particular, since the latter are one-generated, quadratic, and Koszul, it follows from [33, Proposition 3.2.1] that the algebra $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$ is also one-generated, quadratic, and Koszul. Clearly, the set $\{x_i \otimes y_\alpha \mid i \in \{0, \dots, n\}, \alpha \in \{0, \dots, m\}\}$ of cardinality $N+1 = (n+1)(m+1)$ is a generating set of $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$.

Lemma 6.6. *The following equalities hold in the Segre product $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$, for all i, j, α, β, such that $0 \le i < j \le n$ and $0 \le \alpha < \beta \le m$:*

$$(x_i \circ y_\alpha)(x_j \circ y_\beta) = (x_i x_j) \circ (y_\alpha y_\beta). \tag{6.5}$$

$$\begin{aligned} (x_j \circ y_\beta)(x_i \circ y_\alpha) &= q_{ji} q'_{\beta\alpha} (x_i x_j) \circ (y_\alpha y_\beta) = q_{ji} q'_{\beta\alpha} (x_i \circ y_\alpha)(x_j \circ y_\beta). \\ (x_j \circ y_\alpha)(x_i \circ y_\beta) &= q_{ji} q'_{\alpha\beta} (x_i x_j) \circ (y_\beta y_\alpha) = q_{ji} q'_{\alpha\beta} (x_i \circ y_\beta)(x_j \circ y_\alpha). \end{aligned} \tag{6.6}$$

$$(x_i \circ y_\beta)(x_j \circ y_\alpha) = x_i x_j \circ y_\beta y_\alpha = q'_{\beta\alpha} (x_i x_j) \circ (y_\alpha y_\beta) = q'_{\beta\alpha} (x_i \circ y_\alpha)(x_j \circ y_\beta) \tag{6.7}$$

$$\begin{aligned} (x_j \circ y_\alpha)(x_i \circ y_\alpha) &= q_{ji} (x_i x_j) \circ (y_\alpha y_\alpha) = q_{ji} (x_i \circ y_\alpha)(x_j \circ y_\alpha) \\ (x_i \circ y_\beta)(x_i \circ y_\alpha) &= q'_{\beta\alpha} (x_i x_i) \circ (y_\alpha y_\beta) = q'_{\beta\alpha} (x_i \circ y_\alpha)(x_i \circ y_\beta) \end{aligned} \tag{6.8}$$

Remark 6.7. (1) The equalities given in Lemma 6.6 imply the following explicit list of defining relations $\mathcal{R}_{\mathbf{g}}$ for the quantum space $\mathcal{A}^N_{\mathbf{g}}$:

$$\begin{array}{lll} z_{j\beta} z_{i\alpha} - q_{ji} q'_{\beta\alpha} z_{i\alpha} z_{j\beta} & \in \mathcal{R}_{\mathbf{g}} & \text{by (6.6)} \\ z_{j\alpha} z_{i\beta} - q_{ji} q'_{\alpha\beta} z_{i\beta} z_{j\alpha} & \in \mathcal{R}_{\mathbf{g}} & \text{by (6.6)} \\ z_{j\alpha} z_{i\alpha} - q_{ji} z_{i\alpha} z_{j\alpha} & \in \mathcal{R}_{\mathbf{g}} & \text{by (6.8)} \\ z_{i\beta} z_{i\alpha} - q'_{\beta\alpha} z_{i\alpha} z_{i\beta} & \in \mathcal{R}_{\mathbf{g}} & \text{by (6.8)} \end{array}, \tag{6.9}$$

for every $0 \le i < j \le n$ and every $0 \le \alpha < \beta \le m$.

(2) The equalities (6.7) imply that the following quadratic binomials in $\mathcal{A}^N_{\mathbf{g}}$ are in the kernel of the Segre map:

$$z_{i\beta} z_{j\alpha} - q'_{\beta\alpha} z_{i\alpha} z_{j\beta} \in \ker s(n,m), \tag{6.10}$$

for every $0 \le i < j \le n$ and every $0 \le \alpha < \beta \le m$.

Notation 6.8. We denote by $\mathrm{MS}(n,m)$ the following collection of quadruples:

$$\mathrm{MS}(n,m) = \{(i,j,\beta,\alpha) \mid 0 \le i < j \le n, 0 \le \alpha < \beta \le m\}. \tag{6.11}$$

Lemma 6.9. *The cardinality of* $\mathrm{MS}(n,m)$ *is*

$$|\mathrm{MS}(n,m)| = \binom{n+1}{2}\binom{m+1}{2}. \tag{6.12}$$

Proof. Clearly, $|\{(i,j) \mid 0 \le i < j \le n\}| = \binom{n+1}{2}$. Moreover, for each fixed pair (i,j), $0 \le i < j \le n$, the number of quadruples $\{(i,j,\beta,\alpha) \mid 0 \le \alpha < \beta \le m\}$ is exactly $\binom{m+1}{2}$, which finishes the proof. □

We keep the notation and conventions of this section, in particular we identify the quantum space $\mathcal{A}^N_{\mathbf{g}}$ with $(\mathbb{C}\mathcal{T}(Z), \bullet)$. Recall that if $P \subset \mathcal{A}^N_{\mathbf{g}}$ is an arbitrary set, then $\mathbf{LM}(P) = \mathbf{LM}_{\prec_0}(P)$ denotes the set of leading monomials

$$\mathbf{LM}(P) = \{\mathbf{LM}_{\prec_0}(f) \mid f \in P\}.$$

A monomial $T \in \mathcal{T}(Z)$ *is normal modulo* P if it does not contain as a subword any $u \in \mathbf{LM}(P)$. The set of all normal mod P monomials in $\mathcal{T}(Z)$ is denoted by $N_{\prec_0}(P)$, so

$$N_{\prec_0}(P) = \{T \in \mathcal{T}(Z) \mid T \text{ is normal mod P}\}.$$

A criterion for a Gröbner basis F of an ideal $\mathfrak{K} = (F)$ in $\mathcal{A}^N_{\mathbf{g}}$ follows straightforwardly as an analogue of Lemma 3.13, in which we only replace Y_N with the set of generators Z, and keep the remaining notation and assumptions.

Theorem 6.10. *The set*

$$\mathcal{R}^{\mathrm{s}}_{\mathbf{q},\mathbf{q}'} := \{z_{i\beta}z_{j\alpha} - \mathbf{q}'_{\beta\alpha}z_{i\alpha}z_{j\beta} \mid 0 \le i < j \le n, 0 < \alpha < \beta \le m\} \subset \mathcal{A}^N_{\mathbf{g}}$$

consisting of $\binom{n+1}{2}\binom{m+1}{2}$ *quadratic binomials is a reduced Gröbner basis for the kernel of the Segre map*

$$s_{n,m} : \mathcal{A}^N_{\mathbf{g}} \to \mathcal{A}^n_{\mathbf{q}} \otimes \mathcal{A}^m_{\mathbf{q}'}.$$

Proof. It is clear that $\left|\mathcal{R}^{\mathrm{s}}_{\mathbf{q},\mathbf{q}'}\right| = |\mathrm{MS}(n,m)| = \binom{n+1}{2}\binom{m+1}{2}$. We set

$$\begin{aligned} \mathfrak{K} &= \ker s_{n,m}, \quad & N(\mathfrak{K}) &= N_{\prec_0}(\mathfrak{K}),\\ \mathcal{R} &= \mathcal{R}^{\mathrm{s}}_{\mathbf{q},\mathbf{q}'}, \quad & N(\mathcal{R}) &= N_{\prec_0}(\mathcal{R}). \end{aligned}$$

By Remark 6.7(2), $\mathcal{R} \subset \mathfrak{K}$. We claim that $\mathcal{R}$ generates $\mathfrak{K}$ as a two-sided ideal of $\mathcal{A}^N_{\mathbf{g}}$.

The image $s_{n,m}(\mathcal{A}^N_{\mathbf{g}})$ is the Segre product $\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}$, which is a quadratic algebra, see Remark 6.5. Therefore the kernel $\mathfrak{K}$ is generated by polynomials of degree two. Moreover, there is an isomorphism of vector spaces

$$\mathbb{C}N(\mathfrak{K}) \cong \mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}.$$

In particular,

$$\dim(\mathbb{C}N(\mathfrak{K}))_2 = \dim((\mathcal{A}^n_{\mathbf{q}})_2)\dim((\mathcal{A}^m_{\mathbf{q}'})_2) = \binom{n+2}{2}\binom{m+2}{2}. \tag{6.13}$$

It is clear that $(\mathcal{A}^N_{\mathbf{g}})_2 = (\mathbb{C}\mathcal{T}(Z))_2 = (\mathfrak{K})_2 \oplus (\mathbb{C}N(\mathfrak{K}))_2$, hence

$$\begin{aligned} \dim(\mathfrak{K})_2 &= \dim(\mathcal{A}^N_{\mathbf{g}})_2 - \dim(\mathbb{C}N(\mathfrak{K}))_2 = \binom{N+2}{2} - \binom{n+2}{2}\binom{m+2}{2}\\ &= \binom{(n+1)(m+1)+1}{2} - \binom{n+2}{2}\binom{m+2}{2} = \binom{n+1}{2}\binom{m+1}{2} = |\mathcal{R}|\,. \end{aligned}$$

Now the equality $|\mathcal{R}| = \dim(\mathfrak{K})_2$, together with the obvious linear independence of the elements of $\mathcal{R}$, imply that $\mathcal{R}$ is a $\mathbb{C}$-basis of $(\mathfrak{K})_2$, so it spans the space $(\mathfrak{K})_2$. But we know that the kernel $\mathfrak{K}$ is generated by polynomials of degree two, hence $\mathfrak{K} = (\mathcal{R})$.

Next we shall prove that $\mathcal{R}$ is a Gröbner basis of the ideal $\mathfrak{K}$. Let $B = \mathcal{A}^N/\mathfrak{K}$. Then

$$B = \mathcal{A}^N/\ker(s_{n,m}) \cong s_{n,m}(\mathcal{A}^N) = \mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'}.$$

Hence

$$\dim B_3 = \dim(\mathcal{A}^n_{\mathbf{q}} \circ \mathcal{A}^m_{\mathbf{q}'})_3 = \dim(\mathcal{A}^n_{\mathbf{q}})_3 \cdot \dim(\mathcal{A}^m_{\mathbf{q}'})_3 = \binom{n+3}{3}\binom{m+3}{3}. \tag{6.14}$$

We claim that $\dim B_3 = |(N(\mathcal{R}))_3|$. Indeed, by the identification $\mathcal{A}^N_{\mathbf{g}} \simeq (\mathbb{C}\mathcal{T}(Z), \bullet)$ we have

$$(\mathcal{A}^N_{\mathbf{g}})_3 = (\mathbb{C}\mathcal{T}(Z))_3 = \mathbb{C}\{z_{i\alpha}z_{j\beta}z_{k\gamma} \mid (i,\alpha) \le (j,\beta) \le (k,\gamma), 0 \le i,j,k \le n, 0 \le \alpha,\beta,\gamma \le m\}.$$

Clearly, a monomial $z_{i\alpha}z_{j\beta}z_{k\gamma} \in (\mathcal{T}(Z))_3$ is normal modulo $\mathcal{R}$ if and only if each of its subwords of length 2, $z_{i\alpha}z_{j\beta}$ and $z_{j\beta}z_{k\gamma}$, is normal modulo $\mathcal{R}$. Moreover,

$$N(\mathcal{R})_2 = \{z_{i\alpha}z_{j\beta} \mid 0 \le i \le j \le n, 0 \le \alpha \le \beta \le m\},$$

therefore

$$N(\mathcal{R})_3 = \{z_{i\alpha}z_{j\beta}z_{k\gamma} \mid 0 \le i \le j \le k \le n, \quad 0 \le \alpha \le \beta \le \gamma \le m\}. \tag{6.15}$$

It follows from (6.15) that

$$|N(\mathcal{R})_3| = \binom{n+3}{3}\binom{m+3}{3},$$

which together with (6.14) give the desired equality $\dim B_3 = |(N(\mathcal{R}))_3|$. Now Lemma 3.14 implies that $\mathcal{R}$ is a Gröbner basis of the ideal $\mathfrak{K}$. It is obvious that $\mathcal{R}$ is a reduced Gröbner basis of $\mathfrak{K}$. □

7. Examples

We present here some example that illustrate the results of our paper.

Example 7.1 (The non-commutative twisted cubic curve)**.** Let $n = 1$ and $d = 3$. Then

$$X = \{x_0, x_1\}, \quad \mathbf{q} = \begin{pmatrix} 1 & q^{-1} \\ q & 1 \end{pmatrix}, \quad \mathcal{A}^1_{\mathbf{q}} = \mathbb{C}\langle x_0, x_1\rangle/(x_1x_0 - qx_0x_1).$$

In this case $N = \binom{1+3}{3} - 1 = 3$ and the corresponding quantum space $\mathcal{A}^3_{\mathbf{g}}$ is defined by the following data

$$Y = \{y_0, y_1, y_2, y_3\}, \quad \mathbf{q} = \begin{pmatrix} 1 & q^{-3} & q^{-6} & q^{-9} \\ q^3 & 1 & q^{-3} & q^{-6} \\ q^6 & q^3 & 1 & q^{-3} \\ q^9 & q^6 & q^3 & 1 \end{pmatrix}.$$

The kernel $\ker(v_{1,3})$ of the Veronese map $v_{1,3} : \mathcal{A}^3_{\mathbf{g}} \to \mathcal{A}^1_{\mathbf{q}}$ has a reduced Gröbner basis G given below

$$G = \{y_1^2 - q^2y_0y_2,\ y_1y_2 - qy_0y_3,\ y_2^2 - q^2y_1y_3\}.$$

We have used the fact that in this case $\mathrm{MV}(1,3) = \{(1,1),(1,2),(2,2)\}$.

Setting $q = 1$ we obtain that the defining ideal for the *commutative* Veronese is generated by the three polynomials $\{y_1^2 - y_0y_2, y_1y_2 - y_0y_3, y_2^2 - y_1y_3\}$. This is exactly the set of generators described and discussed in [24, pp. 23, 51].

Example 7.2 (The non-commutative rational normal curves)**.** Generalising the previous example, we consider $n = 1$ and d arbitrary. In notation as above, we write

$$\mathcal{A}^1_{\mathbf{q}} = \mathbb{C}\langle x_0, x_1\rangle/(x_1x_0 - qx_0x_1).$$

In this case, $N = \binom{d+1}{d} - 1 = d$ and the corresponding quantum space $\mathcal{A}^d_{\mathbf{g}}$ is determined by the data

$$Y = \{y_0, y_1, \dots, y_d\}, \quad \mathbf{q} = \begin{pmatrix} 1 & q^{-d} & q^{-2d} & \cdots & \cdots & \cdots & q^{-d^2} \\ q^d & 1 & q^{-d} & & & & q^{-d(d-1)} \\ q^{2d} & q^d & 1 & \ddots & & & q^{-d(d-2)} \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ q^{d(d-1)} & & & & \ddots & 1 & q^{-d} \\ q^{d^2} & q^{d(d-1)} & q^{d(d-2)} & \cdots & \cdots & q^d & 1 \end{pmatrix}. \tag{7.1}$$

Observe that whenever q is a d-th root of unity, the derived $(1,d)$-quantum space is a commutative algebra.

The kernel $\ker(v_{1,d})$ of the Veronese map $v_{1,d} : \mathcal{A}^d_{\mathbf{g}} \to \mathcal{A}^1_{\mathbf{q}}$ has a reduced Gröbner basis G given by $\binom{d}{2}$ quadratic relations:

$$G = \{y_iy_j - h_{ij} \mid 1 \le i \le j \le d-1\}, \quad h_{ij} = \begin{cases} q^{i(d-j)}y_0y_{i+1} & i+j \le d \\ q^{i(d-j)}y_{i+j-d}y_d & i+j > d \end{cases}.$$

Once again, for $q = 1$ we obtain that a reduced Gröbner basis for the defining ideal of the *commutative* rational normal curve (see [24, Example 1.16]).

Example 7.3 (The non-commutative Veronese surface)**.** Let $n = d = 2$, that is,

$$X = \{x_0, x_1, x_2\}, \quad \mathbf{q} = \begin{pmatrix} 1 & q_{10}^{-1} & q_{20}^{-1} \\ q_{10} & 1 & q_{21}^{-1} \\ q_{20} & q_{21} & 1 \end{pmatrix},$$

$$\mathcal{A}^2_{\mathbf{q}} = \mathbb{C}\langle x_0, x_1, x_2\rangle/(x_1x_0 - q_{10}x_0x_1, x_2x_0 - q_{20}x_0x_2, x_2x_1 - q_{21}x_1x_2).$$

In this case $N = 5$ and the corresponding $(2,2)$-quantum space $\mathcal{A}^5_{\mathbf{g}}$ is completely determined by the data

$$Y = \{y_0, y_1, y_2, y_3, y_4, y_5\}, \quad \mathbf{g} := \begin{pmatrix} 1 & q_{10}^{-2} & q_{20}^{-2} & q_{10}^{-4} & q_{20}^{-2}q_{10}^{-2} & q_{20}^{-4} \\ q_{10}^{2} & 1 & q_{20}^{-1}q_{21}^{-1}q_{10} & q_{10}^{-2} & (q_{10}q_{20}q_{21})^{-1} & q_{20}^{-2}q_{21}^{-2} \\ q_{20}^{2} & q_{20}q_{21}q_{10}^{-1} & 1 & q_{21}^{2}q_{10}^{-2} & q_{21}q_{10}^{-1}q_{20}^{-1} & q_{20}^{-2} \\ q_{10}^{4} & q_{10}^{2} & q_{21}^{-2}q_{10}^{2} & 1 & q_{21}^{-2} & q_{21}^{-4} \\ q_{20}^{2}q_{10}^{2} & q_{10}q_{20}q_{21} & q_{21}^{-1}q_{10}q_{20} & q_{21}^{2} & 1 & q_{21}^{-2} \\ q_{20}^{4} & q_{20}^{2}q_{21}^{2} & q_{20}^{2} & q_{21}^{4} & q_{21}^{2} & 1 \end{pmatrix}$$

Observe that inside the matrix $\mathbf{g}$ we find as submatrices three occurrences of the matrix in (7.1) for $d = 2$ and q equal to one of the three commutation parameters, namely

$$\begin{pmatrix} 1 & q_{10}^{-2} & q_{10}^{-4} \\ q_{10}^{2} & 1 & q_{10}^{-2} \\ q_{10}^{4} & q_{10}^{2} & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & q_{20}^{-2} & q_{20}^{-4} \\ q_{20}^{2} & 1 & q_{20}^{-2} \\ q_{20}^{4} & q_{20}^{2} & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & q_{21}^{-2} & q_{21}^{-4} \\ q_{21}^{2} & 1 & q_{21}^{-2} \\ q_{21}^{4} & q_{21}^{2} & 1 \end{pmatrix}.$$

The kernel of the Veronese map $v_{2,2} : \mathcal{A}^5_{\mathbf{g}} \to \mathcal{A}^2_{\mathbf{q}}$ has a reduced Gröbner basis consisting of six quadratic polynomials

$$G = (y_1^2 - q_{10}y_0y_3,\ y_1y_2 - q_{10}y_0y_4,\ y_2^2 - q_{20}y_0y_5, \\ y_2y_3 - q_{21}^2y_1y_4,\ y_2y_4 - q_{21}y_1y_5,\ y_4^2 - q_{21}y_3y_5).$$

Example 7.4 (The Segre quadric)**.** Let $n = m = 1$. Following the above conventions, we write $\mathcal{A}^1_{\mathbf{q}} = \mathbb{C}\langle x_0, x_1\rangle/(x_1x_0 - qx_0x_1)$ and $\mathcal{A}^1_{\mathbf{q}'} = \mathbb{C}\langle y_0, y_1\rangle/(y_1y_0 - q'y_0y_1)$.

In this case, $N = 3$ and the quantum space $\mathcal{A}^3_{\mathbf{g}}$ is determined by the data

$$Z = \{z_{00}, z_{01}, z_{10}, z_{11}\}, \quad \mathbf{g} = \begin{pmatrix} 1 & q'^{-1} & q^{-1} & (q'q)^{-1} \\ q' & 1 & q^{-1}q' & q^{-1} \\ q & q(q')^{-1} & 1 & (q')^{-1} \\ qq' & q & q' & 1 \end{pmatrix}.$$

The kernel $\ker(s_{1,1})$ of the Segre map $s_{1,1} : \mathcal{A}^3_{\mathbf{g}} \to \mathcal{A}^1_{\mathbf{q}} \otimes \mathcal{A}^1_{\mathbf{q}'}$ has a reduced Gröbner basis consisting of a single quadratic polynomial

$$G = \{z_{01}z_{10} - q'z_{00}z_{11}\}.$$

Example 7.5 (The non-commutative Segre threefold)**.** Let $n = 2$ and $m = 1$. We consider

$$\mathcal{A}^2_{\mathbf{q}} = \mathbb{C}\langle x_0, x_1, x_2\rangle/(x_1x_0 - q_{1,0}x_0x_1, x_2x_0 - q_{2,0}x_0x_2, x_2x_1 - q_{2,1}x_1x_2)$$

and

$$\mathcal{A}^1_{\mathbf{q}'} = \mathbb{C}\langle y_0y_1\rangle/(y_1y_0 - q'y_0y_1).$$

Then $N = 5$ and the corresponding $(2,1)$-derived quantum space is determined by the following data:

$$Z = \{z_{00}, z_{01}, z_{10}, z_{11}, z_{20}, z_{21}\}, \quad \mathbf{g} = \begin{pmatrix} 1 & (q')^{-1} & q_{10}^{-1} & (q_{10}q')^{-1} & q_{20}^{-1} & (q_{20}q')^{-1} \\ q' & 1 & q_{10}^{-1}q' & q_{10}^{-1} & q_{20}^{-1}q' & q_{20}^{-1} \\ q_{10} & q_{10}(q')^{-1} & 1 & (q')^{-1} & q_{21}^{-1} & (q_{21}q')^{-1} \\ q_{10}q' & q_{10} & q' & 1 & q_{21}^{-1}q' & q_{21}^{-1} \\ q_{20} & q_{20}(q')^{-1} & q_{21} & q_{21}(q')^{-1} & 1 & (q')^{-1} \\ q_{20}q' & q_{20} & q_{21}q' & q_{21} & q' & 1 \end{pmatrix}.$$

The kernel $\ker(s_{2,1})$ of the Segre map $s_{2,1} : \mathcal{A}^5_{\mathbf{g}} \to \mathcal{A}^2_{\mathbf{q}} \otimes \mathcal{A}^1_{\mathbf{q}'}$ has a reduced Gröbner basis consisting of three quadratic polynomials

$$G = \{z_{01}z_{10} - q'z_{00}z_{11}, z_{01}z_{20} - q'z_{00}z_{21}, z_{11}z_{20} - q'z_{10}z_{21}\}.$$

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