Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Interactions between discrete and continuous optimization and critical point theory via multi-way Lovasz extensions

by

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Preprint no.: 38 2020

Interactions between discrete and continuous optimization and critical point theory via multi-way Lovász extensions

Jürgen Jost[*] Dong Zhang[†]

We introduce and systematically study general discrete-to-continuous extensions, with several applications for combinatorial problems and discrete mathematics. This provides new perspectives for understanding relations and interactions between discrete and continuous worlds via multi-way extensions.

We propose a family of one-homogeneous and piecewise linear extensions inspired by the Lovász extension in order to systematically derive continuous analogs of problems from discrete mathematics. This will take place in the following context:

- For combinatorial optimization problems, we systematically develop equivalent continuous versions, thereby making tools from convex optimization, fractional programming and more general continuous algorithm like the stochastic subgradient method available for such optimization problems. Among other applications, we present an effective iteration scheme combining the inverse power and the steepest decent method to relax a Dinkelbach-type scheme for solving the equivalent continuous optimization.
- Submodularity and convexity are studied in multi-way settings, in particular, a necessary and sufficient condition for a continuous function to be a multi-way Lovász extension of some function is provided, which generalizes a recent result of Chateauneuf et al [17].
- We establish an equivalence between Forman's discrete Morse theory on a simplicial complex and the continuous Morse theory (in the sense of any known non-smooth Morse theory) on the associated order complex via a Lovász extension. Furthermore, we propose a new version of the Lusternik-Schnirelman category on abstract simplicial complexes to bridge the classical Lusternik-Schnirelman theorem and its discrete analog on finite complexes. More generally, we can suggest a discrete Morse theory on hypergraphs by employing PL Morse theory and Lovász extension, hoping to provide new tools for exploring the structure of hypergraphs.

This theory has several applications to quantitative and combinatorial problems, among them the following:

(1) Resorting to the multi-way extension, the equivalent continuous representations for the max k-cut problem, variant Cheeger sets and isoperimetric constants are constructed. This also initiates a study of Dirichlet and Neumann 1-Laplacians on graphs, in which the nodal domain property and Cheeger-type equalities are presented. Among them, some Cheeger constants using different versions of vertex-boundary introduced in expander graph theory [38], are transformed into continuous forms, which reprove the inequalities and identities on graph Poincare profiles proposed by Hume [27–29].

(2) Also, we derive a new equivalent continuous representation of the graph independence number, which can be compared with the Motzkin-Straus theorem. More importantly, an equivalent continuous optimization for the chromatic number is provided, which seems to be the first continuous representation of the graph vertex coloring number. Graph matching numbers, submodular vertex covers and multiway partition problems can also be studied in our framework.

Keywords: Lovász extension; submodularity; combinatorial optimization; discrete Morse theory; Lusternik-Schnirelman theory; Cheeger inequalities & isoperimetric problems; chromatic number

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1 Introduction and Background

The Lovász extension is a basic tool in discrete mathematics, especially for some combinatorial optimization problems and submodular analysis [1]. It was introduced in the study of submodular functions which appear frequently in many areas like game theory, matroid theory, stochastic processes, electrical networks, computer vision and machine learning [22].

In fact, a special form of the Lovász extension appeared already in the context of the Choquet integral [54] which has fruitful applications in statistical mechanics, potential theory and decision theory. Since the Lovász extension does not require the monotonicity of the set function in finite cases of the Choquet integral, it has a wider range of applications, for instance in combinatorics, for algorithms in computer science. Recent developments include quasi-Lovász extension on some algebraic structures and fuzzy mathematics, applications of Lovász extensions to graph cut problems and computer science, as well as Lovász-softmax loss in deep learning.

We shall start by looking at the original Lovász extension. For simplicity, we shall work throughout this paper with a finite and nonempty set $V = \{1, \cdots, n\}$ and its power set $\mathcal{P}(V)$. Also, we shall sometimes work on $\mathcal{P}(V)^k := \{(A_1, \cdots, A_k) : A_i \subset V, i = 1, \cdots, k\}$ and $\mathcal{P}_k(V) := \{(A_1, \cdots, A_k) \in \mathcal{P}(V)^k : A_i \cap A_j = \varnothing, \forall i \neq j\}$, as well as some restricted family $\mathcal{A} \subset \mathcal{P}(V)^k$. We denote the cardinality of a set A by $\#A$. Given a function $f : \mathcal{P}(V) \to \mathbb{R}$, one identifies every $A \in \mathcal{P}(V)$ with its indicator vector $\mathbf{1}_A \in \mathbb{R}^V = \mathbb{R}^n$. The Lovász extension extends the domain of f to the whole Euclidean space[1] $\mathbb{R}^V$. There are several equivalent expressions:

- For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a bijection such that $x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)}$ and $\sigma(0) = 0$, where $x_0 := 0$. The Lovász extension of f is defined by

$$f^L(\boldsymbol{x}) = \sum_{i=0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V^{\sigma(i)}(\boldsymbol{x})), \tag{1}$$

where $V^0(\boldsymbol{x}) = V$ and $V^{\sigma(i)}(\boldsymbol{x}) := \{j \in V : x_j > x_{\sigma(i)}\}, \quad i = 1, \cdots, n-1$. We can write (1) in an integral form as

$$f^L(\boldsymbol{x}) = \int_{\min\limits_{1\le i\le n} x_i}^{\max\limits_{1\le i\le n} x_i} f(V^t(\boldsymbol{x}))dt + f(V) \min_{1\le i\le n} x_i, \tag{2}$$

[1]Some other versions in the literatures only extend the domain to the cube $[0,1]^V$ or the first quadrant $\mathbb{R}^V_{\ge 0}$. In fact, many works on Boolean lattices identify $\mathcal{P}(V)$ with the discrete cube $\{0,1\}^n$.

and if we add the natural assumption $f(\varnothing) = 0$,

$$f^L(\boldsymbol{x}) = \int_{-\infty}^{0} (f(V^t(\boldsymbol{x})) - f(V))dt + \int_{0}^{+\infty} f(V^t(\boldsymbol{x}))dt, \tag{3}$$

where $V^t(\boldsymbol{x}) = \{i \in V : x_i > t\}$. If we apply a Möbius transformation, this becomes

$$f^L(\boldsymbol{x}) = \sum_{A \subset V} \sum_{B \subset A} (-1)^{\#A - \#B} f(B) \bigwedge_{i \in A} x_i, \tag{4}$$

where $\bigwedge_{i \in A} x_i$ is the minimum over $\{x_i : i \in A\}$.

In the above formulas, f^L is the unique function that is affine on each polyhedral cone $\mathbb{R}^n_\sigma := \{\boldsymbol{x} \in \mathbb{R}^n : x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)}\}$, for any permutation σ on V. It is known that f^L is positively one-homogeneous, PL (piecewise linear) and Lipschitzian continuous [1, 2]. Also, $f^L(\boldsymbol{x} + t\mathbf{1}_V) = f^L(\boldsymbol{x}) + tf(V)$, $\forall t \in \mathbb{R}$, $\forall \boldsymbol{x} \in \mathbb{R}^V$. Moreover, a continuous function $F : \mathbb{R}^V \to \mathbb{R}$ is a Lovász extension of some $f : \mathcal{P}(V) \to \mathbb{R}$ if and only if $F(\boldsymbol{x} + \boldsymbol{y}) = F(\boldsymbol{x}) + F(\boldsymbol{y})$ whenever $(x_i - x_j)(y_i - y_j) \ge 0$, $\forall i, j \in V$.

In this paper, we shall use the Lovász extension and its variants to study the interplay between discrete and continuous aspects in topics such as convexity, optimization and Morse theory.

Submodular and convex functions

Submodular function have emerged as a powerful concept in discrete optimization, see Fujishige's monograph [22]. A Lovász extension turns a submodular into a convex function, and we can hence minimize the former by minimizing the latter:

Theorem 1.1 (Lovász [1]). *$f : \mathcal{P}(V) \to \mathbb{R}$ is submodular $\Leftrightarrow$ f^L is convex.*

$$\boxed{\text{Submodularity}} \xleftrightarrow{\text{Lovász extension}} \boxed{\text{Convexity}}$$

Theorem 1.2 (Lovász [1]). *If $f : \mathcal{P}(V) \to \mathbb{R}$ is submodular, then*

$$\min_{A \subset V} f(A) = \min_{\boldsymbol{x} \in [0,1]^V} f^L(\boldsymbol{x}).$$

$$\boxed{\text{Submodular optimization}} \xleftrightarrow{\text{Lovász extension}} \boxed{\text{Convex programming}}$$

Thus, submodularity can be seen as some kind of 'discrete convexity', and that naturally lead to many generalizations, such as bisubmodular, k-submodular, L-convex and M-convex, see [22, 53]. Recently, a necessary and sufficient condition for a *continuously submodular function*[2] defined on $\mathbb{R}^n$ to be representable as a Lovász extension of a submodular function defined on $\mathcal{P}(V)$ has been obtained [17].

Theorem 1.3 (Chateauneuf & Cornet [17]). *A one-homogeneous function $F : \mathbb{R}^V \to \mathbb{R}$ is a Lovász extension of some submodular function if and only if $F(\boldsymbol{x} + t\mathbf{1}_V) = F(\boldsymbol{x}) + tF(\mathbf{1}_V)$, $\forall t \in \mathbb{R}$, $\forall \boldsymbol{x} \in \mathbb{R}^V$, and $F(\boldsymbol{x}) + F(\boldsymbol{y}) \ge F(\boldsymbol{x} \vee \boldsymbol{y}) + F(\boldsymbol{x} \wedge \boldsymbol{y})$, where the i-th components of $\boldsymbol{x} \vee \boldsymbol{y}$ and $\boldsymbol{x} \wedge \boldsymbol{y}$ are $(\boldsymbol{x} \vee \boldsymbol{y})_i = \max\{x_i, y_i\}$ and $(\boldsymbol{x} \wedge \boldsymbol{y})_i = \min\{x_i, y_i\}$.*

One may want to extend such a result to the bisubmodular or more general cases. In that direction, we shall obtain some results such as Proposition 2.7 and Theorem 2.3 in Section 2.2.

$$\boxed{\text{Discrete convex analysis}} \xleftrightarrow{\text{Lovász extension}} \boxed{\text{Convex analysis}}$$

[2] see (S2) in Subsection 2.2

So far, research has mainly focused on 'discrete convex' functions, leading to 'Discrete Convex Analysis' [52,53], whereas the discrete non-convex setting which is quite popular in modern sciences has not yet received that much attention.

Non-submodular cases

Obviously, the non-convex case is so diverse and general that it cannot be directly studied by standard submodular tools. Although some publications show several results on non-submodular (i.e., non-convex) minimization based on Lovász extension [15], so far, these only work for special minimizations over the whole power set. Here, we shall find applications for discrete optimization and nonlinear spectral graph theory by employing the multi-way Lovász extension on enlarged domains. We shall also study the Lovász extension on restricted domains, leading to a fascinating connection between discrete and continuous Morse theory and Lusternik-Schnirelman theory. Both the enlarged and the restricted version possess the basic feature of Lovász theory, a correspondence between submodularity and convexity.

In summary, we are going to initiate the study of diverse continuous extensions in non-submodular settings. This paper develops a systematic framework for many aspects around the topic. We establish a universal discrete-to-continuous framework via multi-way extensions, by systematically utilizing integral representations. We shall now discuss some connections with various fields.

Connections with combinatorial optimization

Because of the wide range of applications of discrete mathematics in computer sciences, combinatorial optimization has been much studied from the mathematical perspective. It is known that any combinatorial optimization can be equivalently expressed as a continuous optimization via convex (or concave) extension, but often, there is the difficulty that one cannot write down an equivalent continuous object function in closed-form. For practical purposes, it would be very helpful if one could transfer a combinatorial optimization to an explicit and simple equivalent continuous optimization problem. Formally, in many concrete situations, it would be useful if one could get an identity of the form

$$\min_{(A_1,\cdots,A_k)\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{f(A_1,\cdots,A_k)}{g(A_1,\cdots,A_k)}=\inf_{\psi\in\mathcal{D}(\mathcal{A})}\frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}. \tag{5}$$

where $f,g:\mathcal{A}\to[0,\infty)$, $\mathcal{D}(\mathcal{A})$ is a feasible domain determined by $\mathcal{A}$ only, and $\widetilde{f}$ and $\widetilde{g}$ are suitable continuous extensions of f and g.

So far, only situations where $f:\mathcal{P}(V)\to\mathbb{R}$ or $f:\mathcal{P}_2(V)\to\mathbb{R}$ have been investigated, and what is lacking are situations with restrictions, that is, incomplete data.

Also, to the best of our knowledge, the known results in the literature do not work for combinatorial optimization on set-tuples. But most of combinatorial optimization problems should be formalized in the form of set-tuples, and only a few can be represented in set form or disjoint-pair form. Whenever one can find an equivalent Lipschitz function for a combinatorial problem in the field of discrete optimization, this makes useful tools available and leads to new connections. That is, one wishes to establish a *discrete-to-continuous transformation* like the operator $\sim$ in (5). We will show in Section 3 that the Lovász extension and its variants are suitable choices for such a transformation (see Theorems B, 3.1 and Proposition 3.1 for details).

Connections with discrete Morse theory

Forman introduced a discrete Morse theory on simplicial complexes [36,37]. This theory has some deep connections with smooth Morse theory [32–34], as well as practical applications [35], and also admits several slight generalizations. Both this discrete Morse theory and the classical smooth one are simple, since they exclude some complicated cases such as monkey-saddle points. We will construct the relationship between the Morse theory of a discrete Morse function and its Lovász extension in Section 4. Note that the standard ideas and methods cannot be directly applied because the Lovász extension is one-homogeneous and all local flows can go along the rays from the original point and thus all possible critical points are trivial. Therefore, we should restrict the Lovász extension f^L to a subset of its feasible domain. This will lead us to the following result.

Theorem A (Theorems 4.1, 4.2 and 4.3). *For a simplicial complex with vertex set V and face set $\mathcal{K}$, let $f:\mathcal{K}\to\mathbb{R}$ be an injective discrete Morse function. Then the following conditions are equivalent:*

(1) σ is a critical point of f;

(2) $\mathbf{1}_\sigma$ is a critical point of $f^L|_{|S_\mathcal{K}|}$ with index i in the sense of weak slope (metric Morse theory);

(3) $\mathbf{1}_\sigma$ is a critical point of $f^L|_{|S_\mathcal{K}|}$ with index i in the sense of Kühnel (PL Morse theory);

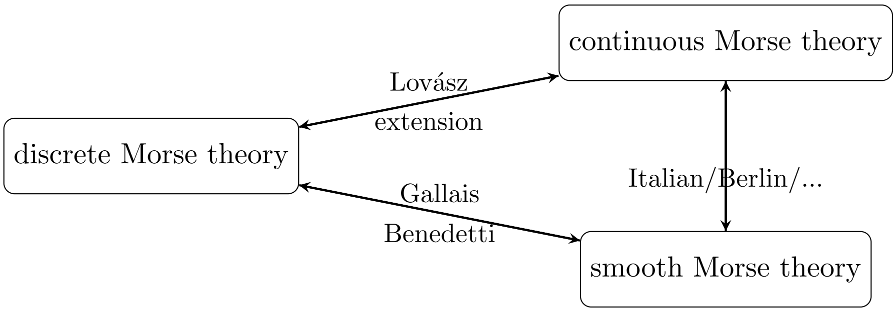
(4) $\mathbf{1}_\sigma$ is a Morse critical point of $f^L|_{|S_\mathcal{K}|}$ with index i in the sense of topological Morse theory;

Here the notation $|S_\mathcal{K}|$ indicates a suitable restriction (see Subsection 4.1) for f^L being well-defined.

Moreover, the discrete Morse vector $(n_0, n_1, \cdots, n_d)$, representing the number n_i of critical points with index i, of f coincides with the continuous Morse vector of $f^L|_{|S_\mathcal{K}|}$.

Moreover, the Lusternik-Schnirelmann category[3] theorem is preserved under Lovász extension:

$$\min_{L\in\mathrm{Cat}_m(\mathcal{K})}\max_{\sigma\in L} f(\sigma) = \inf_{S\in\mathrm{Cat}_m(|S_\mathcal{K}|)}\sup_{\boldsymbol{x}\in S} f^L(\boldsymbol{x}).$$



In summary, Theorem A says that the *Morse structures* of $\mathcal{K}$ and $|S_\mathcal{K}|$ are coarsely equivalent, and one can translate all the results about 'Morse data' of a discrete Morse function f on $\mathcal{K}$ to its Lovász extension f^L restricted on $|S_\mathcal{K}|$. This also reflects the deep result from [33, 34] that smooth Morse theory on a manifold is almost equivalent to the discrete Morse theory on its triangulation. The difference is that we don't assume the complex $|\mathcal{K}|$ to be a topological manifold, so that topological results on manifolds cannot be applied directly. Fortunately, our feasible domain $|S_\mathcal{K}|$ is a piecewise flat geometric complex. Our proofs don't draw heavily on the standard tools in discrete Morse theory.

Connections with Hypergraphs

The idea above allows us to establish a discrete Morse theory on hypergraphs, which helps us to understand the structure of a hypergraph from a Morse theoretical perspective (see Section 4.2).

To reach these goals, we need to systematically study various generalizations of the Lovász extension. More precisely, we shall work with the following two different multi-way forms:

(1) Disjoint-pair version: for a function $f : \mathcal{P}_2(V) \to \mathbb{R}$, its disjoint-pair Lovász extension is defined as
$$f^L(\boldsymbol{x}) = \int_0^{\|\boldsymbol{x}\|_\infty} f(V_+^t(\boldsymbol{x}), V_-^t(\boldsymbol{x}))dt, \tag{6}$$
where $V_\pm^t(\boldsymbol{x}) = \{i \in V : \pm x_i > t\}$, $\forall t \geq 0$. For $\mathcal{A} \subset \mathcal{P}_2(V)$ and $f : \mathcal{A} \to \mathbb{R}$, the feasible domain $\mathcal{D}_\mathcal{A}$ of the disjoint-pair Lovász extension is $\{\boldsymbol{x} \in \mathbb{R}^V : (V_+^t(\boldsymbol{x}), V_-^t(\boldsymbol{x})) \in \mathcal{A}, \forall t \geq 0\}$.

(2) k-way version: for a function $f : \mathcal{P}(V)^k \to \mathbb{R}$, the *simple k-way Lovász extension* $f^L : \mathbb{R}^{kn} \to \mathbb{R}$ is defined as
$$f^L(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k) = \int_{\min \boldsymbol{x}}^{\max \boldsymbol{x}} f(V^t(\boldsymbol{x}^1), \cdots, V^t(\boldsymbol{x}^k))dt + f(V, \cdots, V)\min \boldsymbol{x}, \tag{7}$$
where $V^t(\boldsymbol{x}^i) = \{j \in V : x_j^i > t\}$, $\min \boldsymbol{x} = \min\limits_{i,j} x_j^i$ and $\max \boldsymbol{x} = \max\limits_{i,j} x_j^i$. For $\mathcal{A} \subset \mathcal{P}^k(V)$ and $f : \mathcal{A} \to \mathbb{R}$, we take $\mathcal{D}_\mathcal{A} = \{\boldsymbol{x} \in \mathbb{R}^{kn}_{\geq 0} : (V^t(\boldsymbol{x}^1), \cdots, V^t(\boldsymbol{x}^k)) \in \mathcal{A}, \forall t \in \mathbb{R}\}$ as a feasible domain of the k-way Lovász extension f^L.

[3] The definition of category classes for $\mathcal{K}$ is introduced in Subsection 4.1.

All these multi-way Lovász extensions satisfy the optimal identity Eq. (5):

Theorem B (Theorem 3.1 and Proposition 3.1). *Given two functions* $f, g : \mathcal{A} \to [0, +\infty)$*, let* $\tilde{f}$ *and* $\tilde{g}$ *be two real functions on* $\mathcal{D}_{\mathcal{A}}$ *satisfying* $\tilde{f}(\mathbf{1}_{A_1,\cdots,A_k}) = f(A_1, \cdots, A_k)$ *and* $\tilde{g}(\mathbf{1}_{A_1,\cdots,A_k}) = g(A_1, \cdots, A_k)$*. Then Eq. (5) holds if* $\tilde{f}$ *and* $\tilde{g}$ *further possess the properties (P1) or (P2) below. Correspondingly, if* $\tilde{f}$ *and* $\tilde{g}$ *fulfil (P1') or (P2), there similarly holds*

$$\max_{(A_1,\cdots,A_k)\in\mathcal{A}\cap\mathrm{supp}(g)} \frac{f(A_1, \cdots, A_k)}{g(A_1, \cdots, A_k)} = \sup_{\psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\widetilde{g})} \frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}.$$

Here the optional additional conditions of $\tilde{f}$ *and* $\tilde{g}$ *are:*

(P1) $\tilde{f} \geq f^L$ *and* $\tilde{g} \leq g^L$*. (P1')* $\tilde{f} \leq f^L$ *and* $\tilde{g} \geq g^L$*.*

(P2) $\tilde{f} = \rho^{-1}((\rho \circ f)^L)$ *and* $\tilde{g} = \rho^{-1}((\rho \circ g)^L)$*, where* $\rho : [0, \infty) \to [0, \infty)$ *is a homeomorphism. For example, one can take* $\tilde{f} = ((f^\alpha)^L)^{\frac{1}{\alpha}}$ *and* $\tilde{g} = ((g^\alpha)^L)^{\frac{1}{\alpha}}$*, where* $\alpha > 0$*.*

Here f^L *is either the original or the disjoint-pair or the* k*-way Lovász extension.*

Theorem B shows that by the multi-way Lovász extension, the combinatorial optimization in quotient form can be transformed to fractional programming. And based on this fractional optimization, we propose an effective local convergence scheme, which relaxes the Dinkelbach-type iterative scheme and mixes the inverse power method and the steepest decent method. Furthermore, many other continuous iterations, such as Krasnoselski-Mann iteration, and stochastic subgradient method, could be directly applied here.

The power of Theorem B is embodied in many new examples and applications including Cheeger-type problems, various isoperimetric constants and max k-cut problems (see Subsections 5.2, 5.3 and 5.5). And moreover, we find that not only combinatorial optimization, but also some combinatorial invariants like the independence number and the chromatic number, can be transformed into a continuous representation by this scheme.

Theorem C (Subsections 5.4 and 5.6). *For an unweighted and undirected simple graph* $G = (V, E)$ *with* $\#V = n$*, its independence number can be represented as*

$$\alpha(G) = \max_{\boldsymbol{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\sum\limits_{\{i,j\}\in E}(|x_i - x_j| + |x_i + x_j|) - 2\sum\limits_{i\in V}(\deg_i - 1)|x_i|}{2\|\boldsymbol{x}\|_\infty},$$

where $\deg_i = \#\{j \in V : \{j, i\} \in E\}$*,* $i \in V$*, and its chromatic number is*

$$\gamma(G) = n^2 - \max_{\boldsymbol{x}\in\mathbb{R}^{n^2}\setminus\{\mathbf{0}\}} \sum_{k\in V} \frac{n\sum\limits_{\{i,j\}\in E}(|x_{ik} - x_{jk}| + |x_{ik} + x_{jk}|) + 2n\|\boldsymbol{x}^k\|_\infty - 2n\deg_k\|\boldsymbol{x}^k\|_1 - 2\|\boldsymbol{x}^{,k}\|_\infty}{2\|\boldsymbol{x}\|_\infty},$$

where $\boldsymbol{x} = (x_{ki})_{k,i\in V}$*,* $\boldsymbol{x}^k = (x_{k1}, \cdots, x_{kn})$ *and* $\boldsymbol{x}^{,k} = (x_{1k}, \cdots, x_{nk})^T$*. The maximum matching number of* G *can be expressed as*

$$\max_{\boldsymbol{y}\in\mathbb{R}^E\setminus\{\mathbf{0}\}} \frac{\|\boldsymbol{y}\|_1^2}{\|\boldsymbol{y}\|_1^2 - 2\sum_{e\cap e'=\varnothing} y_e y_{e'}}.$$

Connections with spectral graph theory

Spectral graph theory aims to derive properties of a (hyper-)graph from its eigenvalues and eigenvectors. Going beyond the linear case, nonlinear spectral graph theory is developed in terms of discrete geometric analysis and difference equations on (hyper-)graphs. Every discrete eigenvalue problem can be formulated as a variational problem for an objective functional, a Rayleigh-type quotient. In some cases, this functional is natural and easy to obtain, since one may compare the discrete version with its original continuous analog in geometric analysis. However, in other situations, there is no such analog. Fortunately, we find a unified framework based on multi-way Lovász extension to produce appropriate objective functions from a combinatorial problem (see Sections 2 and 3).

More precisely, for a combinatorial problem with a discrete objective function of the form $\frac{f(A)}{g(A)}$, we might obtain some correspondences by studying the set-valued eigenvalue problem

$$\nabla f^L(\boldsymbol{x}) \bigcap \lambda \nabla g^L(\boldsymbol{x}) \neq \varnothing.$$

Hereafter we use ∇ to denote the (Clarke) sub-gradient operator acting on Lipschitz functions.

$$\boxed{\text{graph quantities}} \xleftrightarrow{\text{Spectral graph theory}} \boxed{\text{eigenvalue problem}}$$

We shall consider the following three versions:

- *Eigenvectors and eigenvalues*: We have the collection of eigenpairs $\left\{(\lambda, \boldsymbol{x}) \,\middle|\, \boldsymbol{0} \in \nabla f^L(\boldsymbol{x}) - \lambda \nabla g^L(\boldsymbol{x})\right\}$. This enables the definition of the graph 1-Laplacian and its variants (see Section 5.3).
- *Critical points and critical values*: The set of critical points $\left\{\boldsymbol{x} \,\middle|\, 0 \in \nabla \frac{f^L(\boldsymbol{x})}{g^L(\boldsymbol{x})}\right\}$ and the corresponding critical values.
- *Minimax critical values* (i.e., *variational eigenvalues in Rayleigh quotient form*): The Lusternik-Schnirelman theory tells us that the min-max values

$$\lambda_m = \inf_{\Psi \in \Gamma_m} \sup_{\boldsymbol{x} \in \Psi} \frac{f^L(\boldsymbol{x})}{g^L(\boldsymbol{x})}, \tag{8}$$

are critical values of $f^L(\cdot)/g^L(\cdot)$. Here Γ_m is a class of certain topological objects at level m, e.g., the family of subsets with L-S category (or Krasnoselskii's $\mathbb{Z}_2$-genus) not smaller than m.

There are the following relations between these three classes:

$$\{\text{Eigenvalues in Rayleigh quotient}\} \subset \{\text{Critical values}\} \subset \{\text{Eigenvalues}\}.$$

For linear spectral theory, the three classes above coincide. However, for the non-smooth spectral theory derived by Lovász extension, we only have the inclusion relation.

The following picture summarizes the relations between the various concepts developed and studied in this paper.

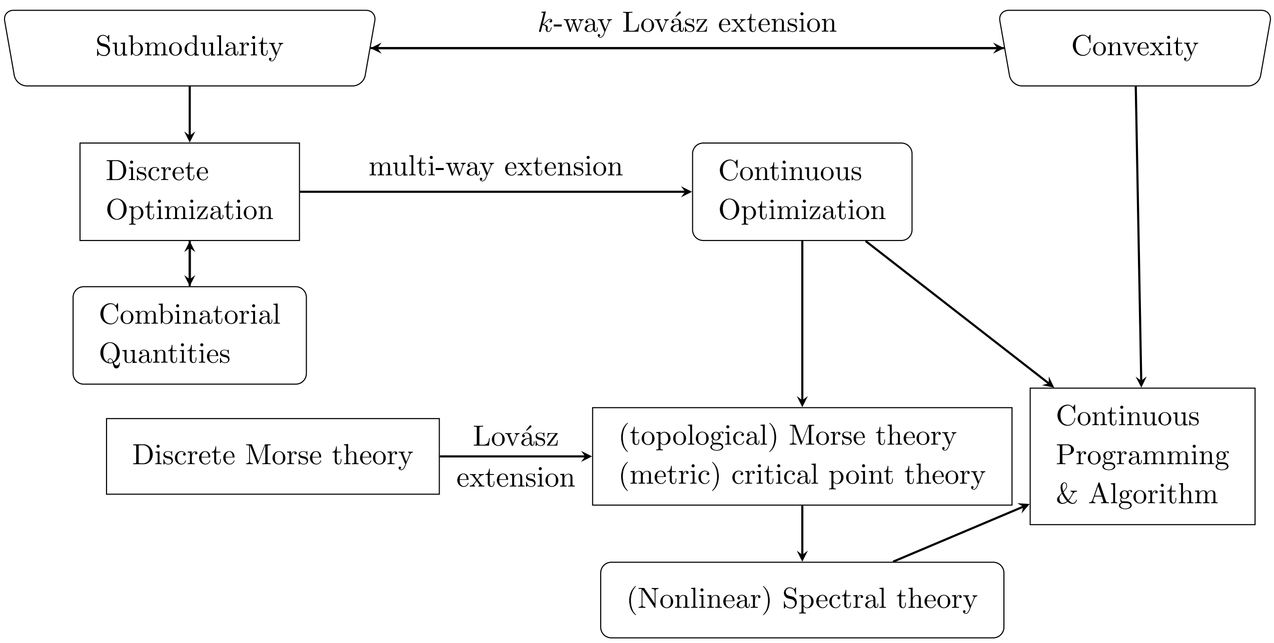


Figure 1: The relationship between the aspects studied in this paper.

Notification 1. *Since this paper contains many interacting parts and relevant results, some notions and concepts may have slightly distinct meanings in different sections, but this will be stated at the beginning of each section.*

2 Multi-way extension

We first formalize some important results about the original Lovász extension.

Definition 2.1. *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are comonotonic if* $(x_i-x_j)(y_i-y_j)\ge 0$, $\forall i,j\in\{1,2,\cdots,n\}$.
A function $F:\mathbb{R}^n\to\mathbb{R}$ *is comonotonic additive if* $F(\boldsymbol{x}+\boldsymbol{y})=F(\boldsymbol{x})+F(\boldsymbol{y})$ *for any comonotonic pair* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

The following proposition shows that a function is comonotonic additive if and only if it can be expressed as the Lovász extension of some function.

Proposition 2.1. $F:\mathbb{R}^n\to\mathbb{R}$ *is the Lovász extension* $F=f^L$ *of some function* $f:\mathcal{P}(V)\to\mathbb{R}$ *if and only if* F *is comonotonic additive.*

Recall the following known results:

Theorem 2.1 (Lovász). *The following conditions are equivalent: (1)* f *is submodular; (2)* f^L *is convex; (3)* f^L *is submodular.*

Theorem 2.2 (Chateauneuf & Cornet). $F:\mathbb{R}^n\to\mathbb{R}$ *is the Lovász extension* $F=f^L$ *of some submodular* $f:\mathcal{P}(V)\to\mathbb{R}$ *if and only if* F *is positively one-homogeneous, submodular and* $F(\boldsymbol{x}+t\boldsymbol{1})=F(\boldsymbol{x})+tF(\boldsymbol{1})$.

We should note that Theorem 2.2 is not a direct consequence of the combination of Proposition 2.1 and Theorem 2.1. We shall establish such results for the disjoint-pair version and the k-way version of the Lovász extension.

2.1 Disjoint-pair and k-way Lovász extensions

Since it is natural to set $f(\varnothing,\varnothing)=0$, one may write (6) as

$$f^L(\boldsymbol{x})=\int_0^\infty f(V_+^t(\boldsymbol{x}),V_-^t(\boldsymbol{x}))dt, \tag{9}$$

$$f^L(\boldsymbol{x})=\sum_{i=0}^{n-1}(|x_{\sigma(i+1)}|-|x_{\sigma(i)}|)f(V_{\sigma(i)}^+(\boldsymbol{x}),V_{\sigma(i)}^-(\boldsymbol{x})), \tag{10}$$

where $\sigma:V\cup\{0\}\to V\cup\{0\}$ is a bijection such that $|x_{\sigma(1)}|\le|x_{\sigma(2)}|\le\cdots\le|x_{\sigma(n)}|$ and $\sigma(0)=0$, where $x_0:=0$, and

$$V_{\sigma(i)}^\pm(\boldsymbol{x}):=\{j\in V:\pm x_j>|x_{\sigma(i)}|\},\quad i=0,1,\cdots,n-1.$$

We regard $\mathcal{P}_2(V)=3^V$ as $\{-1,0,1\}^n$ by identifying the disjoint pair (A,B) with the ternary (indicator) vector $\boldsymbol{1}_A-\boldsymbol{1}_B$.

One may compare the original and the disjoint-pair Lovász extensions by writing (6) as

$$\int_{\min_i|x_i|}^{\max_i|x_i|} f(V_+^t(\boldsymbol{x}),V_-^t(\boldsymbol{x}))dt+\min_i|x_i|f(V_+,V_-), \tag{11}$$

where $V_\pm=\{i\in V:\pm x_i>0\}$. Note that (11) is very similar to (2).

Definition 2.2. *Given* $V_i=\{1,\cdots,n_i\}$, $i=1,\cdots,k$, *and a function* $f:\mathcal{P}(V_1)\times\cdots\times\mathcal{P}(V_k)\to\mathbb{R}$, *the* k*-way Lovász extension* $f^L:\mathbb{R}^{V_1}\times\cdots\times\mathbb{R}^{V_k}\to\mathbb{R}$ *can be written as*

$$\begin{aligned}f^L(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^k)&=\int_{\min\boldsymbol{x}}^{\max\boldsymbol{x}} f(V_1^t(\boldsymbol{x}^1),\cdots,V_k^t(\boldsymbol{x}^k))dt+f(V_1,\cdots,V_k)\min\boldsymbol{x}\\&=\int_{-\infty}^0(f(V_1^t(\boldsymbol{x}^1),\cdots,V_k^t(\boldsymbol{x}^k))-f(V_1,\cdots,V_k))dt+\int_0^{+\infty} f(V_1^t(\boldsymbol{x}^1),\cdots,V_k^t(\boldsymbol{x}^k))dt\end{aligned}$$

where $V_i^t(\boldsymbol{x}^i)=\{j\in V_i:x_j^i>t\}$, $\min\boldsymbol{x}=\min_{i,j}x_j^i$ *and* $\max\boldsymbol{x}=\max_{i,j}x_j^i$.

Definition 2.3 (k-way analog for disjoint-pair Lovász extension)**.** *Given $V_i = \{1, \cdots, n_i\}$, $i = 1, \cdots, k$, and a function $f : \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k) \to \mathbb{R}$, define $f^L : \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k} \to \mathbb{R}$ by*

$$f^L(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k) = \int_0^{\|\boldsymbol{x}\|_\infty} f(V_{1,t}^+(\boldsymbol{x}^1), V_{1,t}^-(\boldsymbol{x}^1), \cdots, V_{k,t}^+(\boldsymbol{x}^k), V_{k,t}^-(\boldsymbol{x}^k))dt$$

where $V_{i,t}^{\pm}(\boldsymbol{x}^i) = \{j \in V_i : \pm x_j^i > t\}$, $\|\boldsymbol{x}\|_\infty = \max\limits_{i=1,\cdots,k} \|\boldsymbol{x}^i\|_\infty$. We can replace $\|\boldsymbol{x}\|_\infty$ by $+\infty$ if we set $f(\varnothing, \cdots, \varnothing) = 0$.

Some basic properties of the multi-way Lovász extension are shown below.

Proposition 2.2. *For the multi-way Lovász extension $f^L(\boldsymbol{x})$, we have*

(a) $f^L(\cdot)$ is positively one-homogeneous, piecewise linear, and Lipschitz continuous.

(b) $(\lambda f)^L = \lambda f^L$, $\forall \lambda \in \mathbb{R}$.

Proposition 2.3. *For the disjoint-pair Lovász extension $f^L(\boldsymbol{x})$, we have*

(a) f^L is Lipschitz continuous, and $|f^L(x) - f^L(y)| \le 2 \max\limits_{(A,B)\in\mathcal{P}_2(V)} f(A,B)\|x - y\|_1$, $\forall x, y \in \mathbb{R}^n$. Also, $|f^L(x) - f^L(y)| \le 2 \sum\limits_{(A,B)\in\mathcal{P}_2(V)} f(A,B)\|x - y\|_\infty$, $\forall x, y \in \mathbb{R}^n$.

(b) $f^L(-\boldsymbol{x}) = \pm f^L(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \mathbb{R}^V$ if and only if $f(A,B) = \pm f(B,A)$, $\forall (A,B) \in \mathcal{P}_2(V)$.

(c) $f^L(\boldsymbol{x}+\boldsymbol{y}) = f^L(\boldsymbol{x})+f^L(\boldsymbol{y})$ whenever $V_\pm(\boldsymbol{y}) \subset V_\pm(\widetilde{\boldsymbol{x}})$, where $\widetilde{\boldsymbol{x}}$ has components $\widetilde{x}_i = \begin{cases} x_i, & \text{if } |x_i| = \|\boldsymbol{x}\|_\infty, \\ 0, & \text{otherwise.} \end{cases}$

Proof. (a) and (b) are actually known results and their proofs are elementary. (c) can be derived from the definition (10). □

Definition 2.4. *Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be absolutely comonotonic if $x_i y_i \ge 0$, $\forall i$, and $(|x_i| - |x_j|)(|y_i| - |y_j|) \ge 0$, $\forall i, j$.*

Proposition 2.4. *A continuous function F is a disjoint-pair Lovász extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$, if and only if $F(\boldsymbol{x}) + F(\boldsymbol{y}) = F(\boldsymbol{x}+\boldsymbol{y})$ whenever $\boldsymbol{x}$ and $\boldsymbol{y}$ are absolutely comonotonic.*

Proof. By the definition of the disjoint-pair Lovász extension (see (10)), we know that F is a disjoint-pair Lovász extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$ if and only if $\lambda F(\boldsymbol{x}) + (1-\lambda)F(\boldsymbol{y}) = F(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})$ for all absolutely comonotonic vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\forall \lambda \in [0,1]$. Therefore, we only need to prove the sufficiency part.

For $\boldsymbol{x} \in \mathbb{R}^V$, since $s\boldsymbol{x}$ and $t\boldsymbol{x}$ with $s, t \ge 0$ are absolutely comonotonic, $F(s\boldsymbol{x}) + F(t\boldsymbol{x}) = F((s+t)\boldsymbol{x})$, which yields a Cauchy equation on the half-line. Thus the continuity assumption implies the linearity of F on the ray $\mathbb{R}^+\boldsymbol{x}$, which implies the property $F(t\boldsymbol{x}) = tF(\boldsymbol{x})$, $\forall t \ge 0$, and hence $\lambda F(\boldsymbol{x}) + (1-\lambda)F(\boldsymbol{y}) = F(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})$ for any absolutely comonotonic vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\forall \lambda \in [0,1]$. This completes the proof. □

For relations between the original and the disjoint-pair Lovász extensions, we further have

Proposition 2.5. *For $h : \mathcal{P}(V) \to [0, +\infty)$ with $h(\varnothing) = 0$, and $f : \mathcal{P}_2(V) \to [0, +\infty)$ with $f(\varnothing, \varnothing) = 0$* [4]*, we have:*

(a) If $f(A,B) = h(A) + h(V \setminus B) - h(V)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L = h^L$.

(b) If $f(A,B) = h(A) + h(B)$ and $h(A) = h(V \setminus A)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L = h^L$.

(c) If $f(A,B) = h(A)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L(\boldsymbol{x}) = h^L(\boldsymbol{x})$, $\forall \boldsymbol{x} \in [0,\infty)^V$.

(d) If $f(A,B) = h(A \cup B)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L(\boldsymbol{x}) = h^L(\boldsymbol{x}^+ + \boldsymbol{x}^-)$.

[4] In fact, if $h(\varnothing) \ne 0$ or $f(\varnothing, \varnothing) \ne 0$, one may change the value and it does not affect the related Lovász extension.

(e) If $f(A,B)=h(A)\pm h(B)$, $\forall(A,B)\in\mathcal{P}_2(V)$, then $f^L(\boldsymbol{x})=h^L(\boldsymbol{x}^+)\pm h^L(\boldsymbol{x}^-)$.

Here $\boldsymbol{x}^\pm:=(\pm\boldsymbol{x})\vee\boldsymbol{0}$.

In the sequel, we will not distinguish the original and the disjoint-pair Lovász extensions, since the reader can infer it from the domains ($\mathcal{P}(V)$ or $\mathcal{P}_2(V)$). Sometime we work on $\mathcal{P}(V)$ only, and in this situation, the disjoint-pair Lovász extension acts on the redefined $f(A,B)=h(A\cup B)$ as Proposition 2.5 states.

The next result is useful for the application on graph coloring.

Proposition 2.6. *For the simple k-way Lovász extension of $f:\mathcal{P}(V_1)\times\cdots\times\mathcal{P}(V_k)\to\mathbb{R}$ with the separable summation form $f(A_1,\cdots,A_k):=\sum_{i=1}^k f_i(A_i)$, $\forall(A_1,\cdots,A_k)\in\mathcal{P}(V)^k$, we have $f^L(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^k)=\sum_{i=1}^k f_i^L(\boldsymbol{x}^i)$, $\forall(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^k)$.*

For $f:\mathcal{P}_2(V_1)\times\cdots\times\mathcal{P}_2(V_k)\to\mathbb{R}$ with the form $f(A_1,B_1\cdots,A_k,B_k):=\sum_{i=1}^k f_i(A_i,B_i)$, $\forall(A_1,B_1,\cdots,A_k,B_k)\in\mathcal{P}_2(V_1)\times\cdots\times\mathcal{P}_2(V_k)$, there similarly holds $f^L(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^k)=\sum_{i=1}^k f_i^L(\boldsymbol{x}^i)$.

2.2 Submodularity and Convexity

In this subsection, we give new analogs of Theorems 2.1 and 2.2 for the disjoint-pair Lovász extension and the k-way Lovász extension. The major difference to existing results in the literature is that we work with the restricted or the enlarged domain of a function.

Let's first recall the standard concepts of submodularity:

(S1) A discrete function $f:\mathcal{A}\to\mathbb{R}$ is submodular if $f(A)+f(B)\ge f(A\cup B)+f(A\cap B)$, $\forall A,B\in\mathcal{A}$, where $\mathcal{A}\subset\mathcal{P}(V)$ is an algebra.

(S2) A continuous function $F:\mathbb{R}^n\to\mathbb{R}$ is submodular if $F(\boldsymbol{x})+F(\boldsymbol{y})\ge F(\boldsymbol{x}\vee\boldsymbol{y})+F(\boldsymbol{x}\wedge\boldsymbol{y})$, where $(\boldsymbol{x}\vee\boldsymbol{y})_i=\max\{x_i,y_i\}$ and $(\boldsymbol{x}\wedge\boldsymbol{y})_i=\min\{x_i,y_i\}$, $i=1,\cdots,n$. For a sublattice $\mathcal{D}\subset\mathbb{R}^n$ that is closed under $\vee$ and $\wedge$, one can define submodularity in the same way.

Notification 2. *All discussions about algebras of sets can be reduced to lattices. Classical submodular functions on a sublattice of the Boolean lattice $\{0,1\}^n$ and their continuous versions on $\mathbb{R}^n$ are presented in (S1) and (S2), respectively, while bisubmodular functions on a sublattice of the lattice $\{-1,0,1\}^n$ are defined in* (12).

Now, we recall the concept of bisubmodularity and introduce its continuous version.

(BS1) A discrete function $f:\mathcal{P}_2(V)\to\mathbb{R}$ is bisubmodular if $\forall\,(A,B),(C,D)\in\mathcal{P}_2(V)$

$$f(A,B)+f(C,D)\ge f((A\cup C)\setminus(B\cup D),(B\cup D)\setminus(A\cup C))+f(A\cap C,B\cap D). \tag{12}$$

One can denote $A\vee B=((A_1\cup B_1)\setminus(A_2\cup B_2),(A_2\cup B_2)\setminus(A_1\cup B_1))$ and $A\wedge B=(A_1\cap B_1,A_2\cap B_2)$, where $A=(A_1,A_2)$, $B=(B_1,B_2)$. For a sublattice $\mathcal{A}\subset\mathcal{P}_2(V)$ that is closed under $\vee$ and $\wedge$, the bisubmodularity of $f:\mathcal{A}\to\mathbb{R}$ can be expressed as $f(A)+f(B)\ge f(A\vee B)+f(A\wedge B)$, $\forall A,B\in\mathcal{A}$.

If we were to continue the definition of submodularity stated in (S2), we would obtain nothing new. Hence, the proof of Theorem 2.2 cannot directly apply to our situation. To overcome this issue, we need to provide a matched definition of bisubmodularity for functions on $\mathbb{R}^n$, and an appropriate and careful modification of the translation linearity condition.

(BS2) A continuous function $F:\mathbb{R}^n\to\mathbb{R}$ is bisubmodular if $F(x)+F(y)\ge F(x\vee y)+F(x\wedge y)$, where

$$(x\vee y)_i=\begin{cases}\max\{x_i,y_i\}, & \text{if } x_i,y_i\ge 0,\\ \min\{x_i,y_i\}, & \text{if } x_i,y_i\le 0,\\ 0, & \text{if } x_iy_i<0,\end{cases}\qquad (x\wedge y)_i=\begin{cases}\min\{x_i,y_i\}, & \text{if } x_i,y_i\ge 0,\\ \max\{x_i,y_i\}, & \text{if } x_i,y_i\le 0,\\ 0, & \text{if } x_iy_i<0.\end{cases}$$

Proposition 2.7. *A function $F:\mathbb{R}^V\to\mathbb{R}$ is a disjoint-pair Lovász extension of a bisubmodular function if and only if F is (continuously) bisubmodular (in the sense of (BS2)) and for any $\boldsymbol{x}\in\mathbb{R}^V$, $t\ge 0$,*

$F(t\boldsymbol{x})=tF(\boldsymbol{x})$ (positive homogeneity);

$F(\boldsymbol{x}+t\mathbf{1}_{V^+,V^-})\ge F(\boldsymbol{x})+F(t\mathbf{1}_{V^+,V^-})$ for some[5] $V^\pm\supset V^\pm(\boldsymbol{x})$ with $V^+\cup V^-=V$.

Henceforth, $\mathbf{1}_{A,B}$ is defined as $\mathbf{1}_A-\mathbf{1}_B$ for simplicity.

Proof. Take the discrete function f defined as $f(A_1,A_2)=F(\mathbf{1}_{A_1,A_2})$. One can check the bisubmodularity of f directly. Fix an $\boldsymbol{x}\in\mathbb{R}^n$ and let $\sigma:V\cup\{0\}\to V\cup\{0\}$ be a bijection such that $|x_{\sigma(1)}|\le|x_{\sigma(2)}|\le\cdots\le|x_{\sigma(n)}|$ and $\sigma(0)=0$, where $x_0:=0$, and

$$V^\pm_{\sigma(i)}=V^\pm_{\sigma(i)}(\boldsymbol{x}):=\{j\in V:\pm x_j>|x_{\sigma(i)}|\},\quad i=0,1,\cdots,n-1.$$

Also, we denote $\boldsymbol{x}_{V^+_{\sigma(i)},V^-_{\sigma(i)}}=\boldsymbol{x}*\mathbf{1}_{V^+_{\sigma(i)}\cup V^-_{\sigma(i)}}$ (i.e., the restriction of $\boldsymbol{x}$ onto $V^+_{\sigma(i)}\cup V^-_{\sigma(i)}$, with other components 0), where $\boldsymbol{x}*\boldsymbol{y}:=(x_1y_1,\cdots,x_ny_n)$.

For simplicity, in the following formulas, we identify $\sigma(i)$ with i for all $i=0,\cdots,n$.

It follows from $|x_{i+1}|\mathbf{1}_{V_i^+,V_i^-}\bigvee\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}=\boldsymbol{x}_{V_i^+,V_i^-}$ and

$$|x_{i+1}|\mathbf{1}_{V_i^+,V_i^-}\bigwedge\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}=|x_{i+1}|\mathbf{1}_{V^+_{i+1},V^-_{i+1}}$$

that

$$\begin{aligned}f^L(\boldsymbol{x})&=\sum_{i=0}^{n-1}(|x_{i+1}|-|x_i|)f(V_i^+,V_i^-)\\&=\sum_{i=0}^{n-1}|x_{i+1}|\left(f(V_i^+,V_i^-)-f(V^+_{i+1},V^-_{i+1})\right)\\&=\sum_{i=0}^{n-1}\left\{F\left(|x_{i+1}|\mathbf{1}_{V_i^+,V_i^-}\right)-F\left(|x_{i+1}|\mathbf{1}_{V^+_{i+1},V^-_{i+1}}\right)\right\}\\&\ge\sum_{i=0}^{n-1}\left\{F\left(\boldsymbol{x}_{V_i^+,V_i^-}\right)-F\left(\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}\right)\right\}=F(\boldsymbol{x}).\end{aligned}$$

On the other hand,

$$\begin{aligned}f^L(\boldsymbol{x})&=\sum_{i=0}^{n-1}(|x_{i+1}|-|x_i|)f(V_i^+,V_i^-)=\sum_{i=0}^{n-1}F\left((|x_{i+1}|-|x_i|)\mathbf{1}_{V_i^+,V_i^-}\right)\\&=\sum_{i=0}^{n-2}\left\{F((|x_{i+1}|-|x_i|)\mathbf{1}_{V_i^+,V_i^-})-F((|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-})\right\}\\&\quad+\left\{\sum_{i=0}^{n-2}F((|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-})\right\}+F\left((|x_n|-|x_{n-1}|)\mathbf{1}_{V^+_{n-1},V^-_{n-1}}\right)\\&\le\sum_{i=0}^{n-2}\left\{F\left(\boldsymbol{x}_{V_i^+,V_i^-}-|x_i|\mathbf{1}_{V^+,V^-}\right)-F\left(\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}-|x_{i+1}|\mathbf{1}_{V^+,V^-}+(|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-}\right)\right\}\\&\quad+\left\{\sum_{i=0}^{n-2}(|x_{i+1}|-|x_i|)F(\mathbf{1}_{V^+,V^-})\right\}+F\left((|x_n|-|x_{n-1}|)\mathbf{1}_{V^+_{n-1},V^-_{n-1}}\right)\\&\le\sum_{i=0}^{n-2}\left(F(\boldsymbol{x}_{V_i^+,V_i^-}-|x_i|\mathbf{1}_{V^+,V^-})-F(\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}-|x_{i+1}|\mathbf{1}_{V^+,V^-})\right)+F\left((|x_n|-|x_{n-1}|)\mathbf{1}_{V^+_{n-1},V^-_{n-1}}\right)\\&=F(\boldsymbol{x})\end{aligned}$$

according to $(|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-}\bigwedge(\boldsymbol{x}_{V_i^+,V_i^-}-|x_i|\mathbf{1}_{V^+,V^-})=(|x_{i+1}|-|x_i|)\mathbf{1}_{V_i^+,V_i^-}$ and

$$(|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-}\bigvee(\boldsymbol{x}_{V_i^+,V_i^-}-|x_i|\mathbf{1}_{V^+,V^-})=\boldsymbol{x}_{V^+_{i+1},V^-_{i+1}}-|x_{i+1}|\mathbf{1}_{V^+,V^-}+(|x_{i+1}|-|x_i|)\mathbf{1}_{V^+,V^-}$$

[5]This is some kind of 'translation linearity' if we adopt the assumption $F(\boldsymbol{x}+t\mathbf{1}_{V^+,V^-})=F(\boldsymbol{x})+F(t\mathbf{1}_{V^+,V^-})$.

for $i = 0, \cdots, n-2$, as well as $\boldsymbol{x}_{V^+_{n-1},V^-_{n-1}} - |x_{n-1}|\mathbf{1}_{V^+,V^-} = (|x_n| - |x_{n-1}|)\mathbf{1}_{V^+_{n-1},V^-_{n-1}}$. Therefore, we have $F(\boldsymbol{x}) = f^L(\boldsymbol{x})$. The proof is completed. □

Proposition 2.8. *A continuous function F is a disjoint-pair Lovász extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$ if and only if $F(\boldsymbol{x} \vee c\mathbf{1}_{V^+,V^-}) + F(\boldsymbol{x} - \boldsymbol{x} \vee c\mathbf{1}_{V^+,V^-}) = F(\boldsymbol{x})$ (or $F(\boldsymbol{x} \wedge c\mathbf{1}_{V^+,V^-}) + F(\boldsymbol{x} - \boldsymbol{x} \wedge c\mathbf{1}_{V^+,V^-}) = F(\boldsymbol{x})$), for some $V^\pm \supset V^\pm(\boldsymbol{x})$ with $V^+ \cup V^- = V$, $\forall c \geq 0$ and $\boldsymbol{x} \in \mathbb{R}^n$.*

Proof. We only need to prove that the condition implies the absolutely comonotonic additivity of F, and then apply Proposition 2.4. Note that the property $F(\boldsymbol{x} \vee c\mathbf{1}) + F(\boldsymbol{x} - \boldsymbol{x} \vee c\mathbf{1}) = F(\boldsymbol{x})$ implies a summation form of F which agrees with the form of the disjoint-pair Lovász extension. Then using the absolutely comonotonic additivity, we get the desired result. □

The k-way submodularity can be naturally defined as (S1) and (S2):

(KS) Given a tuple $V = (V_1, \cdots, V_k)$ of finite sets and $\mathcal{A} \subset \{(A_1, \cdots, A_k) : A_i \subset V_i, i = 1, \cdots, k\}$, a discrete function $f : \mathcal{A} \to \mathbb{R}$ is k-way submodular if $f(A) + f(B) \geq f(A \vee B) + f(A \wedge B)$, $\forall A, B \in \mathcal{A}$, where $\mathcal{A}$ is a lattice under the corresponding lattice operations join $\vee$ and meet $\wedge$ defined by $A \vee B = (A_1 \cup B_1, \cdots, A_k \cup B_k)$ and $A \wedge B = (A_1 \cap B_1, \cdots, A_k \cap B_k)$.

Theorem 2.3. *Under the assumptions and notations in (KS) above, $\mathcal{D}_\mathcal{A}$ is also closed under $\wedge$ and $\vee$, with $\wedge$ and $\vee$ as in (S2). Moreover, the following statements are equivalent:*

a) f is k-way submodular on $\mathcal{A}$;

b) the k-way Lovász extension f^L is convex on each convex subset of $\mathcal{D}_\mathcal{A}$;

c) the k-way Lovász extension f^L is submodular on $\mathcal{D}_\mathcal{A}$.

If one replaces (KS) and (S2) by (BS1) and (BS2) respectively for the bisubmodular setting, then all the above results hold analogously.

Proof. Note that $V^t(\boldsymbol{x}) \vee V^t(\boldsymbol{y}) = V^t(\boldsymbol{x} \vee \boldsymbol{y})$ and $V^t(\boldsymbol{x}) \wedge V^t(\boldsymbol{y}) = V^t(\boldsymbol{x} \wedge \boldsymbol{y})$, where $V^t(\boldsymbol{x}) := (V^t(\boldsymbol{x}^1), \cdots, V^t(\boldsymbol{x}^k))$, $\forall t \in \mathbb{R}$. Since $\boldsymbol{x} \in \mathcal{D}_\mathcal{A}$ if and only if $V^t(\boldsymbol{x}) \in \mathcal{A}$, $\forall t \in \mathbb{R}$, and $\mathcal{A}$ is a lattice, $\mathcal{D}_\mathcal{A}$ must be a lattice that is closed under the operations $\wedge$ and $\vee$. According to the k-way Lovász extension (7), we may write

$$f^L(\boldsymbol{x}) = \int_{-N}^{N} f(V^t(\boldsymbol{x}))dt - Nf(V)$$

where $N > 0$ is a sufficiently large number[6]. Note that $\mathbf{1}_A \vee \mathbf{1}_B = \mathbf{1}_{A \vee B}$ and $\mathbf{1}_A \wedge \mathbf{1}_B = \mathbf{1}_{A \wedge B}$. Combining the above results, we immediately get

$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B) \;\Leftrightarrow\; f^L(\boldsymbol{x}) + f^L(\boldsymbol{y}) \geq f^L(\boldsymbol{x} \vee \boldsymbol{y}) + f^L(\boldsymbol{x} \wedge \boldsymbol{y}),$$

which proves (a) $\Leftrightarrow$ (c). Note that for $\boldsymbol{x} \in \mathcal{D}_\mathcal{A}$, $f^L(\boldsymbol{x}) = \sum\limits_{A \in \mathcal{C}(\boldsymbol{x})} \lambda_A f(A)$ for a unique chain $\mathcal{C}(\boldsymbol{x}) \subset \mathcal{A}$ that is determined by $\boldsymbol{x}$ only, and the extension $f^{\text{convex}}(\boldsymbol{x}) := \inf\limits_{\{\lambda_A\}_{A \in \mathcal{A}} \in \Lambda(\boldsymbol{x})} \sum\limits_{A \in \mathcal{A}} \lambda_A f(A)$ is convex on each convex subset of $\mathcal{D}_\mathcal{A}$, where $\Lambda(\boldsymbol{x}) := \{\{\lambda_A\}_{A \in \mathcal{A}} \in \mathbb{R}^\mathcal{A} : \sum\limits_{A \in \mathcal{A}} \lambda_A \mathbf{1}_A = \boldsymbol{x}, \lambda_A \geq 0 \text{ whenever } A \neq V\}$. We only need to prove $f^L(\boldsymbol{x}) = f^{\text{convex}}(\boldsymbol{x})$ if and only if f is submodular. In fact, along a standard idea proposed in Lovasz's original paper [1], one could prove that for a (strictly) submodular function, the set $\{A : \lambda^*_A \neq 0\}$ must be a chain, where $\sum\limits_{A \in \mathcal{A}} \lambda^*_A f(A) = f^{\text{convex}}(\boldsymbol{x})$ achieves the minimum over $\Lambda(\boldsymbol{x})$, and one can then easily check that it agrees with f^L. The converse can be proved in a standard way: $f(A) + f(B) = f^L(\mathbf{1}_A) + f^L(\mathbf{1}_B) \geq 2f^L(\frac{1}{2}(\mathbf{1}_A + \mathbf{1}_B)) = f(\mathbf{1}_A + \mathbf{1}_B) = f(\mathbf{1}_{A \vee B} + \mathbf{1}_{A \wedge B}) = f(\mathbf{1}_{A \vee B}) + f(\mathbf{1}_{A \wedge B}) = f(A \vee B) + f(A \wedge B)$. Now, the proof is completed.

For the bisubmodular case, the above reasoning can be repeated with minor differences. □

[6] Here we set $f(\varnothing, \cdots, \varnothing) = 0$

3 Combinatorial and continuous optimization

As we have told in the introduction, the application of the Lovász extension to non-submodular optimization meets with several difficulties, and in this section, we start attacking those. First, we set up some useful results.

Notification 3. *In this section, $\mathbb{R}_{\ge 0} := [0,\infty)$ is the set of all non-negative numbers. We use f^L to denote the multi-way Lovász extension which can be either the original or the disjoint-pair or the k-way Lovász extension.*

Theorem 3.1. *Given set functions $f_1,\cdots,f_n : \mathcal{A}\to\mathbb{R}_{\ge 0}$, and a zero-homogeneous function $H : \mathbb{R}^m_{\ge 0}\setminus\{\mathbf{0}\}\to\mathbb{R}\cup\{+\infty\}$ with $H(\boldsymbol{a}+\boldsymbol{b})\ge\min\{H(\boldsymbol{a}),H(\boldsymbol{b})\}$, $\forall\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^m_{\ge 0}\setminus\{\mathbf{0}\}$, we have*

$$\min_{A\in\mathcal{A}'} H(f_1(A),\cdots,f_n(A)) = \inf_{\boldsymbol{x}\in\mathcal{D}'} H(f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x})), \tag{13}$$

where $\mathcal{A}'=\{A\in\mathcal{A} : (f_1(A),\cdots,f_n(A))\in\mathrm{Dom}(H)\}$, $\mathcal{D}'=\{\boldsymbol{x}\in\mathcal{D}_{\mathcal{A}}\cap\mathbb{R}^V_{\ge 0} : (f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x}))\in\mathrm{Dom}(H)\}$ and $\mathrm{Dom}(H)=\{\boldsymbol{a}\in\mathbb{R}^m_{\ge 0}\setminus\{\mathbf{0}\} : H(\boldsymbol{a})\in\mathbb{R}\}$.

Proof. By the property of H, $\forall t_i\ge 0\,, n\in\mathbb{N}^+,\ a_{i,j}\ge 0, i=1,\cdots,m,\ j=1,\cdots,n$,

$$\begin{aligned} H\left(\sum_{i=1}^m t_i a_{i,1},\cdots,\sum_{i=1}^m t_i a_{i,n}\right) &= H\left(\sum_{i=1}^m t_i\boldsymbol{a}^i\right)\ge\min_{i=1,\cdots,m} H(t_i\boldsymbol{a}^i)\\ &=\min_{i=1,\cdots,m} H(\boldsymbol{a}^i)=\min_{i=1,\cdots,m} H(a_{i,1},\cdots,a_{i,n}).\end{aligned}$$

Therefore, in the case of the original Lovász extension, for any $\boldsymbol{x}\in\mathcal{D}'$,

$$\begin{aligned} & H\left(f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x})\right) && (14)\\ =& H\left(\int_{\min\boldsymbol{x}}^{\max\boldsymbol{x}} f_1(V^t(\boldsymbol{x}))dt+f_1(V(\boldsymbol{x}))\min\boldsymbol{x},\cdots,\int_{\min\boldsymbol{x}}^{\max\boldsymbol{x}} f_n(V^t(\boldsymbol{x}))dt+f_n(V(\boldsymbol{x}))\min\boldsymbol{x}\right)\\ =& H\left(\sum_{i=1}^m(t_i-t_{i-1})f_1(V^{t_{i-1}}(\boldsymbol{x}),\cdots,\sum_{i=1}^m(t_i-t_{i-1})f_n(V^{t_{i-1}}(\boldsymbol{x}))\right)\\ \ge&\min_{i=1,\cdots,m} H\left(f_1(V^{t_{i-1}}(\boldsymbol{x}),\cdots,f_n(V^{t_{i-1}}(\boldsymbol{x}))\right)\\ \ge&\min_{A\in\mathcal{A}'} H(f_1(A),\cdots,f_n(A)) && (15)\\ =&\min_{A\in\mathcal{A}'} H(f_1^L(\mathbf{1}_A),\cdots,f_n^L(\mathbf{1}_A))\\ \ge&\inf_{\boldsymbol{x}\in\mathcal{D}'} H(f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x})). && (16)\end{aligned}$$

Combining (14) with (15), we have $\inf_{\boldsymbol{x}\in\mathcal{D}'} H(f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x}))\ge\min_{A\in\mathcal{A}'} H(f_1(A),\cdots,f_n(A))$, and then together with (15) and (16), we get the reversed inequality. Hence, (13) is proved for the original Lovász extension f^L. For multi-way settings, the proof is similar. □

Remark 1. *Duality: If one replaces $H(\boldsymbol{a}+\boldsymbol{b})\ge\min\{H(\boldsymbol{a}),H(\boldsymbol{b})\}$ by $H(\boldsymbol{a}+\boldsymbol{b})\le\max\{H(\boldsymbol{a}),H(\boldsymbol{b})\}$, then*

$$\max_{A\in\mathcal{A}'} H(f_1(A),\cdots,f_n(A)) = \sup_{\boldsymbol{x}\in\mathcal{D}'} H(f_1^L(\boldsymbol{x}),\cdots,f_n^L(\boldsymbol{x})). \tag{17}$$

The proof of identity (17) is similar to that of (13), and thus we omit it.

Remark 2. *A function $H:[0,+\infty)^n\to\overline{\mathbb{R}}$ has the (MIN) property if*

$$H\left(\sum_{i=1}^m t_i\boldsymbol{a}^i\right)\ge\min_{i=1,\cdots,m} H(\boldsymbol{a}^i),\ \forall t_i>0\,, m\in\mathbb{N}^+,\ \boldsymbol{a}^i\in[0,+\infty)^n.$$

The (MAX) property is formulated analogously.

We can verify that the (MIN) property is equivalent to the zero-homogeneity and $H(\boldsymbol{x}+\boldsymbol{y})\ge\min\{H(\boldsymbol{x}),H(\boldsymbol{y})\}$. A similar correspondence holds for the (MAX) property.

Remark 3. *Theorem 3.1 shows if H has the (MIN) or (MAX) property, then a corresponding combinatorial optimization is equivalent to a continuous optimization by means of the multi-way Lovász extension.*

Taking $n=2$ and $H(f_1,f_2)=\frac{f_1}{f_2}$ in Theorem 3.1, then such an H satisfies both (MIN) and (MAX) properties. So, we get

$$\min_{A\in\mathcal{A}'}\frac{f_1(A)}{f_2(A)}=\inf_{\psi\in\mathcal{D}'}\frac{f_1^L(\psi)}{f_2^L(\psi)},\quad \text{and}\quad \max_{A\in\mathcal{A}'}\frac{f_1(A)}{f_2(A)}=\sup_{\psi\in\mathcal{D}'}\frac{f_1^L(\psi)}{f_2^L(\psi)}.$$

In fact, we can get more:

Proposition 3.1. *Given two set functions $f,g:\mathcal{A}\to[0,+\infty)$, let $\tilde{f},\tilde{g}:\mathcal{D}_{\mathcal{A}}\to\mathbb{R}$ satisfy $\tilde{f}\ge f^L$, $\tilde{g}\le g^L$, $\tilde{f}(\mathbf{1}_A)=f(A)$ and $\tilde{g}(\mathbf{1}_A)=g(A)$. Then*

$$\min_{A\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{f(A)}{g(A)}=\inf_{\psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})}\frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}.$$

If we replace the condition $\tilde{f}\ge f^L$ and $\tilde{g}\le g^L$ by $\tilde{f}\le f^L$ and $\tilde{g}\ge g^L$, then

$$\max_{A\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{f(A)}{g(A)}=\sup_{\psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})}\frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}.$$

If $\rho:[0,\infty)\to[0,\infty)$ is a homeomorphism, then $\tilde{f}:=\rho^{-1}((\rho\circ f)^L)$ and $\tilde{g}:=\rho^{-1}((\rho\circ g)^L)$ satisfy the above two identities.

Proof. It is obvious that

$$\inf_{\psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})}\frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}\le\min_{A\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{\widetilde{f}(\mathbf{1}_A)}{\widetilde{g}(\mathbf{1}_A)}=\min_{A\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{f(A)}{g(A)}.$$

On the other hand, for any $\psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})$, $g^L(\psi)\ge\tilde{g}(\psi)>0$. Hence, there exists $t\in(\min\widetilde{\beta}\psi-1,\max\widetilde{\beta}\psi+1)$ satisfying $g(V^t(\psi))>0$. Here $\widetilde{\beta}\psi=\psi$ (resp., $|\psi|$), if f^L represents either the original or the k-way Lovasz extension of f (resp., either the disjoint-pair or the k-way disjoint-pair Lovasz extension). So, the set $W(\psi):=\{t\in\mathbb{R}:g(V^t(\psi))>0\}$ is nonempty. Since $\{V^t(\psi):t\in W(\psi)\}$ is finite, there exists $t_0\in W(\psi)$ such that $\frac{f(V^{t_0}(\psi))}{g(V^{t_0}(\psi))}=\min\limits_{t\in W(\psi)}\frac{f(V^t(\psi))}{g(V^t(\psi))}$. Accordingly, $f(V^t(\psi))\ge\frac{f(V^{t_0}(\psi))}{g(V^{t_0}(\psi))}g(V^t(\psi))$ for any $t\in W(\psi)$, and thus

$$f(V^t(\psi))\ge Cg(V^t(\psi)),\quad \text{with}\quad C=\min_{t\in W(\psi)}\frac{f(V^t(\psi))}{g(V^t(\psi))}\ge 0,$$

holds for any $t\in\mathbb{R}$ (because $g(V^t(\psi))=0$ for $t\in\mathbb{R}\setminus W(\psi)$ which means that the above inequality automatically holds). Consequently,

$$\begin{aligned}&\tilde{f}(\psi)\ge f^L(\psi)\\ =&\int_{\min\widetilde{\beta}\psi}^{\max\widetilde{\beta}\psi}f(V^t(\psi))dt+f(V(\psi))\min\widetilde{\beta}\psi\\ \ge& C\int_{\min\widetilde{\beta}\psi}^{\max\widetilde{\beta}\psi}g(V^t(\psi))dt+g(V(\psi))\min\widetilde{\beta}\psi.\\ =&Cg^L(\psi)\ge C\tilde{g}(\psi).\end{aligned}$$

It follows that

$$\frac{\widetilde{f}(\psi)}{\widetilde{g}(\psi)}\ge C\ge\min_{A\in\mathcal{A}\cap\mathrm{supp}(g)}\frac{f(A)}{g(A)}$$

and thus the proof is completed. The dual case is similar.

For a homeomorphism $\rho : [0, \infty) \to [0, \infty)$,

$$\min_{A \in \mathcal{A}} \frac{f(A)}{g(A)} = \min_{A \in \mathcal{A}} \frac{\rho^{-1} \circ \rho \circ f(A)}{\rho^{-1} \circ \rho \circ g(A)} = \rho^{-1} \left(\min_{A \in \mathcal{A}} \frac{\rho \circ f(A)}{\rho \circ g(A)} \right)$$
$$= \rho^{-1} \left(\inf_{\psi \in \mathcal{D}_{\mathcal{A}}} \frac{(\rho \circ f)^L(\psi)}{(\rho \circ g)^L(\psi)} \right) = \inf_{\psi \in \mathcal{D}_{\mathcal{A}}} \frac{\rho^{-1}(\rho \circ f)^L(\psi)}{\rho^{-1}(\rho \circ g)^L(\psi)}.$$

This completes the proof. $\square$

Similarly, we have:

Proposition 3.2. *Let $f, g : \mathcal{A} \to [0, +\infty)$ be two set functions and $f := f_1 - f_2$ and $g := g_1 - g_2$ be decompositions of differences of submodular functions.*

Let $\widetilde{f}_2, \widetilde{g}_1$ be the restriction of positively one-homogeneous convex functions onto $\mathcal{D}_{\mathcal{A}}$, with $f_1(A) = \widetilde{f}_1(\mathbf{1}_A)$ and $g_2(A) = \widetilde{g}_2(\mathbf{1}_A)$. Define $\widetilde{f} = f_1^L - \widetilde{f}_2$ and $\widetilde{g} = \widetilde{g}_1 - g_2^L$. Then,

$$\min_{A \in \mathcal{A} \cap \mathrm{supp}(g)} \frac{f(A)}{g(A)} = \min_{\boldsymbol{x} \in \mathcal{D}_{\mathcal{A}} \cap \mathrm{supp}(\widetilde{g})} \frac{\widetilde{f}(\boldsymbol{x})}{\widetilde{g}(\boldsymbol{x})}.$$

Remark 4. *Hirai et al introduce the generalized Lovász extension of $f : \mathcal{L} \to \overline{\mathbb{R}}$ on a graded set $\mathcal{L}$ (see [39, 40]). Since $f^L(\boldsymbol{x}) = \sum_i \lambda_i f(\boldsymbol{p}_i)$ for $\boldsymbol{x} = \sum_i \lambda_i \boldsymbol{p}_i$ lying in the orthoscheme complex $K(\mathcal{L})$, the same results as stated in Theorem 3.1 and Proposition 3.1 hold for such a generalized Lovász extension f^L.*

3.1 A relaxation of a Dinkelbach-type scheme

We would like to establish an iteration framework for finding minimum and maximum eigenvalues. These extremal eigenvalues play significant roles in optimization theory. They can be found via the so-called Dinkelbach iterative scheme [18]. This will provide a good starting point for an appropriate iterative algorithm for the resulting fractional programming. Actually, the equivalent continuous optimization has a fractional form, but such kind of fractions have been hardly touched in the field of fractional programming [19], where optimizing the ratio of a concave function to a convex one is usually considered. For convenience, we shall work in a normed space X in this subsection.

For a convex function $F : X \to \mathbb{R}$, its sub-gradient (or sub-derivative) $\nabla F(\boldsymbol{x})$ is defined as the collection of $\boldsymbol{u} \in X^*$ satisfying $F(\boldsymbol{y}) - F(\boldsymbol{x}) \geq \langle \boldsymbol{u}, \boldsymbol{y} - \boldsymbol{x} \rangle$, $\forall \boldsymbol{y} \in X$, where X^* is the dual of X and $\langle \boldsymbol{u}, \boldsymbol{y} - \boldsymbol{x} \rangle$ is the action of $\boldsymbol{u}$ on $\boldsymbol{y} - \boldsymbol{x}$. The concept of a sub-gradient has been extended to Lipschitz functions. This is called the Clarke derivative [5]:

$$\nabla F(\boldsymbol{x}) = \left\{ \boldsymbol{u} \in X^* \,\middle|\, \limsup_{\boldsymbol{y} \to \boldsymbol{x}, t \to 0^+} \frac{F(\boldsymbol{y} + t\boldsymbol{h}) - F(\boldsymbol{y})}{t} \geq \langle \boldsymbol{u}, \boldsymbol{h} \rangle, \forall \boldsymbol{h} \in X \right\}.$$

And it can even be generalized to the class of lower semi-continuous functions [11, 12].

Theorem 3.2 (Global convergence of a Dinkelbach-type scheme)**.** *Let S be a compact set and let $F, G : S \to \mathbb{R}$ be two continuous functions with $G(\boldsymbol{x}) > 0$, $\forall \boldsymbol{x} \in S$. Then the sequence $\{r^k\}$ generated by the two-step iterative scheme*

$$\begin{cases} \boldsymbol{x}^{k+1} = \underset{\boldsymbol{x} \in S}{\arg\,\mathrm{opti}} \{F(\boldsymbol{x}) - r^k G(\boldsymbol{x})\}, & (18) \\ r^{k+1} = \dfrac{F(\boldsymbol{x}^{k+1})}{G(\boldsymbol{x}^{k+1})}, & (19) \end{cases}$$

from any initial point $\boldsymbol{x}^0 \in S$, converges monotonically to a global optimum of $F(\cdot)/G(\cdot)$, where 'opti' is 'min' or 'max'.

Corollary 3.1. *If F/G is a zero-homogeneous continuous function, then the iterative scheme* (18)(19) *from any initial point $\boldsymbol{x}^0$ converges monotonically to a global optimum on the cone spanned by S (i.e., $\{t\boldsymbol{x} : t > 0, \boldsymbol{x} \in S\}$).*

We note that Theorem 3.2 generalizes Theorem 3.1 in [7] and Theorem 2 in [10]. Since it is a Dinkelbach-type iterative algorithm in the field of fractional programming, we omit the proof.

Many minimization problems in the field of fractional programming possess the form

$$\min \frac{\text{convex } F}{\text{concave } G},$$

which is not necessary to be convex programming. The original Dinkelbach iterative scheme turns the ratio form to the inner problem (18) with the form like

$$\min\,(\text{convex } F - \text{concave } \tilde{G}),$$

which is indeed a convex programming problem. However, most of our examples are in the form

$$\min \frac{\text{convex } F}{\text{convex } G},$$

i.e., both the numerator and the denominator of the fractional object function are convex. Since the difference of two convex functions may not be convex, the inner problem (18) is no longer a convex optimization and hence might be very difficult to solve.

In other practical applications, we may encounter optimization problems of the form

$$\min \frac{\text{convex } F_1 - \text{convex } F_2}{\text{convex } G_1 - \text{convex } G_2}.$$

This is NP-hard in general. Fortunately, we can construct an effective relaxation of (18).

The starting point of the relaxation step is the following fact:

Proposition 3.3. *For any function $f : \mathcal{A} \to \mathbb{R}$, there are two submodular functions f_1 and f_2 on $\mathcal{A}$ such that $f = f_1 - f_2$.*

Proof. Taking g to be a strict submodular function and letting

$$\delta = \min_{A,A'\in\mathcal{A}} \big(g(A) + g(A') - g(A \vee A') - g(A \wedge A')\big) > 0.$$

Set $f_2 = Cg$ and $f_1 = f + f_2$ for a sufficiently large $C > 0$. It is clear that f_2 is strict submodular and f_1 is submodular. So, $f = f_1 - f_2$, which completes the proof. $\square$

Thanks to Proposition 3.3, any discrete function can be expressed as the difference of two submodular functions. Since the Lovász extension of a submodular function is convex, every Lovász extension function is the difference of two convex functions.

Now, we begin to establish a method based on convex programming for solving $\min \frac{F(\boldsymbol{x})}{G(\boldsymbol{x})}$ with $F = F_1 - F_2$ and $G = G_1 - G_2$ being two nonnegative functions, where F_1, F_2, G_1, G_2 are four nonnegative convex functions on X. Let $\{H_{\boldsymbol{y}}(\boldsymbol{x}) : \boldsymbol{y} \in X\}$ be a family of convex differentiable functions on X with $H_{\boldsymbol{y}}(\boldsymbol{x}) \geq H_{\boldsymbol{y}}(\boldsymbol{y})$, $\forall \boldsymbol{x} \in X$. Consider the following three-step iterative scheme

$$\begin{cases} \boldsymbol{x}^{k+1} \in \arg\min\limits_{\boldsymbol{x}\in\mathbb{B}}\{F_1(\boldsymbol{x}) + r^k G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^k, \boldsymbol{x}\rangle + r^k\langle \boldsymbol{v}^k, \boldsymbol{x}\rangle) + H_{\boldsymbol{x}^k}(\boldsymbol{x})\}, & \text{(20a)} \\ r^{k+1} = F(\boldsymbol{x}^{k+1})/G(\boldsymbol{x}^{k+1}), & \text{(20b)} \\ \boldsymbol{u}^{k+1} \in \nabla F_2(\boldsymbol{x}^{k+1}),\ \boldsymbol{v}^{k+1} \in \nabla G_1(\boldsymbol{x}^{k+1}), & \text{(20c)} \end{cases}$$

where $\mathbb{B}$ is a convex body containing $\boldsymbol{0}$ as its inner point. Such a scheme mixing the inverse power (IP) method and steepest decent (SD) method can be well used in computing special eigenpairs of (F, G). Note that the inner problem (20a) is a convex optimization and thus many algorithms in convex programming are applicable.

Theorem 3.3 (Local convergence for a mixed IP-SD scheme)**.** *The sequence $\{r^k\}$ generated by the iterative scheme* (20) *from any initial point $\boldsymbol{x}^0 \in \mathrm{supp}(G)\cap\mathbb{B}$ converges monotonically, where* $\mathrm{supp}(G)$ *is the support of G.*

If X is further assumed to be finite-dimensional, F_1 and G_2 are p-homogeneous with $p \geq 1$, then the limit $\lim_{k\to+\infty} r^k$ is an eigenvalue of (F, G).

Theorem 3.3 partially generalize Theorem 3.4 in [7], Theorem 6 in [8] and the first part of Theorem 3 in [10]. It is indeed an extension of both the IP and the SD method [3,4,16].

Proof of Theorem 3.3. It will be helpful to divide this proof into several parts and steps:

Step 1. We may assume $G(\boldsymbol{x}^k) > 0$ for any k. In fact, the initial point $\boldsymbol{x}^0$ satisfies $G(\boldsymbol{x}^0) > 0$. We will show $F(\boldsymbol{x}^1) = 0$ if $G(\boldsymbol{x}^1) = 0$ and thus the iteration should be terminated at $\boldsymbol{x}^1$. This tells us that we may assume $G(\boldsymbol{x}^k) > 0$ for all k before the termination of the iteration.

Note that

$$\begin{aligned} & F_1(\boldsymbol{x}^1) + r^0 G_2(\boldsymbol{x}^1) - (\langle \boldsymbol{u}^0, \boldsymbol{x}^1\rangle + r^0\langle \boldsymbol{v}^0, \boldsymbol{x}^1\rangle) + H_{\boldsymbol{x}^0}(\boldsymbol{x}^1) \\ \le\ & F_1(\boldsymbol{x}^0) + r^0 G_2(\boldsymbol{x}^0) - (\langle \boldsymbol{u}^0, \boldsymbol{x}^0\rangle + r^0\langle \boldsymbol{v}^0, \boldsymbol{x}^0\rangle) + H_{\boldsymbol{x}^0}(\boldsymbol{x}^0), \end{aligned}$$

which implies

$$\begin{aligned} & F_1(\boldsymbol{x}^1) - F_1(\boldsymbol{x}^0) + r^0(G_2(\boldsymbol{x}^1) - G_2(\boldsymbol{x}^0)) + H_{\boldsymbol{x}^0}(\boldsymbol{x}^1) - H_{\boldsymbol{x}^0}(\boldsymbol{x}^0) \\ \le\ & \langle \boldsymbol{u}^0, \boldsymbol{x}^1 - \boldsymbol{x}^0\rangle + r^0\langle \boldsymbol{v}^0, \boldsymbol{x}^1 - \boldsymbol{x}^0\rangle \le F_2(\boldsymbol{x}^1) - F_2(\boldsymbol{x}^0) + r^0(G_1(\boldsymbol{x}^1) - G_1(\boldsymbol{x}^0)), \end{aligned}$$

i.e.,

$$\begin{aligned} F(\boldsymbol{x}^1) - F(\boldsymbol{x}^0) + H_{\boldsymbol{x}^0}(\boldsymbol{x}^1) - H_{\boldsymbol{x}^0}(\boldsymbol{x}^0) &\le r^0(G(\boldsymbol{x}^1) - G(\boldsymbol{x}^0)) \\ &= -r^0 G(\boldsymbol{x}^0) = -F(\boldsymbol{x}^0). \end{aligned} \tag{21}$$

Since the equality holds, we have $F(\boldsymbol{x}^1) = 0$, $H_{\boldsymbol{x}^0}(\boldsymbol{x}^1) = H_{\boldsymbol{x}^0}(\boldsymbol{x}^0)$, $\langle \boldsymbol{u}^0, \boldsymbol{x}^1 - \boldsymbol{x}^0\rangle = F_2(\boldsymbol{x}^1) - F_2(\boldsymbol{x}^0)$ and $\langle \boldsymbol{v}^0, \boldsymbol{x}^1 - \boldsymbol{x}^0\rangle = G_1(\boldsymbol{x}^1) - G_1(\boldsymbol{x}^0)$. So this step is finished.

Step 2. $\{r^k\}_{k=1}^{\infty}$ is monotonically decreasing and hence convergent.

Similar to (21) in Step 1, we can arrive at

$$F(\boldsymbol{x}^{k+1}) - F(\boldsymbol{x}^k) + H_{\boldsymbol{x}^k}(\boldsymbol{x}^{k+1}) - H_{\boldsymbol{x}^k}(\boldsymbol{x}^k) \le r^k(G(\boldsymbol{x}^{k+1}) - G(\boldsymbol{x}^k)),$$

which leads to

$$F(\boldsymbol{x}^{k+1}) \le r^k G(\boldsymbol{x}^{k+1}).$$

Since $G(\boldsymbol{x}^{k+1})$ is assumed to be positive, $r^{k+1} = F(\boldsymbol{x}^{k+1})/G(\boldsymbol{x}^{k+1}) \le r^k$. Thus, there exists $r^* \in [r_{\min}, r^0]$ such that $\lim\limits_{k\to+\infty} r^k = r^*$.

In the sequel, we assume that the dimension of X is finite.

Step 3. $\{\boldsymbol{x}^k\}$, $\{\boldsymbol{u}^k\}$ and $\{\boldsymbol{v}^k\}$ are sequentially compact.

In this setting, $\mathbb{B}$ must be compact. In consequence, there exist k_i, r^*, $\boldsymbol{x}^*$, $\boldsymbol{x}^{**}$, $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ such that $\boldsymbol{x}^{k_i} \to \boldsymbol{x}^*$, $\boldsymbol{x}^{k_i+1} \to \boldsymbol{x}^{**}$, $\boldsymbol{u}^{k_i} \to \boldsymbol{u}^*$ and $\boldsymbol{v}^{k_i} \to \boldsymbol{v}^*$, as $i \to +\infty$.

Step 4. $\boldsymbol{x}^*$ is a minimum of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x})$ on $\mathbb{B}$.

Let $g(r, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}) = \min\limits_{\boldsymbol{x}\in B}\{F_1(\boldsymbol{x}) + rG_2(\boldsymbol{x}) - (\langle \boldsymbol{u}, \boldsymbol{x}\rangle + r\langle \boldsymbol{v}, \boldsymbol{x}\rangle) + H_{\boldsymbol{y}}(\boldsymbol{x})\}$. It is standard to verify that $g(r, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v})$ is continuous on $\mathbb{R}^1 \times X \times X^* \times X^*$ according to the compactness of $\mathbb{B}$.

Since $g(r^{k_i}, \boldsymbol{x}^{k_i}, \boldsymbol{u}^{k_i}, \boldsymbol{v}^{k_i}) = r^{k_i+1}$, taking $i \to +\infty$, one obtains $g(r^*, \boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{v}^*) = r^*$.

By Step 3, $\boldsymbol{x}^{**}$ attains the minimum of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x})$ on $\mathbb{B}$. Suppose the contrary, that $\boldsymbol{x}^*$ is not a minimum of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x})$ on $\mathbb{B}$. Then

$$\begin{aligned} & F_1(\boldsymbol{x}^{**}) + r^* G_2(\boldsymbol{x}^{**}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}^{**}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}^{**}\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x}^{**}) \\ <\ & F_1(\boldsymbol{x}^*) + r^* G_2(\boldsymbol{x}^*) - (\langle \boldsymbol{u}^*, \boldsymbol{x}^*\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}^*\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x}^*), \end{aligned}$$

and thus $F(\boldsymbol{x}^{**}) < r^* G(\boldsymbol{x}^{**})$ (similar to Step 1), which implies $G(\boldsymbol{x}^{**}) > 0$ and $F(\boldsymbol{x}^{**})/G(\boldsymbol{x}^{**}) < r^*$. This is a contradiction. Consequently, $\boldsymbol{x}^*$ is a minimizer of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle) + H_{\boldsymbol{x}^*}(\boldsymbol{x})$ on $\mathbb{B}$.

Step 5. $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle) \ge 0$, $\forall \boldsymbol{x} \in \mathbb{B}$, and the equality holds when $x = x^*$.

In fact, a small modification of Step 4 shows that $\boldsymbol{x}^*$ is also a minimizer of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle)$ on $\mathbb{B}$, and the minimum value is 0.

We now add the further assumption that F_1 and G_2 are p-homogeneous with $p \ge 1$.

Step 6. $(r^*, \boldsymbol{x}^*)$ is an eigenpair.

Since $\mathbb{B}$ contains 0 as its inner point, we have $\{\alpha \boldsymbol{x} : \boldsymbol{x} \in \mathbb{B}, \alpha \ge 1\} = X$. Keeping $\alpha \ge 1$ and $p \ge 1$ in mind, for any $\alpha \ge 1$ and $\boldsymbol{x} \in \mathbb{B}$,

$$\begin{aligned} & F_1(\alpha\boldsymbol{x}) + r^* G_2(\alpha\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \alpha\boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \alpha\boldsymbol{x}\rangle) \\ = \; & \alpha \left(F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle)\right) + (\alpha^p - \alpha)(F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x})) \\ \text{(by Step 5)} \; \ge \; & (\alpha^p - \alpha)(F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x})) \ge 0. \end{aligned}$$

Consequently, $\boldsymbol{x}^*$ is a minimizer of $F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle)$ on X, and thus

$$\begin{aligned} 0 &\in \nabla|_{\boldsymbol{x}=\boldsymbol{x}^*} \left(F_1(\boldsymbol{x}) + r^* G_2(\boldsymbol{x}) - (\langle \boldsymbol{u}^*, \boldsymbol{x}\rangle + r^*\langle \boldsymbol{v}^*, \boldsymbol{x}\rangle)\right) \\ &= \nabla F_1(\boldsymbol{x}^*) + r^*\nabla G_2(\boldsymbol{x}^*) - \boldsymbol{u}^* - r^*\boldsymbol{v}^* \\ &\subset \nabla F_1(\boldsymbol{x}^*) - \nabla F_2(\boldsymbol{x}^*) + r^*\nabla G_2(\boldsymbol{x}^*) - r^*\nabla G_1(\boldsymbol{x}^*) \\ &= \nabla F(\boldsymbol{x}^*) - r^*\nabla G(\boldsymbol{x}^*), \end{aligned}$$

which implies that $(r^*, \boldsymbol{x}^*)$ is an eigenpair of (F, G).

□

Another solver for the continuous optimization $\min \frac{F(\boldsymbol{x})}{G(\boldsymbol{x})}$ is the stochastic subgradient method:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \alpha_k(\boldsymbol{y}^k + \boldsymbol{\xi}^k), \quad \boldsymbol{y}^k \in \nabla\frac{F(\boldsymbol{x}^k)}{G(\boldsymbol{x}^k)},$$

where $\{\alpha_k\}_{k\ge1}$ is a step-size sequence and $\{\boldsymbol{\xi}^k\}_{k\ge1}$ is now a sequence of random variables (the "noise") on some probability space. Theorem 4.2 in [44] shows that under some natural assumptions, almost surely, every limit point of the stochastic subgradient iterates $\{\boldsymbol{x}^k\}_{k\ge1}$ is critical for F/G, and the function values $\{\frac{F}{G}(\boldsymbol{x}^k)\}_{k\ge1}$ converge.

4 Discrete Morse theory and its Lovász extension

Morse theory [20, 21] enables us to analyze the topology of an object M by studying functions $f : M \to \mathbb{R}$. In the classical case, M is a manifold and f is generic and differentiable. There are, however, many extensions of Morse theory in modern mathematics that do not require a smooth structure, such as the metric and topological Morse theory by the Italian school [11–14], the PS (piecewise smooth) or stratified Morse theory by Thom, Goresky and MacPherson [51], the PL Morse theory by Banchoff [45], Kühnel [47, 48] and the Berlin school, as well as the discrete Morse theory by Forman [36, 37].

In all such cases, a typical function f on M will reflect the topology quite directly, allowing one to find CW structures on M and to obtain information about their homology. The following results embody the abstract content of Morse theory, and they hold in continuous as well as in discrete cases.

Morse fundamental theorem. If f has n_i critical points of index i, $i = 0, 1, \cdots, d$, then M is homotopy equivalent to a cell complex (called Morse complex) with n_i cells of dimension i. One can write it as

$$M \simeq \text{cell complex with } n_i \text{ cells of } \dim i$$

Morse relation. Denote by $P(X,A)(\cdot)$ the Poincare polynomial[7] of the pair of topological spaces (X,A) over a given field $\mathbb{F}$, where $X \supset A$. Then

$$\sum_{a<f(x)<b} P(\{f \le f(x)\}, \{f \le f(x)\} \setminus \{x\})(t) = P(\{f < b\}, \{f \le a\})(t) + (1+t)Q(t)$$

where $a < b$, $Q(\cdot)$ is a polynomial with nonnegative coefficients.

The main aim of this section is to study the Lovász extension of a discrete Morse function on a simplicial complex, and to provide equivalences between discrete Morse theory and its Lovász extension.

For this purpose, we first clarify the notions and concepts and summarize the various Morse theories mentioned above.

- **Metric Morse theory**: Let M be a metric space and F a continuous function on M. For a point $\boldsymbol{a} \in M$, there exists $\epsilon \ge 0$ such that there exist $\delta > 0$ and a continuous map

  $$\mathcal{H} : B_\delta(\boldsymbol{a}) \times [0, \delta] \to M$$

  satisfying

  $$F(\mathcal{H}(\boldsymbol{x}, t)) \le F(\boldsymbol{x}) - \epsilon t, \ \ \mathrm{dist}(\mathcal{H}(\boldsymbol{x}, t), \boldsymbol{x}) \le t$$

  for any $\boldsymbol{x} \in B_\delta(\boldsymbol{a})$ and $t \in [0, \delta]$. The *weak slope* [12–14] denoted by $|dF|(\boldsymbol{a})$ is defined as the supremum of such ϵ above. A point $\boldsymbol{a}$ is called a critical point of F on M, if it has vanishing weak slope, i.e., $|dF|(\boldsymbol{a}) = 0$.

  The local behaviour of F near $\boldsymbol{a}$ is described by the so-called *critical group* $C_q(F, \boldsymbol{a}) := H_q(\{F \le c\} \cap U_{\boldsymbol{a}}, \{F \le c\} \cap U_{\boldsymbol{a}} \setminus \{\boldsymbol{a}\})$, $q \in \mathbb{Z}$, where $H_*(\cdot, \cdot)$ is the singular relative homology. So the Morse polynomial $p(F, \boldsymbol{a})(t) := \sum_{q=0}^{d} \mathrm{rank}\, C_q(F, \boldsymbol{a}) t^q$ can be defined. If $C_q(F, \boldsymbol{a})$ is non-vanishing, then we say q is an index of a metric critical point $\boldsymbol{a}$, and the number $p(F, \boldsymbol{a})(1)$ is called the *total multiplicity* of $\boldsymbol{a}$.

- **Topological Morse theory**: Let M be a topological space and F a continuous function on M. A point $\boldsymbol{a} \in M$ is a *Morse regular point* of F if there exist a neighborhood U and a continuous map

  $$\mathcal{H} : U \times [0, 1] \to M, \ \ \mathcal{H}(\boldsymbol{x}, 0) = \boldsymbol{x}$$

  satisfying

  $$F(\mathcal{H}(\boldsymbol{x}, t)) < F(\boldsymbol{x}),$$

  for any $\boldsymbol{x} \in U$ and $t > 0$. We say $\boldsymbol{a}$ is a *Morse critical point* of F on M if it is not Morse regular. The index with multiplicity of a critical point is same as in the metric setting above [11].

  A *symmetric homological critical value* [49] of F is a real number c for which there exists an integer such that for all sufficiently small $\epsilon > 0$, the map $H_k(\{F \le c - \epsilon\}) \hookrightarrow H_k(\{F \le c + \epsilon\})$ induced by inclusion is not an isomorphism [50]. Here H_k denotes the k-th singular homology (possibly with coefficients in a field).

  A real number c is a *homological regular value* of the function F if there exists $\epsilon > 0$ such that for each pair of real numbers $t_1 < t_2$ on the interval $(c-\epsilon, c+\epsilon)$, the inclusion $\{F \le t_1\} \hookrightarrow \{F \le t_2\}$ induces isomorphisms on all homology groups [50]. A real number that is not a homological regular value of F is called a *homological critical value* of F.

- **Piecewise-Linear Morse theory**: Similar to the smooth setting, the PL (piecewise linear) Morse theory introduced by Banchoff requires working with a *combinatorial manifold* which is both a PL manifold and a simplicial complex. Here we will use the notions developed by Kühnel [47] and later by Edelsbrunner [48].

  Denote by $\mathrm{star}_-(v)$ the subset of the star of v on which the PL function F takes values not greater than $F(v)$. Similarly, one can define $\mathrm{link}_-(v)$.

  Let M be a combinatorial manifold, and let F be a PL (piecewise linear) function on M.

[7] Formally, $P(X,A)(t) := \sum_{n\ge 0} \mathrm{rank}\, H^n(X,A) t^n$, where $H^n(X,A)$ is the relative cohomology of the pair (X,A).

Definition 4.1 (Kühnel [47])**.** *A vertex v of M is said to be a PL critical point of F with index i and multiplicity k_i if $\beta_i(\overline{\mathrm{star}_-(v)}, \mathrm{link}_-(v)) = k_i$, where β_i is the i-th Betti number of the relative homology group.*

Equivalently, let β'_j be the rank of the reduced j-th homology group of $\mathrm{link}_-(v)$. Using this notation, we have

Definition 4.2 (Edelsbrunner [48])**.** *A vertex v is a PL critical point of F with index i and multiplicity k_i if $\beta'_{i-1} = k_i$.*

Clearly, a PL critical point may have many indices and multiplicities. A vertex v is called non-degenerate critical if its total multiplicity $\sum_{i=0}^{d} k_i$ is equal to 1. The PL function F is called a PL Morse function if all critical vertices are non-degenerate.

- **Discrete Morse theory**: A discrete Morse function on an abstract simplicial complex $(V, \mathcal{K})$ is a function $f : K \to \mathbb{R}$ satisfying for any p-dimensional simplex $\sigma \in K_p$, $\#U(\sigma) \le 1$ and $\#L(\sigma) \le 1$, where

$$U(\sigma) := \{\tau^{p+1} \supset \sigma : f(\tau) \le f(\sigma)\} \;\text{ and }\; L(\sigma) := \{\nu^{p-1} \subset \sigma : f(\nu) \ge f(\sigma)\}.$$

Definition 4.3 (Forman [36,37])**.** *We say that $\sigma \in K_p$ is a critical point of f on K if $\#U(\sigma) = 0$ and $\#L(\sigma) = 0$. The index of a critical point σ is defined to be* $\dim \sigma$.

The main results in this section can be summarized by:

Discrete Morse data of a typical discrete Morse function f on a finite simplicial complex $\mathcal{K}$ $\xleftrightarrow{\text{equivalent}}$ Continuous Morse data of the Lovász extension f^L restricted on a suitable domain

While the discrete Morse data are taken here in the sense of Forman, the continuous Morse data can be in the metric, topological or PL category as described above.

Precise statements are presented in the following subsection.

4.1 Relations between discrete Morse theory and its continuous extension

A finite simplicial complex $\mathcal{K}$ with vertex set V can be the power set $\mathcal{P}(V)$. But this case is trivial. For simplicity, we always assume $\{\{i\} : i \in V\} \subset \mathcal{K} \subsetneqq \mathcal{P}(V)$ in this section.

Definition 4.4. *The order complex of $\mathcal{K}$ is defined by*

$$S_{\mathcal{K}} := \{\mathcal{C} \subset \mathcal{K} : \mathcal{C} \text{ is a chain}\},$$

where $\mathcal{C}$ is a chain if for any $\sigma_1, \sigma_2 \in \mathcal{C}$, either $\sigma_1 \subset \sigma_2$ or $\sigma_2 \subset \sigma_1$. It is clear that $S_{\mathcal{K}}$ is a simplicial complex with the vertex set $\mathcal{K}$. Define the special geometric realization of $S_{\mathcal{K}}$ by

$$|S_{\mathcal{K}}| = \bigcup_{\mathcal{C} \in S_{\mathcal{K}}} \mathrm{conv}(\mathbf{1}_\sigma : \sigma \in \mathcal{C}).$$

Fact: For any function $f : \mathcal{K} \to \mathbb{R}$, the feasible domain $\mathcal{D}_{\mathcal{K}}$ of its Lovász extension f^L is $\bigcup\limits_{t \ge 0} t|S_{\mathcal{K}}|$. It means that the Lovász extension f^L is well-defined on $|S_{\mathcal{K}}|$.

Observation:

$$|S_{\mathcal{K}}| = \mathcal{D}_{\mathcal{K}} \cap S_\infty = \bigcup_{\text{maximal chain } \mathcal{C} \subset \mathcal{K}} \mathrm{conv}(\mathbf{1}_\sigma : \sigma \in \mathcal{C}),$$

where $S_\infty = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty = 1\}$ is the unit l^∞-sphere. Maximal chains from $\mathcal{K}$ have a one-to-one correspondence with facets of $|S_{\mathcal{K}}|$.

Lemma 4.1. *Given a discrete Morse function f on a finite simplicial complex $\mathcal{K}$, we have:*

(1) If σ is critical, then $f(\tau) > f(\sigma) > f(\nu)$, whenever $\tau \supset \sigma \supset \nu$.

(2) If f is an injective Morse function and (σ, τ) is a regular pair (i.e., $\sigma \subset \tau$ with $\dim \tau = \dim \sigma + 1$ and $f(\sigma) > f(\tau)$), then

(2.1) for all $\tau' \supset \sigma$ with $\tau' \not\supset \tau \setminus \sigma$, $f(\tau') > f(\sigma)$;

(2.2) for any $\sigma' \subset \tau$ with $\sigma' \supset \tau \setminus \sigma$, $f(\sigma') < f(\tau)$;

(2.3) for each $\sigma'' \subset \sigma$, $f(\sigma'') < f(\sigma)$.

Proof of Lemma 4.1. Let $c = f(\sigma)$. Note that $f(\nu^{p-1}) < c$ for all $\nu^{p-1} \subset \sigma$. If there exists $\nu^{p-2} \subset \sigma$ such that $f(\nu^{p-2}) \geq c$, then $f(\nu^{p-2}) \geq f(\sigma) > f(\nu^{p-1})$ for all $\nu^{p-1} \supset \nu^{p-2}$ with $\nu^{p-1} \subset \sigma$. Since there are two ν^{p-1} in σ containing ν^{p-2}, this is not compatible with the definition of a discrete Morse function. In this way, we can prove by induction on the dimension of faces of σ that every face $\nu \subset \sigma$ satisfies $f(\nu) < c$.

The other proofs are similar. □

Notification 4. *We use $\cong$ and $\simeq$ to express homeomorphism equivalence and homotopy equivalence, respectively. The link and star of some $\sigma \in \mathcal{K}$ will be taken on $S_{\mathcal{K}}$. The operation $*$ is the geometric join operator.*

Lemma 4.2. *Given an injective Morse function, we have:*

$$\mathrm{link}_-(\sigma) \simeq \begin{cases} \mathbb{S}^{\dim \sigma - 1}, & \text{if } \sigma \text{ is critical}, \\ \mathrm{pt}, & \text{if } \sigma \text{ is regular}. \end{cases}$$

Proof. The link of σ in the order complex $|S_{\mathcal{K}}|$ is the geometric join of

$$\mathbb{S}_-(\mathbf{1}_\sigma) := \bigcup_{\text{chain } \mathcal{C} \subset \mathcal{P}(\sigma) \setminus \{\sigma\}} \mathrm{conv}(\mathbf{1}_\nu : \nu \in \mathcal{C}) \cong \mathbb{S}^{\dim \sigma - 1}$$

and

$$\bigcup_{\text{chain } \mathcal{C} \subset \{\tau \in \mathcal{K} | \tau \supsetneqq \sigma\}} \mathrm{conv}(\mathbf{1}_\tau : \tau \in \mathcal{C}).$$

According to Lemma 4.1 and the definition of $\mathrm{link}_-(\sigma)$, we obtain that if σ is critical, then

$$\mathrm{link}_-(\sigma) := \mathrm{link}_-(\mathbf{1}_\sigma) = \mathbb{S}_-(\mathbf{1}_\sigma) \cong \mathbb{S}^{\dim \sigma - 1}.$$

If (σ, τ) is a regular pair, we note that $\mathrm{link}_-(\sigma)$ is the join of $\mathbb{S}_-(\mathbf{1}_\sigma)$ and

$$\bigcup_{\text{chain } \mathcal{C} \in [\sigma, \tau]_f} \mathrm{conv}(\mathbf{1}_{\tau'} : \tau' \in \mathcal{C}) \simeq \mathbf{1}_\tau,$$

where $[\sigma, \tau]_f := \{\tau' \supset \tau : f(\tau') < f(\sigma)\}$. That means, $\mathrm{link}_-(\sigma) \simeq \mathbb{S}^{\dim \sigma - 1} * \mathbf{1}_\tau \cong \mathbb{B}^{\dim \sigma} \simeq \mathrm{pt}$.

Similarly, one can check that $\mathrm{link}_-(\tau) \cong \mathbb{B}^{\dim \tau - 1} \simeq \mathrm{pt}$. The proof is completed. □

Lemma 4.3 (Kühnel [46]). *Given a PL function f^{PL} on a simplicial complex $|\mathcal{K}|$, then the induced subcomplex of $\mathcal{K}$ on $\{v \in \mathcal{K}_0 : f^{PL}(v) \leq t\}$ is homotopic to the sublevel set $\{f^{PL} \leq t\}$.*

Lemma 4.4. *Given an injective Morse function, denote by $\epsilon_0 = \min\{|f(\sigma) - f(\sigma')| : \sigma \neq \sigma'\} > 0$.*

If σ is critical, then

$$|S_{\mathcal{K}}| \cap \{f^L \leq t\} \cap \mathbb{B}_{\mathbf{1}_\sigma} \simeq \begin{cases} \mathbb{S}^{\dim \sigma - 1}, & \text{if } f(\sigma) - \epsilon_0 < t < f(\sigma), \\ \mathbb{B}^{\dim \sigma}, & \text{if } f(\sigma) \leq t < f(\sigma) + \epsilon_0. \end{cases}$$

And $\mathbf{1}_\sigma$ is a topological/metric critical point of $f^L|_{|S_{\mathcal{K}}|}$, and $f(\sigma)$ is a (symmetric) homological critical value.

Proof. Denote by

$$|S_\sigma| = \bigcup_{\text{maximal chain } \mathcal{C}\subset\mathcal{P}(\sigma)} \mathrm{conv}(\mathbf{1}_\nu : \nu \in \mathcal{C}).$$

Then it can be checked that $|S_\sigma|$ is homeomorphism to the closed geometric simplex $|\overline{\sigma}|$ in $|\mathcal{K}|$, and thus it is homotopic to the disc $\mathbb{B}^{\dim\sigma}$. Hence, $|S_\sigma| \cap \mathbb{B}_{\mathbf{1}_\sigma} \cap \{f^L < t\}$ is homotopic to $\mathbb{S}^{\dim\sigma-1}$. Together with the piecewise linearity of f^L, one gets that $|S_\mathcal{K}| \cap \mathbb{B}_{\mathbf{1}_\sigma} \cap \{f^L < t\}$ is homotopic to $|S_\sigma| \cap \mathbb{B}_{\mathbf{1}_\sigma} \cap \{f^L < t\}$ and thus the proof is completed.

For more details, we may apply Lemma 4.3 to f^L on $|S_\mathcal{K}|$. Then we only need to check the homotopy type of $\mathrm{star}_-(\sigma)$ in $|S_\mathcal{K}|$ for $t \ge f(\sigma)$ and $\mathrm{link}_-(\sigma)$ for $t < f(\sigma)$. According to Lemma 4.1 and similar to the proof of Lemma 4.2, we obtain that for a critical point σ, $\mathrm{star}_-(\sigma)$ is

$$\bigcup_{\text{chain } \mathcal{C}\subset\mathcal{P}(\sigma)} \mathrm{conv}(\mathbf{1}_\nu : \nu \in \mathcal{C}) \cong \mathbb{B}^{\dim\sigma},$$

and $\mathrm{link}_-(\sigma) \cong \mathbb{S}^{\dim\sigma-1}$. The proof is completed. □

Lemma 4.5. *If (σ,τ) is a regular pair, then $\left|df^L|_{|S_\mathcal{K}|}\right|(\mathbf{1}_\sigma) > 0$ and $\left|df^L|_{|S_\mathcal{K}|}\right|(\mathbf{1}_\tau) > 0$.*

Proof. By the definition of weak slope, we should construct a locally decreasing flow from a neighborhood of $\mathbf{1}_\sigma$ to a neighborhood of $\mathbf{1}_\tau$.

Case 1. Locally decreasing flow near $\mathbf{1}_\sigma$: For any chain containing the pair (σ,τ), consider the decreasing vector $\overrightarrow{\mathbf{1}_\sigma\mathbf{1}_\tau}$. Then with the help of Lemma 4.1 (2), the neighborhood of $\mathbf{1}_\sigma$ on $|S_\mathcal{K}|$ can be decreased uniformly along the direction $\overrightarrow{\mathbf{1}_\sigma\mathbf{1}_\tau}$ with a small modification. Slight perturbations and concrete approximations in the construction of the locally decreasing flow are necessary, but we omit the tedious and elementary process.

Case 2. Locally decreasing flow near $\mathbf{1}_\tau$: The construction depends on Lemma 4.1 (2), as in Case 1.

By the deformation lemma, $\mathbf{1}_\sigma$ is Morse regular, and by the piecewise linearity of f^L, $\mathbf{1}_\sigma$ is not a critical point in the sense of weak slope. Moreover, points on $|S_\mathcal{K}|$ other than vertices of $|S_\mathcal{K}|$ cannot be critical points of f^L if f is injective. □

Theorem 4.1. *Given a finite simplicial complex with vertex set V and face set $\mathcal{K}$, let $f : \mathcal{K} \to \mathbb{R}$ be a discrete Morse function.*

If σ is a critical point of f, then $\mathbf{1}_\sigma$ is a critical point of $f^L|_{|S_\mathcal{K}|}$ with the same index in the sense of topological/metric/PL critical point theory, and the converse holds if f is further assumed to be injective.

Proof. The proof is a combination of Lemmas 4.2, 4.4 and 4.5. □

Definition 4.5. *If a generic discrete Morse function $f : \mathcal{K} \to \mathbb{R}$ has n_i critical points of index i, we say that $\mathcal{K}$ has* discrete Morse vector *$\boldsymbol{c} = (n_0, n_1, \cdots, n_d)$. Similarly, for a generic Lipschitz function on a piecewise flat metric space M having n_i critical points of index i, we say that M has* Morse vector *$\boldsymbol{c} = (n_0, n_1, \cdots, n_d)$.*

Now we verify that the discrete Morse structure on a simplicial complex is equivalent to the continuous Morse structure on the restricted domain of its Lovász extension. The key idea is to translate it into PL Morse theory by barycentric subdivision. This discovers the relation between the discrete Morse vectors of $\mathcal{K}$ and the Morse vectors of $|S_\mathcal{K}|$. Such a result is relevant for the main results in [34], but we develop it here in a wider context.

Theorem 4.2. *Given a finite simplicial complex with vertex set V and face set $\mathcal{K}$, let $f : \mathcal{K} \to \mathbb{R}$ be an injective discrete Morse function. Then the discrete Morse vector of f agrees with the continuous Morse vector of $f^L|_{|S_\mathcal{K}|}$.*

Proof. It can be checked that the simplicial complex $(\mathcal{K}, S_{\mathcal{K}})$ is simplicially equivalent to the simplicial complex obtained by the barycentric subdivision of $(V, \mathcal{K})$:

$$(\mathcal{K}, S_{\mathcal{K}}) \underset{\text{equivalent}}{\overset{\text{simplicially}}{\Longleftrightarrow}} \mathrm{sd}(V, \mathcal{K})$$

where $\mathrm{sd}(V, \mathcal{K})$ is the barycentric subdivision of the complex $(V, \mathcal{K})$. Here two complexes are called *simplicially equivalent* (or combinatorial equivalent) if their face posets[8] are isomorphic as posets.

Thus, one may redefine a discrete function $\hat{f}$ on the vertex set of the barycentric subdivision

$$\boxed{\hat{f} : \mathcal{V}(\mathrm{sd}(\mathcal{K})) \to \mathbb{R}} \overset{\text{equivalent}}{\longleftrightarrow} \boxed{f : \mathcal{K} \to \mathbb{R}}$$

via $\hat{f}(v_\sigma) = f(\sigma)$, $\forall \sigma \in \mathcal{K}$, where $\mathcal{V}(\mathrm{sd}(\mathcal{K}))$ is the vertex set of $\mathrm{sd}(\mathcal{K})$.

Then the Lovász extension f^L is piecewise-linearly equivalent to the piecewise linear extension $\hat{f}^{PL}$ defined by

$$\hat{f}^{PL}(\sum_{v \in F} t_v v) = \sum_{v \in F} t_v \hat{f}(v)$$

for any face F of the refined barycentric complex and any $t_v \geq 0$ with $\sum_{v \in F} t_v = 1$. Combining the above observations, we get the following commutative diagram:

$$\begin{tikzcd}
f \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow, "\text{Lovász extension}"'] & & \hat{f} \arrow[d, Rightarrow, "\text{PL extension}"] \\
f^L|_{|S_{\mathcal{K}}|} \arrow[rr, Leftrightarrow, "\text{PL equivalent}"'] & & \hat{f}^{PL}
\end{tikzcd}$$

from which we derive that the Morse data of $f^L|_{|S_{\mathcal{K}}|}$ and $\hat{f}^{PL}$ are entirely equivalent, and furthermore, the (continuous) Morse structures of $|S_{\mathcal{K}}|$ and $|\mathrm{sd}(\mathcal{K})|$ essentially agree with each other.

It is clear that $\{f^L|_{|S_{\mathcal{K}}|} \leq t\}$ is homeomorphic to $\{\hat{f}^{PL} \leq t\}$. Applying Lemma 4.3, $\{\hat{f}^{PL} \leq t\}$ is homotopic to the induced subcomplex on the sublevel set $\{\hat{f} \leq t\}$. Note that the level subcomplex induced by $\{f \leq t\}$ could collapse onto the induced subcomplex on the sublevel set $\{\hat{f} \leq t\}$. So we have

$$|\mathcal{K}(\{f \leq t\})| \simeq |\mathrm{Sd}_{\mathcal{K}}(\{\hat{f} \leq t\})| \simeq \{\hat{f}^{PL} \leq t\} \cong \{f^L|_{|S_{\mathcal{K}}|} \leq t\}$$

and thus the statement is proved. $\square$

Theorems 4.1 and 4.2 establish a correspondence between the geometric data of a discrete Morse function and the geometric information of its Lovász extension.

We shall now introduce the concept of a category in the sense of critical point theory on an abstract simplicial complex $(V, \mathcal{K})$ at level m. We recall the classical Lusternik-Schnirelman category

$$\mathrm{cat}(S) := \min\{k \in \mathbb{N}^+ : \exists k+1 \text{ contractible sets } U_0, U_1, \cdots, U_k \text{ with } \cup_{i=0}^k U_i \supset S\},$$

where we call U_i contractible if the inclusion map $U_i \hookrightarrow |S_{\mathcal{K}}|$ is null-homotopic. We then put

$$\mathrm{Cat}_m(\mathcal{K}) = \{L \subset \mathcal{K} : \ \mathrm{cat}(|S_{\mathcal{K}}(L)|) \geq m\}$$

where $S_{\mathcal{K}}(L)$ is the induced subcomplex of $S_{\mathcal{K}}$ on L. Note that this is a family of subsets of $\mathcal{P}(\mathcal{K})$ (not $\mathcal{P}(V)$!). Similarly,

$$\mathrm{Cat}_m(|S_{\mathcal{K}}|) = \{S \subset |S_{\mathcal{K}}| : \ \mathrm{cat}(S) \geq m\}$$

We are now ready to establish a Lusternik-Schnirelman category theorem relating a discrete Morse function and its Lovász extension:

[8] The *face poset* of a complex is the set of all of its simplices, ordered by inclusion.

Theorem 4.3 (L-S category theorem for a discrete Morse function and its Lovász extension)**.** *Let $f : \mathcal{K} \to \mathbb{R}$ be an injective discrete Morse function. Then we have a sequence of critical values:*

$$\min_{L\in\mathrm{Cat}_m(\mathcal{K})}\max_{\sigma\in L} f(\sigma) = \inf_{S\in\mathrm{Cat}_m(|S_{\mathcal{K}}|)}\sup_{\boldsymbol{x}\in S} f^L(\boldsymbol{x}), \quad m = 0, 1, \cdots, \dim\mathcal{K}.$$

Proof. For any $S \in \mathrm{Cat}_m(|S_{\mathcal{K}}|)$, f^L achieves a maximum on S at some point s, that is, $f^L(\boldsymbol{s}) = \sup_{\boldsymbol{x}\in S} f^L(\boldsymbol{x})$. If $\boldsymbol{s}$ does not belong to the vertex set of $|S_{\mathcal{K}}|$, then $\boldsymbol{s}$ is not an inner point of S according to the definition of f^L. So $\boldsymbol{s} \in \partial S \setminus \mathrm{Vertex}(|S_{\mathcal{K}}|)$, and thus we can take a small perturbation S' of S such that $S' \in \mathrm{Cat}_m(|S_{\mathcal{K}}|)$ and $\sup f^L(S') < f(\boldsymbol{s}) = \sup f^L(S)$. Therefore, we only need to consider such S with the property that $\max_{\boldsymbol{x}\in S} f^L(\boldsymbol{x})$ is achieved at some vertex points of $|S_{\mathcal{K}}|$. For such S, there exists $\boldsymbol{v} \in S$ satisfying $f^L(\boldsymbol{v}) \geq f^L(\boldsymbol{x})$, $\forall \boldsymbol{x} \in S$. Consider the sublevel set $\{f^L \leq f(\boldsymbol{v})\}$. It is clear that $\mathrm{cat}(\{f^L \leq f(\boldsymbol{v})\}) \geq \mathrm{cat}(S) \geq m$ and $\max f^L(\{f^L \leq f(\boldsymbol{v})\}) = f(\boldsymbol{v}) = \max f^L(S)$.

Claim. $\mathrm{cat}(\{f^L \leq a\}) = \mathrm{cat}(S_{\mathcal{K}}|_{\{\sigma\in\mathcal{K}:f(\sigma)\leq a\}})$, where $S_{\mathcal{K}}|_{\{\sigma\in\mathcal{K}:f(\sigma)\leq a\}}$ is the induced closed subcomplex of $S_{\mathcal{K}}$ on the vertices $\{\sigma \in \mathcal{K} : f(\sigma) \leq a\}$ of $S_{\mathcal{K}}$.

Proof. In fact, by Lemma 4.3, there is a homotopy equivalence between $\{f^L \leq a\}$ and $S_{\mathcal{K}}|_{\{\sigma\in\mathcal{K}:f(\sigma)\leq a\}}$.

By the above claim, we establish the following identities

$$\begin{aligned}\inf_{S\in\mathrm{Cat}_m(|S_{\mathcal{K}}|)}\sup_{\boldsymbol{x}\in S} f^L(\boldsymbol{x}) &= \inf_{a\in\mathbb{R}\text{ s.t. }\{f^L\leq a\}\in\mathrm{Cat}_m(|S_{\mathcal{K}}|)}\ \sup_{\boldsymbol{x}\in\{f^L\leq a\}} f^L(\boldsymbol{x}) \\ &= \min_{a\in\mathbb{R}\text{ s.t. }S_{\mathcal{K}}|_{\{\sigma\in\mathcal{K}:f(\sigma)\leq a\}}\in\mathrm{Cat}_m(\mathcal{K})}\ \max_{\sigma\in S_{\mathcal{K}}|_{\{\sigma\in\mathcal{K}:f(\sigma)\leq a\}}} f(\sigma) \\ &= \min_{L\in\mathrm{Cat}_m(\mathcal{K})}\max_{\sigma\in L} f(\sigma).\end{aligned}$$

□

We point out that our notion of discrete Lusternik-Schnirelman category for abstract simplicial complexes is different from that of Definition 4.3 in [55]. We also remark that other recent Lusternik-Schnirelman category theorems for discrete Morse theory do not lead to a result like Theorem 4.3.

4.2 Discrete Morse theory on hypergraphs

In the preceding, we have established a correspondence between discrete Morse theory on a simplicial complex $\mathcal{K}$ with vertex set V and continuous Morse theory on the associated order complex $S_{\mathcal{K}}$. Now, since the order complex $S_{\mathcal{E}}$ is still a simplicial complex when $\mathcal{E}$ is only a hypergraph with vertex set V, we can use the continuous Morse theory on that complex to define a discrete Morse theory on $\mathcal{E}$. That is what we shall now do.

A *hypergraph* is a pair $(V, \mathcal{E})$ with $\mathcal{E} \subset \mathcal{P}(V)$. In other words, $\mathcal{E}$ is a general set family on V. We consider the combinatorial structure of a hypergraph from a topological perspective.

Topologies on hypergraph. There are several ways to endow a finite hypergraph $(V, \mathcal{E})$ with a topology.

1) The finite topology $(\mathcal{E}, \mathcal{T})$ is generated by the base $\{U_e\}_{e\in\mathcal{E}}$, where $U_e = \{e' \in \mathcal{E} : e' \subset e\}$.

2) The associated simplicial complex $(V, \mathcal{K}_{\mathcal{E}})$ is the smallest simplicial complex $\mathcal{K}_{\mathcal{E}} \supset \mathcal{E}$. Note that each edge e corresponds to an open simplex $|e|$ in the geometric realization $|\mathcal{K}_{\mathcal{E}}|$. Hence, we can define the geometric realization $|\mathcal{E}|$ as $\bigcup_{e\in E} |e|$ in the geometric simplicial complex $|\mathcal{K}_{\mathcal{E}}|$.

3) The order complex $S_{\mathcal{E}}$ and its geometric realization $|S_{\mathcal{E}}|$ are defined by replacing the simplicial complex $(V, \mathcal{K})$ by the hypergraph $(V, \mathcal{E})$ in Definition 4.4.

Fact: $(\mathcal{E}, \mathcal{T}) \overset{weak}{\simeq} |\mathcal{E}| \simeq |S_{\mathcal{E}}|$.

Here two topological spaces are *weakly homotopy equivalent* (denoted by $\overset{weak}{\simeq}$) if there exists a continuous map between these topological spaces which induces isomorphisms between all homotopy groups. The Lovász extension f^L is well-defined on $|S_{\mathcal{E}}|$ for any $f : \mathcal{E} \to \mathbb{R}$.

Remark 5. *Indeed, the original Lovász extension f^L of $f : \mathcal{E} \to \mathbb{R}$ is always well-defined on $\bigcup\limits_{t\ge0} t|S_{\mathcal{E}}| \subset \mathbb{R}^V_{\ge0}$. Precisely, the domain of f^L is $\bigcup\limits_{t\ge0} t|S_{\mathcal{E}}|$ if the set V is not a hyperedge (otherwise, the domain of f^L could be $\bigcup\limits_{t\ge0} t|S_{\mathcal{E}}| + \mathbb{R}\mathbf{1}_V$).*

Definition 4.6. *An edge pair (e', e) is called* sequential *if $e' \subsetneqq e$ and there is no other e'' with $e' \subsetneqq e'' \subsetneqq e$. A function $f : \mathcal{E} \to \mathbb{R}$ is a simple discrete Morse function if it has the property that for any $e \in \mathcal{E}$, $\#\{$sequential pair $(e', e) : f(e') \ge f(e)\} \le 1$ and $\#\{$sequential pair $(e, \tilde{e}) : f(e) \ge f(\tilde{e})\} \le 1$. An edge e is called a critical point of a simple discrete Morse function f if $\{$sequential pair $(e', e) : f(e') \ge f(e)\} = \varnothing = \{$sequential pair $(e, \tilde{e}) : f(e) \ge f(\tilde{e})\}$. We say that e has height k if there are at most k edges, $e^1, \cdots, e^k$, in a chain of the form $e^1 \subsetneqq e^2 \subsetneqq \cdots \subsetneqq e^k \subsetneqq e$. A critical point e of f has* index *k if the height of e is k.*

We have a preliminary result for special hypergraphs and the corresponding typical functions, which is a straightforward generalization of Forman's discrete Morse theory.

Theorem 4.4. *For a finite hypergraph $(V, \mathcal{E})$, assume that $\mathcal{E}$ has the properties that the geometric realization $|\{e' \in \mathcal{E} : e' \subsetneqq e\}|$ is homotopic to a sphere for any e, and the geometric realization $|\{e'' \in \mathcal{E} : e'' \subset e, e'' \notin \{e', e\}\}|$ is contractible for any sequential edge pair (e', e). Let $f : \mathcal{E} \to \mathbb{R}$ be a simple discrete Morse function with a critical point of index k. Then the geometric realization $|\mathcal{E}|$ is homotopy equivalent to a CW-complex with one k-cell.*

Let

$$\mathrm{Cat}_m(\mathcal{E}) = \{E' \subset \mathcal{E} : \ \mathrm{cat}(|S_{\mathcal{E}}(E')|) \ge m\}$$

where $S_{\mathcal{E}}(E')$ is the induced subcomplex of $S_{\mathcal{E}}$ on E'. Other results like Theorems 4.1, 4.2 and 4.3 can also be generalized to this setting:

Theorem 4.5. *For a hypergraph $(V, \mathcal{E})$ under the assumptions of Theorem 4.4, let $f : \mathcal{E} \to \mathbb{R}$ be an injective discrete Morse function. Then the following conditions are equivalent:*

(1) e is a critical point of f;

(2) $\mathbf{1}_e$ is a critical point of $f^L|_{|S_{\mathcal{E}}|}$ with index i in the sense of weak slope (metric Morse theory);

(3) $\mathbf{1}_e$ is a critical point of $f^L|_{|S_{\mathcal{E}}|}$ with index i in the sense of Kühnel (PL Morse theory);

(4) $\mathbf{1}_e$ is a Morse critical point of $f^L|_{|S_{\mathcal{E}}|}$ with index i in the sense of topological Morse theory;

Moreover, the discrete Morse vector $(n_0, n_1, \cdots, n_d)$, representing the number n_i of critical points with index i, of f coincides with the continuous Morse vector of $f^L|_{|S_{\mathcal{E}}|}$.

Moreover, the Lusternik-Schnirelmann category theorem is preserved under Lovász extension:

$$\min_{E' \in \mathrm{Cat}_m(\mathcal{E})} \max_{e \in E'} f(e) = \inf_{S \in \mathrm{Cat}_m(|S_{\mathcal{E}}|)} \sup_{\boldsymbol{x} \in S} f^L(\boldsymbol{x}),$$

The details of a general Morse theory on hypergraphs and applications will be developed in [57]. The key idea is that the definition of critical points of a general function f on $\mathcal{E}$ is translated into the PL critical point theory of its restricted Lovász extension $f^L|_{|S_{\mathcal{E}}|}$.

Definition 4.7. *Given a finite hypergraph $(V, \mathcal{E})$ and a function $f : \mathcal{E} \to \mathbb{R}$, we say that $e \in \mathcal{E}$ is a critical point of f if $\mathbf{1}_e$ is a critical point of $f^L|_{|S_{\mathcal{E}}|}$ in the sense of PL Morse theory.*

Table 1: Original Lovász extension of some objective functions.

Set function $f(A) =$	Lovász extension $f^L(\boldsymbol{x}) =$
$\#E(A, V \setminus A)$	$\sum\limits_{\{i,j\}\in E} \lvert x_i - x_j\rvert$
C	$C \max_i x_i$
$\mathrm{vol}(A)$	$\sum_i \deg_i x_i$
$\min\{\mathrm{vol}(A), \mathrm{vol}(V \setminus A)\}$	$\min\limits_{t\in\mathbb{R}} \lVert \boldsymbol{x} - t\mathbf{1}\rVert_1$
$\#A \cdot \#(V \setminus A)$	$\sum\limits_{i,j\in V} \lvert x_i - x_j\rvert$
$\#V(E(A, V \setminus A))$	$\sum\limits_{i=1}^{n}(\max\limits_{j\in N(i)} x_j - \min\limits_{j\in N(i)} x_j)$

Table 2: Set-pair Lovász extension of several objective functions.

Objective function $f(A, B) =$	Set-pair Lovász extension $f^L(\boldsymbol{x}) =$
$\#E(A, V \setminus A) + \#E(B, V \setminus B)$	$\sum\limits_{\{i,j\}\in E} \lvert x_i - x_j\rvert$
$\#E(A, B)$	$\frac{1}{2}\left(\sum\limits_{i\in V} \deg_i \lvert x_i\rvert - \sum\limits_{\{i,j\}\in E} \lvert x_i + x_j\rvert\right)$
C	$C\lVert\boldsymbol{x}\rVert_\infty$
$\mathrm{vol}(A) + \mathrm{vol}(B)$	$\sum\limits_{i\in V} \deg_i \lvert x_i\rvert$
$\min\{\mathrm{vol}(A), \mathrm{vol}(V \setminus A)\} + \min\{\mathrm{vol}(B), \mathrm{vol}(V \setminus B)\}$	$\min\limits_{\alpha\in\mathbb{R}} \lVert(\lvert x_1\rvert, \cdots, \lvert x_n\rvert) - \alpha\mathbf{1}\rVert$
$\#E(A \cup B, A \cup B)$	$\sum_{i\sim j} \min\{\lvert x_i\rvert, \lvert x_j\rvert\}$
$\#(A \cup B) \cdot \#E(A \cup B, A \cup B)$	$\sum_{k\in V, i\sim j} \min\{\lvert x_k\rvert, \lvert x_i\rvert, \lvert x_j\rvert\}$
$\#(A \cup B) \cdot \#(V \setminus (A \cup B))$	$\sum_{i>j} \lvert\lvert x_i\rvert - \lvert x_j\rvert\rvert$

5 Examples and Applications

Tables 1 and 2 and Propositions 5.1, 5.2 and 5.3 present a general correspondence between set or set-pair functions and their Lovász extensions. We shall make use of several of those in this section. Note that the first four lines in Table 1 for the original Lovász extension, and the first five lines in Table 2 for the disjoint-pair Lovász extension are known (see [10, 15]).

Proposition 5.1. *Suppose $f, g : \mathcal{P}(V) \to [0, +\infty)$ are two set functions with $g(A) > 0$ for any $A \in \mathcal{P}(V) \setminus \{\varnothing\}$. Then*

$$\min_{A\in\mathcal{P}(V)\setminus\{\varnothing\}} \frac{f(A)}{g(A)} = \min_{(A,B)\in\mathcal{P}(V)^2\setminus\{(\varnothing,\varnothing)\}} \frac{f(A) + f(B)}{g(A) + g(B)} = \min_{(A,B)\in\mathcal{P}_2(V)\setminus\{(\varnothing,\varnothing)} \frac{f(A) + f(B)}{g(A) + g(B)},$$

where the right identity needs additional assumptions like $f(\varnothing) = g(\varnothing) = 0$[9] or the symmetric property[10] of f and g. Replacing $f(B)$ and $g(B)$ by $f(V \setminus B)$ and $g(V \setminus B)$, all the above identities hold without any additional assumption. Clearly, replacing 'min' by 'max', all statements still hold.

Proposition 5.2. *Suppose $f, g : \mathcal{P}(V) \to [0, +\infty)$ are two set functions with $g(A) > 0$ for any $A \in \mathcal{P}(V) \setminus \{\varnothing\}$. Then*

$$\min_{A\in\mathcal{P}(V)} \frac{f(A)}{g(A)} = \min_{(A_1,\cdots,A_k)\in\mathcal{P}(V)^k} \frac{\sum_{i=1}^k f(A_i)}{\sum_{i=1}^k g(A_i)} = \min_{(A_1,\cdots,A_k)\in\mathcal{P}(V)^k} \sqrt[k]{\frac{\prod_{i=1}^k f(A_i)}{\prod_{i=1}^k g(A_i)}} = \min_{(A_1,\cdots,A_k)\in\mathcal{P}_k(V)} \frac{\sum_{i=1}^k f(A_i)}{\sum_{i=1}^k g(A_i)},$$

where the last identity needs additional assumptions like $f(\varnothing) = g(\varnothing) = 0$.

[9] This setting is natural, as the Lovász extension doesn't use the datum on $\varnothing$.

[10] A function $f : \mathcal{P}(V) \to \mathbb{R}$ has *symmetric property* if $f(A) = f(V \setminus A)$, $\forall A \subset V$.

Proposition 5.3. *Suppose $f, g : \mathcal{P}_2(V) \to [0, +\infty)$ are two set functions with $g(A, B) > 0$ for any $(A, B) \in \mathcal{P}_2(V) \setminus \{(\varnothing, \varnothing)\}$. Then*

$$\min_{A\in\mathcal{P}_2(V)} \frac{f(A,B)}{g(A,B)} = \min_{(A_1,B_1\cdots,A_k,B_k)\in\mathcal{P}_2(V)^k} \frac{\sum_{i=1}^k f(A_i,B_i)}{\sum_{i=1}^k g(A_i,B_i)} = \min_{(A_1,B_1,\cdots,A_k,B_k)\in\mathcal{P}_{2k}(V)} \frac{\sum_{i=1}^k f(A_i,B_i)}{\sum_{i=1}^k g(A_i,B_i)},$$

where the last identity needs additional assumptions like $f(\varnothing, \varnothing) = g(\varnothing, \varnothing) = 0$[11].

Together with Propositions 2.5 and 5.1, one may directly transfer the data from Table 1 to Table 2. Similarly, by employing Propositions 2.6, 5.2 and 5.3, the k-way Lovász extension of some special functions can be transformed to the original and the disjoint-pair versions.

5.1 Submodular vertex cover and multiway partition problems

As a first immediate application of Theorem B, we obtain an easy way to rediscover the famous identity by Lovász, and the two typical submodular optimizations – submodular vertex cover and multiway partition problems.

Example 5.1. *The identity $\min\limits_{A\in\mathcal{P}(V)} f(A) = \min\limits_{\boldsymbol{x}\in[0,1]^V} f^L(\boldsymbol{x})$ discovered by Lovász in his original paper [1] can be obtained by our result. In fact,*

$$\min_{A\in\mathcal{P}(V)} f(A) = \min_{A\in\mathcal{P}(V)} \frac{f(A)}{1} = \min_{\boldsymbol{x}\in[0,\infty)^V} \frac{f^L(\boldsymbol{x})}{\max\limits_{i\in V} x_i} = \min_{\boldsymbol{x}\in[0,1]^V} \frac{f^L(\boldsymbol{x})}{\max\limits_{i\in V} x_i} = \min_{\boldsymbol{x}\in[0,1]^V, \max\limits_i x_i=1} f^L(\boldsymbol{x}).$$

Checking this is easy: if $f \ge 0$, then $\min\limits_{\boldsymbol{x}\in[0,1]^V,\max\limits_i x_i=1} f^L(\boldsymbol{x}) = 0$; if $f(A) < 0$ for some $A \subset V$, then $\min\limits_{\boldsymbol{x}\in[0,1]^V,\max\limits_i x_i=1} f^L(\boldsymbol{x}) = \min\limits_{\boldsymbol{x}\in[0,1]^V} f^L(\boldsymbol{x})$.

Vertex cover number A vertex cover (or node cover) of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The vertex cover number is the minimal cardinality of a vertex cover. Similarly, the independence number of a graph is the maximal number of vertices not connected by edges. The sum of the vertex cover number and the independence number is the cardinality of the vertex set.

By a variation of Motzkin-Straus theorem and Theorem C, the vertex cover number thus has at least two equivalent continuous representations similar to the independence number.

Submodular vertex cover problem Given a graph $G = (V, E)$, and a submodular function $f : \mathcal{P}(V) \to [0, \infty)$, find a vertex cover $S \subset V$ minimizing $f(S)$.

By Theorem B,

$$\min\{f(S) : S \subset V,\ S \text{ is a vertex cover}\} = \min_{\boldsymbol{x}\in\mathcal{D}} \frac{f^L(\boldsymbol{x})}{\|\boldsymbol{x}\|_\infty} = \min_{\boldsymbol{x}\in\widetilde{\mathcal{D}}} f^L(\boldsymbol{x})$$

where $\mathcal{D} = \{\boldsymbol{x} \in [0,\infty)^V : V^t(\boldsymbol{x}) \text{ vertex cover}, \forall t \ge 0\} = \{\boldsymbol{x} \in [0,\infty)^V : x_i + x_j > 0, \forall\{i,j\} \in E,\ \{i : x_i = \max_j x_j\} \text{ vertex cover}\}$, and $\widetilde{\mathcal{D}} = \{\boldsymbol{x} \in \mathcal{D} : \|\boldsymbol{x}\|_\infty = 1\} = \{\boldsymbol{x} \ge \mathbf{0} : x_i + x_j \ge 1, \forall\{i,j\} \in E,\ \{i : x_i = \max_j x_j\} \text{ vertex cover}\}$. Note that

$$\mathrm{conv}(\widetilde{\mathcal{D}}) = \{\boldsymbol{x} : x_i + x_j \ge 1, \forall\{i,j\} \in E,\ x_i \ge 0, \forall i \in V\}.$$

Therefore, $\min\limits_{\boldsymbol{x}\in\mathrm{conv}(\widetilde{\mathcal{D}})} f^L(\boldsymbol{x}) \le \min\{f(S) :\ \text{vertex cover } S \subset V\}$, which rediscovers the convex programming relaxation.

[11]This setting is natural, as the disjoint-pair Lovász extension doesn't use the information on $(\varnothing, \varnothing)$.

Submodular multiway partition problem This problem is about to minimize $\sum_{i=1}^k f(V_i)$ subject to $V = V_1 \cup \cdots \cup V_k$, $V_i \cap V_j = \varnothing$, $i \neq j$, $v_i \in V_i$, $i = 1, \cdots, k$, where $f : \mathcal{P}(V) \to \mathbb{R}$ is a submodular function.

Letting $\mathcal{A} = \{$ partition $(A_1, \cdots, A_k)$ of $V : A_i \ni a_i$, $i = 1, \cdots, k\}$, by Theorem B,

$$\min_{(A_1,\cdots,A_k)\in\mathcal{A}} \sum_{i=1}^k f(A_i) = \inf_{\boldsymbol{x}\in\mathcal{D}_\mathcal{A}} \frac{\sum_{i=1}^k f^L(\boldsymbol{x}^i)}{\|\boldsymbol{x}\|_\infty} = \inf_{\boldsymbol{x}\in\mathcal{D}'} \sum_{i=1}^k f^L(\boldsymbol{x}^i),$$

where $\mathcal{D}_\mathcal{A} = \{\boldsymbol{x} \in [0,\infty)^{kn} : (V^t(\boldsymbol{x}^1), \cdots, V^t(\boldsymbol{x}^k))$ is a partition, $V^t(\boldsymbol{x}^i) \ni a_i, \forall t \geq 0\} = \{\boldsymbol{x} \in [0,\infty)^{kn} : \boldsymbol{x}^i = t1_{A_i}, A_i \ni a_i, \forall t \geq 0\}$, and $\mathcal{D}' = \{(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k) : \boldsymbol{x}^i \in [0,\infty)^V, \boldsymbol{x}^i = \mathbf{1}_{A_i}, A_i \ni a_i\}$. Note that

$$\mathrm{conv}(\mathcal{D}') = \{(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k) : \sum_{v\in V} x_v^i = 1, x_{a_i}^i = 1, x_v^i \geq 0\}.$$

So one rediscovers the corresponding convex programming relaxation $\min\limits_{\boldsymbol{x}\in\mathrm{conv}(\mathcal{D}')} \sum_{i=1}^k f^L(\boldsymbol{x}^i)$.

5.2 Max k-cut problem

The max k-cut problem is to determine a graph k-cut by solving

$$\mathrm{MaxC}_k(G) = \max_{\text{partition } (A_1,A_2,\ldots,A_k) \text{ of } V} \sum_{i\neq j} |E(A_i, A_j)| = \max_{(A_1,A_2,\ldots,A_k)\in\mathcal{C}_k(V)} \sum_{i=1}^k |\partial A_i|, \tag{22}$$

where $\mathcal{C}_k(V) = \{(A_1, \ldots, A_k) \big| A_i \cap A_j = \varnothing, \bigcup_{i=1}^k A_i = V\}$. We may write (22) as

$$\mathrm{MaxC}_k(G) = \max_{(A_1,A_2,\ldots,A_{k-1})\in\mathcal{P}_{k-1}(V)} \sum_{i=1}^{k-1} |\partial A_i| + |\partial(A_1 \cup \cdots \cup A_{k-1})|.$$

Taking $f_k(A_1, \cdots, A_k) = \sum_{i=1}^k |\partial A_i| + |\partial(A_1 \cup \cdots \cup A_k)|$, the k-way Lovász extension is

$$f_k^L(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k) = \sum_{i=1}^k \sum_{i\sim j} |x_i^k - x_j^k| + \sum_{j\sim j'} \left| \max_{i=1,\cdots,k} x_j^i - \max_{i=1,\cdots,k} x_{j'}^i \right|.$$

Applying Theorem B, we have

$$\mathrm{MaxC}_{k+1}(G) = \max_{\boldsymbol{x}^i\in\mathbb{R}^n_{\geq 0}\setminus\{\mathbf{0}\},\, \mathrm{supp}(\boldsymbol{x}^i)\cap\mathrm{supp}(\boldsymbol{x}^j)=\varnothing} \frac{\sum_{i=1}^k \sum_{i\sim j} |x_i^k - x_j^k| + \sum_{j\sim j'} \left| \max\limits_{i=1,\cdots,k} x_j^i - \max\limits_{i=1,\cdots,k} x_{j'}^i \right|}{\max\limits_{i,j} x_j^i}$$

5.3 Relative isoperimetric constants on subgraph with boundary

Given a finite graph $G = (V, E)$ and a subgraph, we consider the Dirichlet and Neumann eigenvalue problems for the corresponding 1-Laplacian. For $A \subset V$, put $\overline{A} = A \cup \delta A$, where δA is the set of points in A^c that are adjacent to some points in A (see Fig. 2).

Given $S \subset \overline{A}$, denote the boundary of S relative to A by

$$\partial_A S = \{(u, v) \in E : u \in S \cap A, v \in \delta A \setminus S \text{ or } u \in S, v \in A \setminus S\}.$$

If $S \subset A$, then $\partial_A S = \{(u, v) \in E : u \in S, v \in \overline{A} \setminus S\}$.

The Cheeger (cut) constant of the subgraph A of G is defined as

$$h(A) = \min_{S\subset\overline{A}} \frac{|\partial_A S|}{\min\{\mathrm{vol}(A \cap S), \mathrm{vol}(A \setminus S)\}}.$$

A set pair $(S, \overline{A} \setminus S)$ that achieves the Cheeger constant is called a Cheeger cut.

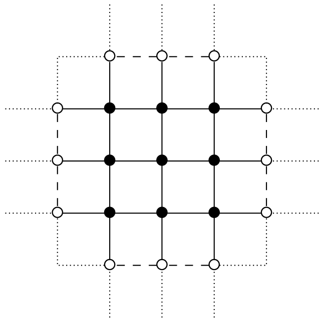


Figure 2: In this graph, let A be the set of solid points, δA the set of hollow points. We only consider the edges between A and $\overline{A}$ (solid lines). We will ignore the dashed lines in δA, and the dotted lines outside $\overline{A}$.

The Cheeger isoperimetric constant[12] of A is defined as

$$h_1(A) = \min_{S \subset A} \frac{|\partial_A S|}{\mathrm{vol}(S)},$$

where a set S achieving the Cheeger isoperimetric constant is called a Cheeger set.

According to our generalized Lovász extension, we have

$$h_1(G) = \inf_{\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{i \sim j} |x_i - x_j| + \sum_{i \in A} p_i |x_i|}{\sum_{i \in A} d_i |x_i|} \tag{23}$$

and

$$h(G) = \inf_{\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{i \sim j, i,j \in A} |x_i - x_j| + \sum_{i \sim j, i \in A, j \in \delta A} |x_i - x_j|}{\inf_{c \in \mathbb{R}} \sum_{i \in A} d_i |x_i - c|}.$$

Note that the term on the right hand side of (23) can be written as

$$\inf_{\boldsymbol{x}|_{V \setminus S} = 0,\, \boldsymbol{x} \neq 0} \mathcal{R}_1(x)$$

which is called the *Dirichlet* 1-*Poincare constant* (see [56]) over S, where

$$\mathcal{R}_1(\boldsymbol{x}) := \frac{\sum\limits_{\{i,j\} \in E} |x_i - x_j|}{\sum_i d_i |x_i|}$$

is called the 1-Rayleigh quotient of $\boldsymbol{x}$.

We can consider the corresponding spectral problems.

- Dirichlet eigenvalue problem:

$$\begin{cases} \Delta_1 \boldsymbol{x} \cap \mu D \,\mathrm{Sgn}\, \boldsymbol{x} \neq \varnothing, & \text{in } A \\ \boldsymbol{x} = 0, & \text{on } \delta A \end{cases}$$

  that is,

$$\begin{cases} (\Delta_1 \boldsymbol{x})_i - \mu d_i \,\mathrm{Sgn}\, x_i \ni 0, & i \in A \\ x_i = 0, & i \in \delta A \end{cases}$$

  whose component form is: $\exists\ c_i \in \mathrm{Sgn}(x_i)$, $z_{ij} \in \mathrm{Sgn}(x_i - x_j)$ satisfying $z_{ji} = -z_{ij}$ and

$$\sum_{j \sim i} z_{ij} + p_i c_i \in \mu d_i \,\mathrm{Sgn}(x_i),\ i \in A,$$

  in which p_i is the number of neighbors of i in δA.

[12] Some authors call it the Dirichlet isoperimetric constant.

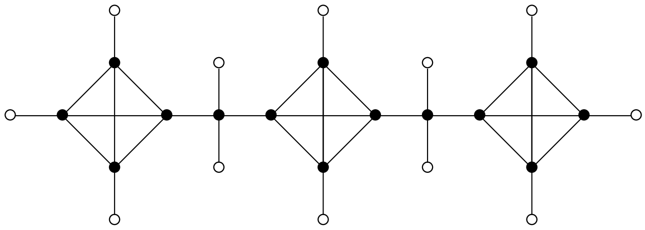


Figure 3: In this example, there are 3 nodal domains of an eigenvector corresponding to the first Dirichlet eigenvalue of the graph 1-Laplacian. Each nodal domain is the vertex set of the 4-order complete subgraph shown in the figure.

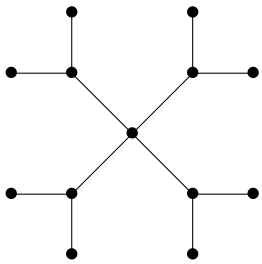


Figure 4: In this example, there are 4 nodal domains of an eigenvector corresponding to the second Neumann eigenvalue of the graph 1-Laplacian. Each nodal domain is the vertex set of the 3-order subgraph after removing the center vertex and its edges.

- Neumann eigenvalue problem: There exists $c_i \in \mathrm{Sgn}(x_i)$, $z_{ij} \in \mathrm{Sgn}(x_i - x_j)$ with $z_{ji} = -z_{ij}$ such that

$$\begin{cases} \sum_{j\sim i, j\in \overline{A}} z_{ij} - \mu d_i c_i = 0, & i \in A \\ \sum_{j\sim i, j\in A} z_{ij} = 0, & i \in \delta A. \end{cases}$$

For a graph G with boundary, we use $\Delta_1^D(G)$ and $\Delta_1^N(G)$ to denote the Dirichlet 1-Laplacian and the Neumann 1-Laplacian, respectively. Then

Proposition 5.4.

$$h_1(G) = \lambda_1(\Delta_1^D(G)) \quad \textit{and} \quad h(G) = \lambda_2(\Delta_1^N(G)).$$

For a connected graph, the first eigenvector of $\Delta_1^N(G)$ is constant and it has only one nodal domain while the first eigenvector of $\Delta_1^D(G)$ may have any number of nodal domains.

Proposition 5.5. *For any $k \in \mathbb{N}^+$, there exists a connected graph G with boundary such that its Dirichlet 1-Laplacian $\Delta_1^D(G)$ has a first eigenvector (corresponding to $\lambda_1(\Delta_1^D(G))$) with exactly k nodal domains; and its Neumann 1-Laplacian $\Delta_1^N(G)$ possesses a second eigenvector (corresponding to $\lambda_2(\Delta_1^N(G))$) with exactly k nodal domains.*

5.4 Independence number

The independence number $\alpha(G)$ of an unweighted and undirected simple graph G is the largest cardinality of a subset of vertices in G, no two of which are adjacent. It can be seen as an optimization problem $\max\limits_{S\subset V \text{ s.t. } E(S)=\varnothing} \#S$. However, such a graph optimization is not global, and the feasible domain seems to be very complicated. But we may simply multiply by a truncated term $(1-\#E(S))$. The independence number can then be expressed as a global optimization on the power set of vertices:

$$\alpha(G) = \max_{S\subset V} \#S(1 - \#E(S)), \tag{24}$$

and thus the Lovász extension can be applied.

Proof of Eq. (24). Since G is simple, $\#S$ and $\#E(S)$ take values in the natural numbers. Therefore,

$$\#S(1-\#E(S)) \begin{cases} \le 0, & \text{if } E(S)\ne\varnothing \text{ or } S=\varnothing, \\ \ge 1, & \text{if } E(S)=\varnothing \text{ and } S\ne\varnothing. \end{cases}$$

Thus, $\max\limits_{S\subset V} \#S(1-\#E(S)) = \max\limits_{S\subset V \text{ s.t. } E(S)=\varnothing} \#S = \alpha(G)$. □

However, Eq. (24) is difficult to calculate. By the disjoint-pair Lovász extension, it equals to

$$\alpha(G) = \max_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{\|\boldsymbol{x}\|_1 - \sum\limits_{k\in V, i\sim j} \min\{|x_k|,|x_i|,|x_j|\}}{\|\boldsymbol{x}\|_\infty},$$

but we don't know how to further simplify it.

So, we provide a simpler optimization which is a tight relaxation:

Proposition 5.6. *The independence number $\alpha(G)$ of a finite simple graph $G=(V,E)$ satisfies*

$$\alpha(G) = \max_{S\subset V} \left(\#S - \#E(S)\right). \tag{25}$$

Proof. Let A be an independent set of G, then $\alpha(G) = \#A = \#A - \#E(A) \le \max\limits_{S\subset V}(\#S-\#E(S))$ because there is no edge connecting points in A.

Let $B\subset V$ satisfy $\#B-\#E(B) = \max\limits_{S\subset V}(\#S-\#E(S))$. Assume the induced subgraph $(B,E(B))$ has k connected components, $(B_i,E(B_i))$, $i=1,\cdots,k$. Then $B=\sqcup_{i=1}^k B_i$ and $E(B)=\sqcup_{i=1}^k E(B_i)$. Since $(B_i,E(B_i))$ is connected, $\#B_i \le \#E(B_i)+1$ and equality holds if and only if $(B_i,E(B_i))$ is a tree. Now taking $B'\subset B$ such that $\#(B'\cap B_i)=1$, $i=1,\cdots,k$, then B' is an independent set and thus

$$\begin{aligned}\alpha(G) \ge \#B' = k = \sum_{i=1}^k 1 \ge \sum_{i=1}^k (\#B_i - \#E(B_i)) = \sum_{i=1}^k \#B_i - \sum_{i=1}^k \#E(B_i) \\ = \#(\cup_{i=1}^k B_i) - \#(\cup_{i=1}^k E(B_i)) = \#B - \#E(B) = \max_{S\subset V}(\#S-\#E(S)).\end{aligned}$$

As a result, Eq. (25) is proved. □

According to Lovász extension, we get

$$\alpha(G) = \max_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{\|\boldsymbol{x}\|_1 - \sum\limits_{i\sim j}\min\{|x_i|,|x_j|\}}{\|\boldsymbol{x}\|_\infty}. \tag{26}$$

By the elementary identities: $\sum_{i\sim j}|x_i+x_j| + \sum_{i\sim j}|x_i-x_j| = 2\sum_{i\sim j}\max\{|x_i|,|x_j|\} = \sum_{i\sim j}||x_i|-|x_j|| + \sum_i \deg_i|x_i|$ and $\sum_i \deg_i|x_i| = \sum_{i\sim j}\max\{|x_i|,|x_j|\} + \sum_{i\sim j}\min\{|x_i|,|x_j|\}$, Eq. (26) can be reduced to

$$\alpha(G) = \max_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{2\|\boldsymbol{x}\|_1 + I^-(\boldsymbol{x}) + I^+(\boldsymbol{x}) - 2\|\boldsymbol{x}\|_{1,\deg}}{2\|\boldsymbol{x}\|_\infty}, \tag{27}$$

where $I^\pm(\boldsymbol{x}) = \sum_{i\sim j}|x_i\pm x_j|$ and $\|\boldsymbol{x}\|_{1,\deg} = \sum_i \deg_i|x_i|$. One would like to write Eq. (27) as

$$\alpha(G) = \max_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{I^-(\boldsymbol{x}) + I^+(\boldsymbol{x}) - 2\|\boldsymbol{x}\|_{1,\deg'}}{2\|\boldsymbol{x}\|_\infty}, \tag{28}$$

where $\|\boldsymbol{x}\|_{1,\deg'} = \sum\limits_{i\in V}(\deg_i-1)|x_i|$.

Chromatic number of a perfect graph Berge's strong perfect graph conjecture has been proved in [23]. A graph G is perfect if for every induced subgraph H of G, the chromatic number of H equals the size of the largest clique of H. The complement of every perfect graph is perfect.

So for a perfect graph, we have an easy way to calculate the chromatic number. In a general simple graph, we refer to Section 5.6 for transforming the chromatic number.

Maximum matching A matching M in G is a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share a common vertex. A maximal matching is one with the largest possible number of edges.

Consider the line graph (E, R) whose vertex set E is the edge set of G, and whose edge set is $R = \{\{e, e'\} : e \cap e' \neq \varnothing, e, e' \in E\}$. Then the maximum matching number of (V, E) coincides with the independence number of (E, R). So, we have an equivalent continuous optimization for a maximum matching problem.

Hall's Marriage Theorem provides a characterization of bipartite graphs which have a perfect matching and the Tutte theorem provides a characterization for arbitrary graphs.

The Tutte-Berge formula says that the size of a maximum matching of a graph is

$$\frac{1}{2}\min_{U\subset V}\left(\#V + \#U - \#\text{ odd connected components of } G|_{V\setminus U}\right).$$

Can one transform the above discrete optimization problem into an explicit continuous optimization via some extension?

k-independence number The independence number admits several generalizations: the maximum size of a set of vertices in a graph whose induced subgraph has maximum degree $(k-1)$ [41]; the size of the largest k-colourable subgraph [42]; the size of the largest set of vertices such that any two vertices in the set are at short-path distance larger than k (see [43]). For the k-independence number involving short-path distance, one can easily transform it into the following two continuous representations:

$$\alpha_k = \max_{\boldsymbol{x}\in\mathbb{R}^V\setminus\{\boldsymbol{0}\}} \frac{\|\boldsymbol{x}\|_1^2}{\|\boldsymbol{x}\|_1^2 - 2\sum\limits_{\mathrm{dist}(i,j)\ge k+1} x_i x_j} = \max_{\boldsymbol{x}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}} \frac{\sum\limits_{\mathrm{dist}(i,j)\le k}(|x_i - x_j| + |x_i + x_j|) - 2\sum\limits_{i\in V}(\deg_k(i) - 1)|x_i|}{2\|\boldsymbol{x}\|_\infty},$$

where $\deg_k(i) = \#\{j \in V : \mathrm{dist}(j, i) \le k\}$, $i = 1, \cdots, n$.

5.5 Various and variant Cheeger problems

Several Cheeger-type constants on graphs have been proposed that are different from the classical one.

Multiplicative Cheeger constant For instance

$$h = \min_{\varnothing\neq A\subsetneq V} \frac{\#E(A, V\setminus A)}{\#A\cdot\#(V\setminus A)}.$$

By Proposition 3.1, it equals to

$$\min_{\langle\boldsymbol{x},\boldsymbol{1}\rangle=0,\boldsymbol{x}\neq\boldsymbol{0}} \frac{\sum_{i\sim j}|x_i - x_j|}{\sum_{i<j}|x_i - x_j|}.$$

Isoperimetric profile The isoperimetric profile $IP : \mathbb{N} \to [0, \infty)$ is defined by

$$IP(k) = \inf_{A\subset V, \#A\le k} \frac{\#E(A, V\setminus A)}{\#A}.$$

Then by Lovász extension, it equals to

$$\inf_{\boldsymbol{x}\in\mathbb{R}^V,\, 1\le\#\mathrm{supp}(\boldsymbol{x})\le k} \frac{\sum_{\{i,j\}\in E}|x_i - x_j|}{\|\boldsymbol{x}\|_1} = \min_{\boldsymbol{x}\in CH_k(\mathbb{R}^V)} \frac{\sum_{\{i,j\}\in E}|x_i - x_j|}{\|\boldsymbol{x}\|_1},$$

where $CH_n := \{\boldsymbol{x} \in \mathbb{R}^V,\ \#\mathrm{supp}(\boldsymbol{x}) \le k\}$ is the union of all k-dimensional coordinate hyperplanes in $\mathbb{R}^V$.

Modified Cheeger constant On a graph $G=(V,E)$, there are three definitions of the vertex-boundary of a subset $A\subset V$:

$$\partial_{\rm ext}A:=\{j\in V\setminus A\mid \{j,i\}\in E\text{ for some }i\in A\}\tag{29}$$
$$\partial_{\rm int}A:=\{i\in A\mid \{i,j\}\in E\text{ for some }j\in V\setminus A\}\tag{30}$$
$$\partial_{\rm ver}A:=\partial_{\rm out}A\cup\partial_{\rm int}A=V(E(A,V\setminus A))=V(\partial_{\rm edge}A)\tag{31}$$

The *external vertex boundary* (29) and the *internal vertex boundary* (30) are introduced and studied recently in [30,31]. Researches on metric measure space [28] suggest to consider the *vertex boundary* (31).

Denote by $N(i)=\{i\}\cup\{j\in V:\{i,j\}\in E\}$ the 1-neighborhood of i. Then the Lovász extensions of $\#\partial_{\rm ext}A$, $\#\partial_{\rm int}A$ and $\#\partial_{\rm ver}A$ are

$$\sum_{i=1}^n(\max_{j\in N(i)}x_j-x_i),\quad \sum_{i=1}^n(x_i-\min_{j\in N(i)}x_j)\quad\text{and}\quad \sum_{i=1}^n(\max_{j\in N(i)}x_j-\min_{j\in N(i)}x_j),$$

respectively. They can be seen as the 'total variation' of $\boldsymbol{x}$ with respect to V in G, while the usual *edge boundary* leads to $\sum\limits_{\{i,j\}\in E}|x_i-x_j|$ which is regarded as the total variation of $\boldsymbol{x}$ with respect to E in G. Their disjoint-pair Lovász extensions are

$$\sum_{i=1}^n\max_{j\in N(i)}|x_j|-\|\boldsymbol{x}\|_1,\quad \|\boldsymbol{x}\|_1-\sum_{i=1}^n\min_{j\in N(i)}|x_j|,\quad \sum_{i=1}^n\left(\max_{j\in N(i)}|x_j|-\min_{j\in N(i)}|x_j|\right).$$

Comparing with the graph 1-Poincare profile (see [27–29])

$$P^1(G):=\inf_{\langle\boldsymbol{x},\mathbf{1}\rangle=0,\boldsymbol{x}\neq\mathbf{0}}\frac{\sum_{i\in V}\max\limits_{j\sim i}|x_i-x_j|}{\|\boldsymbol{x}\|_1},$$

we easily get the following

Proposition 5.7.

$$\frac12\max\{h_{\rm int}(G),h_{\rm ext}(G)\}\le P^1(G)\le h_{\rm ver}(G):=\min_{A\in\mathcal{P}(V)\setminus\{\varnothing,V\}}\frac{\#\partial_{\rm ver}A}{\min\{\#(A),\#(V\setminus A)\}}$$

where $h_{\rm int}(G)$, $h_{\rm ext}(G)$ and $h_{\rm ver}(G)$ are modified Cheeger constants w.r.t. the type of vertex-boundary.

Cheeger-like constant Some further recent results [25] can be also rediscovered via Lovász extension.

A main equality in [25] can be absorbed into the following identities:

$$\begin{aligned}\max_{\text{edges }(v,w)}\left(\frac{1}{\deg v}+\frac{1}{\deg w}\right)&=\max_{\gamma:E\to\mathbb{R}}\frac{\sum_{v\in V}\frac{1}{\deg v}\cdot\left|\sum_{e_{\rm in}:v\text{ input}}\gamma(e_{\rm in})-\sum_{e_{\rm out}:v\text{ output}}\gamma(e_{\rm out})\right|}{\sum_{e\in E}|\gamma(e)|}\\&=\max_{\hat\Gamma\subset\Gamma\text{ bipartite}}\frac{\sum_{v\in V}\frac{\deg_{\hat\Gamma}(v)}{\deg_\Gamma(v)}}{|E(\hat\Gamma)|},\end{aligned}\tag{32}$$

where the left quantity is called a Cheeger-like constant [25].

In fact, given $c_i\ge0$, $i\in V$,

$$\max_{\{i,j\}\in E}(c_i+c_j)=\max_{E'\subset E}\frac{\sum_{\{i,j\}\in E'}(c_i+c_j)}{\#E'},$$

and then via Lovász extension, one immediately gets that the above constant equals to

$$\max_{\boldsymbol{x}\in[0,\infty)^E\setminus\{\mathbf{0}\}}\frac{\sum\limits_{e=\{i,j\}\in E}x_e(c_i+c_j)}{\sum_{e\in E}x_e}=\max_{\boldsymbol{x}\in[0,\infty)^E\setminus\{\mathbf{0}\}}\frac{\sum_{i\in V}c_i\sum_{e\ni i}x_e}{\sum_{e\in E}x_e}=\max_{\boldsymbol{x}\in\mathbb{R}^E\setminus\{\mathbf{0}\}}\frac{\sum_{i\in V}c_i\left|\sum_{e\ni i}x_e\right|}{\sum_{e\in E}|x_e|}$$

Thus, for any family $\mathcal{E} \subset \mathcal{P}(E)$ such that $E' \in \mathcal{E} \Rightarrow E' \supset \{\{e\} : e \in E\}$, we have

$$\max_{\{i,j\}\in E}(c_i + c_j) = \max_{\boldsymbol{x}\in\mathbb{R}^E\setminus\{\boldsymbol{0}\}} \frac{\sum_{i\in V} c_i \left|\sum_{e\ni i} x_e\right|}{\sum_{e\in E}|x_e|} = \max_{E'\in\mathcal{E}} \frac{\sum_{\{i,j\}\in E'}(c_i + c_j)}{\#E'},$$

which recovers the interesting equality (32) by taking $c_i = \frac{1}{\deg i}$ and $\mathcal{E}$ the collections of all edge sets of bipartite subgraphs.

A similar simple trick gives

$$\min_{(v,w)} \frac{\left|\mathcal{N}(v) \cap \mathcal{N}(w)\right|}{\max\{\deg v, \deg w\}} = \min_{\boldsymbol{x}\in\mathbb{R}^E\setminus\{\boldsymbol{0}\}} \frac{\sum_{i\in V}\sum_{e\ni i}|x_e| \cdot \#\text{ triangles containing } e}{\sum_{e=\{i,j\}\in E}|x_e|\max\{d_i, d_j\}}.$$

5.6 Chromatic number

The chromatic number (i.e., the smallest vertex coloring number) of a graph is the smallest number of colors needed to color the vertices so that no two adjacent vertices share the same color. Given a simple connected graph $G = (V, E)$ with $\#V = n$, its chromatic number $\gamma(G)$ can be expressed as a global optimization on the n-power set of vertices:

$$\gamma(G) = \min_{(A_1,\cdots,A_n)\in\mathcal{P}_n(V)} \left\{ n\sum_{i=1}^n \#E(A_i) + \sum_{i=1}^n \mathrm{sign}(\#A_i) + n\left(n - \sum_{i=1}^n \#A_i\right)^2 \right\} \tag{33}$$

and similarly, we get the following

Proposition 5.8. *The chromatic number $\gamma(G)$ of a finite simple graph $G = (V, E)$ satisfies*

$$\gamma(G) = \min_{(A_1,\cdots,A_n)\in\mathcal{P}(V)^n} \left\{ n\sum_{i=1}^n \#E(A_i) + \sum_{i=1}^n \mathrm{sign}(\#A_i) + n\left(n - \#\bigcup_{i=1}^n A_i\right) \right\} \tag{34}$$

Proof. Let $f : \mathcal{P}(V)^n \to \mathbb{R}$ be defined by

$$f(A_1, \cdots, A_n) = n\sum_{i=1}^n \#E(A_i) + \sum_{i=1}^n \mathrm{sign}(\#A_i) + n\left(n - \#\bigcup_{i=1}^n A_i\right).$$

Let $\{C_1, \cdots, C_{\gamma(G)}\}$ be a proper coloring class of G, and set $C_{\gamma(G)+1} = \cdots = C_n = \varnothing$. Then we have $E(C_i) = \varnothing$, $\# \cup_{i=1}^n C_i = n$, $\#C_i \ge 1$ for $1 \le i \le \gamma(G)$, and $\#C_i = 0$ for $i > \gamma(G)$. In consequence, $f(C_1, \cdots, C_n) = \gamma(G)$. Thus, it suffices to prove $f(A_1, \cdots, A_n) \ge \gamma(G)$ for any $(A_1, \cdots, A_n) \in \mathcal{P}(V)^n$.

If $\bigcup_{i=1}^n A_i \ne V$, then $f(A_1, \cdots, A_n) \ge n + 1 > \gamma(G)$.

If there exist at least $\gamma(G)+1$ nonempty sets $A_1, \cdots, A_{\gamma(G)+1}$, then $f(A_1, \cdots, A_n) \ge \gamma(G)+1 > \gamma(G)$.

So we focus on the case that $\bigcup_{i=1}^n A_i = V$ and $A_{\gamma(G)+1} = \cdots = A_n = \varnothing$. If there further exists $i \in \{1, \cdots, \gamma(G)\}$ such that $A_i = \varnothing$, then by the definition of the chromatic number, there is $j \in \{1, \cdots, \gamma(G)\} \setminus \{i\}$ with $E(A_j) \ne \varnothing$. So $f(A_1, \cdots, A_n) \ge n + 1 > \gamma(G)$. Accordingly, each of $A_1, \cdots, A_{\gamma(G)}$ must be nonempty, and thus $f(A_1, \cdots, A_n) \ge \gamma(G)$.

Also, when the equality $f(A_1, \cdots, A_n) = \gamma(G)$ holds, one may see from the above discussion that $A_1, \cdots, A_{\gamma(G)}$ are all independent sets of G with $\bigcup_{i=1}^n A_i \ne V$. $\square$

Note that

$$\#\bigcup_{i=1}^n V^t(\boldsymbol{x}^i) = \#\{j \in V : \exists i \text{ s.t. } x_{i,j} > t\} = \sum_{j=1}^n \max_{i=1,\cdots,n} 1_{x_{i,j}>t} = \sum_{j=1}^n 1_{\max\limits_{i=1,\cdots,n} x_{i,j}>t}$$

So the n-way Lovász extension of $\#\bigcup_{i=1}^n A_i$ is

$$\int_{\min \boldsymbol{x}}^{\max \boldsymbol{x}} \#\bigcup_{i=1}^n V^t(\boldsymbol{x}^i)dt + \min \boldsymbol{x}\#\bigcup_{i=1}^n V(\boldsymbol{x}^i) = \sum_{j=1}^n \int_{\min \boldsymbol{x}}^{\max \boldsymbol{x}} 1_{\max\limits_{i=1,\cdots,n} x_{i,j}>t}dt + \min \boldsymbol{x}\#V$$

$$= \sum_{j=1}^{n}(\max_{i=1,\cdots,n} x_{i,j} - \min \boldsymbol{x}) + n \min \boldsymbol{x}$$
$$= \sum_{j=1}^{n} \max_{i=1,\cdots,n} x_{i,j}$$

And the n-way disjoint-pair Lovász extension of $\# \bigcup_{i=1}^n A_i$ is $\sum_{j=1}^n \max\limits_{i=1,\cdots,n} |x_{i,j}| = \sum_{j=1}^n \|\boldsymbol{x}^{,j}\|_\infty$.

The n-way Lovász extension of $\mathrm{sign}(\#A_i)$ is

$$\int_{\min \boldsymbol{x}}^{\max \boldsymbol{x}} \mathrm{sign}(\#V^t(\boldsymbol{x}^i))dt + \min \boldsymbol{x}\,\mathrm{sign}(\#V(\boldsymbol{x}^i)) = \int_{\min \boldsymbol{x}}^{\max \boldsymbol{x}^i} 1dt + \min \boldsymbol{x}\,\mathrm{sign}(\#V)$$
$$= \max_{j=1,\cdots,n} x_{i,j} - \min \boldsymbol{x} + \min \boldsymbol{x} = \max_{j=1,\cdots,n} x_{i,j}$$

and the n-way disjoint-pair Lovász extension of $\mathrm{sign}(\#A_i)$ is $\|\boldsymbol{x}^i\|_\infty$. Similarly, the n-way disjoint-pair Lovász extension of $\#E(A_i)$ is $\sum_{j\sim j'} \min\{|x_{i,j}|, |x_{i,j'}|\}$. Thus

$$f^L(\boldsymbol{x}) = n \sum_{i=1}^n \sum_{j\sim j'} \min\{|x_{i,j}|, |x_{i,j'}|\} + \sum_{i=1}^n \|\boldsymbol{x}^i\|_\infty + n \left(n\|\boldsymbol{x}\|_\infty - \sum_{j=1}^n \|\boldsymbol{x}^{,j}\|_\infty \right)$$
$$= n \sum_{i=1}^n \left(\|\boldsymbol{x}^i\|_{1,\deg} - (I^+(\boldsymbol{x}^i) + I^-(\boldsymbol{x}^i))/2\right) + \sum_{i=1}^n \|\boldsymbol{x}^i\|_\infty + n \left(n\|\boldsymbol{x}\|_\infty - \sum_{j=1}^n \|\boldsymbol{x}^{,j}\|_\infty \right)$$
$$= n^2\|\boldsymbol{x}\|_\infty + n\|\boldsymbol{x}\|_{1\text{-}deg,1} + \|\boldsymbol{x}\|_{\infty,1} - nI_{\pm,1}(\boldsymbol{x}) - n\|\boldsymbol{x}\|^{\infty,1}.$$

According to Proposition 3.1 on the multi-way Lovász extension, we get

$$\gamma(G) = n^2 - \sup_{\boldsymbol{x}\in\mathbb{R}^{n^2}\setminus\{\boldsymbol{0}\}} \frac{nI_{\pm,1}(\boldsymbol{x}) + n\|\boldsymbol{x}\|^{\infty,1} - n\|\boldsymbol{x}\|_{1\text{-}deg,1} - \|\boldsymbol{x}\|_{\infty,1}}{\|\boldsymbol{x}\|_\infty}. \tag{35}$$

Clique covering number The clique covering number of a graph G is the minimal number of cliques in G needed to cover the vertex set. It is equal to the chromatic number of the graph complement of G. Consequently, we can explicitly write down the continuous representation of a clique covering number by employing Theorem C.

Acknowledgement

Dong Zhang would like to thank Professor Kung-Ching Chang for his long-term guidance, encouragement and support in mathematics.

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