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Quantitative homogenization for the case of an interface between two heterogeneous media

by

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Abstract

In this article we are interested in quantitative homogenization results for linear elliptic equations in the non-stationary situation of a straight interface between two heterogenous media. This extends the previous work [Josien, 2019] to a substantially more general setting, in which the surrounding heterogeneous media may be periodic or random stationary and ergodic. Our main result is a quantification of the sublinearity of a homogenization corrector adapted to the interface, which we construct using an improved version of the method developed in [Fischer and Raithel, 2017]. This quantification is optimal up to a logarithmic loss and allows to derive almost-optimal convergence rates.

Keywords homogenization, interfaces, correctors, Lipschitz estimates, convergence rate

AMS classification 35B27, 35J15, 74A40, 74A50.

1 Introduction

In this article we construct and estimate the growth rate of homogenization correctors associated to linear elliptic operators in divergence form in the context of a flat interface between two heterogenous media (see, *e.g.*, Figure 1). It is a continuation of the previous work of the first author [17], inspired by [9], which studies the case of an interface between two periodic media. We refer the reader to [17], which is more elementary than the present study. There, definitions for the homogenization correctors and 2-scale expansion adapted to the interface are designed, motivated and proved to produce an accurate approximation of the solution of the multiscale problem. Equipped with these algebraic definitions, we explore here a substantially broader framework, in which we do not assume any structure on the two surrounding heterogeneous media, but only that each

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of them admits a constant homogenized matrix and correctors with a controlled growth rate. Under these assumptions, we build adapted correctors satisfying suboptimal sublinearity estimates by taking advantage of the techniques developed in [12] by Fischer and the second author. In our main theorem, we use Green's function estimates to obtain an almost-optimal control of the growth rate of the correctors.

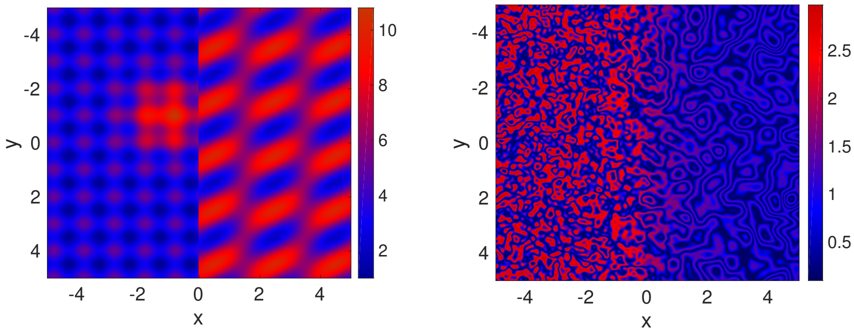

Figure 1: On the left, a sharp interface between two periodic media with a defect; on the right, a smooth interface between two random media generated from independent Gaussian fields. The colors indicate the value of a (which is here assumed to be scalar).

1.1 Motivation and related works

General theory of homogenization Consider a linear elliptic equation in divergence form

$$-\mathrm{div}\,(a(x)\nabla u(x)) = f(x). \tag{1}$$

Such equations play a central role in many branches of material physics; *e.g.* in elasticity, electrostatics, and thermostatics. We refer to [1] for a didactic introduction to homogenization and its applications. The coefficient field a typically represents local characteristics of a sample: elasticity, electrical conductance, or thermal conductance (depending on the context). Here, as in the classical theory of homogenization, the coefficient a is assumed to be varying at a characteristic small scale, which is of order 1 (by a change of variables). When this small scale vanishes (or equivalently, on infinitely large scales), equation (1) may be approximated by the following homogenized equation:

$$-\mathrm{div}\,(\bar{a}(x)\nabla \bar{u}(x)) = f(x), \tag{2}$$

where the so-called *homogenized matrix* $\bar{a}$ is usually simpler that the original matrix a.

In most works, the coefficient a is assumed to have stationarity properties. Roughly speaking, the behavior of the medium is shift-invariant; *e.g.* a might be periodic [1], almost-periodic [3], or random stationary and ergodic [2, 15, 16]. In those cases, the homogenized matrix $\bar{a}$ is constant. This, in particular, shows that homogenization is an efficient tool for approximating (1), which would be very costly to solve numerically. Nevertheless, even though stationarity –in all its aforementioned expressions– is a convenient mathematical tool, it may not always be a realistic hypothesis.

Beyond stationary coefficient fields Quite recently, in [9], there was a deliberate attempt to study theoretically more general structures. More precisely, two cases were proposed: The case of a defect in a periodic structure and the case of an interface between two periodic media.

The first case breaks stationarity, but only on the microscopic level, for the defect has no macroscopic impact (at least at the main order). Thus, once the corrector is built and estimated [9,10], classical approaches in periodic homogenization (namely Avellaneda and Lin's [4], later improved in [19]) are sufficient to obtain accurate convergence rates [7,8].

The second case not only breaks stationarity at the microscopic level, but also at the macroscopic level. Indeed, the interface plays a role at any scale: The homogenized matrix $\bar{a}$ is generically piecewise constant with a discontinuity through the interface. Notice that the book [5, Chap. 9 p. 312], which predates [9,17] and inspired [26], proposes another point of view on interfaces, with slightly different –however consistent– definitions for the correctors and asymptotic expansion than we give below.

The case of an interface between periodic media The case of an interface requires adapted definitions of correctors [17]. These correctors ϕ_j, for $j \in [\![1,d]\!]$, are strictly sublinear (see (6)) solutions to the following equation:

$$-\mathrm{div}\left(a\nabla\left(P_j+\phi_j\right)\right)=0 \quad \text{in} \quad \mathbb{R}^d. \tag{3}$$

where the piecewise affine functions P_j span the space of non-constant and strictly subquadratic $\overline{a}$-harmonic functions[1]. Namely, the functions P_j solve

$$-\mathrm{div}\left(\overline{a}(x)\nabla P_j(x)\right)=0 \quad \text{in} \quad \mathbb{R}^d. \tag{4}$$

In [17], in a specific case of periodic media, these correctors were actually built and an almost-optimal convergence rate for the gradient of the adapted 2-scale expansion[2],

$$\widetilde{u} := \overline{u} + \phi\cdot\left(\nabla P\right)^{-1}\nabla\overline{u}, \tag{5}$$

was obtained. The techniques of [17], however, were crucially based on some periodic structures of the underlying heterogeneous media.

The case of general interface In the current contribution we consider a more general case of flat interface between two media. We do not assume any joint structure on them, but only that each of them admits a constant homogenized matrix and correctors with a controlled growth rate. Our main result is that the global medium– which consists of the two heterogeneous parts glued along the interface– enjoys the *same* quantitative homogenization properties as

[1]By the Liouville principle for piecewise constant coefficient fields, this space has dimension d.

[2]Interestingly, such an expansion in only required when considering the gradient in the vicinity of the interface, which may be relevant in elasticity in the context of fractures. See Figure 3.

the two components, up to a logarithmic loss. In particular, the growth rate of the global corrector is essentially bounded by the maximum of the growth rates of the correctors associated with each of the heterogeneous media (see Theorem 1 below).

To obtain this result, we first rely on the approach of [12,24], which construct correctors for the half-space with homogeneous Dirichlet or Neumann boundary conditions. These articles provide a robust way to build correctors for simple geometries, but with a suboptimal growth rate. Other than the existence of correctors on the whole space satisfying a weak quantified sublinearity condition, there are no other structural assumptions made on the coefficient fields. Then, capitalizing on estimates for the heterogeneous Green's function provided by large-scale Lipschitz regularity, we prove an almost optimal growth rate. We remark that the strategy for proving the large-scale Lipschitz regularity is to transfer large-scale regularity properties from the homogenized to the heterogeneous problem –here we adapt the strategy of [15]. However, since now the homogenized problem involves a piecewise continuous coefficient, we make use of the results of [20, 21].

Last, as is classical in homogenization (see, *e.g.* [15] or the introductory course [18]), our estimate for the growth rate of the correctors produces, in turn, a convergence rate on the level of the adapted 2-scale expansion.

1.2 Precise mathematical setting

In this section we fix the model for a flat interface between two heterogeneous media that we will consider throughout this paper.

General notations Let d be the dimension and $(e_i)_{i\in[\![1,d]\!]}$ be the canonical basis of $\mathbb{R}^d$. In this paper we always assume that $d \geq 2$. If $x \in \mathbb{R}^d$, we define

$$x^{\perp} := x \cdot e_1 \in \mathbb{R} \quad \text{and} \quad x^{\parallel} := (x \cdot e_2, \cdots, x \cdot e_d) \in \mathbb{R}^{d-1},$$

so that $x = (x^{\perp}, x^{\parallel})$. If $R > 0$, we denote by $\mathrm{Q}_R(x) \subset \mathbb{R}^d$ the cube of side length R centered at x; also $\mathrm{B}_R(x) \subset \mathbb{R}^d$ is the ball of radius R centered at x. When $x = 0$ or $R = 1$, the parameters might be omitted (for example, we denote $\mathrm{B} = \mathrm{B}_1(0)$).

We highlight that throughout this paper we make use of the Einstein summation convention.

We say that a function f is *sublinear* if it satisfies the following condition:

$$\limsup_{r\uparrow\infty} \frac{1}{r}\Big(\fint_{\mathrm{B}_r} \Big|f - \fint_{\mathrm{B}_r} f\Big|^2\Big)^{\frac{1}{2}} < \infty. \tag{6}$$

It is said to be *strictly sublinear* if the above limit is equal to 0.

Definition of the interface We define a coefficient field a on $\mathbb{R}^d$ by

$$a(x) = \begin{cases} a_-(x) & \text{if} \quad x^{\perp} < -1, \\ a_\circ(x) & \text{if} \quad -1 < x^{\perp} < 1, \\ a_+(x) & \text{if} \quad x^{\perp} > 1. \end{cases} \tag{7}$$

The interface is defined by $\mathcal{I} := \{0\} \times \mathbb{R}^{d-1}$. In our model, the thin layer $[-1,1]\times\mathbb{R}^{d-1}$ allows for a transition between the surrounding media represented by $a_\pm$. Our running assumption on every coefficient field a is that they are uniformly elliptic and bounded; namely, there exists a fixed constant $\lambda > 0$ such that, for every $x, \xi \in \mathbb{R}^d$, there holds

$$\lambda |\xi|^2 \leq \xi \cdot a(x)\xi \quad \text{and} \quad \lambda |\xi|^2 \leq \xi \cdot a(x)^{-1}\xi. \tag{8}$$

In order to describe random media, we assume that we have an ensemble $\langle\cdot\rangle$ on the space Ω (with the topology of H-convergence), which we define as follows:

$$\Omega := \left\{(a_+, a_-, a_\circ) \,|\, a_\pm, a_\circ : \mathbb{R}^d \to \mathbb{R}^{d\times d} \text{ satisfy (8)}\right\}. \tag{9}$$

In a deterministic case, the measure of the ensemble $\langle\cdot\rangle$ concentrates on one specific coefficient field.

Our first hypothesis is that the coefficient field a, $\langle\cdot\rangle$-almost surely, admits the following piecewise constant (deterministic) homogenized matrix $\bar{a}$:

$$\overline{a}(x) = \begin{cases} \overline{a}_+ & \text{if} \quad x^\perp > 0, \\ \overline{a}_- & \text{if} \quad x^\perp < 0. \end{cases} \tag{10}$$

(By local properties of H-convergence, $\overline{a}_-$ and $\overline{a}_+$ depend only on a_- and a_+ respectively.) We also assume that, $\langle\cdot\rangle$-almost surely, there exist generalized homogenization correctors

$$\Phi_\pm := (\Phi_-, \Phi_+) \quad \text{for} \quad \Phi_- := (\phi_-, \phi_-^*, \sigma_-, \sigma_-^*) \quad \text{and} \quad \Phi_+ := (\phi_+, \phi_+^*, \sigma_+, \sigma_+^*).$$

Here, (ϕ_+, σ_+) are strictly sublinear functions satisfying[3]

$$-\mathrm{div}\left(a_+\left(\nabla\left(\phi_+\right)_i + e_i\right)\right) = 0 \quad \text{and} \quad \left(\sigma_+\right)_{ijk} := \partial_i\left(N_+\right)_{jk} - \partial_j\left(N_+\right)_{ik}, \tag{11}$$

in $\mathbb{R}^d$, where $(N_+)_{jk}$ is a strictly subquadratic solution of the following equation

$$\Delta\left(N_+\right)_{jk} = \left(\overline{a}_+\right)_{jk} - \left(a_+\right)_{jl}\left(\delta_{lk} + \partial_l\left(\phi_+\right)_k\right) \quad \text{in} \quad \mathbb{R}^d. \tag{12}$$

(The other correctors (ϕ_-, σ_-), $\left(\phi_-^*, \sigma_-^*\right)$ and $\left(\phi_+^*, \sigma_+^*\right)$ correspondingly satisfy similar equations, where the coefficients fields $(a_+, \bar{a}_+)$ should be respectively replaced by $(a_-, \bar{a}_-)$, and the transposed coefficient fields $\left(a_-^*, \overline{a}_-^*\right)$ and $\left(a_+^*, \overline{a}_+^*\right)$.)

Our second and main hypothesis about the two heterogeneous media is that the correctors related to $a_\pm$ and $a_\pm^*$ are strongly sublinear in the following annealed way:

$$\sup_{x,y\in\mathbb{R}^d, |x-y|\leq r} \left\langle\left(\int_{\mathrm{Q}} |\Phi_\pm(x+z) - \Phi_\pm(y+z)|^2 \mathrm{d}z\right)^{\frac{p}{2}}\right\rangle^{\frac{1}{p}} \leq c_p r^{1-\nu}, \tag{13}$$

for every $r \geq 2$ and $p < \infty$, and for a given exponent $\nu \in (0,1]$ and constant $c_p \geq 1$. (Without loss of generality –by the Hölder inequality– we may assume that the constants c_p are increasing in p.) As will be seen below, the case $\nu = 1$ gives rise to logarithmic terms in some estimates. For the sake of simplicity, we make use of the notation $\lfloor\nu\rfloor$, which is equal to 0, respectively 1, if $\nu < 1$, respectively $\nu = 1$.

[3] Notice that our indexing convention for the flux corrector σ is different from [15].

Remark 1. While we assume that $a_\pm$ are coefficient fields on $\mathbb{R}^d$ with corresponding generalized homogenization correctors, it would suffice to have these coefficient fields and generalized correctors defined on $\mathbb{R}_\pm \times \mathbb{R}^{d-1}$ with an accordingly modified assumption (13). Also, the coefficient field $a_\circ$ might only be defined on the layer $[-1,1] \times \mathbb{R}^{d-1}$. We define the space Ω by (9) for simplicity.

1.3 Definition of adapted correctors and 2-scale expansion

Following [17], we introduce a basis for the space of strictly subquadratic $\overline{a}$-harmonic functions (see (4)): For $j \in [\![1,d]\!]$ we define

$$P_j(x) = P(x) \cdot e_j := \begin{cases} x \cdot e_j & \text{if} \quad x^\perp < 0, \\ x \cdot e_j + \frac{(\overline{a}_-)_{1j} - (\overline{a}_+)_{1j}}{(\overline{a}_+)_{11}} x \cdot e_1 & \text{if} \quad x^\perp > 0, \end{cases} \tag{14}$$

where the bottom line corresponds to the transmission condition through the interface. This prompts us to define the *generalized correctors* (ϕ, σ) associated to a coefficient field a of the form (7) as follows:

Definition 1 (Generalized Correctors)**.** *The correctors ϕ_j, for $j \in [\![1,d]\!]$, are strictly sublinear solutions to* (3)*. Simultaneously, the flux correctors σ_{ijk}, for $i,j,k \in [\![1,d]\!]$ are defined as*

$$\sigma_{ijk} := \partial_i N_{jk} - \partial_j N_{ik}, \tag{15}$$

where N_{jk} is a strictly subquadratic solution of the following equation:

$$\Delta N_{jk} = \overline{a}_{jl} \partial_l P_k - a_{jl} \left(\partial_l P_k + \partial_l \phi_k \right) \quad \mathit{in} \quad \mathbb{R}^d. \tag{16}$$

At this point we make an important distinction: Notice that (15) and (16) imply that the flux corrector σ satisfies the familiar identity (see also [15, (7)])

$$\partial_i \sigma_{ijk} = \overline{a}_{jl} \partial_l P_k - a_{jl} \left(\partial_l P_k + \partial_l \phi_k \right) \quad \text{in} \quad \mathbb{R}^d \tag{17}$$

along with the skew-symmetry constraint

$$\sigma_{ijk} = -\sigma_{jik}. \tag{18}$$

It turns out that the two latter identities are sufficient for many purposes (*e.g.* to obtain large-scale Lipschitz estimates, that is Theorem 2 below). Functions σ^{u} that are strictly sublinear and satisfy (17) and (18) are called *ungauged flux correctors*; we use the superscript "u" to indicate it. The main difference is that, in contrast to the gauged flux correctors of Definition 1, the ungauged flux correctors are not unique, which becomes an issue in the proof of Theorem 1.

In our setting with the interface we need a modification of the standard 2-scale expansion, namely (5). With this definition of $\widetilde{u}$ we find that

$$-\operatorname{div}(a \cdot \nabla(u - \widetilde{u})) = \partial_i((a_{ij}\phi_k - \sigma_{ijk}) \partial_j \overline{\partial}_k \overline{u}), \tag{19}$$

where we denote $\overline{\nabla} \overline{u} := (\nabla P)^{-1} \nabla \overline{u}$ and $\overline{\partial}_k \overline{u} := e_k \cdot \overline{\nabla} \overline{u}$. The motivation for (5) and the detailed calculation leading to (19) lie in [17, Section 3.3]. We underline that the function $\overline{\nabla} \overline{u}$ is continuous through the interface: Thus, its gradient $\nabla \overline{\nabla} \overline{u}$ lies in $L^\infty_{\mathrm{loc}}\left(\mathbb{R}^d\right)$, so that the terms on the right-hand side of (19) are well-defined (see Lemma 2 below).

1.4 Theorem 1: Main result

The main contribution of this article is the following:

Theorem 1. *Let $d \geq 2$ and $\langle\cdot\rangle$ be an ensemble on Ω defined in* (9) *that satisfies the conditions given in Section 1.2. Then, $\langle\cdot\rangle$-almost surely there exists a unique (up to addition of a random constant*[4]*) generalized corrector (ϕ, σ) associated to a such that for every $\nu_0 < \nu$, $2 \leq p < \infty$ and $r \geq 2$, the following estimates hold:*

$$\begin{aligned}&\sup_{x,y\in\mathbb{R}^d,|x-y|\le r}\Big\langle\Big(\int_{\mathrm{B}}|\phi(x+z)-\phi(y+z)|^2\,\mathrm{d}z\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\\&\qquad\lesssim_{d,\lambda,\nu,\nu_0,p} C_{2pd/\nu}^{d/\nu_0+1}\begin{cases}r^{1-\nu} & \text{if } \nu<1,\\ \ln(r) & \text{if } \nu=1,\end{cases}\end{aligned}\tag{20}$$

$$\begin{aligned}&\sup_{x,y\in\mathbb{R}^d,|x-y|\le r}\Big\langle\Big(\int_{\mathrm{B}}|\sigma(x+z)-\sigma(y+z)|^2\,\mathrm{d}z\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\\&\qquad\lesssim_{d,\lambda,\nu,\nu_0,p} C_{2pd/\nu}^{d/\nu_0+1}\begin{cases}r^{1-\nu}\ln(r) & \text{if } \nu<1,\\ \ln^3(r) & \text{if } \nu=1.\end{cases}\end{aligned}\tag{21}$$

Above and in the sequel, the symbol "$\lesssim_\delta$" reads "$\leq C$, for a constant C depending only on the tuple δ of previously defined parameters" (for simplicity, throuhgout the course of the proofs, the subscript might be omitted).

In words, as previously advertised, we learn from Theorem 1 that the correctors adapted to the interface enjoy the same quantified sublinearity properties as the correctors on the left and on the right of the interface (possibly up to a logarithmic correction).

In the case $\nu = 1$, the assumption (13) is quite well-motivated. In particular, it is satisfied if $a_\pm$ are both periodic. Of course, the assumption (13) is less common for $\nu < 1$. It has, however, been shown that such a growth rate naturally arises when studying periodic media perturbed by a defect that is quite spread (see Example 1 below and [7, 8]). It also arises in stochastic homogenization when considering a general random field satisfying a log-Sobolev inequality (see [15, Th. 3]). As a consequence, Theorem 1 may be applied in various frameworks, as illustrated in Section 1.5 below. We also remark that, from a practical point of view, it may happen that correctors related to some heterogeneous materials can be computed numerically. Thus, condition (13) would be easier to check than a structure assumption.

In light of [9, 17], it is a bit surprising that no structural relationship between the coefficients $a_\pm$ is assumed. Indeed, in [17, Prop. 5.4] the coefficients $a_\pm$ have a common periodic cell in the directions of the interface $\mathcal{I}$ and in [9, Th. 5.7] a diophantine condition relating the periods of the coefficients $a_\pm$ is assumed. The more general statement in Theorem 1, however, does come at a cost. In particular, defining the glued composite correctors

$$\check{\phi}(x) := \begin{cases}\phi_+(x) & \text{if } x^\perp > 0,\\ \phi_-(x) & \text{if } x^\perp < 0\end{cases} \quad\text{and}\quad \check{\sigma}(x) := \begin{cases}\sigma_+(x) & \text{if } x^\perp > 0,\\ \sigma_-(x) & \text{if } x^\perp < 0,\end{cases}\tag{22}$$

[4] We use here the quite paradoxical words "random constant" to designate a random field that is constant in space.

the estimates in [17, Prop. 5.4] provide an exponential decay of $\nabla\phi(x) - \nabla\check{\phi}(x)$ (and accordingly of $\nabla\sigma(x) - \nabla\check{\sigma}(x)$) in the distance to the interface $|x^\perp|$. In contrast, the methods used to prove Theorem 1 only yield a decay as the inverse of this distance.

Following our proof of Theorem 1, we show that enforcing a structural assumption between the two surrounding media may lead to stronger estimates than (20) or (21), and not only in the periodic case [17]. In particular, we prove in Theorem 3 that, in a special stochastic setting where both heterogeneous media are *independently* generated from two Gaussian fields with integrable correlation functions, we obtain that all the stochastic moments of the generalized corrector are uniformly bounded in $\mathbb{R}^d$.

As a counterpart to Theorem 3, we justify that, under the assumptions outlined in Section 1.2, the rate (20) is optimal.[5] (The only non-obvious case is $\nu = 1$.) In particular, in Proposition 2 we give an example of coefficients $a_\pm$ that admit bounded correctors and a uniformly elliptic and bounded $a_\circ$ such that the global corrector for the medium with interface displays a logarithmic growth.

We lastly underline that, apart from boundedness and uniform ellipticity, no further assumptions are imposed on $a_\circ$ inside the layer of width 2 along the interface. This is indeed a zone that we need to "sacrifice" in the proof of Theorem 1 because of our use of cut-off functions –we cannot take advantage of any good behavior of a in this zone, but we also do not suffer from any bad behavior. As can be seen in (20), the presence of this zone does not worsen the growth rate when $\nu < 1$, but its influence is felt when $\nu = 1$. (This is also apparent in Proposition 2 below.) In the terminology of [9], the layer $a_\circ$ could be seen as a *defect* since it it appears as "microscopic" when zooming out. However, it is only in $L^\infty(\mathbb{R}^d)$ and not in any $L^r(\mathbb{R}^d)$ for $r < \infty$.

1.5 Examples

In this section, we propose three different simple, but representative, examples of interfaces between heterogeneous media (see Figures 1 and 2) satisfying (13), that will be further discussed in specific results. The first example is deterministic, the second one is stochastic, and the third example does not require homogenization theory, but it illustrates that (20) is optimal.

Example 1. The matrices $a_\pm$ represent periodic media perturbed by defects:

$$a_- = a_{\mathrm{per},-} + \widetilde{a}_- \quad \text{and} \quad a_+ = a_{\mathrm{per},+} + \widetilde{a}_+,$$

where the coefficient fields $a_{\mathrm{per},\pm}$ are both periodic (with possibly different periods) and Hölder continuous. Moreover, the defects are localized in the sense of $\widetilde{a}_\pm \in L^\infty\left(\mathbb{R}^d\right) \cap L^r\left(\mathbb{R}^d\right)$, for $r \in [1,\infty)$, and uniformly Hölder continuous. The coefficient fields $a_\pm$, $a_{\mathrm{per},\pm}$ satisfy (8). The only constraint on the layer is that $a_\circ$ satisfies (8).

In such a case, by [9, Th. 4.1], (13) is satisfied for $\nu := \min\left(1, \frac{d}{r}\right)$ if $r \neq d$ (the special case $r = d$ can be treated in a suboptimal way by artificially increasing r), and for a trivial ensemble $\langle\cdot\rangle$. Such an example is illustrated on the left-hand side of Figure 1, and might be a realistic model for an interface between two crystals.

[5] Nevertheless, we do not claim that the exponent in the logarithm of (21) is optimal.

Example 2. Let $d \geq 2$, $0 < \lambda < 1$, $\kappa > 0$ and $\alpha \in (0,1)$. Let c_-, c_+ and $c_\circ : \mathbb{R}^d \to \mathbb{R}$ be covariance functions such that their Fourier transforms satisfy

$$|\mathcal{F}c_-(k)| + |\mathcal{F}c_+(k)| + |\mathcal{F}c_\circ(k)| \leq \kappa\,(1+|k|)^{-d-2\alpha} \qquad \text{for any } k \in \mathbb{R}^d, \quad (23)$$

and let the deterministic matrix-valued functions A_-, A_+, $A_\circ : \mathbb{R} \to \mathbb{R}^{d\times d}$ be such that each element in the range of A_-, A_+, and $A_\circ$ satisfies (8). Assume that these functions are uniformly Lipschitz continuous, *i.e.*,

$$\|A_-'\|_{L^\infty} + \|A_+'\|_{L^\infty} + \|A_\circ'\|_{L^\infty} \leq \kappa. \quad (24)$$

The coefficient fields a_+, a_- and $a_\circ$ are generated from independent vectorial stationary Gaussian fields g_-, g_+ and $g_\circ : \mathbb{R}^d \to \mathbb{R}$ with covariance functions c_-, c_+ and $c_\circ$ in the following sense:

$$a_-(x) := A_-(g_-(x)), \quad a_+(x) := A_+(g_+(x)) \quad \text{and} \quad a_\circ(x) := A_\circ(g_\circ(x)). \quad (25)$$

We denote by $\langle\cdot\rangle$ the ensemble induced by the joint laws of g_-, g_+ and $g_\circ$.

By [18, Prop. 3.2] (see also [15]), estimate (13) is satisfied for $\nu = 1$ in Example 2. Of course, it might be more realistic to make use of the layer coefficient $a_\circ$ to have a smooth transition between the two surrounding media as in Figure 1.

Remark 2. There is no need to assume independence between all the media in order to apply Theorem 1. However, as will be seen in Theorem 3 below, this produces more refined estimates.

Example 3. Let η be a function on $\mathbb{R}^d$ defined by

$$\eta(x) := \eta_1(x^\perp)\eta_2\,(x\cdot e_2)\,, \quad (26)$$

where η_1 and $\eta_2 : \mathbb{R} \to [0,1]$ are two smooth functions such that

$$\begin{cases} [1,+\infty) \subset \{t\,:\,\eta_1(t)=1\} \subset \mathrm{Supp}(\eta_1) \subset [0,+\infty), \\ [-1/2,1/2] \subset \{t\,:\,\eta_2(t)=1\} \subset \mathrm{Supp}(\eta_2) \subset [-1,1]. \end{cases}$$

By definition (26), the support of η lies in the strip D defined by

$$D := [0,+\infty) \times [-1,1] \times \mathbb{R}^{d-2}. \quad (27)$$

We then define the symmetric coefficient field a as

$$a(x) := \mathrm{I} + \eta(x) e_1 \otimes e_1. \quad (28)$$

In Example 3, one may write a in the form (7) for $a_-(x) := \mathrm{I}$, $a_+(x) := \mathrm{I} + \eta(x)e_1 \otimes e_1$, and $a_\circ = \mathrm{I} + \eta(x)e_1 \otimes e_1$. Hence, a admits the matrix $\overline{a} = \mathrm{I}$ as its homogenized matrix. Moreover, we easily derive that

$$\phi_- = \phi_+ = 0, \quad \sigma_- = 0, \quad \text{and} \quad (\sigma_+)_{ijk} = (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})\,\delta_{k1}\int_0^{x\cdot e_2} \eta_2.$$

Thus the correctors $(\phi_\pm, \sigma_\pm)$ are uniformly bounded in $\mathbb{R}^d$. However, we show in Proposition 2 that a admits a global corrector ϕ with an unbounded growth rate.

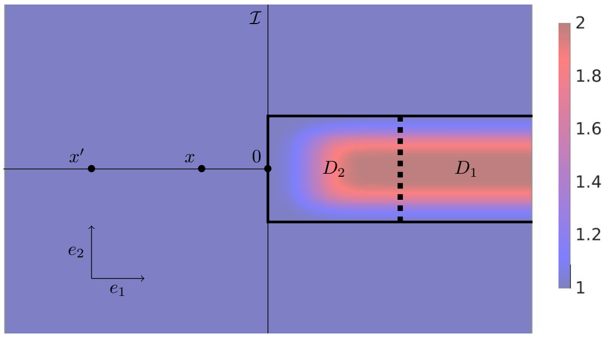

Figure 2: Value of $a_{11}(x)$. The support of η corresponds to the zone $D = D_1 \cup D_2$.

1.6 Outline

The article is organized as follows: In Section 2, we sketch the main steps leading to Theorem 1. Then, we state a few additional results: We deduce from Theorem 1 an almost-optimal convergence rate for the modified 2-scale expansion; we show that the rate (20) is attained in Example 3; and in a special stochastic case, namely Example 2, we get a slightly better growth rate for the generalized correctors. Sections 3 - 8 are devoted to the proofs. Namely, Sections 3, 4 and 5 contain the proof of Theorem 1, each of them corresponding to an intermediate result, whereas Sections 6, 7 and 8 contain the proofs of the additional results. Last, we state and prove in Appendix A a useful result on harmonic functions.

2 Strategy of proof and additional results

2.1 Strategy for the proof of Theorem 1

We go through the following sequence of steps: First, in Theorem 2, we assume access to a strictly sublinear generalized ungauged corrector (see Definition 1) and obtain an averaged Lipschitz estimate for a-harmonic functions above some minimal radius $r^* > 0$. Then, in Proposition 1, we show that, assuming the existence of generalized correctors $\Phi_\pm$ corresponding to $a_\pm$ satisfying (13), we can construct the generalized ungauged corrector needed as input in Theorem 2. Therefore, we obtain a large-scale Lipschitz estimate for a-harmonic functions. In turn, the latter yields annealed estimates for the first and second mixed derivatives of the Green's function associated to $-\mathrm{div}\,(a\nabla)$ (as shown in [6]). These Green's function estimates are a main ingredient to get the almost-optimal growth rates in Theorem 1. Their use is complemented by Lemma 7, in which we go from the ungauged flux corrector that comes out of Proposition 1 to a unique (up to addition of a random constant) flux corrector satisfying the same sublinearity properties, and Lemma 8, in which we control the moments of the minimal radius r^*.

2.2 Theorem 2: Large-scale Lipschitz estimate

Our Theorem 2 generalizes the previous result [17, Th. 4.1] by adapting the proof of [15, Lem. 2]. It takes as input strictly sublinear ungauged generalized correctors and yields a large-scale Lipschitz estimate for a-harmonic functions. The method in [15] is inspired by the earlier work of Avellaneda and Lin in the setting of periodic coefficients [4]. The main idea is to transfer regularity properties from the constant-coefficient homogenized operator to the heterogenous operator at large scales. In our case, to overcome the discontinuity of the homogenized matrix at the interface, we need to use the modified 2-scale expansion (5).

We use the convention that the *excess of an a-harmonic function on the ball of radius $r > 0$ centered around $x_0 \in \mathbb{R}$* is given by:

$$\mathcal{E}_r(x_0)[u] = \inf_{\xi \in \mathbb{R}^d} \fint_{\mathrm{B}_r(x_0)} |\nabla u - (\nabla P + \nabla \phi) \cdot \xi|^2 . \tag{29}$$

For this definition of the excess we obtain the following large-scale regularity result:

Theorem 2. *Assume that the coefficient field a has the form* (7) *and satisfies* (8)*, the homogenized matrix $\overline{a}$ has the form* (10)*, and the $\overline{a}$-harmonic coordinates are defined by* (14)*. We let $(\phi, \sigma^{\mathrm{u}})$ denote an associated generalized ungauged corrector. Then, for any Hölder exponent $\alpha \in (0,1)$, there exists a constant $\delta = \delta(d, \lambda, \alpha)$ such that the following properties hold:*

Let $x_0 \in \mathbb{R}^d$ and $r_{\max} > r^* > 0$. Assume that $(\phi, \sigma^{\mathrm{u}})$ satisfy the sublinearity condition

$$\sup_{r \in [r^*, r_{\max}]} \frac{1}{r} \Big(\fint_{\mathrm{B}_r(x_0)} \Big| (\phi, \sigma^{\mathrm{u}}) - \fint_{\mathrm{B}_r(x_0)} (\phi, \sigma^{\mathrm{u}}) \Big|^2 \Big)^{\frac{1}{2}} \leq \delta. \tag{30}$$

Then, for $R \in [r^*, r_{\max}]$ and a function u that is a-harmonic in $\mathrm{B}_R(x_0)$, the excess $\mathcal{E}$ defined by (29) *satisfies*

$$\mathcal{E}_r(x_0)[u] \leq \delta^{-1} \left(\frac{r}{R} \right)^{2\alpha} \mathcal{E}_R(x_0)[u] \qquad \text{for any } r \in [r^*, R]. \tag{31}$$

Moreover, the correctors have the following non-degeneracy property:

$$\delta |\xi|^2 \leq \fint_{\mathrm{B}_r(x_0)} |\nabla P \cdot \xi + \nabla \phi \cdot \xi|^2 \leq \delta^{-1} |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^d, r \in [r^*, r_{\max}/2]. \tag{32}$$

Finally, the following large-scale Lipschitz estimate holds:

$$\fint_{\mathrm{B}_r(x_0)} |\nabla u|^2 \leq \delta^{-1} \fint_{\mathrm{B}_R(x_0)} |\nabla u|^2 \qquad \text{for any } r^* \leq r \leq R \leq r_{\max}. \tag{33}$$

Remark 3 (Liouville theorem). As a consequence of Theorem 2, the space of a-harmonic functions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ that are strictly subquadratic is of dimension $d+1$. More precisely, such functions u can be written as

$$u(x) = c + b \cdot (P(x) + \phi(x)) \qquad \text{for constants } c \in \mathbb{R}, b \in \mathbb{R}^d.$$

Here, u is said to be subquadratic if there exists $\beta \in (0,1)$ such that

$$\lim_{R \uparrow \infty} R^{-(1+\beta)} \Big(\fint_{\mathrm{B}_R} \Big| u - \fint_{\mathrm{B}_R} u \Big|^2 \Big)^{\frac{1}{2}} = 0.$$

2.3 Proposition 1: Construction of generalized *ungauged* correctors

For the construction of the generalized ungauged corrector $(\phi, \sigma^{\mathrm{u}})$ that we take as input for Theorem 2, we adapt the method used in [12, 24] to construct Dirichlet and Neumann correctors. The general iterative scheme of [12,24] was first introduced to build higher order correctors in [11].

In [11,12,24] it is sufficient to assume existence of a whole-space (first order) corrector satisfying a quantified sublinearity condition. For simplicity, here, we restrict ourselves to a slightly less general sublinearity condition[6]. Indeed, we assume that there exist whole-space correctors (ϕ_-, σ_-) and (ϕ_+, σ_+) associated with $a_\pm$ such that

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} |(\phi_-, \sigma_-, \phi_+, \sigma_+)|^2\Big)^{\frac{1}{2}} \le r^{-\nu} \qquad \text{for any } r \ge 1, \tag{34}$$

for a given exponent $\nu \in (0,1]$ and for $x_0 \in \mathbb{R}^d$. Note that if (13) is satisfied, then, $\langle\cdot\rangle$-almost surely, for any $x_0 \in \mathbb{R}^d$ there exist (ϕ_-, σ_-) and (ϕ_+, σ_+) such that (34) holds (up to a uniform multiplicative constant).

The basic strategy of the construction we use here is to iteratively, on increasingly large scales, correct the glued composite correctors $(\check{\phi}, \check{\sigma})$ defined in (22). The intuition is that $(\nabla\phi, \nabla\sigma^{\mathrm{u}})$ should behave like $(\nabla\phi_\pm \cdot \nabla P, \nabla\sigma_\pm \cdot \nabla P)$ far from the interface on the right/left. This naturally leads to the ansatz:

$$\phi_k = (1-\chi)\check{\phi}_j \partial_j P_k + \widetilde{\phi}_k \qquad \text{and} \qquad \sigma^{\mathrm{u}}_{ijk} = (1-\chi)\check{\sigma}_{ijl}\partial_l P_k + \widetilde{\sigma}_{ijk}, \tag{35}$$

where the function χ is smooth, equals 1 on a narrow layer along the interface (containing the interface layer $[-1,1]\times\mathbb{R}^{d-1}$), and vanishes far from the interface (it will specified precisely in Section 4.1). The functions $\widetilde{\phi}$ and $\widetilde{\sigma}$ correspond to layer corrections along the interface.

More formally, we decompose $\mathbb{R}^d$ into dyadic annuli and solve the corrector equations (3) and (16) in the associated increasing balls by using the ansatz (35). We thus obtain a sequence $\big\{\big(\phi^M, \sigma^{\mathrm{u},M}\big)\big\}_{M\in\mathbb{N}}$ of "local generalized ungauged correctors". An induction argument yields the convergence of this sequence. Indeed, by appealing to the large-scale Lipschitz estimate of Theorem 2 in the M^{th} step, we ascertain a sublinearity estimate on the local generalized ungauged correctors $\big(\phi^M, \sigma^{\mathrm{u},M}\big)$. This latter property is then used in the $M+1^{\text{th}}$ step in order to invoke Theorem 2 again. Last, using these sublinearity estimates, we find that $\big(\phi^M, \sigma^{\mathrm{u},M}\big)$ converges to solutions $(\phi, \sigma^{\mathrm{u}})$ of (3) and (17) on the whole-space $\mathbb{R}^d$.

Using the above strategy, we obtain the following result:

Proposition 1. *Let a be defined in* (7) *and satisfy* (8)*, and $\overline{a}_\pm$ be the constant homogenized matrices associated with $a_\pm$. Assume that there exist generalized correctors (ϕ_-, σ_-) and (ϕ_+, σ_+) associated with $a_\pm$ such that* (34) *holds for $\nu \in (0,1]$ and $x_0 \in \mathbb{R}^d$. Then, there exists a generalized ungauged corrector $(\phi, \sigma^{\mathrm{u}})$ associated with the coefficient field a that satisfies*

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big| (\phi, \sigma^{\mathrm{u}}) - \fint_{\mathrm{B}(x_0)} (\phi, \sigma^{\mathrm{u}}) \Big|^2\Big)^{\frac{1}{2}} \le \kappa r^{-\widetilde{\nu}} \quad \textit{for any } r \ge 1, \tag{36}$$

[6] An inspection of the proof of Proposition 1 should convince the reader that the result, in fact, holds under the direct analogue of the quantified sublinearity condition [12, (11)].

for the exponent $\widetilde{\nu} := \nu/3$, and a constant κ depending on d, λ and ν.

Remark 4. Proposition (1) also applies if the input correctors are ungauged.

Notice that the sublinearity condition (34) involves an anchoring point $x_0 \in \mathbb{R}^d$. As we will see in Section 4.1, this affects the definition of the cut-off functions defining the various local generalized ungauged correctors. In particular, in the method that we have described above, the successively large annuli were implicitly centered at 0. Of course, the output $(\phi, \sigma^{\mathrm{u}})$ of the proposition apriori depends on the anchoring point x_0. However, we see in Section 5.1 that the gradient of the corrector $\nabla\phi$ is unique and thus independent of x_0, whereas $\nabla\sigma^{\mathrm{u}}$ generally depends on x_0 ($\nabla\sigma^{\mathrm{u}}$ is not unique since σ^{u} is ungauged).

As already observed in [12] for the boundary correctors, the estimate (36) is suboptimal in terms of the exponent $\widetilde{\nu}$. Actually, even if the generalized correctors $(\phi_\pm, \sigma_\pm)$ were uniformly bounded, optimizing this method would only upgrade (36) to the exponent $\widetilde{\nu} = 1/2$ (whereas one may hope for $\widetilde{\nu} = 1$). The non-optimality of this estimate is an inherent feature of the method, which relies on energy estimates to capture the "smallness" of the layer around the interface. This strategy is predestined to be suboptimal: Indeed, the normalized L^2-energy corresponding to the layer around the interface and inside a ball of radius r scales like $r^{-1/2}$, whereas the normalized L^1-norm of the same domain scales like r^{-1}. In particular, we formally have that

$$\Big(\fint_{\mathrm{B}_r} \big(\mathbb{1}_{[-1,1]\times\mathbb{R}^{d-1}}\big)^2\Big)^{\frac{1}{2}} \simeq r^{-1/2} \gg r^{-1} \simeq \fint_{\mathrm{B}_r} \mathbb{1}_{[-1,1]\times\mathbb{R}^{d-1}} \quad \text{for} \quad r \gg 1.$$

Thus, the energy norm is not the best way to account for the smallness of the layer. This observation advocates for using more refined tools, namely estimates for the mixed gradient of the Green's function. The latter will transfer the L^1 optimal bound corresponding to the layer to an L^∞ bound for the growth rate of the generalized correctors (up to logarithmic losses). This remark is at the core of the proof of Theorem 1 and a key observation of this paper.

2.4 Theorem 3: Improved rates via independence

To demonstrate the price that we pay in Theorem 1 due to a lack of joint structure assumptions on the media, we take a closer look at the situation of Example 2 in Section 1.5. Here, a relationship between the two media and the layer is enforced by assuming independence of their laws. As an analogue of [18, Sec. 3.2], we obtain:

Theorem 3. *Assuming the situation described in Example 2 of Section 1.5, there exists a unique (up to addition of a random constant) generalized corrector (ϕ, σ) associated with the coefficient field a that satisfies the estimate:*

$$\langle |(\phi,\sigma)(x) - (\phi,\sigma)(y)|^p\rangle^{\frac{1}{p}} \lesssim_{d,\lambda,\kappa,\alpha,p} \begin{cases} \ln^{\frac{1}{2}}(2+|x-y|) & \text{if} \quad d=2, \\ 1 & \text{if} \quad d>2. \end{cases} \tag{37}$$

Our argument for Theorem 3 is essentially a corollary of the techniques used in [18, Sec. 3.2] (where there is no interface), which rely on the availability of a powerful tool: a spectral gap estimate. This ingredient is actually available in our current setting. Indeed, thanks to the independence assumption[7] and (23), the ensemble $\langle\cdot\rangle$ is such that, for any functional $F = F(a)$ and $p \in [1,\infty)$ the following spectral gap holds:

$$\Big\langle\, |F - \langle F\rangle|^{2p} \,\Big\rangle \lesssim_{d,\lambda,\kappa,\alpha,p} \Big\langle \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial F}{\partial a(z)} \Big|^2 \mathrm{d}z \Big)^p \Big\rangle, \tag{38}$$

where we make use of the functional derivative defined by

$$\lim_{\varepsilon\to 0} \frac{F(a+\varepsilon\delta a) - F(a)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\partial F(a)}{\partial a(z)} \, (\delta a(z)) \, \mathrm{d}z.$$

Moreover, the Gaussian fields $g = g_-$, $g = g_+$, and $g = g_\circ$ are smooth in the following sense [18, Sec. 5.2]: For any $0 < \alpha' < \alpha$ and $p \in [1,\infty)$, there holds

$$\Big\langle\, \|g\|^p_{C^{0,\alpha'}(\mathrm{B})} \,\Big\rangle \lesssim_{d,\lambda,\kappa,\alpha,\alpha',p} 1. \tag{39}$$

2.5 Proposition 2: Example for optimality of Theorem 1

We show that the coefficient a given by Example 3 admits a global corrector ϕ with an unbounded growth rate:

Proposition 2. *Assume that $d > 2$. Let the coefficient field a be defined by (28) and ϕ be the associated unique (up to addition of a constant) strictly sublinear corrector. Then, there exists a constant $C > 0$ such that for any $r \geq 1$ there exist points $x, x' \in \mathbb{R}^d$ satisfying*

$$|\phi(x) - \phi(x')| \geq C \ln(r) \quad \textit{and} \quad |x - x'| = r. \tag{40}$$

2.6 Corollary 1: A convergence rate

As an application of Theorem 1, we obtain optimal convergence rates (up to powers of a logarithm). In particular, we prove the following corollary (which, for simplicity, is deterministic):

Corollary 1. *Let $d > 2$, $\nu \in (0,1]$ and $\varepsilon > 0$. Assume that the deterministic coefficient field a is defined by (7), satisfies (8), is uniformly α-Hölder continuous for a fixed exponent $\alpha \in (0,1)$, and satisfies:*

$$\|a\|_{C^{0,\alpha}(\mathbb{R}^d)} \leq \kappa.$$

Suppose that the underlying coefficient fields $a_\pm$ admit constant homogenized matrices $\overline{a}_\pm$, and that there exist generalized correctors associated with $a_\pm$ that satisfy:

$$|\Phi_\pm(x) - \Phi_\pm(y)| \leq \kappa |x-y|^{1-\nu} \qquad \textit{for any } x, y \in \mathbb{R}^d.$$

Let $f \in L^p(\mathbb{R}^d)$ with support inside $\mathrm{B}(x_0)$, for $p > d$. Assume that the functions u^ε and $\overline{u}$ are the zero-mean Lax-Milgram solutions to

$$-\mathrm{div}\,(a\,(x/\varepsilon)\,\nabla u^\varepsilon(x)) = -\mathrm{div}\,(\overline{a}(x)\nabla\overline{u}(x)) = f(x) \quad \textit{in} \quad \mathbb{R}^d. \tag{41}$$

[7] Independence is a sufficient –but not necessary– condition for deriving a spectral gap for the global medium.

Then, there there holds

$$\|u^\varepsilon - \overline{u}\|_{L^\infty(\mathbb{R}^d)} \lesssim_{d,\lambda,\kappa,\alpha,\nu,p} \varepsilon^\nu \ln^{1+2\lfloor\nu\rfloor}\left(2+\varepsilon^{-1}\right)\|f\|_{L^p(\mathbb{R}^d)}. \tag{42}$$

Moreover, if $f \in L^\infty\left(\mathbb{R}^d\right)$, then:

$$\left\|\nabla u^\varepsilon - \nabla\overline{u} - \nabla\phi\left(\frac{\cdot}{\varepsilon}\right)\cdot\overline{\nabla}\overline{u}\right\|_{L^\infty(\mathbb{R}^d)} \lesssim_{d,\lambda,\kappa,\alpha,\nu} \varepsilon^\nu \ln^{2+3\lfloor\nu\rfloor}(2+\varepsilon^{-1})\|f\|_{L^\infty(\mathbb{R}^d)}. \tag{43}$$

Remark 5. The assumptions of Corollary 1 encompass special cases of Example 1.

Remark 6. We assume in Corollary 1 that the dimension $d > 2$ and that f has compact support in order to ascertain that $f \in H^{-1}(\mathbb{R}^d)$. (Relaxations of these assumptions are possible, but we do not consider these subtleties for simplicity.)

In our proof of Corollary 1, we make use of the generalized 2-scale expansion (5). In Figure 3 we illustrate its accuracy in comparison with a naive glued 2-scale expansion. For this, we define the local errors $E^{\varepsilon,1}$ and $E^{\varepsilon,2}$ respectively corresponding to the left-hand side of (43) and to the glued 2-scale expansion:

$$E^{\varepsilon,1} := \nabla u^\varepsilon - \nabla\overline{u} - \nabla\phi\left(\frac{\cdot}{\varepsilon}\right)\cdot\overline{\nabla}\overline{u},$$
$$E^{\varepsilon,2} := \begin{cases} \left|\nabla u^\varepsilon - \left(\mathrm{I} + \nabla\phi_+\left(\frac{\cdot}{\varepsilon}\right)\right)\cdot\nabla\overline{u}\right| & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^{d-1}, \\ \left|\nabla u^\varepsilon - \left(\mathrm{I} + \nabla\phi_-\left(\frac{\cdot}{\varepsilon}\right)\right)\cdot\nabla\overline{u}\right| & \text{in} \quad \mathbb{R}_-\times\mathbb{R}^{d-1}. \end{cases}$$

As expected, we observe that our approximation performs well uniformly on the space, whereas the glued 2-scale expansion is accurate far from the interface, but useless in the vicinity of it. While letting $\varepsilon \downarrow 0$, we observe that the convergence rate is approximately linear in ε, as predicted by Corollary 1.

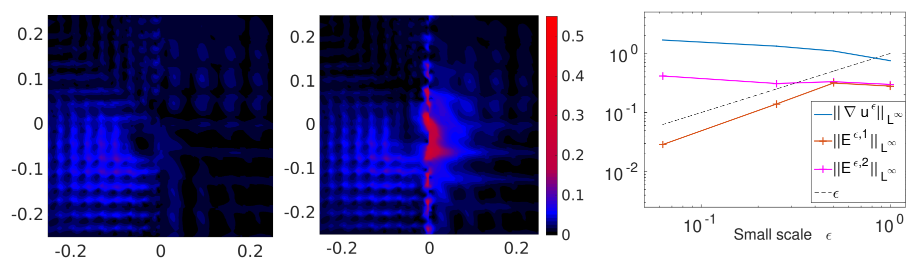

Figure 3: Consider a coefficient field a as on the left of Figure 1 and a smooth forcing term f with compact support. On the left of this figure we have the local error $E^{\varepsilon,1}$, in the middle is the local error $E^{\varepsilon,2}$, and on the right we see the L^∞-norms of the aforementioned quantities when $\varepsilon \downarrow 0$.

2.7 Further extensions

Corners and boundary In this contribution, we only consider equations (41) posed on the whole-space $\mathbb{R}^d$. This has the beneficial consequence of avoiding the problem of boundaries. However, this is *not* only a convenient simplification. Indeed, if we would replace $\mathbb{R}^d$ by a smooth bounded domain and complement the equation with homogeneous Dirichlet boundary conditions, we would face the geometrical problem of the crossing between the interface and the boundary. There, the regularity results for discontinuous coefficients that we are using from [20,21,22] do not apply. This geometrical problem is not unrelated with the case of a non-flat interface with a corner –which might be relevant when modeling interfaces between crystals. We intend to explore these issues in upcoming works.

From another perspective, let us emphasize that the techniques used here could improve the results of [12, 24]. In particular, one could obtain almost-optimal growth rates for correctors on the half-space with homogeneous Dirichlet and Neumann boundary conditions.

The case of a wide interface layer In this article, we have assumed that the layer around the interface has a width $2L = 2$ (see (7)). In general, this width $L \geq 1$ might be much larger than the characteristic scale (here 1) of the oscillations of the coefficients $a_{\pm}$. Thus, it might be useful to track the dependence in L in the estimates (20) and (21).

To handle this situation, we perform the rescaling $\widetilde{a}(y) := a(Ly)$. Notice that the width of the interface layer of $\widetilde{a}$ is now 2, so that we may apply our results to $\widetilde{a}$. Also, remark that the correctors $\widetilde{\phi}$, $\widetilde{\sigma}$, $\widetilde{\phi}_{\pm}$, *etc.* associated to $\widetilde{a}$ are obtained from the correctors ϕ, σ, $\phi_{\pm}$ associated to a by the following rescaling: $\widetilde{\phi}(y) = L^{-1}\phi(Ly)$. Performing the arguments for Theorem 1 (see Section 5) in this special context produces (after rescaling back) the following estimates

$$\Big\langle \Big(\int_{\mathrm{B}} |\phi(x+z) - \phi(y+z)|^2 \,\mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \lesssim L \ln\Big(2 + \frac{|r|}{L}\Big) + |r|^{1-\nu} \ln^{\lfloor \nu \rfloor}\Big(2 + \frac{|r|}{L}\Big),$$

$$\Big\langle \Big(\int_{\mathrm{B}} |\sigma(x+z) - \sigma(y+z)|^2 \,\mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \lesssim L \ln^3\Big(2 + \frac{|r|}{L}\Big) + |r|^{1-\nu} \ln^{1+2\lfloor \nu \rfloor}\Big(2 + \frac{|r|}{L}\Big),$$

for any $x, y \in \mathbb{R}^d$ with $|x - y| = r$.

Systems In all of this article, we consider a scalar equation. However, we do not use any elliptic tool specific to scalar equations (*e.g.*, the maximum principle). Therefore, our theorems may extend to the case of systems, that is replacing (1) by

$$-\mathrm{div}\,(A : \nabla u) = f \quad \text{in the sense of} \quad -\partial_i A_{ij}^{\alpha\beta} \partial_j u^\beta = f^\alpha,$$

where the coefficient $A = \Big(A_{ij}^{\alpha\beta}\Big)$, for $i, j \in [\![1, d]\!]$ and $\alpha, \beta \in [\![1, m]\!]$, $m \in \mathbb{N}$, is elliptic in the following sense[8]:

$$\lambda |\xi|^2 \leq A_{ij}^{\alpha\beta}(x) \xi_i^\alpha \xi_j^\beta \leq |\xi|^2 \quad \text{for any } x \in \mathbb{R}^d, \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times m}. \tag{44}$$

[8] Notice that *elastic* systems do not satisfy fully (44), but a weaker version of it, where the vector-valued vectors ξ in (44) shall be symmetric in the sense of $\xi_i^\alpha = \xi_\alpha^i$ (in such a case, $m = d$) –see [25, Sec. 2.4 p. 26].

3 Argument for Theorem 2

In all of this section, we place ourselves under the assumptions of Theorem 2. We follow the strategy of [15, Prop. 1] and for brevity, whenever an argument does not see substantial alteration in its passage to the current setting of an interface, we keep details sparse and refer the reader to the latter contribution. The main idea is to take advantage of the sublinearity of the (ungauged) generalized corrector (30) in oder to control the excess. This is rephrased in the following lemma, which is an adaptation of [15, Lem. 3]:

Lemma 1. *Assume that the function u is a-harmonic in $\mathrm{B}_R(x_0)$ and that δ defined by*

$$\delta := \frac{1}{R}\Big(\fint_{\mathrm{B}_R(x_0)} \Big|(\phi,\sigma^{\mathrm{u}}) - \fint_{\mathrm{B}_R(x_0)} (\phi,\sigma^{\mathrm{u}})\Big|^2\Big)^{\frac{1}{2}} \tag{45}$$

satisfies $\delta \leq 1$. Then, there exists a constant $\varepsilon = \varepsilon(d,\lambda) > 0$ such that, for any $0 < r \leq R$, the following estimate holds:

$$\mathcal{E}_r(x_0)[u] \lesssim_{d,\lambda} \Big(\Big(\frac{r}{R}\Big)^2 + \delta^{2\varepsilon}\Big(\frac{R}{r}\Big)^{d+2}\Big)\fint_{\mathrm{B}_R(x_0)} |\nabla u|^2 . \tag{46}$$

We remark that the difference between our proof of Lemma 1 and [15, Lem. 3] is that we use the generalized 2-scale expansion (5) instead of the classical one. This results in the standard algebraical identity involving the difference between the solution of the oscillating problem and the 2-scale expansion (see [16, p. 26-27] or [15, (79)]) being replaced by (19); and the regularity estimate for $\nabla^2\overline{u}$ used in the original proof (that is unavailable in the case of an interface) is replaced by a corresponding estimate for $\nabla\overline{\nabla}\overline{u}$, which is given in Lemma 2.

Before proceeding with the proof of Lemma 1, we emphasize two technical lemmas: Lemma 2, which concerns the regularity of $\nabla\overline{\nabla}\overline{u}$ and Lemma 3, which provides a weighted energy estimate.

Lemma 2. *Let $x_0 \in \mathbb{R}^d$ and $\rho \in (0,1)$. Assume that $\overline{u} \in H^1(\mathrm{B}(x_0))$ is $\overline{a}$-harmonic in $\mathrm{B}(x_0)$, where $\overline{a}$ is defined in* (10)*. Then, we have that*

$$\sup_{x\in \mathrm{B}_{1-\rho}(x_0)\setminus\mathcal{I}} \rho^2|\nabla\overline{\nabla}\overline{u}(x)|^2 + \sup_{x\in\mathrm{B}_{1-\rho}(x_0)} |\overline{\nabla}\overline{u}(x)|^2 \lesssim_{d,\lambda} \rho^{-d}\fint_{\mathrm{B}(x_0)} |\overline{\nabla}\overline{u}|^2. \tag{47}$$

Lemma 3. *Assume that a satisfies* (8) *and let $u \in H^1(\mathrm{B})$ satisfy*

$$\begin{cases} -\mathrm{div}\,(a\nabla u) = \mathrm{div}(g) & in \quad \mathrm{B}, \\ u = 0 & on \quad \partial\mathrm{B}. \end{cases} \tag{48}$$

Then, there exists an exponent $\beta = \beta(d,\lambda) \in (0,1)$ such that the following estimate holds:

$$\int_{\mathrm{B}} (1-|x|)^{\beta}\,|\nabla u(x)|^2\,\mathrm{d}x \lesssim_{d,\lambda} \int_{\mathrm{B}} (1-|x|)^{\beta}\,|g(x)|^2\,\mathrm{d}x. \tag{49}$$

As both of the above lemmas are classical, we do not prove them here. We refer to [21, Prop. 2.1] for Lemma 2, and to the lecture notes [14, Chap. 3] for Lemma 3 (see also the Hardy inequality from [23, Th. 1.6]).

Proof of Lemma 1. Let β be given by Lemma 3. Without loss of generality, we may assume that $4r \leq R$ and $R = 1$. Furthermore, we may assume that ϕ and σ^{u} are of zero mean on $\mathrm{B}(x_0)$.

Step 1: Set-up We define the function $\overline{u} \in H^1(\mathrm{B}(x_0))$ as the Lax-Milgram solution of

$$\begin{cases} -\mathrm{div}\,(\overline{a}\nabla\overline{u}) = 0 & \text{in} \quad \mathrm{B}(x_0), \\ \overline{u} = u & \text{on} \quad \partial\mathrm{B}(x_0), \end{cases} \tag{50}$$

and denote the corresponding homogenization error as

$$w := u - \overline{u} - \eta\phi \cdot \overline{\nabla}\overline{u}, \tag{51}$$

where $\eta \in C^\infty_{\mathrm{c}}(\mathrm{B}(x_0))$ will be fixed later. Notice that $w \equiv 0$ on $\partial\mathrm{B}(x_0)$, and, using the same computations as in Step 3 of [15, Prop. 1], w solves the following cut-off version of (19):

$$-\mathrm{div}\,(a\nabla w) = \mathrm{div}(g) \quad \text{in} \quad \mathrm{B}(x_0), \tag{52}$$

for $g_i := \big(a_{ij}\phi_k - \sigma^{\mathrm{u}}_{ijk}\big)\partial_j\left(\eta\overline{\partial}_k\overline{u}\right) + (1-\eta)\left(a_{ij} - \overline{a}_{ij}\right)\partial_j\overline{u}$.

Step 2: Estimate for g We claim that the following inequality holds:

$$\int_{\mathrm{B}(x_0)} (1 - |x - x_0|)^\beta |g|^2 \lesssim \left(\rho^\beta + \rho^{-d-2}\delta^2\right) \int_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2, \tag{53}$$

where we assume now that $\eta = 1$ in $\mathrm{B}_{1-2\rho}(x_0)$, $\mathrm{Supp}(\eta) \subset \mathrm{B}_{1-\rho}(x_0)$, and $|\nabla\eta| \lesssim \rho^{-1}$. We remark that (53) is the counterpart of [15, (96)] and is proven in the same way, using Lemma 2.

Step 3: Excess decay estimate We claim the following:

$$\mathcal{E}_r(x_0)[u] \lesssim r^{-d-2} \fint_{\mathrm{B}(x_0)} (1 - |x - x_0|)^\beta |\nabla w|^2 + \left(r^2 + r^{-d}\delta^2\right) \fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2. \tag{54}$$

To obtain (54), we set

$$\xi := \fint_{\mathrm{B}_r(x_0)} \overline{\nabla}\overline{u} \qquad \text{and} \quad \widetilde{w} := u - (P + \phi - P(x_0)) \cdot \xi - \overline{u}(x_0). \tag{55}$$

By definition (29) of the excess, there holds:

$$\mathcal{E}_r(x_0)[u] \le \fint_{\mathrm{B}_r(x_0)} |\nabla\widetilde{w}|^2. \tag{56}$$

Since the function $\widetilde{w}$ is a-harmonic in $\mathrm{B}(x_0)$, the Caccioppoli estimate yields

$$\fint_{\mathrm{B}_r(x_0)} |\nabla\widetilde{w}|^2 \lesssim r^{-2} \fint_{\mathrm{B}_{2r}(x_0)} |\widetilde{w}|^2. \tag{57}$$

Applying the triangle inequality to the definition (55) of $\widetilde{w}$, we may decompose in $\mathrm{B}_{2r}(x_0)$ (recall the definition (51) of w):

$$|\widetilde{w}| \le |w| + |\phi| \left|\xi - \overline{\nabla}\overline{u}\right| + |\overline{u} - (P - P(x_0)) \cdot \xi - \overline{u}(x_0)|.$$

The second and third terms on the right-hand side may then be handled by appealing to Lemma 2 (see [17, Lem. 5.3] for more details), to the effect of:

$$
\begin{aligned}
&\left\|\xi - \overline{\nabla}\overline{u}\right\|_{L^\infty(\mathrm{B}_{2r}(x_0))} \lesssim r\Big(\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\Big)^{\frac{1}{2}},\\
&\|\overline{u} - (P - P(x_0))\cdot\xi - \overline{u}(x_0)\|_{L^\infty(\mathrm{B}_{2r}(x_0))} \lesssim r^2\Big(\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Combining these observations with (57), we find that

$$
\begin{aligned}
\fint_{\mathrm{B}_r(x_0)} |\nabla\widetilde{w}|^2 &\lesssim r^{-2}\fint_{\mathrm{B}_{2r}(x_0)} |w|^2 + \Big(r^2 + \fint_{\mathrm{B}_{2r}(x_0)} |\phi|^2\Big)\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\\
&\lesssim r^{-d-2}\fint_{\mathrm{B}(x_0)} (1-|x-x_0|)^{\beta-2}\,|\nabla w|^2 + \big(r^2 + r^{-d}\delta^2\big)\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\,.
\end{aligned}
\tag{58}
$$

This entails (54).

Step 4: Conclusion Applying Lemma 3 to w, which satisfies (52), and invoking (53), we deduce that

$$
\begin{aligned}
\int_{\mathrm{B}(x_0)} (1-|x-x_0|)^\beta\,|\nabla w|^2 &\lesssim \int_{\mathrm{B}(x_0)} (1-|x-x_0|)^\beta\,|g|^2\\
&\lesssim \big(\rho^\beta + \rho^{-d-2}\delta^2\big)\int_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\,.
\end{aligned}
$$

We optimize the above estimate by setting $\rho := \delta^{\frac{2}{d+2+\beta}}$. Thus,

$$
\int_{\mathrm{B}(x_0)} (1-|x-x_0|)^\beta\,|\nabla w|^2 \lesssim \delta^{2\varepsilon}\int_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2 \tag{59}
$$

for $\varepsilon := \beta/(d+2+\beta)$. By (54), this entails

$$
\mathcal{E}_r(x_0)[u] \lesssim \big(r^{-d-2}\delta^{2\varepsilon} + r^2 + r^{-d}\delta^2\big)\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2\,. \tag{60}
$$

Then, since δ, r, and ε are smaller than 1, we deduce that $\delta^2 \le r^{-2}\delta^{2\varepsilon}$. Moreover, by (50), there holds:

$$
\fint_{\mathrm{B}(x_0)} |\nabla\overline{u}|^2 \lesssim \fint_{\mathrm{B}(x_0)} |\nabla u|^2\,.
$$

As a consequence, (46) is obtained from (60) by restoring the scale R. □

With Lemma 1 in-hand, we are now in a position to give the argument for Theorem 2. Since it closely follows the argument for [15, Th. 1], we do not detail every step.

Proof of Theorem 2. Let $R \in [r^*, r_{\max}]$ and $\theta \in (0,1)$. Since $u - (P+\phi)\cdot\xi$ is a-harmonic for any $\xi \in \mathbb{R}^d$, we deduce from Lemma 1 that

$$
\mathcal{E}_{\theta R}(x_0)[u] \le C\left(\theta^2 + \delta^{2\varepsilon}\theta^{-d-2}\right)\mathcal{E}_R(x_0)[u]. \tag{61}
$$

We set θ and then δ (in that order) sufficiently small so that $C\theta^2 \le \theta^{2\alpha}/2$ and $C\delta^{2\varepsilon}\theta^{-d-2} \le \theta^{2\alpha}/2$. Then (61) reads:

$$\mathcal{E}_{\theta R}(x_0)[u] \le \theta^{2\alpha}\mathcal{E}_R(x_0)[u]. \tag{62}$$

Up to setting δ smaller, (31) follows by a standard iteration argument (see Step 1 of [15, Th. 1]).

The proofs of (32) and (33) are similar to [15, (33) & (16)]. In particular, (32) follows from the Caccioppoli and Poincaré inequalities (see Step 7 of [15, Prop. 1]) and (33) is a corollary of (46) and (32) (see Step 2 of [15, Th. 1]). $\square$

4 Argument for Proposition 1

We begin by introducing the geometric objects needed in our argument and constructing local ungauged generalized correctors. In the second subsection, we show that we can inductively use the large-scale Lipschitz estimate (33) from Theorem 2 to, given a local ungauged generalized corrector at some scale, obtain another local ungauged generalized corrector on a larger scale that still satisfies the same sublinearity property. In the final subsection we show that we can pass to the limit in this procedure to obtain a global ungauged generalized corrector that is strictly sublinear.

4.1 Construction of local ungauged generalized correctors

We use a geometrical construction involving dyadic balls and a narrow layer along the interface (see Figure 4) according to which we decompose the interface layer corrections $\widetilde{\phi}$ and $\widetilde{\sigma}$ that we have already introduced in Section 2.3.

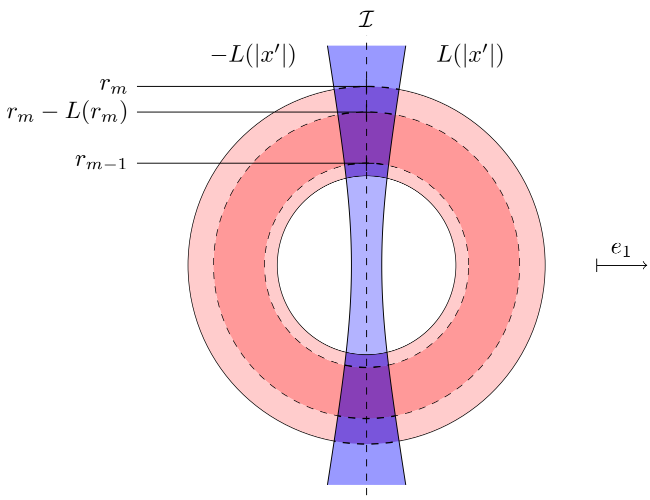

Figure 4: In this figure, we set $x_0 = 0$. Here the blue area represents the support of the cut-off function χ along the interface, *i.e.* the "trumpet", and the pink areas denotes the support of $\eta_m - \eta_{m-1}$. Notice that $\nabla(\eta_m - \eta_{m-1})$ is supported in the lighter pink area and $\eta_m - \eta_{m-1} = 1$ in the darker pink area.

Geometrical setting Fix $x_0 \in \mathbb{R}^d$. For $r_0 \geq 1$ to be fixed later, we set a dyadic sequence $r_m := 2^m r_0$ for $m \geq 0$ and $r_m = 0$ if $m < 0$. In order to define the interface layer, we introduce the function

$$L(r) := r^{-2\nu/3+1} + 2, \tag{63}$$

which, as will be seen in our proof, actually turns out to be optimal for our estimates. We now let S_m be functions of x depending only on $x^\perp$ and satisfying the following constraints:

$$\begin{cases} \left\{x \in \mathbb{R}^d : S_m(x) = 1\right\} \subseteq \left\{x \in \mathbb{R}^d : \left|x^\perp\right| \leq \frac{L(r_m)}{2}\right\}, \\ \mathrm{Supp}(S_m) \subseteq \left\{x \in \mathbb{R}^d : \left|x^\perp\right| \leq L(r_m)\right\}, \\ \|\nabla S_m\|_{L^\infty(\mathbb{R}^d)} \lesssim (L(r_m))^{-1}, \text{ and } \|S_m\|_{L^\infty(\mathbb{R}^d)} \leq 1. \end{cases} \tag{64}$$

We remark that the "$+2$" in (63) is included to ensure that $S_m \equiv 1$ on $[-1,1] \times \mathbb{R}^{d-1}$. We also define the cut-off functions η_m associated with the nested balls

$$\mathrm{B}_{r_m - L(r_m)}(x_0) \subseteq \left\{x \in \mathbb{R}^d : \eta_m(x) = 1\right\} \subseteq \mathrm{Supp}(\eta_m) \subseteq \mathrm{B}_{r_m}(x_0) \tag{65}$$

such that $|\nabla \eta_m| \lesssim |L(r_m)|^{-1}$. To compliment this definition, we use the convention that a ball of negative radius is the empty set. Finally, we introduce the notation

$$\chi_m := S_m(\eta_m - \eta_{m-1}) \qquad \text{and} \quad \chi := \sum_{m=0}^{+\infty} \chi_m$$

and notice that χ equals 1 in a layer along the interface and vanishes far from it. This layer is thin along the interface, but is shaped like a trumpet and becomes wider far from the origin.

Local ungauged generalized corrector $\left(\phi^M, \sigma^{\mathrm{u},M}\right)$ In this subsection, for simplicity, we set $x_0 = 0$ (the cases $x_0 \neq 0$ can be dealt with similarly). For $m \geq 0$, we first derive an equation for the m^{th} contribution to the interface layer correction $\widetilde{\phi}^m$. For each $M \geq 0$ we sum these contributions up to the scale r_M in order to obtain the local corrector ϕ^M. In particular, we have the following result:

Lemma 4. *For every $m \geq 0$ and $k \in [\![1,d]\!]$, there exists $\widetilde{\phi}_k^m \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ that is a weak solution of*

$$-\mathrm{div}\big(a \nabla \widetilde{\phi}_k^m\big) = \partial_i (g_{ik}^m) \quad \textit{in} \quad \mathbb{R}^d, \tag{66}$$

for the vector g_k^m defined by

$$g_{ik}^m := \big(\chi_m (a_{ij} - \overline{a}_{ij}) + \big(a_{il}\check{\phi}_j - \check{\sigma}_{ilj}\big)\, \partial_l\left((\eta_m - \eta_{m-1}) - \chi_m\right)\big)\, \partial_j P_k, \tag{67}$$

such that

$$\int_{\mathbb{R}^d} \left|\nabla \widetilde{\phi}^m\right|^2 \lesssim_{d,\lambda} \int_{\mathbb{R}^d} |g^m|^2. \tag{68}$$

Furthermore, the function $\phi_k^M \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *defined by*

$$\phi_k^M := \sum_{m=0}^{M} \widetilde{\phi}_k^m + \Big(\eta_M - \sum_{m=0}^{M} \chi_m\Big)\check{\phi} \cdot \nabla P_k \tag{69}$$

is a "local corrector" in the sense that it satisfies

$$-\mathrm{div}\left(a\left(\nabla \phi_k^M + \eta_M \nabla P_k\right) - \eta_M \overline{a} \nabla P_k\right) = 0 \quad \textit{in} \quad \mathbb{R}^d, \tag{70}$$

whence

$$-\mathrm{div}\left(a\left(\nabla \phi_k^M + \nabla P_k\right)\right) = 0 \quad \textit{in} \quad \mathrm{B}_{r_{M-1}}. \tag{71}$$

Remark 7 (Discontinuities through the interface). In order to make sense of g^m in (67), we notice that the functions $\check{\phi}$ and $\check{\sigma}$ are (generically) discontinuous through the interface $\mathcal{I}$. This is, however, not a problem as $(1-\chi)$ and $\nabla\chi$ vanish in a neighborhood of the interface and, therefore, possible singularities of distributions multiplied by these functions in this neighborhood are not important. To illustrate this, we remark that apriori the formal product $\partial_l \check{\phi}_j \partial_j P_k$ has no mathematical significance for $x^\perp = 0$ (even in the sense of distributions), but the expression

$$\left((1-\chi)\partial_l \check{\phi}_j \partial_j P_k\right)(x) := \begin{cases} 0 & \text{if} \quad x \notin \mathrm{Supp}\left((1-\chi)\right), \\ \left((1-\chi)\partial_l \check{\phi}_j \partial_j P_k\right)(x) & \text{if} \quad x \in \mathrm{Supp}\left((1-\chi)\right) \end{cases}$$

is well-defined. From now on, we will use this and all analogous conventions without further notice.

Proof of Lemma 4. The matter of the existence of $\widetilde{\phi}_k^m$ and the energy estimate (68) can be settled using a standard Lax-Milgram argument.

By the definition (65) of η_M, (71) is an obvious consequence of (70) and (14). Then, by (66) and (67), there holds:

$$\begin{aligned} &- \mathrm{div}\Big(a\nabla\Big(\sum_{m=0}^{M} \widetilde{\phi}_k^m\Big)\Big) \\ &= \partial_i\Big(\sum_{m=0}^{M} \chi_m \left(a_{ij} - \overline{a}_{ij}\right)\partial_j P_k + \left(a_{il}\check{\phi}_j - \check{\sigma}_{ilj}\right)\partial_l\Big(\eta_M - \sum_{m=0}^{M}\chi_m\Big)\partial_j P_k\Big). \end{aligned} \tag{72}$$

To finish, we show that with the ansatz (69), the relations (70) and (72) are equivalent. By plugging the ansatz (69) in (70), we obtain:

$$\begin{aligned} -\partial_i\Big(a_{ij}\sum_{m=0}^{M}\partial_j\widetilde{\phi}_k^M\Big) &= \partial_i\Big(\Big(\eta_M - \sum_{m=0}^{M}\chi_m\Big)\left(a_{il}\partial_l\check{\phi}_j + a_{ij} - \overline{a}_{ij}\right)\partial_j P_k \\ &\quad + a_{il}\partial_l\Big(\eta_M - \sum_{m=0}^{M}\chi_m\Big)\check{\phi}_j\partial_j P_k + \sum_{m=0}^{M}\chi_m\left(a_{ij} - \overline{a}_{ij}\right)\partial_j P_k\Big). \end{aligned}$$

By the definition and skew-symmetry of $\check{\sigma}$, the first term on the right-hand side reads

$$\partial_i\left(\theta^M\left(a_{il}\partial_l\check{\phi}_j + a_{ij} - \overline{a}_{ij}\right)\partial_j P_k\right) = -\partial_i\left(\theta^M\partial_l\check{\sigma}_{lij}\partial_j P_k\right) = -\partial_l\left(\check{\sigma}_{lij}\partial_i\theta^M\partial_j P_k\right),$$

where we used $\theta^M := \eta_M - \sum_{m=0}^{M}\chi_m$ for brevity. Whence (72) is established. $\square$

With the local correctors ϕ^M from Lemma 4, we can now build local ungauged flux correctors:

Lemma 5. *Let g^m, $\widetilde{\phi}^m$ and ϕ^M be defined as in Lemma 4. For $M \geq 0$ and $j, k \in [\![1, d]\!]$, assume that there exists a function $\widetilde{N}^M_{jk} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ that satisfies*

$$\Delta \widetilde{N}^M_{jk} = -\sum_{m=0}^{M} \left(g^m_{jk} + a_{jl}\partial_l \widetilde{\phi}^m_k \right) \quad \text{in} \quad \mathbb{R}^d. \tag{73}$$

If we, furthermore, assume $\nabla \widetilde{N}^M$ to be strictly sublinear, then $\sigma^{\mathrm{u},M}$ defined by

$$\sigma^{\mathrm{u},M}_{ijk} = (\partial_i \widetilde{N}^M_{jk} - \partial_j \widetilde{N}^M_{ik}) + \Big(\eta_M - \sum_{m=0}^{M} \chi_m \Big) \check{\sigma}_{ijl} \partial_l P_k \tag{74}$$

is a "local ungauged flux corrector associated with ϕ^M". In other words, it satisfies (18) in $\mathbb{R}^d$ and is a weak solution of

$$\partial_i \sigma^{\mathrm{u},M}_{ijk} = \eta_M \overline{a}_{jl} \partial_l P_k - a_{jl} \left(\partial_l \phi^M_k + \eta_M \partial_l P_k \right) \quad \text{in} \quad \mathbb{R}^d, \tag{75}$$

which implies that

$$\partial_i \sigma^{\mathrm{u},M}_{ijk} = \overline{a}_{jl} \partial_l P_k - a_{jl} \left(\partial_l P_k + \partial_l \phi^M_k \right) \quad \text{in} \quad \mathrm{B}_{r_{M-1}}. \tag{76}$$

Proof of Lemma 5. Notice that $\sigma^{\mathrm{u},M}$ defined by (74) naturally satisfies (18). The only part that remains to be checked is (75). For this, we recall from Lemma 4 that $g^m + a\nabla\widetilde{\phi}^m$ is divergence-free, which, when combined with (73), implies that $\Delta \partial_j \widetilde{N}^M_{jk} = 0$. As a consequence, by the first order Liouville principle for harmonic functions, $\partial_j \widetilde{N}^M_{jk}$ is constant. We then have that

$$\partial_{ii} \widetilde{N}^M_{jk} - \partial_j \partial_i \widetilde{N}^M_{ik} = -\sum_{m=0}^{M} \left(g^m_{jk} + a_{jl} \partial_l \widetilde{\phi}^m_k \right). \tag{77}$$

Whence, by the definitions of $\sigma^{\mathrm{u},M}$ and $\check{\sigma}$, and as a consequence of (77), we obtain that

$$\begin{aligned}
\partial_i \sigma^{\mathrm{u},M}_{ijk} =& \partial_{ii} \widetilde{N}^M_{jk} - \partial_j \partial_i \widetilde{N}^M_{ik} + \partial_i \Big(\Big(\eta_M - \sum_{m=0}^{M} \chi_m \Big) \check{\sigma}_{ijl} \partial_l P_k \Big) \\
=& \sum_{m=0}^{M} \Big(- g^m_{jk} - a_{jl} \partial_l \widetilde{\phi}^m_k \Big) + \partial_i \Big(\eta_M - \sum_{m=0}^{M} \chi_m \Big) \check{\sigma}_{ijl} \partial_l P_k \\
&+ \Big(\eta_M - \sum_{m=0}^{M} \chi_m \Big) \left(\overline{a}_{jl} - a_{jl} - a_{jh} \partial_h \check{\phi}_l \right) \partial_l P_k.
\end{aligned}$$

By definition (67) of g, (69), and the antisymmetry of $\check{\sigma}$:

$$
\begin{aligned}
\partial_i \sigma^{\mathrm{u},M}_{ijk} =& \Big(- \sum_{m=0}^{M} \chi_m \left(a_{jl} - \overline{a}_{jl}\right) - \left(a_{ji}\check{\phi}_l - \check{\sigma}_{jil}\right) \partial_i \Big(\eta_M - \sum_{m=0}^{M} \chi_m\Big)\Big) \partial_l P_k \\
& - a_{ji}\partial_i \Big(\phi_k^M - \Big(\eta_M - \sum_{m=0}^{M} \chi_m\Big)\check{\phi}_l \partial_l P_k\Big) - \check{\sigma}_{jil}\partial_i \Big(\eta_M - \sum_{m=0}^{M} \chi_m\Big)\partial_l P_k \\
& + \Big(\eta_M - \sum_{m=0}^{M} \chi_m\Big) \left(\overline{a}_{jl} - a_{jl} - a_{jh}\partial_h \check{\phi}_l\right) \partial_l P_k \\
=& \eta_M \left(\left(\overline{a}_{jl} - a_{jl}\right) - a_{jl}\partial_l \phi_k^M\right) \partial_l P_k,
\end{aligned}
$$

which establishes (75). □

4.2 Inductive use of large-scale Lipschitz regularity

Lemma 6. *Let $\alpha = 1/2$, $M \geq 0$, $\nu \in (0,1]$, and $x_0 \in \mathbb{R}^d$. Assume that there exists a local ungauged generalized corrector $\left(\phi^M, \sigma^{\mathrm{u},M}\right)$ on $\mathrm{B}_{r_{M-1}}(x_0)$ that simultaneously satisfies the growth conditions (34), and (30) for $r^* = r_0 \geq 1$ and $r_{\max} = r_{M-1}$ and for $\delta := \delta(d, \lambda, 1/2)$ as in Theorem 2.*

Under these assumptions, there exist a local corrector ϕ^{M+1}, namely a solution of (70), and a local ungauged flux corrector $\sigma^{\mathrm{u},M+1}$, namely a solution of (75) satisfying (18), such that for any $r \geq r_0$ the following estimates hold:

$$
\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\phi^{M+1} - \fint_{\mathrm{B}_r(x_0)} \phi^{M+1}\Big|^2\Big)^{\frac{1}{2}} \leq C(d,\lambda) r^{-\nu/3}, \tag{78}
$$

$$
\text{and} \quad \frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\sigma^{\mathrm{u},M+1} - \fint_{\mathrm{B}_r(x_0)} \sigma^{\mathrm{u},M+1}\Big|^2\Big)^{\frac{1}{2}} \leq C(d,\lambda) r^{-\nu/3}, \tag{79}
$$

for a constant $C(d, \lambda)$ only depending on d and λ.

Proof. Using the objects that we have defined in Lemmas 4 and 5, we proceed in three steps: We first establish an estimate on $\nabla\widetilde{\phi}^m$ by appealing to Theorem 2, from which we deduce (78) in a second step. Then, to finish, we show (79) by using the result of the first step and basic facts about harmonic functions.

Step 1: Energy estimate We first show that, for any $r \geq r_0$ and $m \leq M+1$, the functions $\widetilde{\phi}^m$ from Lemma 4 satisfy the following estimate

$$
\fint_{\mathrm{B}_r(x_0)} \left|\nabla\widetilde{\phi}^m\right|^2 \lesssim_{d,\lambda} \min\Big(1, \Big(\frac{r_m}{r}\Big)^d\Big) r_m^{-2\nu/3}. \tag{80}
$$

To obtain (80), we use (67) to write:

$$
\int_{\mathbb{R}^d} |g^m|^2 \lesssim \int_{\mathbb{R}^d} \chi_m^2 + \int_{\mathbb{R}^d} \left|(\check{\phi}, \check{\sigma})\right|^2 \left|\nabla\left((\eta_m - \eta_{m-1}) - \chi_m\right)\right|^2. \tag{81}
$$

Since the function χ_m has a localized support, *i.e.* we have that

$$
\mathrm{Supp}(\chi_m) \subset \mathrm{B}_{r_m}(x_0) \backslash \mathrm{B}_{r_{m-2}}(x_0) \cap \left\{x \in \mathbb{R}^d \,:\, \left|x^\perp\right| \leq L(r_m)\right\},
$$

the first integral on the right-hand side of (81) is bounded as

$$\int_{\mathbb{R}^d} \chi_m^2 \lesssim r_m^{d-1} L(r_m).$$

For the second integral we use the decomposition

$$\nabla\left(\eta_m - \eta_{m-1} - \chi_m\right) = (1 - S_m)\left(\nabla\eta_m - \nabla\eta_{m-1}\right) - (\eta_m - \eta_{m-1})\,\nabla S_m$$

and properties of the cut-off functions defined in Section 4.1 to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|(\check{\phi}, \check{\sigma})\right|^2 \left|\nabla\left((\eta_m - \eta_{m-1}) - \chi_m\right)\right|^2 \\
&\lesssim \int_{\mathrm{B}_{r_m}(x_0)} \left|(\check{\phi}, \check{\sigma})\right|^2 L(r_m)^{-2} + \int_{\mathrm{B}_{r_m}(x_0)} \left|(\check{\phi}, \check{\sigma})\right|^2 L(r_{m-1})^{-2} \\
&\lesssim r_m^{d+2(1-\nu)} L(r_m)^{-2} + r_m^{d+2(1-\nu)} L(r_{m-1})^{-2}.
\end{aligned}$$

Notice that we have also used the growth condition (34).

As a consequence,

$$\int_{\mathbb{R}^d} |g^m|^2 \lesssim r_m^d \Big(\frac{L(r_m)}{r_m} + \frac{r_m^{2(1-\nu)}}{L(r_m)^2} \Big) + r_{m-1}^d \frac{r_{m-1}^{2(1-\nu)}}{L(r_{m-1})^2}. \tag{82}$$

Recalling that $r_m = 2^m r_0$, (63) appears to be the optimal choice and plugging it in (82) yields:

$$\int_{\mathbb{R}^d} |g^m|^2 \lesssim r_m^d r_m^{-2\nu/3}. \tag{83}$$

Therefore, from (68) and (83), we obtain (80) for $r \geq r_{m-2}$.

For the case $r \leq r_{m-2}$, we can use Theorem 2 for which, up to the scale $r_0 2^{M-1}$, it is sufficient to have a local ungauged generalized corrector in $\mathrm{B}_{r_{M-1}}(x_0)$. Since $\widetilde{\phi}^m$ is a-harmonic in $\mathrm{B}_{r_{m-2}}(x_0) \subset \mathrm{B}_{r_{M-1}}(x_0)$, this entails

$$\fint_{\mathrm{B}_r(x_0)} \left|\nabla\widetilde{\phi}^m\right|^2 \lesssim \fint_{\mathrm{B}_{r_{m-2}}(x_0)} \left|\nabla\widetilde{\phi}^m\right|^2 \lesssim r_m^{-2\nu/3},$$

where we have used (80) for $r = r_{m-2}$. This finishes the argument for (80) for all $r \geq r_0$.

Step 2: Argument for (78) Once (80) is established, applying the Poincaré-Wirtinger inequality yields

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\widetilde{\phi}^m - \fint_{\mathrm{B}_r(x_0)} \widetilde{\phi}^m\Big|^2\Big)^{\frac{1}{2}} \lesssim \min\Big(1, \Big(\frac{r_m}{r}\Big)^{d/2}\Big) r_m^{-\nu/3}.$$

Whence, by definition (69) combined with the triangle inequality, (34), and recalling $r_m = 2^m r_0$, we obtain (78) as follows:

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\phi^{M+1} - \fint_{\mathrm{B}_r(x_0)} \phi^{M+1}\Big|^2\Big)^{\frac{1}{2}} \lesssim r^{-\nu} + \sum_{m=0}^{M+1} \min\Big(1, \Big(\frac{r_m}{r}\Big)^{d/2}\Big) r_m^{-\nu/3} \lesssim r^{-\nu/3}.$$

Step 3: Argument for (79) Since the right-hand term of (73) is divergence-free, we can rewrite it as

$$\Delta \widetilde{N}_{jk}^{M+1} = -\,\partial_i\Big(\,(x-x_0)\cdot e_j \sum_{m=0}^{M+1} \big(g_{ik}^m + a_{il}\partial_l\widetilde{\phi}_k^m\big)\Big).$$

Now, we would like to invoke Lemma 10. To do this we first notice that, for $r \geq 1$, we have:

$$\fint_{\mathrm{B}_r(x_0)} \Big|\,(x-x_0)\cdot e_j \sum_{m=0}^{M+1} \big(g_{ik}^m + a_{il}\partial_l\widetilde{\phi}_k^m\big)\Big|^2 \lesssim r \sum_{m=0}^{M+1} \Big(\fint_{\mathrm{B}_r(x_0)} |g^m|^2 + \big|\nabla\widetilde{\phi}^m\big|^2\Big).$$

By applying (80) this becomes:

$$r\sum_{m=0}^{M+1}\Big(\fint_{\mathrm{B}_r(x_0)}\big|\nabla\widetilde{\phi}^m\big|^2\Big)^{\frac{1}{2}} \lesssim r\sum_{m=0}^{M+1}\min\Big(1,\Big(\frac{r_m}{r}\Big)^{d/2}\Big)r_m^{-\nu/3} \lesssim r^{1-\nu/3}.$$

On the other hand, since $g_m = 0$ outside of $\mathrm{B}_{r_m}(x_0)\backslash\mathrm{B}_{r_{m-2}}(x_0)$, (83) implies that

$$\begin{aligned} r\sum_{m=0}^{M+1}\Big(\fint_{\mathrm{B}_r(x_0)}|g^m|^2\Big)^{\frac{1}{2}} &\lesssim r \sum_{m=0}^{\min(M+1,\lceil r\rceil+2)} r^{-d/2}\Big(\int_{\mathbb{R}^d}|g^m|^2\Big)^{\frac{1}{2}} \\ &\lesssim r \sum_{m=0}^{\min(M+1,\lceil r\rceil+2)} \Big(\frac{r_m}{r}\Big)^{d/2} r_m^{-\nu/3} \lesssim r^{1-\nu/3}. \end{aligned}$$

Therefore, we obtain:

$$\Big(\fint_{\mathrm{B}_r(x_0)}\Big|\,(x-x_0)\cdot e_j \sum_{m=0}^{M+1}\big(g_{ik}^m + a_{il}\partial_l\widetilde{\phi}_k^m\big)\Big|^2\Big)^{\frac{1}{2}} \lesssim r^{1-\nu/3}.$$

As a consequence, Lemma 10 produces a solution $\widetilde{N}^{M+1}$ to (73). Moreover, (139) implies that, for any $r \geq 1$, there holds:

$$r^{-1}\Big(\fint_{\mathrm{B}_r(x_0)}\Big|\nabla\widetilde{N}^{M+1} - \fint_{\mathrm{B}_r(x_0)}\nabla\widetilde{N}^{M+1}\Big|^2\Big)^{\frac{1}{2}} \lesssim r^{-\nu/3}. \tag{84}$$

Finally, we define $\sigma^{\mathrm{u},M+1}$ by (74). Then, since (73) is satisfied and (84) implies that $\nabla\widetilde{N}^{M+1}$ is strictly sublinear, by Lemma 5 we know that $\sigma^{\mathrm{u},M+1}$ is a local ungauged flux corrector and solves (75). Last, summing up (84) and the estimate satisfied by $\breve{\sigma}$ yields (79). □

4.3 Proof of Proposition 1

Proof. As in the previous works [11, 12], the proof is done by induction.

Step 1: Induction Note that the local ungauged generalized corrector $\left(\phi^0, \sigma^0\right)$ satisfies (78) and (79), as a straightforward corollary of the energy estimate – possibly at the price of taking a larger uniform constant. We then set $r_0 \geq 1$ such that

$$C(d,\lambda) r_0^{-\nu/3} \leq \delta(d,\lambda,1/2), \tag{85}$$

where $C(d,\lambda)$ refers to the common constant of (78) and (79) and $\delta(d,\lambda,1/2)$ is fixed in Theorem 2.

Next, assume that the local ungauged generalized corrector $\left(\phi^M, \sigma^{\mathrm{u},M}\right)$ satisfies (78) and (79) for a given $M \in \mathbb{N}$. Therefore, by (85), this local generalized corrector also satisfies the growth condition (30) for $r^* = r_0$, $r_{\max} = r_{M-1}$, and $\delta := \delta(d,\lambda,1/2)$. Whence, applying Lemma 6, we obtain that $\left(\phi^{M+1}, \sigma^{\mathrm{u},M+1}\right)$ also satisfy (78) and (79).

As a conclusion of the inductive proof, $\left(\phi^M, \sigma^{\mathrm{u},M}\right)$ satisfy (78) and (79) for any $M \in \mathbb{N}$.

Step 2: Limit $M \uparrow \infty$ By a compactness argument in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, the following convergences hold up to a subsequence:

$$\left(\phi^M, \sigma^{\mathrm{u},M}\right) \underset{M\uparrow\infty}{\rightharpoonup} (\phi, \sigma^{\mathrm{u}}) \quad \text{and} \quad \nabla\phi^M \underset{M\uparrow\infty}{\rightharpoonup} \nabla\phi \quad \text{in} \quad L^2_{\mathrm{loc}}(\mathbb{R}^d).$$

By taking the limit of the weak formulations of (71) and (76), we deduce that ϕ and σ^{u} respectively satisfy (3) and (17). Also, the generalized ungauged correctors $(\phi, \sigma^{\mathrm{u}})$ inherit (78) and (79). As a consequence, we have established

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x_0)} \Big| (\phi,\sigma^{\mathrm{u}}) - \fint_{\mathrm{B}_r(x_0)} (\phi,\sigma^{\mathrm{u}}) \Big|^2\Big)^{\frac{1}{2}} \leq \widetilde{\kappa} r^{-\widetilde{\nu}} \quad \text{for any } r \geq 1. \tag{86}$$

Step 3: Post-processing (86) We use a standard argument to get (36) from (86). Applying the Cauchy-Schwarz inequality and (86), there holds:

$$\begin{aligned}\Big|\fint_{\mathrm{B}_r(x_0)} (\phi,\sigma^{\mathrm{u}}) - \fint_{\mathrm{B}_{2r}(x_0)} (\phi,\sigma^{\mathrm{u}})\Big| &\leq \Big(\fint_{\mathrm{B}_{2r}(x_0)} \Big| (\phi,\sigma^{\mathrm{u}}) - \fint_{\mathrm{B}_{2r}(x_0)} (\phi,\sigma^{\mathrm{u}}) \Big|^2\Big)^{\frac{1}{2}} \\ &\lesssim_d \widetilde{\kappa} r^{1-\widetilde{\nu}},\end{aligned}$$

for any $r \geq 1$. Thus, using a dyadic sequence of increasing balls, for $2^{n-1} < r \leq 2^n$, we get

$$\Big|\fint_{\mathrm{B}_r(x_0)} (\phi,\sigma^{\mathrm{u}}) - \fint_{\mathrm{B}(x_0)} (\phi,\sigma^{\mathrm{u}})\Big| \lesssim_d \widetilde{\kappa} \sum_{j=1}^{n+1} 2^{j(1-\widetilde{\nu})} \lesssim_d \widetilde{\kappa} \frac{1}{2^{1-\widetilde{\nu}} - 1} r^{1-\widetilde{\nu}}.$$

As a conclusion, combining the above inequality and (86), we obtain (36). □

5 Argument for Theorem 1

As already discussed in Section 2, we first collect some peripheral results and then combine these in Section 5.4, which contains the proof of Theorem 1.

5.1 Enforcing the gauge on the flux corrector

Here, we assume that we are given an ungauged generalized corrector satisfying a quantified sublinearity estimate (*e.g.* the output of Proposition 1) and use Lemma 10 to obtain a generalized corrector with the same sublinearity properties. We also show that strictly sublinear generalized correctors are unique up to the addition of a random constant. Here is the result of this subsection:

Lemma 7. *Let $\overline{a}_+$ and $\overline{a}_- \in \mathbb{R}^{d\times d}$ be fixed, and let $\langle\cdot\rangle$ be an ensemble on Ω such that, $\langle\cdot\rangle$-almost surely, a admits $\overline{a}$ defined by (10) as its homogenized matrix (this ensemble does not necessarily satisfy the assumptions of Section 1.2).*

Assume that $(\phi, \sigma^{\mathrm{u}})$ is an ungauged generalized corrector such that for fixed $x_0 \in \mathbb{R}^d$ and $p \in [2,\infty)$, and for any $r \geq 2$ we have that

$$\frac{1}{r}\Big\langle\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\phi(x) - \fint_{\mathrm{B}(x_0)} \phi\Big|^2 \mathrm{d}x\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \leq \kappa r^{-\nu} \ln^{\beta}(r), \tag{87}$$

$$\text{and} \quad \frac{1}{r}\Big\langle\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\sigma^{\mathrm{u}}(x) - \fint_{\mathrm{B}(x_0)} \sigma^{\mathrm{u}}\Big|^2 \mathrm{d}x\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \leq \kappa r^{-\nu} \ln^{\widetilde{\beta}}(r), \tag{88}$$

for $\nu \in (0,1]$, $\beta \geq 0$, $\widetilde{\beta} \geq 0$, and $\kappa > 0$. Under these conditions, ϕ is $\langle\cdot\rangle$-almost surely unique up to the addition of a constant in the class of strictly sublinear correctors. Furthermore, $\langle\cdot\rangle$-almost surely, there exists a strictly sublinear flux corrector σ that is unique up to the addition of a constant and that satisfies, for any $r \geq 2$,

$$\frac{1}{r}\Big\langle\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\sigma(x) - \fint_{\mathrm{B}(x_0)} \sigma\Big|^2 dx\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim_{d,\nu,\widetilde{\beta}} \kappa r^{-\nu} \ln^{\widetilde{\beta}+\lfloor\nu\rfloor}(r). \tag{89}$$

Remark 8. An easy consequence of Proposition 1 and Lemma 7 is that for an ensemble $\langle\cdot\rangle$ on Ω satisfying the conditions of Section 1.2, there $\langle\cdot\rangle$-almost surely exist generalized correctors (ϕ, σ) and (ϕ^*, σ^*) respectively associated to a and a^* that are strictly sublinear and, therefore, unique (up to the addition of random constants).

Proof of Lemma 7. We first focus on ϕ. By the Markov inequality and Borel-Cantelli lemma (using the same arguments as in Step 2 of Lemma 10), ϕ is $\langle\cdot\rangle$-almost surely strictly sublinear. Now, assume that, given a realization a, there are two strictly sublinear solutions to (3). By definition, their difference u is a-harmonic in $\mathbb{R}^d$. Yet, by Theorem 2, there exists $r^* > 1$ which is $\langle\cdot\rangle$ almost-surely finite, such that, for any $R \geq r \geq r^*$, there holds

$$\fint_{\mathrm{B}_r(x_0)} |\nabla u|^2 \lesssim \fint_{\mathrm{B}_R(x_0)} |\nabla u|^2 .$$

Therefore, using the Caccioppoli estimate and the strict sublinearity of u, we get

$$\Big(\fint_{\mathrm{B}_r(x_0)} |\nabla u|^2\Big)^{\frac{1}{2}} \lesssim R^{-1}\Big(\fint_{\mathrm{B}_{2R}(x_0)} \Big|u - \fint_{\mathrm{B}_{2R}(x_0)} u\Big|^2\Big)^{\frac{1}{2}} \underset{R\uparrow\infty}{\longrightarrow} 0.$$

Since $r \geq r^*$ is arbitrary, we deduce that $\nabla u = 0$. Therefore, $\langle\cdot\rangle$-almost surely, u is constant in the space, so that ϕ is unique up to the addition of a random constant.

Next, we apply Lemma 10 with $f = \sigma^{\mathrm{u}}_{ijk} - \fint_{\mathrm{B}(x_0)} \sigma^{\mathrm{u}}_{ijk}$, which we may do since (136) is satisfied. We obtain a solution N_{jk} to (16) such that ∇N_{jk} satisfies (140) and σ as defined in (15), *i.e.* $\sigma_{ijk} := \partial_i N_{jk} - \partial_j N_{ik}$, satisfies (89). The $\langle\cdot\rangle$-almost sure uniqueness (up to the addition of constants) of σ follows from the uniqueness of the N_{jk} up to the addition of affine functions (shown in Lemma 10). □

5.2 Control of the stochastic moments of r^*

We define the minimal radius r^* (inspired from [15, Th. 1]), above which Lipschitz regularity is available for the operator $-\nabla\cdot a\nabla$ and prove that its stochastic moments are bounded:

Lemma 8. *Let the assumptions of Section 1.2 hold and $\Phi(a) := (\phi, \phi^*, \sigma, \sigma^*)$ be composed of strictly sublinear generalized correctors associated to a and a^* respectively. Given $\delta_0 > 0$ and $x \in \mathbb{R}^d$, we define $r^*(x) \geq 1$ as the minimal radius such that there exists an ungauged corrector Φ^{u} associated with a with*

$$\sup_{r \geq r^*(x)} \frac{1}{r} \Big(\fint_{\mathrm{B}_r(x)} \Big| \Phi^{\mathrm{u}} - \fint_{\mathrm{B}_r(x)} \Phi^{\mathrm{u}} \Big|^2 \Big)^{\frac{1}{2}} \leq \delta_0. \tag{90}$$

Then, for any $\nu_0 < \nu$ and $p \in [2, \infty)$, the p^{th} moment of $r^*(x)$ is bounded as

$$\langle |r^*(x)|^p \rangle^{\frac{1}{p}} \lesssim_{d,\lambda,\delta_0,\nu,\nu_0,p} c_{p/\nu_0}^{1/\nu_0}. \tag{91}$$

Proof. Notice first that the generalized corrector $\Phi = (\phi, \phi^*, \sigma, \sigma^*)$ and thus the minimal radius $r^*(x)$ are $\langle\cdot\rangle$-almost surely well-defined thanks to Remark 8. For brevity, we use the notations

$$\check{\delta}(x, r) := \frac{1}{r} \Big(\int_{\mathrm{B}_r(x)} |\Phi_\pm|^2 \Big)^{\frac{1}{2}} \tag{92}$$

for $x \in \mathbb{R}^d$ and $r > 0$.

Since $p \geq 2$, by the Cauchy-Schwarz and the Bochner inequalities, we may post-process (13) to obtain, for any $x, y \in \mathbb{R}^d$,

$$\Big\langle \Big(\int_{\mathrm{Q}} \Big| \Phi_\pm(y+z) - \fint_{\mathrm{Q}(x)} \Phi_\pm \Big|^2 \mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \lesssim c_p \left(1 + |y - x|\right)^{1-\nu}. \tag{93}$$

In the sequel, we anchor $\Phi_\pm$ at x, in the sense of $\int_{\mathrm{Q}(x)} \Phi_\pm = 0$. Whence, by the above inequality combined with the Bochner estimate, we deduce that

$$\big\langle \check{\delta}(x, r)^p \big\rangle^{\frac{1}{p}} \lesssim_d c_p r^{-\nu} \qquad \text{for any } r \geq 1. \tag{94}$$

We set $0 < \nu_0 < \nu$ and

$$\varepsilon^{-1} := 1 + \sup_{r \geq 1} \left(r^{\nu_0} \check{\delta}(x, r) \right)^{1/\nu_0}, \tag{95}$$

so that we have $\check{\delta}(x,r) \leq (\varepsilon r)^{-\nu_0}$ for any $r \geq 1$. Notice that, by a simple scaling argument, the generalized corrector associated to $a^\varepsilon_\pm := a_\pm\left(\frac{\cdot}{\varepsilon}\right)$ is given by $\Phi^\varepsilon_\pm = \varepsilon\Phi_\pm\left(\frac{\cdot}{\varepsilon}\right)$. This means that

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x)} \left|\Phi^\varepsilon_\pm\right|^2\Big)^{\frac{1}{2}} = \frac{\varepsilon}{r}\Big(\fint_{\mathrm{B}_{\varepsilon^{-1}r}(x)} |\Phi_\pm|^2\Big)^{\frac{1}{2}} \overset{(95)}{\leq} r^{-\nu_0},$$

which allows us to apply Proposition 1 to obtain an *ungauged* generalized corrector $\Phi^{\varepsilon,\mathrm{u}} = (\phi^\varepsilon, \sigma^{\varepsilon,\mathrm{u}})$ associated with the coefficient field[9] $a^\varepsilon := a\left(\frac{\cdot}{\varepsilon}\right)$. By (36) applied to $\Phi^{\mathrm{u},\varepsilon}$ we obtain:

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x)} |\Phi^{\mathrm{u},\varepsilon}|^2\Big)^{\frac{1}{2}} \lesssim r^{-\nu_0/3} \qquad \text{for any } r \geq 1. \tag{96}$$

Rescaling in the opposite direction as before, we notice that $\Phi^{\mathrm{u}} := \varepsilon^{-1}\Phi^{\mathrm{u},\varepsilon}(\varepsilon\,\cdot)$ is a generalized ungauged corrector associated to the original coefficient field a, so that (96) translates into

$$\frac{1}{r}\Big(\fint_{\mathrm{B}_r(x)} \Big|\Phi^{\mathrm{u}} - \fint_{\mathrm{B}(x)} \Phi^{\mathrm{u}}\Big|^2\Big)^{\frac{1}{2}} \leq C(\kappa, d, \nu_0)\varepsilon^{-\nu_0/3} r^{-\nu_0/3} \qquad \text{for any } r \geq \varepsilon^{-1}, \tag{97}$$

where $C(\kappa, d, \nu_0)$ is a constant depending only on κ, d, ν_0. By (97) and by definition (95) of ε, it appears that

$$r^*(x) \leq \max\left(\varepsilon^{-1}, \varepsilon^{-1}\big(C(\kappa,d,\nu_0)\delta_0^{-1}\big)^{\frac{3}{\nu_0}}\right) \lesssim 1 + \sup_{r\geq 1}\left(r^{\nu_0}\check{\delta}(x,r)\right)^{1/\nu_0}.$$

Therefore, taking the p^{th} moment, using a dyadic argument and (94), we get

$$\langle |r^*(x)|^p\rangle^{\frac{1}{p}} \lesssim 1 + \sum_{j=1}^{+\infty}\Big\langle 2^{jp}\big(\check{\delta}(x,2^j)\big)^{\frac{p}{\nu_0}}\Big\rangle^{\frac{1}{p}} \lesssim 1 + \sum_{j=1}^{+\infty} c_{p/\nu_0}^{\frac{1}{\nu_0}} 2^j 2^{-j\frac{\nu}{\nu_0}} \lesssim c_{p/\nu_0}^{1/\nu_0}.$$

This concludes the proof of Lemma 8. □

5.3 Estimate for the Green's function

We now use the uniform control of the p^{th} stochastic moments of $r^*(x)$ for $x \in \mathbb{R}^d$ from the previous subsection in order to bound the p^{th} stochastic moments of local L^2-averages of the second mixed derivatives of the heterogeneous Green's function. In particular, we find:

Lemma 9. *Assume that $d \geq 2$. Let the assumptions of Section 1.2 hold. Then, the mixed second derivatives of the Green's function G associated with the operator $-\mathrm{div}\,(a\cdot\nabla)$ in $\mathbb{R}^d$ satisfy, for any $\nu_0 < \nu$, $p \in [2, +\infty)$, and $|x-y| \geq 3$,*

$$\Big\langle\Big(\fint_{\mathrm{B}(x)}\fint_{\mathrm{B}(y)} |\nabla_x\nabla_y G(x',y')|^2\,\mathrm{d}y'\mathrm{d}x'\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim_{d,\lambda,\nu,\nu_0,p} c_{dp/\nu_0}^{d/\nu_0}|x-y|^{-d}. \tag{98}$$

Remark 9. If the coefficient field a is uniformly Hölder continuous in $\mathbb{R}^d$, then estimate (98) can be upgraded to a pointwise estimate by Schauder theory.

[9] By definition, the interface layer of a^ε has a width $2\varepsilon < 2$.

Proof of Lemma 9. Invoking the result [6, Th. 1] (and [6, Cor. 1] for $d = 2$) we obtain: For a well-chosen constant $\delta_0 > 0$ depending on d and λ, and for the minimal radius $r^*(x) \geq 1$ associated with the condition (90), the mixed derivatives of the Green's function G satisfy

$$\Big(\fint_{\mathrm{B}(x)}\fint_{\mathrm{B}(y)} |\nabla_x\nabla_y G(x',y')|^2\,\mathrm{d}y'\mathrm{d}x'\Big)^{\frac{1}{2}} \lesssim_{d,\lambda} \Big(\frac{(r^*(x)r^*(y))^{\frac{1}{2}}}{|x-y|}\Big)^{d}. \tag{99}$$

Hence, (98) is a direct consequence of (99) and Lemma 8, which yields:

$$\big\langle\, (r^*(x)r^*(y))^{dp/2}\,\big\rangle^{\frac{1}{p}} \lesssim \big(\langle r^*(x)^{dp}\rangle^{1/dp}\langle r^*(y)^{dp}\rangle^{1/dp}\big)^{d/2} \lesssim_{d,\lambda,\nu,\nu_0,p,\delta} c_{dp/\nu_0}^{d/\nu_0}.$$

□

5.4 Proof of Theorem 1

Equipped with Lemmas 7, 8 and 9, we are in a position to proceed with the:

Proof of Theorem 1. Throughout our argument we fix $p \in [2, \infty)$ and use the notation " $\lesssim$ " to denote " $\lesssim_{d,\lambda,\nu,\nu_0,p}$ ".

Strategy of proof By Remark 8, we already have the $\langle\cdot\rangle$-almost sure existence and the uniqueness up to a random constant of a strictly sublinear generalized corrector (ϕ, σ). Therefore, the following proof is devoted to establishing the improved sublinearity estimates (20) and (21). We use a similar construction as that in Section 4, but replacing the use of energy estimates with that of the Green's function estimates provided by Lemma 9.

More precisely, we set a smooth function $\chi(x)$ only depending on $x^\perp$ such that

$$[-2,2]\times\mathbb{R}^{d-1} \subseteq \{x : \chi(x) = 1\} \subseteq \mathrm{Supp}(\chi) \subseteq [-3,3]\times\mathbb{R}^{d-1}.$$

Then, imposing the anchoring relation $\fint_{\mathrm{Q}(x_0)} \Phi_\pm = 0$ for $x_0 \in \mathbb{R}^d$, we propose a decomposition of ϕ in the spirit of Proposition 1:

$$\phi_k = (1-\chi)\check{\phi}_j[x_0]\partial_j P_k + \widetilde{\phi}_k[x_0], \tag{100}$$

where the dependence of $\check{\phi}[x_0]$, $\widetilde{\phi}[x_0]$, and $\check{\sigma}[x_0]$ in the anchoring is made explicit. Moreover, we define an ungauged corrector $\sigma^{\mathrm{u}}[x_0]$ by:

$$\begin{cases} \sigma^{\mathrm{u}}_{ijk}[x_0] = (1-\chi)\check{\sigma}_{ijl}[x_0]\partial_l P_k + \partial_i\widetilde{N}_{jk}[x_0] - \partial_j\widetilde{N}_{ik}[x_0], \\ \Delta\widetilde{N}_{jk}[x_0] = -g_{jk}[x_0] - a_{jl}\partial_l\widetilde{\phi}_k[x_0], \\ g_{ik}[x_0] = \big(\chi\,(a_{ij}-\overline{a}_{ij}) - \big(a_{il}\check{\phi}_j[x_0] - \check{\sigma}_{ilj}[x_0]\big)\,\partial_l\chi\big)\,\partial_j P_k. \end{cases} \tag{101}$$

Then, we prove estimates on $\widetilde{\phi}[x_0]$ and $\sigma^{\mathrm{u}}[x_0]$ that may be transfered to ϕ and σ, either directly using (100) or indirectly through Lemma 7 (note that the uniqueness of (ϕ, σ) plays a fundamental role).

Our argument has five steps: In Step 1, we use Lemma 9 to show that

$$\Big\langle\Big(\int_{\mathrm{B}(x_0)} \big|\nabla\widetilde{\phi}[x_0]\big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_{2dp/\nu_0}^{1+d/\nu_0}\,\big(1+\big|x_0^\perp\big|\big)^{-\nu} \tag{102}$$

for any $p < \infty$. In Step 2, using the uniqueness $\nabla\phi$ and changing the anchoring point x_0, we extend the above estimate as follows:

$$\Big\langle\Big(\int_{\mathrm{B}(x)} \big|\nabla\widetilde{\phi}[x_0]\big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} \begin{cases} \big(1+\big|x^\perp\big|\big)^{-\nu} & \text{if} \quad \big|x^\perp\big| \geq 4, \\ (1+|x-x_0|)^{1-\nu} & \text{if} \quad \big|x^\perp\big| \leq 4. \end{cases} \tag{103}$$

From this, we deduce in Step 3 that the corrector ϕ satisfies (20). In Step 4, defining $x_0' = (0, x_0^\parallel)$, we show that the ungauged flux corrector $\sigma^{\mathrm{u}}[x_0']$ satisfies

$$\Big\langle\Big(\fint_{\mathrm{B}_r(x_0)} \Big|\sigma^{\mathrm{u}}\left[x_0'\right] - \fint_{\mathrm{B}(x_0)} \sigma^{\mathrm{u}}\left[x_0'\right]\Big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} r^{1-\nu} \ln^{1+\lfloor\nu\rfloor}(r) \tag{104}$$

for any $r \geq 2$. In Step 5, by appealing to Lemma 7, we obtain that the (gauged) flux corrector σ satisfies

$$\Big\langle\Big(\fint_{\mathrm{B}_r(x)} \Big|\sigma - \fint_{\mathrm{B}(x)} \sigma\Big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} r^{1-\nu} \ln^{1+2\lfloor\nu\rfloor}(r) \tag{105}$$

for any $x \in \mathbb{R}^d$ and $r \geq 2$. To finish, we convert this into (21).

Step 1 Set $x_0 \in \mathbb{R}^d$. We reinterpret (66) and (67) in the easier context of (100) to write:

$$-\mathrm{div}\big(a\nabla\widetilde{\phi}_k[x_0]\big) = \partial_i\left(g_{ik}[x_0]\right) \quad \text{in} \quad \mathbb{R}^d \tag{106}$$

with g defined by (101). By (93), we have:

$$\Big\langle\Big(\int_{\mathrm{Q}(x)} \big|\big(\check{\phi}[x_0], \check{\sigma}[x_0]\big)\big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_p(1+|x-x_0|)^{1-\nu} \tag{107}$$

for any $p < \infty$ and $x \in \mathbb{R}^d$. The definition (101) then yields that

$$\Big\langle\Big(\int_{\mathrm{Q}(x)} |g[x_0]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim \begin{cases} c_p\left(1+|x-x_0|\right)^{1-\nu} & \text{if} \quad x^\perp \in [-4,4], \\ 0 & \text{if} \quad x^\perp \notin [-4,4]. \end{cases} \tag{108}$$

To obtain (102), we decompose $\widetilde{\phi}[x_0] = \widetilde{\phi}^1[x_0] + \widetilde{\phi}^2[x_0]$, where for each $k \in [\![1,d]\!]$ the function $\widetilde{\phi}_k^1[x_0] \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a Lax-Milgram solution of

$$-\mathrm{div}\big(a\nabla\widetilde{\phi}_k^1[x_0]\big) = \partial_i\left(\mathbb{1}_{\mathrm{Q}_3(x_0)} g_{ik}[x_0]\right) \qquad \text{in} \quad \mathbb{R}^d \tag{109}$$

and $\widetilde{\phi}_k^2[x_0] \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a strictly sublinear solution to

$$-\mathrm{div}\big(a\nabla\widetilde{\phi}_k^2[x_0]\big) = \partial_i\left(\left(1-\mathbb{1}_{\mathrm{Q}_3(x_0)}(y)\right) g_{ik}[x_0]\right) \qquad \text{in} \quad \mathbb{R}^d. \tag{110}$$

Combining the energy estimate with (108), we get

$$\Big\langle\Big(\int_{\mathbb{R}^d} \big|\nabla\widetilde{\phi}^1[x_0]\big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_p \Big\langle\Big(\int_{\mathrm{Q}_3(x_0)} |g[x_0]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}} \lesssim c_p. \tag{111}$$

To obtain estimates for $\nabla\widetilde{\phi}^2[x_0]$, we differentiate the Green's function representation and decompose it on cubes. In particular, for $x \in \mathrm{Q}(x_0)$, we write:

$$\nabla\widetilde{\phi}^2[x_0](x) = \int_{\mathbb{R}^d} \nabla_x\nabla_y G(x,y) \cdot g[x_0](y)\left(1-\mathbb{1}_{\mathrm{Q}_3(x_0)}(y)\right) \mathrm{d}y.$$

Combining this representation with the triangle inequality and Hölder's inequality, we obtain:

$$\begin{aligned}
&\Big\langle\Big(\int_{Q(x_0)}\Big|\nabla\widetilde{\phi}^2[x_0]\Big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\\
&\lesssim\sum_{k\in\mathbb{Z}^d\backslash Q_3}\Big\langle\Big(\int_{Q(x_0)}\Big|\int_{Q(k+x_0)}\nabla_x\nabla_y G(x,y)\cdot g[x_0](y)\mathrm{d}y\Big|^2\mathrm{d}x\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\\
&\lesssim\sum_{k\in\mathbb{Z}^d\backslash Q_3}\Big\langle\Big(\int_{Q(k+x_0)}|g[x_0]|^2\Big)^{p}\Big\rangle^{\frac{1}{2p}}\Big\langle\Big(\int_{Q(x_0)}\int_{Q(k+x_0)}|\nabla_x\nabla_y G(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\Big)^{p}\Big\rangle^{\frac{1}{2p}}.
\end{aligned}$$

We treat the second term on the right-hand side of the above estimate with Lemma 9 to the effect of

$$\Big\langle\Big(\int_{Q(x_0)}\int_{Q(k+x_0)}|\nabla_x\nabla_y G(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\Big)^{p}\Big\rangle^{\frac{1}{2p}}\lesssim c_{2dp/\nu_0}^{d/\nu_0}|k|^{-d}.$$

Since $g[x_0]$ is supported inside $[-3,3]\times\mathbb{R}^{d-1}$ and satisfies (108), we have:

$$\begin{aligned}
\Big\langle\Big(\int_{Q(x_0)}|\nabla\widetilde{\phi}^2[x_0](x)|^2\mathrm{d}x\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}&\lesssim c_{2p}c_{2dp/\nu_0}^{d/\nu_0}\sum_{\substack{k\,\in\,\mathbb{Z}^d\backslash Q_3,\\ x_0^\perp\,+\,k^\perp\,\in\,[-4,4]}}|k|^{-d}|k|^{1-\nu}\\
&\lesssim c_{2dp/\nu_0}^{1+d/\nu_0}\sum_{k^\parallel\in\mathbb{Z}^{d-1}}\big(1+|k^\parallel|+|x_0^\perp|\big)^{-d+1-\nu}\\
&\lesssim c_{2dp/\nu_0}^{1+d/\nu_0}\big(1+|x_0^\perp|\big)^{-\nu}.
\end{aligned}\tag{112}$$

(By monotonicity, we have $c_{2dp/\nu_0}>c_{2p}$.) Thus, we have established (102).

Step 2 Here comes the argument for (103). Recall that $\nabla\phi$ is uniquely defined (almost surely). Therefore, by (100), changing the anchoring of $\phi_\pm$ in the sense of $x_0\rightsquigarrow x\in\mathbb{R}^d$, implies replacing $\nabla\widetilde{\phi}[x_0]\rightsquigarrow\nabla\widetilde{\phi}[x]$ as follows:

$$\begin{aligned}
\nabla\widetilde{\phi}_k[x]=&\nabla\widetilde{\phi}_k[x_0]+\mathbb{1}_{\mathbb{R}_+\times\mathbb{R}^{d-1}}\Big(\int_{Q(x_0)}\phi_+-\int_{Q(x)}\phi_+\Big)\cdot\nabla P_k\nabla\chi\\
&+\mathbb{1}_{\mathbb{R}_-\times\mathbb{R}^{d-1}}\Big(\int_{Q(x_0)}\phi_--\int_{Q(x)}\phi_-\Big)\cdot\nabla P_k\nabla\chi.
\end{aligned}\tag{113}$$

In particular, changing the anchoring point x_0 does not change the value of $\nabla\widetilde{\phi}[x_0]$ outside of $[-3,3]\times\mathbb{R}^{d-1}$. Hence, for any $x\in\mathbb{R}^d\backslash([-4,4]\times\mathbb{R}^{d-1})$, (102) yields that

$$\Big\langle\Big(\int_{Q(x)}|\nabla\widetilde{\phi}[x_0]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}=\Big\langle\Big(\int_{Q(x)}|\nabla\widetilde{\phi}[x]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\lesssim c_{2dp/\nu_0}^{1+d/\nu_0}(1+|x^\perp|)^{-\nu}.\tag{114}$$

Moreover, when $x\in[-4,4]\times\mathbb{R}^{d-1}$, using (113), by assumption (13) in the form (93) and by (102), we obtain:

$$\begin{aligned}
\Big\langle\Big(\int_{Q(x)}|\nabla\widetilde{\phi}[x_0]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}&\lesssim\Big\langle\Big(\int_{Q(x)}|\nabla\widetilde{\phi}[x]|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}+\Big\langle\Big|\int_{Q(x)}\phi_\pm-\int_{Q(x_0)}\phi_\pm\Big|^p\Big\rangle^{\frac{1}{p}}\\
&\lesssim c_{2dp/\nu_0}^{1+d/\nu_0}\left(1+|x_0-x|\right)^{1-\nu}.
\end{aligned}$$

As a consequence, (103) holds.

Step 3 We now establish (20). To estimate ϕ, we set a suitable anchoring point $x_0 \in \mathbb{R}^d$ and apply the triangle inequality on (100): The part $(1-\chi)\check{\phi}[x_0]$ is treated with (13) whereas $\widetilde{\phi}[x_0]$ is handled by integrating $\nabla\widetilde{\phi}[x_0]$, which is controlled by (103), along a suitable path Γ.

Fix $x, y \in \mathbb{R}^d$; we assume that $x^\perp \le -4 < 4 \le y^\perp$. (It is easy to check that our method extends to the general case.) We then introduce the points

$$x_0 := (0, x^\parallel), \quad x_1 := (2|x-y|, x^\parallel), \quad \text{and } x_2 := (2|x-y|, y^\parallel). \tag{115}$$

From these points we draw the path $\Gamma = [x, x_1] \cup [x_1, x_2] \cup [x_2, y]$ parametrized by $\gamma : [0,1] \to \Gamma$ (that is a renormalized natural parametrization).

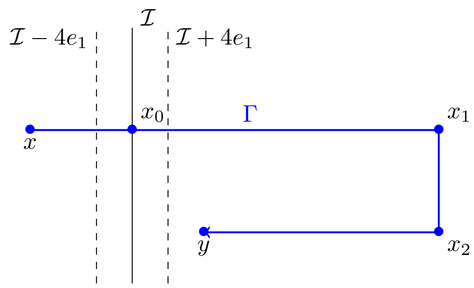

Figure 5: Path of integration Γ.

By (93), using x_0 as the reference point, there holds:

$$\Big\langle \Big(\int_{\mathrm{Q}} \big|\check{\phi}[x_0](x+z) - \check{\phi}[x_0](y+z)\big|^2 \mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \lesssim_d c_p(1 + |x-y|^{1-\nu}).$$

Combining (93) with the splitting (100) and an application of the triangle inequality, we obtain:

$$\begin{aligned} &\Big\langle \Big(\int_{\mathrm{Q}} |\phi(x+z) - \phi(y+z)|^2 \,\mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\ &\qquad \lesssim c_p(1 + |x-y|^{1-\nu}) + \Big\langle \Big(\int_{\mathrm{Q}} \big|\widetilde{\phi}[x_0](x+z) - \widetilde{\phi}[x_0](y+z)\big|^2 \mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}}, \end{aligned}$$

which implies that proving (20) amounts to establishing

$$\begin{aligned} &\Big\langle \Big(\int_{\mathrm{Q}} \Big|\widetilde{\phi}[x_0](x+z) - \widetilde{\phi}[x_0](y+z)\Big|^2 \,\mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\ &\qquad \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} (1 + |x-y|)^{1-\nu} \ln^{\lfloor \nu \rfloor}(2 + |x-y|). \end{aligned} \tag{116}$$

For (116), we first use the Bochner inequality:

$$\begin{aligned} \Big(\int_{\mathrm{Q}} \big|\widetilde{\phi}[x_0](x+z) - \widetilde{\phi}[x_0](y+z)\big|^2 \mathrm{d}z \Big)^{\frac{1}{2}} &= \Big(\int_{\mathrm{Q}} \Big| \int_0^1 \nabla\widetilde{\phi}[x_0](z + \gamma(t))\gamma'(t) \,\mathrm{d}t \Big|^2 \mathrm{d}z \Big)^{\frac{1}{2}} \\ &\le \int_0^1 \Big(\int_{\mathrm{Q}(\gamma(t))} \big|\nabla\widetilde{\phi}[x_0](z)\big|^2 \mathrm{d}z \Big)^{\frac{1}{2}} |\gamma'(t)| \,\mathrm{d}t. \end{aligned}$$

As a consequence, using once more the Bochner inequality, we get from (103) that

$$\begin{aligned}
&\Big\langle \Big(\int_{\mathrm{Q}} \Big|\widetilde{\phi}[x_0](x+z) - \widetilde{\phi}[x_0](y+z)\Big|^2 \mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\
&\quad \lesssim \int_0^1 \Big\langle \Big(\int_{\mathrm{Q}(\gamma(t))} \big|\nabla \widetilde{\phi}[x_0](z)\big|^2 \mathrm{d}z \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} |\gamma'(t)| \, \mathrm{d}t \\
&\quad \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} \int_0^1 \big(1 + |\gamma(t)^{\perp}|\big)^{-\nu} |\gamma'(t)| \, \mathrm{d}t \\
&\quad \lesssim c_{dp/\nu_0}^{1+d/\nu_0} (1+|x-y|)^{1-\nu} \ln^{\lfloor \nu \rfloor}(2+|x-y|)
\end{aligned}$$

This establishes (116), so that (20) is proved.

Step 4 We now consider the ungauged corrector $\sigma^{\mathrm{u}}[x_0']$ defined by (101), for $x_0' = (0, x_0^{\parallel})$.

By splitting $\widetilde{N}[x_0']$ into a far-field and near-field contribution (depending on the support of the right-hand side), we can use the Green's function associated to the Laplacian for the far-field piece and the standard energy estimate for the near-field piece to obtain:

$$\begin{aligned}
\int_{\mathrm{B}(x)} \big|\nabla^2 \widetilde{N}[x_0']\big|^2 \lesssim & \int_{\mathrm{Q}_2(x)} \big(|g[x_0']|^2 + \big|\nabla \widetilde{\phi}[x_0']\big|^2\big) \\
& + \Big(\int_{\mathbb{R}^d \setminus \mathrm{Q}_2(x)} \frac{|g[x_0'](z)| + |\nabla \widetilde{\phi}[x_0'](z)|}{|x-z|^d} \mathrm{d}z \Big)^2 .
\end{aligned}$$

Using (103) and (108), we can decompose the above estimate to find that

$$\begin{aligned}
&\Big\langle \Big(\int_{\mathrm{B}(x)} \big|\nabla^2 \widetilde{N}[x_0']\big|^2 \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\
&\lesssim \sum_{k \in \mathbb{Z}^d} \frac{1}{(|k-x|+1)^d} \Big\langle \Big(\int_{\mathrm{Q}(k)} |g[x_0']|^2 + \big|\nabla \widetilde{\phi}[x_0']\big|^2 \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\
&\lesssim c_{2dp/\nu_0}^{1+d/\nu_0} \Big(\sum_{k \in \mathbb{Z}^d, |k^{\perp}| \le 4} \frac{(1+|k-x_0'|)^{1-\nu}}{(|k-x|+1)^d} + \sum_{k \in \mathbb{Z}^d, |k^{\perp}| \ge 4} \frac{|k^{\perp}|^{-\nu}}{(|k-x|+1)^d} \Big).
\end{aligned} \tag{117}$$

Now, up to a multiplicative constant, we bound the first right-hand term of (117) by

$$\begin{aligned}
\sum_{k^{\parallel} \in \mathbb{Z}^{d-1}} \frac{\Big(1+|k^{\parallel} - x_0^{\parallel}|\Big)^{1-\nu}}{\big(1+|k^{\parallel} - x^{\parallel}| + |x^{\perp}|\big)^d} &= \sum_{k^{\parallel} \in \mathbb{Z}^{d-1}} \frac{\Big(1+|k^{\parallel} - x_0^{\parallel} + x^{\parallel}|\Big)^{1-\nu}}{\big(1+|k^{\parallel}| + |x^{\perp}|\big)^d} \\
&\lesssim \big(1+|x^{\parallel} - x_0^{\parallel}|\big)^{1-\nu} \big(1+|x^{\perp}|\big)^{-1} + \big(1+|x^{\perp}|\big)^{-\nu}
\end{aligned}$$

and the second right-hand term of (117) by

$$\sum_{k^{\perp} \in \mathbb{Z} \setminus \{0\}} \frac{|k^{\perp}|^{-\nu}}{|k^{\perp} - x^{\perp}| + 1} \lesssim \big(1+|x^{\perp}|\big)^{-\nu} \ln\big(2+|x^{\perp}|\big) .$$

By summation, we deduce from the three above inequalities that

$$\Big\langle \Big(\int_{\mathrm{B}(x)} |\nabla^2 \widetilde{N}[x_0']|^2 \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} \big(1 + |x^{\|} - x_0^{\|}|\big)^{1-\nu} \big(1 + |x^{\perp}|\big)^{-1} \\ + c_{2dp/\nu_0}^{1+d/\nu_0} \big(1 + |x^{\perp}|\big)^{-\nu} \ln\big(2 + |x^{\perp}|\big).$$

Integrating the previous estimate along a path Γ similar to Step 3, we obtain that

$$\Big\langle \Big(\int_{\mathrm{B}} \big|\nabla \widetilde{N}[x_0'](x_0 + \cdot) - \nabla \widetilde{N}[x_0'](x + \cdot)\big|^2 \Big)^{\frac{p}{2}} \Big\rangle^{\frac{1}{p}} \\ \lesssim c_{2dp/\nu_0}^{1+d/\nu_0} \left|x - x_0\right|^{1-\nu} \ln^{1+\lfloor \nu \rfloor} \left(2 + |x - x_0|\right).$$

As a consequence, for any $R \geq 1$ and $\nu \in (0, 1]$, we find:

$$\Big\langle \Big(\fint_{\mathrm{B}_R(x_0)} \Big|\nabla \widetilde{N}[x_0] - \fint_{\mathrm{B}(x_0)} \nabla \widetilde{N}[x_0]\Big|^2 \Big)^{\frac{p}{2}} \Big\rangle^{2/p} \\ \lesssim \big(c_{2dp/\nu_0}^{1+d/\nu_0}\big)^2 R^{-d} \int_{\mathrm{B}_R} |z|^{2(1-\nu)} \ln^{2(1+\lfloor \nu \rfloor)} \left(2 + |z|\right) \mathrm{d}z \\ \lesssim \big(c_{2dp/\nu_0}^{1+d/\nu_0}\big)^2 R^{2(1-\nu)} \ln^{2(1+\lfloor \nu \rfloor)}(2 + R).$$

Then, recalling (13) in the form (93), we obtain that σ^{u} defined by (101) satisfies (104).

Step 5 From the previous step, we have a family of ungauged flux correctors $\sigma^{\mathrm{u}}[x_0']$ depending on the anchoring point $x_0' = (0, x_0^{\|}) \in \mathbb{R}^d$ and satisfying (104). Thus, we may apply Lemma 7 to each $\sigma^{\mathrm{u}}[x_0']$ and obtain a new estimate on the $\langle \cdot \rangle$-almost surely unique (up to the addition of a random constant) flux corrector σ. Namely, the latter satisfies (105), for any $x \in \mathbb{R}^d$.

Finally, we notice that, for fixed $x, y \in \mathbb{R}^d$ such that $|x - y| = r \geq 1$, we may apply (105) for x and y with $r = 1$ and then again for y, replacing $r \rightsquigarrow 2r$, in order to obtain the desired (21) by the triangle inequality. $\square$

6 Proof of Corollary 1

Since a very similar version of Corollary 1 has already been proved in [17], we only emphasize the main steps. Indeed, the only technical difference between the setting in [17] and here is that the generalized correctors considered here are not bounded, but only satisfy (20). We refer the interested reader to [17] for the technicalities due to the interface, and to [7] for the technicalities due to the growth rate (20) of the generalized correctors. We also refer to [19], from which the main ideas of the aforementioned articles are borrowed.

Proof of Corollary 1. We first justify (42), which corresponds to [17, Prop. 4.3]. It is a consequence of (19), that can be reformulated thanks to the Green's function G^ε associated with the operator $-\mathrm{div}\,(a(\cdot/\varepsilon)\nabla \cdot)$ in $\mathbb{R}^d$ as

$$u^\varepsilon(x) - \overline{u}(x) - \varepsilon \phi\big(\frac{x}{\varepsilon}\big) \cdot \overline{\nabla} \overline{u}(x) = -\varepsilon \int_{\mathbb{R}^d} \partial_{y_i} G^\varepsilon(x, y) \Big((a_{ij}\phi_k - \sigma_{ijk})\big(\frac{y}{\varepsilon}\big) \partial_j \overline{\partial}_k \overline{u}(y) \Big) \mathrm{d}y.$$

By applying a Hölder inequality on the above right-hand term, in which we inject the regularity of $\overline{\nabla}\overline{u}$ (see [22, Th.]), the estimates on the generalized correctors provided by (20) and the following estimate on the Green's function

$$|\nabla_y G^\varepsilon(x,y)| \lesssim |x-y|^{-d+1}, \tag{118}$$

(which follows from [6, Th. 1]) we obtain (42).

By a duality argument detailed in [7, Th. 4.6] and in [17, Prop. 4.4], we deduce from (42) that, for any $x \neq y$, the following estimate hold:

$$\left|G^\varepsilon(x,y) - \bar{G}(x,y)\right| \lesssim \varepsilon^\nu \frac{\ln^{1+2\lfloor\nu\rfloor}\left(2+\frac{|x-y|}{\varepsilon}\right)}{|x-y|^{d-2+\nu}}, \tag{119}$$

where $\bar{G}$ is the Green's function of the homogenized operator $-\mathrm{div}\,(\overline{a}\nabla\cdot)$ in $\mathbb{R}^d$.

As in [17, Th. 4.5], it is deduced from Lipschitz regularity (*i.e.* Theorem 2) that the previous estimate (119) can be upgraded to the level of the gradients:

$$\left|\nabla_x G^\varepsilon(x,y) - \nabla_x \bar{G}(x,y) - \nabla\phi\big(\frac{x}{\varepsilon}\big)\cdot\overline{\nabla}_x\bar{G}(x,y)\right| \leq C\varepsilon^\nu \frac{\ln^{2+2\lfloor\nu\rfloor}\left(2+\frac{|x-y|}{\varepsilon}\right)}{|x-y|^{d-1+\nu}}. \tag{120}$$

Finally, as in the proof of [17, Cor. 4.6], we express

$$\begin{aligned}&\nabla u^\varepsilon(x) - \nabla\overline{u}(x) - \nabla\phi\big(\frac{x}{\varepsilon}\big)\cdot\overline{\nabla}\overline{u}(x)\\ &= \int_{\mathrm{B}(x_0)}\Big(\nabla_x G^\varepsilon(x,y) - \nabla\bar{G}(x,y) - \nabla\phi\big(\frac{x}{\varepsilon}\big)\cdot\overline{\nabla}\bar{G}(x,y)\Big)f(y)\mathrm{d}y.\end{aligned}$$

Then, by using a simple Hölder inequality involving the previous estimate, we obtain the desired result (43). (In the case $\nu = 1$, an additional decomposition is required to avoid the singularity $x = y$ for $\nu = 1$; it is detailed in [17, Cor. 4.6].) □

7 Proof of Proposition 2

Proof. In the sequel, the symbol "$\lesssim$" will be used for "$\lesssim_{d,\eta_1,\eta_2}$". The equation on ϕ_1 reads:

$$-\,\mathrm{div}\,(a\nabla\phi_1) = \mathrm{div}\,(ae_1) = \mathrm{div}\,(\eta e_1) \qquad \text{in} \quad \mathbb{R}^d.$$

The strategy of the proof is to make use of the Green's function G associated with the operator $-\mathrm{div}\,(a\cdot\nabla)$ to express the growth rate of ϕ_1 between two points x and $x' \in \mathbb{R}^d$:

$$\phi_1(x') - \phi_1(x) = \int_{[x',x]}\Big(\int_D \eta(y)\nabla_x\nabla_y G(x'',y)\cdot e_1\mathrm{d}y\Big)\cdot\mathrm{d}x''. \tag{121}$$

The proof then falls in two steps. Step 1 is devoted to deriving an asymptotic estimate for $\nabla_y G(x,y)$ when y is far from the interface $\mathcal{I}$. Equipped with this, in Step 2, we choose suitable points x and $x' \in \mathbb{R}^d$ and explicitly compute the leading order of (121), thus obtaining (40).

Step 1 We prove an approximation of $\nabla_y G(x,y)$ in the form of

$$\left|\nabla_y G(x,y) + \nabla \bar{G}(x-y)\right| \lesssim \frac{|x-y|^{-\frac{1}{2}} + (1+|y^{\perp}|)^{-1}}{|x-y|^{d-1}}. \tag{122}$$

Recall that $\phi_{\pm} = 0$ and that the flux correctors $\sigma_{\pm}$ are bounded. Therefore, we may apply Theorem 1 to obtain strictly sublinear generalized correctors (ϕ, σ). By regularity of a, we may turn (102) into a pointwise estimate, to the effect of:

$$|\nabla \phi(y)| \lesssim (1+|y^{\perp}|)^{-1} \qquad \text{for any } y \in \mathbb{R}^d. \tag{123}$$

Also, the Green's function G satisfies (120). Thus, we may deduce from the symmetry of a the following (suboptimal but convenient) estimate:

$$\left|\nabla_y G(x,y) + (\mathrm{I} + \nabla \phi\,(y)) \cdot \nabla \bar{G}(x-y)\right| \lesssim |x-y|^{-d+\frac{1}{2}}, \tag{124}$$

for any $x, y \in \mathbb{R}^d$, $|x-y| \geq 1$, where $\bar{G}$ is the Green's function of the operator $-\Delta$. The latter function satisfies

$$\nabla \bar{G}(x) = -C_d |x|^{-d} x, \tag{125}$$

for a universal constant $C_d > 0$ (see [13, (4.1) Chap. 4 p. 51]). By a triangle inequality involving (123), (124), and (125), for any $x, y \in \mathbb{R}^d$ with $|x-y| \geq 1$, we obtain (122).

Step 2 We now show that, given $r \geq 2$ sufficiently large, estimate (40) holds for $x := -e_1$ and $x' := -(r+1)e_1$.

We split the strip D, defined in (27), into a far domain and a near domain:

$$D_1 := [r, +\infty) \times [-1,1] \times \mathbb{R}^{d-2} \quad \text{and} \quad D_2 := [0,r] \times [-1,1] \times \mathbb{R}^{d-2} \tag{126}$$

(see Figure 2). The identity (121) is accordingly reinterpreted as

$$\phi_1(x') - \phi_1(x) = \int_{[x',x]} \Big(\Big(\int_{D_1} + \int_{D_2} \Big) \eta(y) \nabla_x \partial_{y_1} G(x'', y) \mathrm{d}y \Big) \cdot \mathrm{d}x'' =: I_1 + I_2. \tag{127}$$

By Lemma 9 combined with (118) and the regularity of a, the Green's function G satisfies the following estimates:

$$|\nabla_y G(x,y)| \lesssim |x-y|^{-d+1} \quad \text{and} \quad |\nabla_x \nabla_y G(x,y)| \lesssim |x-y|^{-d}, \tag{128}$$

for any $x, y \in \mathbb{R}^d$ with $|x-y| \geq 1$. Whence, we may estimate I_1 defined in (127) by

$$|I_1| \lesssim \int_{[x',x]} \int_{D_1} \frac{1}{|x''-y|^d} \mathrm{d}y\, |\mathrm{d}x''| \lesssim r \int_{[-1,1]\times\mathbb{R}_+\times\mathbb{R}^{d-2}} \frac{1}{(r+|y|)^d} \mathrm{d}y \lesssim 1, \tag{129}$$

where we used that $|x-x'| = r$.

We now consider I_2 in (127). We first integrate along the x'' variable:

$$I_2 = \int_{D_2} \eta(y)\partial_{y_1} G(x,y)\mathrm{d}y - \int_{D_2} \eta(y)\partial_{y_1} G(x',y)\mathrm{d}y =: I_{21} + I_{22}.$$

On the one hand, we treat the second integral by using (128)

$$|I_{22}| \lesssim \int_{D_2} \frac{1}{|x'-y|^{d-1}}\mathrm{d}y \lesssim \int_0^r \int_{\mathbb{R}^{d-2}} \frac{1}{(r+1+y_1+|\widetilde{y}|)^{d-1}}\mathrm{d}y_1\mathrm{d}\widetilde{y} \lesssim 1. \tag{130}$$

On the other hand, by (122) and since $1 \le |x-y| \lesssim |x'-y|$ if $y \in D_2$, we approximate I_{21} as follows:

$$\left|I_{21} + \int_{D_2} \eta\partial_1\bar{G}(x-\cdot)\right| \lesssim \int_{D_2} \frac{|x-y|^{-1/2} + \left(1+|y^\perp|\right)^{-1}}{|x-y|^{d-1}}\mathrm{d}y \lesssim 1.$$

We further simplify I_{21} by using a Taylor expansion to remove the dependence of $\partial_1\bar{G}(z)$ on $z \cdot e_2$. In particular, this requires the decay of $\nabla^2\bar{G}$ provided by (125). Recalling that η is defined by (26), that $\eta_1 = 1$ in $[1,+\infty)$, substituting $x = -e_1$, injecting (125), we may explicitly compute

$$\begin{aligned}
I_{21} &= -C_0 \int_0^r \int_{\mathbb{R}^{d-2}} \partial_1\bar{G}\left((-1-y_1, 0, \widetilde{y})\right)\mathrm{d}\widetilde{y}\mathrm{d}y_1 + O(1) \\
&= -C_1 \int_0^r \int_{\mathbb{R}^{d-2}} \frac{1+y_1}{\left((1+y_1)^2 + |\widetilde{y}|^2\right)^{\frac{d}{2}}}\mathrm{d}\widetilde{y}\mathrm{d}y_1 + O(1) \\
&= -C_2 \ln(r) + O(1)
\end{aligned} \tag{131}$$

for positive constants $C_0, C_1, C_2 > 0$.

Combining (127), (129), (130) and (131) establishes (40) for $r \ge 1$ sufficiently large and concludes the proof of Proposition 2. □

8 Proof of Theorem 3

Remark 10. For the sake of simplicity, we make use of Theorem 1 to show Theorem 3. However, we only need the uniqueness of correctors and a (suboptimal) growth quantification (the one provided by Proposition 1 is sufficient).

Proof of Theorem 3. Existence and uniqueness (up to the addition of a constant) of strictly sublinear generalized correctors is already provided by Theorem 1. Thus, we only have to check that (37) is satisfied.

We follow the classical approach described in the lecture notes [18]. First, we obtain from Theorem 1 and from the regularity of the coefficient field a a bound on the moments of the gradient of ϕ and σ:

$$\langle |(\nabla\phi, \nabla\sigma)(x)|^p \rangle^{\frac{1}{p}} \lesssim 1 \qquad \text{for any } x \in \mathbb{R}^d \setminus (-e_1 + \mathcal{I} \cup e_1 + \mathcal{I}). \tag{132}$$

Next, we establish that (38) holds. We finally obtain (37) by applying (38) to a suitable functional and by making use of the regularity of a.

For simplicity, we assume that $d > 2$. (The only difference for the case $d = 2$ is located in Step 3, where dimension plays a role in the potential theory).

Step 1 Let $x \in \mathbb{R}^d$. By combining the Caccioppoli estimate, (20), (21), we easily obtain that

$$\Big\langle \Big(\fint_{\mathrm{B}(x)} |(\nabla\phi, \nabla\sigma)|^2 \Big)^p \Big\rangle \lesssim_{d,\lambda,\kappa,\alpha,p} 1 \qquad \text{for any } p \in [1, +\infty).$$

Moreover, by (39) and (24), the coefficient a satisfies

$$\Big\langle \, \|a\|^p_{C^{0,\frac{\alpha}{2}}(\mathrm{B}(x)\setminus(-e_1+\mathcal{I}\cup e_1+\mathcal{I}))} \Big\rangle^{\frac{1}{p}} \lesssim_{d,\lambda,\kappa,\alpha,p} 1 \qquad \text{for any } p \in [1, +\infty).$$

Thus, a local version of the regularity theorem [20, Th. 1.1] successively yields that ϕ and σ satisfy (132).

Step 2 For simplicity, we assume that F only depends on g_- and g_+ (the argument below is easily generalized), and that $\langle F \rangle = 0$. We denote by $\langle \cdot \rangle_\pm$ the ensembles associated with $g_\pm$ respectively. We already know (see [18, Sec. 3.2]), that any functional depending only on $g_\pm$ satisfies (38) with a and $\langle \cdot \rangle$ respectively replaced by $g_\pm$ and $\langle \cdot \rangle_\pm$. By independence of g_- and g_+, we also have that $\langle G \rangle = \big\langle \langle G \rangle_- \big\rangle_+$, for any random variable G depending only on g_- and g_+.

We apply the spectral gap (38) first with respect to $\langle \cdot \rangle_-$, and then with respect to $\langle \cdot \rangle_+$, to the effect of

$$\begin{aligned} \big\langle |F|^{2p} \big\rangle = \big\langle \langle |F|^{2p} \rangle_- \big\rangle_+ &\lesssim \Big\langle \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial F}{\partial g_-(z)} \Big|^2 \mathrm{d}z \Big)^p \Big\rangle + \big\langle | \langle F \rangle_- |^{2p} \big\rangle_+ \\ &\lesssim \Big\langle \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial F}{\partial g_-(z)} \Big|^2 \mathrm{d}z \Big)^p \Big\rangle + \Big\langle \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial F}{\partial g_+(z)} \Big|^2 \mathrm{d}z \Big)^p \Big\rangle, \end{aligned}$$

where we have used the Bochner inequality and $\langle F \rangle = 0$. Finally, by the chain rule (and recalling (24)), we establish (38).

Step 3 For the sake of simplicity, we only show (37) for ϕ. Also, we make take p large (by Jensen's inequality, if (37) holds for $p = p_1$, then it also holds for any $p = p_2 \leq p_1$). Let $x, y \in \mathbb{R}^d$ (we recall that $d > 2$ here). The aim of this step is to establish the following estimate:

$$\Big\langle \Big| \fint_{\mathrm{B}(x)} \phi - \fint_{\mathrm{B}(y)} \phi \Big|^{2p} \Big\rangle \lesssim 1. \tag{133}$$

By the Sobolev embedding (provided $2p > d$), there also holds that

$$\Big| \phi(x) - \fint_{\mathrm{B}(x)} \phi \Big|^{2p} \lesssim \fint_{\mathrm{B}(x)} |\nabla\phi|^{2p}.$$

Thence, taking the expectation and recalling (132), we deduce that

$$\Big\langle \Big| \phi(x) - \fint_{\mathrm{B}(x)} \phi \Big|^{2p} \Big\rangle \lesssim \Big\langle \fint_{\mathrm{B}(x)} |\nabla\phi|^{2p} \Big\rangle \lesssim 1.$$

Combining this estimate with (133) establishes (37) and concludes the proof in the case $d > 2$.

Here comes the argument for (133). We define w and v as the strictly sublinear solutions to

$$-\Delta w = \frac{1}{|\mathrm{B}|}\left(\mathbb{1}_{\mathrm{B}(x)} - \mathbb{1}_{\mathrm{B}(y)}\right) \quad \text{and} \quad - \operatorname{div}\left(a^* \nabla v + \nabla w\right) = 0. \tag{134}$$

We set

$$F := \fint_{\mathrm{B}(x)} \phi - \fint_{\mathrm{B}(y)} \phi = - \int_{\mathbb{R}^d} \Delta w \phi = \int_{\mathbb{R}^d} \nabla w \cdot \nabla \phi. \tag{135}$$

Now, notice that

$$\frac{\partial F}{\partial a(z)} = \int_{\mathbb{R}^d} \nabla w \cdot \nabla \frac{\partial \phi}{\partial a(z)} = - \int_{\mathbb{R}^d} \nabla v \cdot a \nabla \frac{\partial \phi}{\partial a(z)} = \left(\nabla\left(\phi_j + P_j\right) \otimes \nabla v\right)(z),$$

where we used the equation (3) in its differentiated form:

$$-\operatorname{div}\Big(a \cdot \nabla \frac{\partial \phi_j}{\partial a(z)} + \delta_z \nabla\left(\phi_j + P_j\right)^*\Big) = 0.$$

Therefore, by the spectral gap (38) and a duality argument, we deduce that

$$\begin{aligned}\left\langle\, |F|^{2p} \right\rangle &\lesssim \Big\langle \Big(\int_{\mathbb{R}^d} \left|\nabla\left(\phi_j + P_j\right) \otimes \nabla v\right|^2 \Big)^p \Big\rangle \\ &\lesssim \sup_{\langle |G|^{2p'} \rangle \leq 1} \Big\langle \int_{\mathbb{R}^d} \left|\nabla\left(\phi_j + P_j\right) \otimes \nabla v\right|^2 |G|^2 \Big\rangle^p,\end{aligned}$$

where p' is the conjugated exponent of p (*i.e.* $1/p + 1/p' = 1$). By the Hölder inequality and thanks to (132), we estimate the above right-hand side as follows:

$$\begin{aligned}\Big\langle \int_{\mathbb{R}^d} \left|(\nabla \phi_j + P_j) \otimes \nabla v\right|^2 |G|^2 \Big\rangle &\leq \int_{\mathbb{R}^d} \left\langle |\nabla(\phi_j + P_j)|^{2p} \right\rangle^{\frac{1}{p}} \left\langle\, |G \nabla v|^{2p'} \right\rangle^{\frac{1}{p'}} \\ &\lesssim \int_{\mathbb{R}^d} \left\langle\, |G \nabla v|^{2p'} \right\rangle^{\frac{1}{p'}}.\end{aligned}$$

Since G does not depend on space, we may multiply the equation (134) by G. Therefore, if p' is sufficiently small, that is, if p is sufficiently large, we may apply the annealed Meyers estimates (see [18], where it is sufficient that a^* is uniformly elliptic and bounded). Thus, we get

$$\int_{\mathbb{R}^d} \left\langle\, |G \nabla v|^{2p'} \right\rangle^{\frac{1}{p'}} \lesssim \int_{\mathbb{R}^d} \left\langle\, |G \nabla w|^{2p'} \right\rangle^{\frac{1}{p'}} \lesssim \int_{\mathbb{R}^d} |\nabla w|^2 \lesssim 1,$$

since w is deterministic and $\langle |G|^{2p'} \rangle \leq 1$ (the last bound being obtained by the potential theory in dimension $d > 2$). This implies (133). $\square$

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References

[1] G. Allaire. *Shape optimization by the homogenization method*, volume 146 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.

[2] S. Armstrong, T. Kuusi, and J.-C. Mourrat. *Quantitative stochastic homogenization and large-scale regularity*, volume 352 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2019.

[3] S. Armstrong and Z. Shen. Lipschitz estimates in almost-periodic homogenization. *Comm. Pure Appl. Math.*, 69(10):1882–1923, 2016.

[4] M. Avellaneda and F.-H. Lin. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, 40(6):803–847, 1987.

[5] N. Bakhvalov and G. Panasenko. *Homogenisation: averaging processes in periodic media*, volume 36 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989.

[6] P. Bella and A. Giunti. Green's function for elliptic systems: moment bounds. *Netw. Heterog. Media*, 13(1):155–176, 2018.

[7] X. Blanc, M. Josien, and C. Le Bris. Precised approximations in elliptic homogenization beyond the periodic setting. *Asymptotic Analysis*. Under press.

[8] X. Blanc, M. Josien, and C. Le Bris. Approximation locale précisée dans des problèmes multi-échelles avec défauts localisés. *Comptes Rendus Mathématique de l'Académie des Sciences*, 357:167–174, 2019.

[9] X. Blanc, C. Le Bris, and P.-L. Lions. Local profiles for elliptic problems at different scales: defects in, and interfaces between periodic structures. *Comm. Partial Differential Equations*, 40(12):2173–2236, 2015.

[10] X. Blanc, C. Le Bris, and P.-L. Lions. On correctors for linear elliptic homogenization in the presence of local defects. *Comm. Partial Differential Equations*, 43(6):965–997, 2018.

[11] J. Fischer and F. Otto. A higher-order large-scale regularity theory for random elliptic operators. *Comm. Partial Differential Equations*, 41(7):1108–1148, 2016.

[12] J. Fischer and C. Raithel. Liouville principles and a large-scale regularity theory for random elliptic operators on the half-space. *SIAM J. Math. Anal.*, 49(1):82–114, 2017.

[13] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[14] A. Gloria. A quantitative theory of stochastic homogenization: regularity, oscillations, and fluctuations. Lecture notes.

[15] A. Gloria, S. Neukamm, and F. Otto. A regularity theory for random elliptic operators. *ArXiv e-print arXiv:1409.2678v3*, 2015.

[16] V. Jikov, S. Kozlov, and O. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994.

[17] M. Josien. Some quantitative homogenization results in a simple case of interface. *Communications in Partial Differential Equations*, 44(10):907–939, 2019.

[18] M. Josien and F. Otto. Introduction to stochastic homogenization: oscillations and fluctuations. In preparation.

[19] C. Kenig, F.-H. Lin, and Z. Shen. Periodic homogenization of Green and Neumann functions. *Comm. Pure Appl. Math.*, 67(8):1219–1262, 2014.

[20] Y. Y. Li and L. Nirenberg. Estimates for elliptic systems from composite material. *Comm. Pure Appl. Math.*, 56(7):892–925, 2003.

[21] Y. Y. Li and M. Vogelius. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch. Ration. Mech. Anal.*, 153(2):91–151, 2000.

[22] A. Lorenzi. On elliptic equations with piecewise constant coefficients. II. *Ann. Scuola Norm. Sup. Pisa (3)*, 26:839–870, 1972.

[23] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Ann. Scuola Norm. Sup. Pisa (3)*, 16:305–326, 1962.

[24] C. Raithel. A large-scale regularity theory for random elliptic operators on the half-space with homogeneous neumann boundary data. *arXiv preprint arXiv:1703.04328*, 2017.

[25] Z. Shen. *Periodic homogenization of elliptic systems*, volume 269 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, Cham, 2018. Advances in Partial Differential Equations (Basel).

[26] Valentin Vinoles. *Problèmes d'interface en présence de métamatériaux: modélisation, analyse, et simulations*. PhD thesis, Université Paris-Saclay, 2017.

A A technical lemma for harmonic functions

Throughout this paper we make use of the following technical lemma:

Lemma 10. *Let $d \geq 2$, $p \geq 2$, $x_0 \in \mathbb{R}^d$, $\langle\cdot\rangle$ be a given ensemble, and $f \in L^p(\Omega, L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$ be a random vector field. We assume that for any $R \geq 2$ we have:*

$$\left\langle \left(\fint_{\mathrm{B}_R(x_0)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \leq C_0 R^{1-\nu} \ln^\beta(R) \tag{136}$$

for some exponents $\nu \in (0,1]$ and $\beta \geq 0$. Then, $\langle\cdot\rangle$-almost surely, there exists a distributional solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ of

$$-\Delta u = \nabla \cdot f \quad \text{in} \quad \mathbb{R}^d \tag{137}$$

subject to the constraint:

$$\limsup_{R\uparrow\infty} R^{-1}\Big(\fint_{\mathrm{B}_R(x_0)}\Big|\nabla u-\fint_{\mathrm{B}_R(x_0)}\nabla u\Big|^2\Big)^{\frac{1}{2}}=0. \tag{138}$$

The solution u is unique up to the addition of affine functions. Moreover, for any $R\geq 2$ we have that

$$\Big\langle\Big(\fint_{\mathrm{B}_R(x_0)}\Big|\nabla u-\fint_{\mathrm{B}_R(x_0)}\nabla u\Big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\lesssim_{d,\nu,\beta} C_0R^{1-\nu}\ln^{\beta}(R), \tag{139}$$

and $$\Big\langle\Big(\fint_{\mathrm{B}_R(x_0)}\Big|\nabla u-\fint_{\mathrm{B}(x_0)}\nabla u\Big|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\lesssim_{d,\nu,\beta} C_0R^{1-\nu}\ln^{\beta+\lfloor\nu\rfloor}(R). \tag{140}$$

Proof of Lemma 10. The uniqueness of the solution u is a direct application of the Liouville principle for harmonic functions. Our argument then breaks down into three steps: First, in Step 1, we prove the $\langle\cdot\rangle$-almost sure existence of a solution to (137). Then, in Step 2, we show that this solution satisfies (139) for any $R\geq 1$. To finish, in Step 3 we deduce (140) from (139).

Step 1 We decompose

$$f=\sum_{m=1}^{+\infty} f_m \quad \text{for} \quad f_m:=f\mathbb{1}_{\mathrm{B}_{2^m-1}(x_0)\setminus\mathrm{B}_{2^{m-1}-1}(x_0)}$$

and let $u_m\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the Lax-Milgram solutions of

$$\int_{\mathbb{R}^d}\nabla\psi\cdot\nabla u_m=-\int_{\mathbb{R}^d}\nabla\psi\cdot f_m. \tag{141}$$

Taking $\psi=u_m$ in (141) and using Hölder's inequality and (136) implies that

$$\Big\langle\Big(\int_{\mathbb{R}^d}|\nabla u_m|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\leq\Big\langle\Big(\int_{\mathbb{R}^d}|f_m|^2\Big)^{\frac{p}{2}}\Big\rangle^{\frac{1}{p}}\leq(2^m)^{d/2}\,(2^m)^{1-\nu}\ln^{\beta}(2+2^m). \tag{142}$$

We then define:

$$\widetilde{u}_m(x):=\begin{cases}u_m(x) & \text{if} \quad m=1,\\ u_m(x)-u_m(x_0)-x\cdot\nabla u_m(x_0) & \text{if} \quad m>1,\end{cases} \tag{143}$$

and claim that the series

$$u:=\sum_{m=1}^{+\infty}\widetilde{u}_m \tag{144}$$

almost surely defines a distributional solution to (137). In particular, we show that the series in (144) converges in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ $\langle\cdot\rangle$-almost surely.

Let $R:=2^{m_0}$ for $m_0\in\mathbb{N}$. Notice that, to show the desired convergence, we may discard the terms of (144) with $m\in[\![1,m_0+3]\!]$ as the estimate (142) ensures that $\langle\cdot\rangle$-almost surely $\nabla u_m\in L^2(\mathbb{R}^d)$ for any $m\geq 1$. For $m\geq m_0+3$, the function u_m is harmonic in $\mathrm{B}_{2R}(x_0)$, which means that we have access to the mean-value property and the Caccioppoli inequality. We obtain:

$$\begin{aligned}\fint_{\mathrm{B}_R(x_0)} |\nabla \widetilde{u}_m|^2 &\overset{(143)}{\lesssim} R^2 \sup_{\mathrm{B}_R(x_0)} \left|\nabla^2 u_m\right|^2 \lesssim R^2 \fint_{\mathrm{B}_{2R}(x_0)} \left|\nabla^2 u_m\right|^2 \\ &\lesssim R^2 \fint_{\mathrm{B}_{2^{m-1}}(x_0)} \left|\nabla^2 u_m\right|^2 \lesssim R^2 2^{-2m} \fint_{\mathrm{B}_{2^m}(x_0)} |\nabla u_m|^2 .\end{aligned} \tag{145}$$

Therefore, recalling (142) we obtain for any $m \geq m_0 + 3$

$$\left\langle \left(\fint_{\mathrm{B}_R(x_0)} |\nabla \widetilde{u}_m|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim 2^{m_0 - m} 2^{m(1-\nu)} \ln^\beta (2 + 2^m) .$$

Thus, by the triangle inequality, we get

$$\begin{aligned}\left\langle \left(\sum_{m=m_0+3}^{+\infty} \left(\fint_{\mathrm{B}_R(x_0)} |\nabla \widetilde{u}_m|^2 \right)^{\frac{1}{2}} \right)^p \right\rangle^{\frac{1}{p}} &\lesssim 2^{m_0} \sum_{m=m_0+3}^{+\infty} 2^{-m\nu} \ln^\beta (2 + 2^m) \\ &\lesssim 2^{m_0(1-\nu)} \ln^\beta (2 + 2^{m_0}) ,\end{aligned} \tag{146}$$

which shows the desired convergence. Hence, u is a solution to (137).

Step 2 As above, we set $R := 2^{m_0}$. Thus, we obtain

$$\begin{aligned}\left\langle \left(\fint_{\mathrm{B}_R(x_0)} \left| \nabla \widetilde{u}_m - \fint_{\mathrm{B}_R(x_0)} \nabla \widetilde{u}_m \right|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} &\leq \left\langle \left(\fint_{\mathrm{B}_R(x_0)} |\nabla \widetilde{u}_m|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\overset{(142)}{\lesssim} 2^{(m-m_0)d/2} 2^{m(1-\nu)} \ln^\beta (2 + 2^m) .\end{aligned}$$

Whence, by the triangle inequality, we have

$$\begin{aligned}&\left\langle \left(\fint_{\mathrm{B}_R} \left| \sum_{m=1}^{m_0+2} \left(\nabla \widetilde{u}_m - \fint_{\mathrm{B}_R(x_0)} \nabla \widetilde{u}_m \right) \right|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \\ &\qquad \lesssim \sum_{m=1}^{m_0+2} 2^{(m-m_0)d/2} 2^{m(1-\nu)} \ln^\beta (2 + 2^m) \lesssim 2^{m_0(1-\nu)} \ln^\beta (2 + 2^{m_0}) .\end{aligned} \tag{147}$$

Therefore, by a triangle inequality involving (146) and (147) (recall that $R = 2^{m_0}$), we obtain (139). Hence

$$\sum_{m_0=1}^{+\infty} 2^{-m_0} \left\langle \left(\fint_{\mathrm{B}_{2^{m_0}}(x_0)} \left| \nabla u - \fint_{\mathrm{B}_{2^{m_0}}(x_0)} \nabla u \right|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} < +\infty.$$

Therefore, by the Markov inequality and the Borel-Cantelli Lemma, this shows that (138) is $\langle\cdot\rangle$-almost surely satisfied.

Step 3 We finally replace the innermost integral of the left-hand side of (139) by an average over $\mathrm{B}(x_0)$. Indeed, by the Cauchy-Schwarz inequality, there holds:

$$\left| \fint_{\mathrm{B}_{2^{m-1}}(x_0)} \nabla u - \fint_{\mathrm{B}_{2^m}(x_0)} \nabla u \right| \leq 2^{d/2} \left(\fint_{\mathrm{B}_{2^m}(x_0)} \left| \nabla u - \fint_{\mathrm{B}_{2^m}(x_0)} \nabla u \right|^2 \right)^{\frac{1}{2}} .$$

Hence, invoking (139) yields

$$\left\langle \left| \fint_{B(x_0)} \nabla u - \fint_{B_{2^{m_0}}(x_0)} \nabla u \right|^p \right\rangle^{\frac{1}{p}} \lesssim \sum_{m=0}^{m_0} 2^{m(1-\nu)} \ln^\beta (2 + 2^m) \, ,$$

which, by a triangle inequality involving once more (139), produces (140) for $R := 2^{m_0}$ (the general case $R \geq 2$ being a consequence of it). $\square$