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by

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# Tighter constraints of multiqubit entanglement in terms of Rényi- $\alpha$ entropy

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Quantum entanglement plays essential roles in quantum information processing. The monogamy and polygamy relations characterize the entanglement distributions in the multipartite systems. We present a class of monogamy inequalities related to the  $\mu$ th power of the entanglement measure based on Rényi- $\alpha$  entropy, as well as polygamy relations in terms of the  $\mu$ th powered of Rényi- $\alpha$  entanglement of assistance. These monogamy and polygamy relations are shown to be tighter than the existing ones.

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## I. INTRODUCTION

Quantum entanglement is one of the most quintessential features of quantum mechanics [1–3], revealing the basic understanding of the nature of quantum correlations. One distinct property of quantum entanglement is that a quantum system entangled with another system limits its sharing with other systems, known as the monogamy of entanglement [4, 5]. The monogamy of entanglement can be used as a resource to distribute a secret key which is secure against unauthorized parties [6, 7]. It also plays a significant role in many field of physics such as foundations of quantum mechanics [8, 9], condensed matter physics [10, 11], statistical mechanics [8], and even black-hole physics [12, 13].

The monogamy inequality was first established by Coffman-Kundu-Wootters (CKW), using tangle as a bipartite entanglement measure in three-qubit systems [14], and then generalized to multiqubit systems based on various entanglement measure [15]. The assisted entanglement is a dual concept to bipartite entanglement measure, which shows polygamy relations in multiparty quantum systems. For a three-qubit state  $\rho_{ABC}$ , a polygamy inequality was introduced as [16]  $\tau^a(\rho_{ABC}) \leq \tau^a(\rho_{AB}) + \tau^a(\rho_{AC})$ , where  $\tau^a(\rho_{ABC}) = \max \sum_i p_i \tau(|\psi_i\rangle_{A|B})$  is the tangle of assistance [16, 17], with the maximum taking over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ . This tangle-based polygamy inequality was extended to multiqubit systems and also high-dimensional quantum systems in terms of various entropic entanglement measures [18, 19]. General polygamy inequalities of entanglement is also established in arbitrary dimensional multipartite quantum systems [20–25].

In this paper, we investigate the monogamy and polygamy constraints based on the  $\mu$ th power of entanglement measures in terms of the Rényi- $\alpha$  entropy for multiqubit systems. By using the Hamming weight of binary vectors we present a class of monogamy inequalities for multiqubit entanglement based on the  $\mu$ th power of Rényi- $\alpha$  entanglement (R $\alpha$ E) [26] for  $\mu \geq 1$ . For  $0 \leq \mu \leq 1$ , we establish a class of tight polygamy inequalities based on the  $\mu$ th power of the Rényi- $\alpha$  entanglement of assistance (R $\alpha$ EoA). Then, we show that both the monogamy inequalities with  $\mu \geq 1$  and the polygamy inequalities with  $0 \leq \mu \leq 1$  can be further improved to be tighter under certain conditions. These monogamy and polygamy relations are shown to be tighter than the existing ones. Moreover, our monogamy inequality is shown to be more effective for the counterexamples of the CKW monogamy inequality in higher-dimensional systems.

## II. PRELIMINARIES

We first recall the conceptions of Rényi- $\alpha$  entropy, Rényi- $\alpha$  entanglement, and multiqubit monogamy and polygamy inequalities. For any  $\alpha > 0$ ,  $\alpha \neq 1$ , the Rényi- $\alpha$  entropy of a quantum state  $\rho$  is defined as [27]

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log(\text{tr}\rho^\alpha).$$

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$S_\alpha(\rho)$  reduces to the von Neumann entropy when  $\alpha$  tends to 1.

The Rényi- $\alpha$  entanglement (R $\alpha$ E)  $E_\alpha(|\psi\rangle_{AB})$  of a bipartite pure state  $|\psi\rangle_{AB}$  is defined as

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A),$$

where  $\rho_A = \text{Tr}_B|\psi\rangle_{AB}\langle\psi|$  is the reduced state of system  $A$ . For a mixed state  $\rho_{AB}$ , the Rényi- $\alpha$  entanglement is given by

$$E_\alpha(\rho_{A|B}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{A|B}),$$

where the minimum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ .

As a dual concept to R $\alpha$ E, the Rényi- $\alpha$  entanglement of assistance (R $\alpha$ EoA) is introduced as

$$E_\alpha^a(\rho_{A|B}) = \max \sum_i p_i E_\alpha(|\psi_i\rangle_{A|B}), \quad (1)$$

where the maximum is taken over all possible pure state decompositions of  $\rho_{AB}$  [28].

For any multiqubit state  $\rho_{AB_0 \dots B_{N-1}}$ , a monogamous inequality has been presented in Ref. [28] for  $\alpha \geq 2$ ,

$$E_\alpha(\rho_{A|B_0 \dots B_{N-1}}) \geq \sum_{i=0}^{N-1} E_\alpha(\rho_{A|B_i}), \quad (2)$$

where  $E_\alpha(\rho_{A|B_0 \dots B_{N-1}})$  is the R $\alpha$ E of  $\rho_{AB_0 \dots B_{N-1}}$  with respect to the bipartition between  $A$  and  $B_0 \dots B_{N-1}$ , and  $E_\alpha(\rho_{A|B_i})$  is the R $\alpha$ E of the reduced density matrix  $\rho_{AB_i}$ ,  $i = 0, \dots, N-1$ .

In addition, a class of polygamy inequalities of has been obtained for multiqubit systems,

$$E_\alpha^a(\rho_{A|B_0 \dots B_{N-1}}) \leq \sum_{i=0}^{N-1} E_\alpha^a(\rho_{A|B_i}), \quad (3)$$

for  $0 \leq \alpha \leq 2$ ,  $\alpha \neq 1$ , where  $E_\alpha(\rho_{AB_0 \dots B_{N-1}})$  is the R $\alpha$ EoA of  $\rho_{AB_0 \dots B_{N-1}}$  with respect to the bipartition between  $A$  and  $B_0 \dots B_{N-1}$ , and  $E_\alpha^a(\rho_{A|B_i})$  is the R $\alpha$ EoA of the reduced density matrix  $\rho_{AB_i}$ ,  $i = 0, \dots, N-1$ .

In Ref. [29], Kim established a class of tight monogamy inequalities of multiqubit entanglement in terms of Hamming weight. For any nonnegative integer  $j$  with binary expansion  $j = \sum_{i=0}^{n-1} j_i 2^i$ , where  $\log_2 j \leq n$  and  $j_i \in \{0, 1\}$  for  $i = 0, \dots, n-1$ , one can always define a unique binary vector associated with  $j$ ,  $\vec{j} = (j_0, j_1, \dots, j_{n-1})$ . The Hamming weight  $\omega_H(\vec{j})$  of the binary vector  $\vec{j}$  is defined to be the number of 1's in its coordinates [30]. Moreover, the Hamming weight  $\omega_H(\vec{j})$  is bounded above by  $\log_2 j$ ,

$$\omega_H(\vec{j}) \leq \log_2 j \leq j. \quad (4)$$

Kim proposed the tight constraints of multiqubit entanglement based on Hamming weights [29],

$$[E_\alpha(\rho_{A|B_0 B_1 \dots B_{N-1}})]^\mu \geq \sum_{j=0}^{N-1} \mu^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu \quad (5)$$

for  $\mu \geq 1$ , and

$$[E_\alpha^a(\rho_{A|B_0 B_1 \dots B_{N-1}})]^\mu \leq \sum_{j=0}^{N-1} \mu^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu \quad (6)$$

for  $0 \leq \mu \leq 1$ . Inequalities (5) and (6) are then further written as

$$[E_\alpha(\rho_{A|B_0 B_1 \dots B_{N-1}})]^\mu \geq \sum_{j=0}^{N-1} \mu^j [E_\alpha(\rho_{A|B_j})]^\mu$$

for  $\mu \geq 1$ , and

$$[E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu \leq \sum_{j=0}^{N-1} \mu^j [E_\alpha(\rho_{A|B_j})]^\mu$$

for  $0 \leq \mu \leq 1$ .

In the following we show that these inequalities above can be further improved to be much tighter under certain conditions, which provide tighter constraints on the multiqubit entanglement distribution.

### III. TIGHTER CONSTRAINTS OF MULTIQUBIT ENTANGLEMENT IN TERMS OF $R_{\alpha E}$

We first present a class of tighter monogamy and polygamy inequalities of multiqubit entanglement in terms of the  $\mu$ th power of  $R_{\alpha E}$ . We need the following results [33]. Suppose  $k$  is a real number,  $0 < k \leq 1$ . Then for any  $0 \leq x \leq k$ , we have

$$(1+x)^\mu \geq 1 + \frac{(1+k)^\mu - 1}{k^\mu} x^\mu \quad (7)$$

for  $\mu \geq 1$ , and

$$(1+x)^\mu \leq 1 + \frac{(1+k)^\mu - 1}{k^\mu} x^\mu \quad (8)$$

for  $0 \leq \mu \leq 1$ . Based on the inequality (7), we have the following theorem for  $R_{\alpha E}$ .

**Theorem 1** For any multiqubit state  $\rho_{AB_0\dots B_{N-1}}$  and  $\alpha \geq 2$ , we have

$$[E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu \geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu, \quad (9)$$

where  $\mu \geq 1$ ,  $\vec{j} = (j_0, \dots, j_{n-1})$  is the vector from the binary representation of  $j$ , and  $\omega_H(\vec{j})$  is the Hamming weight of  $\vec{j}$ .

[Proof] We first prove that

$$\left[ \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right]^\mu \geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu. \quad (10)$$

Without loss of generality, we assume that the qubit subsystems  $B_0, \dots, B_{N-1}$  are so labeled such that

$$kE_\alpha(\rho_{A|B_j}) \geq E_\alpha(\rho_{A|B_{j+1}}) \geq 0 \quad (11)$$

for  $j = 0, 1, \dots, N-2$  and some  $0 < k \leq 1$ .

We first show that the inequality (10) holds for the case of  $N = 2^n$ . For  $n = 1$ , let  $\rho_{AB_0}$  and  $\rho_{AB_1}$  be the two-qubit reduced density matrices of a three-qubit pure state  $\rho_{AB_0B_1}$ . We obtain

$$[E_\alpha(\rho_{A|B_0}) + E_\alpha(\rho_{A|B_1})]^\mu = [E_\alpha(\rho_{A|B_0})]^\mu \left( 1 + \frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})} \right)^\mu. \quad (12)$$

Combining (7) and (11), we have

$$\left( 1 + \frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})} \right)^\mu \geq 1 + \frac{(1+k)^\mu - 1}{k^\mu} \left( \frac{E_\alpha(\rho_{A|B_1})}{E_\alpha(\rho_{A|B_0})} \right)^\mu. \quad (13)$$

From (12) and (13), we get

$$[E_\alpha(\rho_{A|B_0}) + E_\alpha(\rho_{A|B_1})]^\mu \geq [E_\alpha(\rho_{A|B_0})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha(\rho_{A|B_1})]^\mu.$$

Therefore, the inequality (10) holds for  $n = 1$ .

We assume that the inequality (10) holds for  $N = 2^{n-1}$  with  $n \geq 2$ , and prove the case of  $N = 2^n$ . For an  $(N + 1)$ -qubit pure state  $\rho_{AB_0B_1\dots B_{N-1}}$ , we have  $E_\alpha(\rho_{A|B_{j+2^{n-1}}}) \leq k^{2^{n-1}} E_\alpha(\rho_{A|B_j})$  from (11). Therefore,

$$0 \leq \frac{\sum_{j=2^{n-1}}^{2^n-1} E_\alpha(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1} E_\alpha(\rho_{A|B_j})} \leq k^{2^{n-1}} \leq k,$$

and

$$\left( \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu = \left( \sum_{j=0}^{2^{n-1}-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \left( 1 + \frac{\sum_{j=2^{n-1}}^{2^n-1} E_\alpha(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1} E_\alpha(\rho_{A|B_j})} \right)^\mu.$$

Thus, we have

$$\left( \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \geq \left( \sum_{j=0}^{2^{n-1}-1} E_\alpha(\rho_{A|B_j}) \right)^\mu + \frac{(1+k)^\mu - 1}{k^\mu} \left( \sum_{j=2^{n-1}}^{2^n-1} E_\alpha(\rho_{A|B_j}) \right)^\mu.$$

According to the induction hypothesis, we get

$$\left( \sum_{j=0}^{2^{n-1}-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \geq \sum_{j=0}^{2^{n-1}-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})-1} [E_\alpha(\rho_{A|B_j})]^\mu.$$

By relabeling the subsystems, the induction hypothesis leads to

$$\left( \sum_{j=2^{n-1}}^{2^n-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \geq \sum_{j=2^{n-1}}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})-1} [E_\alpha(\rho_{A|B_j})]^\mu.$$

Thus, we have

$$\left( \sum_{j=0}^{2^n-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \geq \sum_{j=0}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu.$$

Now consider a  $(2^n + 1)$ -qubit state

$$\Gamma_{AB_0B_1\dots B_{2^n-1}} = \rho_{AB_0B_1\dots B_{N-1}} \otimes \sigma_{B_N\dots B_{2^n-1}}, \quad (14)$$

which is the tensor product of  $\rho_{AB_0B_1\dots B_{N-1}}$  and an arbitrary  $(2^n - N)$ -qubit state  $\sigma_{B_N\dots B_{2^n-1}}$ . We have

$$[E_\alpha(\Gamma_{A|B_0B_1\dots B_{2^n-1}})]^\mu \geq \sum_{j=0}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\Gamma_{A|B_j})]^\mu,$$

where  $\Gamma_{A|B_j}$  is the two-qubit reduced density matrix of  $\Gamma_{AB_0B_1\dots B_{2^n-1}}$ ,  $j = 0, 1, \dots, 2^n - 1$ . Therefore,

$$\begin{aligned} [E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu &= [E_\alpha(\Gamma_{A|B_0B_1\dots B_{2^n-1}})]^\mu \\ &\geq \sum_{j=0}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\Gamma_{A|B_j})]^\mu \\ &= \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu, \end{aligned}$$

where  $\Gamma_{A|B_0B_1\dots B_{2^n-1}}$  is separable with respect to the bipartition  $AB_0\dots B_{N-1}$  and  $B_N\dots B_{2^n-1}$ ,  $E_\alpha(\Gamma_{A|B_0B_1\dots B_{2^n-1}}) = E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})$ ,  $E_\alpha(\Gamma_{A|B_j}) = 0$  for  $j = N, \dots, 2^n - 1$ , and  $\Gamma_{AB_j} = \rho_{AB_j}$  for each  $j = 0, \dots, N - 1$ .  $\square$

Since  $\left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} \geq \mu^{\omega_H(\vec{j})}$  for  $\mu \geq 1$ , for any multiqubit state  $\rho_{AB_0B_1\dots B_{N-1}}$  we have the following relation,

$$[E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu \geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu \geq \sum_{j=0}^{N-1} \mu^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu.$$

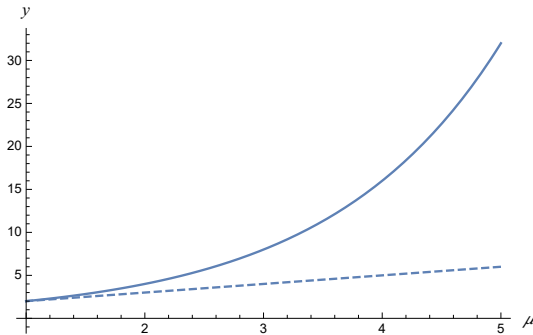


FIG. 1: Rényi- $\alpha$  entanglement with respect to  $\mu$ : the solid line is for  $y_1$  and the dashed line for  $y_2$  from the result in [29].

Therefore, our inequality (9) in Theorem 1 is always tighter than the inequality (5).

In fact, the tighter monogamy inequality (9) holds not only for multiqubit systems, but also for some multipartite higher-dimensional quantum systems, which can be proved in a similar way as in [29]. Here, we show that (9) is also more efficient than (5) for such higher-dimensional quantum systems. Let us consider the counterexample of the CKW inequality in tripartite quantum systems [31],

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle). \quad (15)$$

One has  $E_\alpha(|\psi\rangle_{A|BC}) = S_\alpha(\rho)$ . Taking  $\alpha = 3$ , we have  $E_\alpha(|\psi\rangle_{A|BC}) = \log 3$  and the R $\alpha$ E of the two-qubit reduced density matrices are

$$E_\alpha(\rho_{A|B}) = E_\alpha(\rho_{A|C}) = -\frac{1}{2} \log \text{tr} \sigma_A^3 = 1.$$

In this case  $k = 1$ , for  $\mu \geq 1$ , we have

$$y_1 \equiv [E_\alpha(\rho_{A|B})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha(\rho_{A|C})]^\mu = 1 + \frac{(1+k)^\mu - 1}{k^\mu} = 2^\mu,$$

and

$$y_2 \equiv [E_\alpha(\rho_{A|B})]^\mu + \mu [E_\alpha(\rho_{A|C})]^\mu = 1 + \mu.$$

Therefore, one gets

$$[E_\alpha(\rho_{A|B})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha(\rho_{A|C})]^\mu \geq [E_\alpha(\rho_{A|B})]^\mu + \mu [E_\alpha(\rho_{A|C})]^\mu.$$

where  $\mu \geq 1$ , see Fig. 1. In other words, our new monogamy inequality is indeed tighter than the previous one given in [29].

Under certain conditions, the inequality (9) can even be improved further to become a much tighter inequality.

**Theorem 2** For  $\mu \geq 1$ ,  $\alpha \geq 2$  and real number  $0 < k \leq 1$ , any multiqubit state  $\rho_{AB_0 \dots B_{N-1}}$  satisfies

$$[E_\alpha(\rho_{A|B_0 \dots B_{N-1}})]^\mu \geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j [E_\alpha(\rho_{A|B_j})]^\mu, \quad (16)$$

if

$$k E_\alpha(\rho_{A|B_j}) \geq \sum_{j=i+1}^{N-1} E_\alpha(\rho_{A|B_j}) \quad (17)$$

for  $i = 0, 1, \dots, N-2$ .

[Proof] We need to show

$$\left( \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j (E_\alpha(\rho_{A|B_j}))^\mu. \quad (18)$$

For any multiqubit state  $\rho_{AB_0\dots B_{N-1}}$ , it is easy to show that

$$\left( \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu = (E_\alpha(\rho_{A|B_0}))^\mu \left( 1 + \frac{\sum_{j=1}^{N-1} E_\alpha(\rho_{A|B_j})}{E_\alpha(\rho_{A|B_0})} \right)^\mu$$

and

$$\left( 1 + \frac{\sum_{j=1}^{N-1} E_\alpha(\rho_{A|B_j})}{E_\alpha(\rho_{A|B_0})} \right)^\mu \geq 1 + \left( \frac{(1+k)^\mu - 1}{k^\mu} \right) \left( \frac{\sum_{j=1}^{N-1} E_\alpha(\rho_{A|B_j})}{E_\alpha(\rho_{A|B_0})} \right)^\mu.$$

Thus,

$$\begin{aligned} \left( \sum_{j=0}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu &\geq (E_\alpha(\rho_{A|B_0}))^\mu + \left( \frac{(1+k)^\mu - 1}{k^\mu} \right) \left( \sum_{j=1}^{N-1} E_\alpha(\rho_{A|B_j}) \right)^\mu \\ &\geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j (E_\alpha(\rho_{A|B_j}))^\mu, \end{aligned}$$

where the second inequality is due to the induction hypothesis.  $\square$

In fact, according to (4), for any  $\mu \geq 1$ , one has

$$\begin{aligned} [E_\alpha(\rho_{A|B_0\dots B_{N-1}})]^\mu &\geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j (E_\alpha(\rho_{A|B_j}))^\mu \\ &\geq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha(\rho_{A|B_j})]^\mu. \end{aligned}$$

For the case of  $\mu < 0$ , we can also derive a tighter upper bound of  $E_\alpha^\mu(\rho_{A|B_0B_1\dots B_{N-1}})$ .

**Theorem 3** For any multiqubit state  $\rho_{AB_0\dots B_{N-1}}$  with  $E_\alpha(\rho_{AB_i}) \neq 0$ ,  $i = 0, 1, \dots, N-1$ , we have

$$[E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu \leq \frac{1}{N-1} \sum_{j=0}^{N-1} [E_\alpha(\rho_{A|B_j})]^\mu, \quad (19)$$

for all  $\mu < 0$  and  $\alpha \geq 2$ .

[Proof] Similar to the proof in [32], for arbitrary three-qubit states we have

$$\begin{aligned} [E_\alpha(\rho_{A|B_0B_1})]^\mu &\leq [E_\alpha^2(\rho_{A|B_0}) + E_\alpha^2(\rho_{A|B_1})]^\frac{\mu}{2} \\ &= [E_\alpha(\rho_{A|B_0})]^\mu \left( 1 + \frac{E_\alpha^2(\rho_{A|B_1})}{E_\alpha^2(\rho_{A|B_0})} \right)^\frac{\mu}{2} \\ &< [E_\alpha(\rho_{A|B_0})]^\mu, \end{aligned} \quad (20)$$

where the first inequality is from  $\mu < 0$ , the second inequality is due to  $\left( 1 + \frac{E_\alpha^2(\rho_{A|B_1})}{E_\alpha^2(\rho_{A|B_0})} \right)^\frac{\mu}{2} < 1$ . Moreover, we have

$$[E_\alpha(\rho_{A|B_0B_1})]^\mu < [E_\alpha(\rho_{A|B_1})]^\mu. \quad (21)$$

Combining (20) and (21), we get

$$[E_\alpha(\rho_{A|B_0B_1})]^\mu < \frac{1}{2} \{ [E_\alpha(\rho_{A|B_0})]^\mu + [E_\alpha(\rho_{A|B_1})]^\mu \}.$$

Thus, we obtain

$$\begin{aligned} &[E_\alpha(\rho_{A|B_0B_1\dots B_{N-1}})]^\mu \\ &< \frac{1}{2} \left\{ [E_\alpha(\rho_{A|B_0})]^\mu + [E_\alpha(\rho_{A|B_1\dots B_{N-1}})]^\mu \right\} \\ &< \frac{1}{2} [E_\alpha(\rho_{A|B_0})]^\mu + \left( \frac{1}{2} \right)^2 [E_\alpha(\rho_{A|B_1})]^\mu + \left( \frac{1}{2} \right)^2 [E_\alpha(\rho_{A|B_2\dots B_{N-1}})]^\mu \\ &< \dots \\ &< \frac{1}{2} [E_\alpha(\rho_{A|B_0})]^\mu + \left( \frac{1}{2} \right)^2 [E_\alpha(\rho_{A|B_1})]^\mu + \dots + \left( \frac{1}{2} \right)^{N-2} [E_\alpha(\rho_{A|B_{N-2}})]^\mu + \left( \frac{1}{2} \right)^{N-2} [E_\alpha(\rho_{A|B_{N-1}})]^\mu. \end{aligned} \quad (22)$$



One can get a set of inequalities through the cyclic permutation of the pair indices  $B_0, B_1, \dots, B_{N-1}$  in (22). Summing up these inequalities, we get (19).  $\square$

#### IV. TIGHTER CONSTRAINTS OF MULTIQUBIT ENTANGLEMENT IN TERMS OF $R_\alpha EOA$

We consider now the Rényi- $\alpha$  entanglement of assistance ( $R_\alpha EoA$ ) defined in (1), and provide a class of polygamy inequalities satisfied by the multiqubit entanglement in terms of  $R_\alpha EoA$ .

**Theorem 4** *For any multiqubit state  $\rho_{AB_0 \dots B_{N-1}}$  and  $0 \leq \mu \leq 1$ ,  $0 < \alpha < 2$ ,  $\alpha \neq 1$ , we have*

$$[E_\alpha^a(\rho_{A|B_0 B_1 \dots B_{N-1}})]^\mu \leq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu. \quad (23)$$

[Proof] Similar to proof in Ref. [29], we just need to prove

$$\left[ \sum_{j=0}^{N-1} E_\alpha^a(\rho_{A|B_j}) \right]^\mu \leq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu. \quad (24)$$

Firstly, assume that the qubit subsystems  $B_0, \dots, B_{N-1}$  satisfies

$$kE_\alpha^a(\rho_{A|B_j}) \geq E_\alpha^a(\rho_{A|B_{j+1}}) \geq 0, \quad (25)$$

where  $j = 0, 1, \dots, N-2$  and  $0 < k \leq 1$ . Similar to the proof of Theorem 1, we first show that the inequality (24) holds for a three-qubit pure state  $\rho_{AB_0 B_1}$ . We have

$$\begin{aligned} [E_\alpha^a(\rho_{A|B_0}) + E_\alpha^a(\rho_{A|B_1})]^\mu &= [E_\alpha^a(\rho_{A|B_0})]^\mu \left( 1 + \frac{E_\alpha^a(\rho_{A|B_1})}{E_\alpha^a(\rho_{A|B_0})} \right)^\mu \\ &\leq [E_\alpha^a(\rho_{A|B_0})]^\mu \left[ 1 + \frac{(1+k)^\mu - 1}{k^\mu} \left( \frac{E_\alpha^a(\rho_{A|B_1})}{E_\alpha^a(\rho_{A|B_0})} \right)^\mu \right] \\ &= [E_\alpha^a(\rho_{A|B_0})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha^a(\rho_{A|B_1})]^\mu, \end{aligned}$$

where the inequality is due to (8).

Then we assume that the inequality (24) holds for  $N = 2^{n-1}$  with  $n \geq 2$ . Consider the case of  $N = 2^n$ . For an  $(N+1)$ -qubit pure state  $\rho_{AB_0 B_1 \dots B_{N-1}}$  with its two-qubit reduced density matrices  $\rho_{AB_j}$ ,  $j = 0, 1, \dots, N-1$ , we have  $E_\alpha^a(\rho_{A|B_{j+2^{n-1}}}) \leq k^{2^{n-1}} E_\alpha^a(\rho_{A|B_j})$  due to the ordering of subsystems in the inequality (25). Then, we get

$$0 \leq \frac{\sum_{j=2^{n-1}}^{2^n-1} E_\alpha^a(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1} E_\alpha^a(\rho_{A|B_j})} \leq k^{2^{n-1}} \leq k \leq 1,$$

and

$$\left( \sum_{j=0}^{N-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu = \left( \sum_{j=0}^{2^{n-1}-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu \left( 1 + \frac{\sum_{j=2^{n-1}}^{2^n-1} E_\alpha^a(\rho_{A|B_j})}{\sum_{j=0}^{2^{n-1}-1} E_\alpha^a(\rho_{A|B_j})} \right)^\mu.$$

Hence,

$$\left( \sum_{j=0}^{N-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu \leq \left( \sum_{j=0}^{2^{n-1}-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu + \frac{(1+k)^\mu - 1}{k^\mu} \left( \sum_{j=2^{n-1}}^{2^n-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu.$$

According to the induction hypothesis, we get

$$\left( \sum_{j=0}^{2^{n-1}-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu \leq \sum_{j=0}^{2^{n-1}-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})-1} [E_\alpha^a(\rho_{A|B_j})]^\mu.$$

By relabeling the subsystems, the induction hypothesis leads to

$$\left( \sum_{j=2^{n-1}}^{2^n-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu \leq \sum_{j=2^{n-1}}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})-1} [E_\alpha^a(\rho_{A|B_j})]^\mu.$$

Therefore,

$$\left( \sum_{j=0}^{2^n-1} E_\alpha^a(\rho_{A|B_j}) \right)^\mu \leq \sum_{j=0}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu.$$

Consider the  $(2^n + 1)$ -qubit state (14). We have

$$\begin{aligned} [E_\alpha^a(\rho_{A|B_0 B_1 \dots B_{N-1}})]^\mu &= [E_\alpha^a(\Gamma_{A|B_0 B_1 \dots B_{2^n-1}})]^\mu \\ &\leq \sum_{j=0}^{2^n-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\Gamma_{A|B_j})]^\mu \\ &= \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu. \end{aligned}$$

□

Since  $\frac{(1+k)^\mu - 1}{k^\mu} \leq \mu$  for  $0 \leq \mu \leq 1$ , it is easily seen that (23) is tighter than (6).

As an example, let us consider the three-qubit W-state [34],

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|211\rangle + |121\rangle + |112\rangle). \quad (26)$$

We have  $E_\alpha^a(|\psi\rangle_{A|BC}) = S_\alpha(\rho) = \log 3 - \frac{2}{3}$  and

$$E_\alpha^a(\rho_{A|B}) = E_\alpha^a(\rho_{A|C}) = -\frac{1}{2} \log \text{tr} \sigma_A^3 = \frac{2}{3}.$$

In the case of  $k = 1$  and  $0 \leq \mu \leq 1$ , we have

$$y_3 \equiv [E_\alpha^a(\rho_{A|B})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha^a(\rho_{A|C})]^\mu = \left(\frac{2}{3}\right)^\mu + \frac{(1+k)^\mu - 1}{k^\mu} \left(\frac{2}{3}\right)^\mu = \left(\frac{2}{3}\right)^\mu 2^\mu,$$

and

$$y_4 \equiv [E_\alpha^a(\rho_{A|B})]^\mu + \mu [E_\alpha^a(\rho_{A|C})]^\mu = \left(\frac{2}{3}\right)^\mu (1 + \mu).$$

Therefore, we get

$$[E_\alpha^a(\rho_{A|B})]^\mu + \frac{(1+k)^\mu - 1}{k^\mu} [E_\alpha^a(\rho_{A|C})]^\mu \leq [E_\alpha^a(\rho_{A|B})]^\mu + \mu [E_\alpha^a(\rho_{A|C})]^\mu$$

where  $0 \leq \mu \leq 1$ , see Fig. 2.

Similar to the improvement from the inequality (9) to the inequality (16), we can also improve the polygamy inequality in Theorem 4. The proof is similar to the Theorem 2.

**Theorem 5** For  $0 \leq \mu \leq 1$ ,  $0 < \alpha < 2$ ,  $\alpha \neq 1$  and  $0 < k \leq 1$ , we have for any multiqubit state  $\rho_{AB_0 \dots B_{N-1}}$ ,

$$[E_\alpha^a(\rho_{A|B_0 \dots B_{N-1}})]^\mu \leq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j [E_\alpha^a(\rho_{A|B_j})]^\mu, \quad (27)$$

if

$$k E_\alpha^a(\rho_{A|B_i}) \geq \sum_{j=i+1}^{N-1} E_\alpha^a(\rho_{A|B_j}) \quad (28)$$

for  $i = 0, 1, \dots, N-2$ .

Since  $\omega_H(\vec{j}) \leq j$ , for  $0 \leq \mu \leq 1$  we obtain

$$\begin{aligned} [E_\alpha^a(\rho_{A|B_0 \dots B_{N-1}})]^\mu &\leq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^j (E_\alpha^a(\rho_{A|B_j}))^\mu \\ &\leq \sum_{j=0}^{N-1} \left( \frac{(1+k)^\mu - 1}{k^\mu} \right)^{\omega_H(\vec{j})} [E_\alpha^a(\rho_{A|B_j})]^\mu. \end{aligned}$$

Therefore, for any multiqubit state  $\rho_{AB_0 \dots B_{N-1}}$  satisfying the condition (28), the inequality (27) of Theorem 5 is tighter than the inequality (23) of Theorem 4.

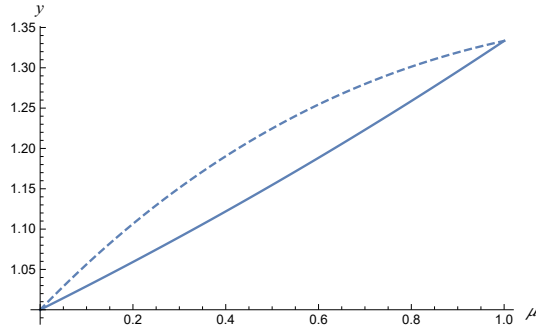


FIG. 2: Rényi- $\alpha$  entanglement  $y$  with respect to  $\mu$ : the solid line is for  $y_3$  and the dashed line for  $y_4$  from the result in [29].

## V. CONCLUSION

Quantum entanglement is the essential resource in quantum information. The monogamy and polygamy relations characterize the entanglement distributions in the multipartite systems. Tighter monogamy and polygamy inequalities give finer characterization of the entanglement distribution. In this article, by using the Hamming weights of binary vectors we have proposed a class of monogamy inequalities related to the  $\mu$ th power of the entanglement measure based on Rényi- $\alpha$  entropy, polygamy relations in terms of the  $\mu$ th powered of of  $R\alpha EoA$  for  $0 \leq \mu \leq 1$ . These new monogamy and polygamy relations are shown to be tighter than the existing ones. Moreover, it has been shown that our monogamy inequality is effective for the counterexamples of the CKW monogamy inequality in higher-dimensional systems. Our results may highlight further investigations on the entanglement distribution in multipartite systems.

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