Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

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by

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Preprint no.: 51 2020

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Quantum coherence is a fundamental property that can emerge within any quantum system. Incoherent operations, defined in terms of the Kraus decomposition, take an important role in state transformation. The maximum number of incoherent Kraus operators has been present in [A. Streltsov, S. Rana, P. Boes, J. Eisert, Phys. Rev. Lett. 119. 140402 (2017)]. In this work, we show that the number of incoherent Kraus operators for a single qubit can be reduce from 5 to 4 by constructing a proper unitary matrix. For qutrit systems we further obtain 32 incoherent Kraus operators, while the upper bound in the research of Sterltsov gives 39 Kraus operators. Besides, we reduce the number of strictly incoherent Kraus operators from more than 15 to 13. And we consider the state transformation via single qutrit strictly incoherent operation and incoherent operation.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

I.INTRODUCTION

Quantum resource theories [1, 2] offer a powerful framework for understanding the natural change of certain physical properties in a physical system and their applications for quantum technology. In recent years, a lot of works on the development of quantum resource theory in different physics fields have been done, such as the quantum resource theory of entanglement [3], the quantum resource theory of thermodynamics [4], the quantum resource theory of coherence [5] and so on. The general structure of quantum resource theory has three ingredients in common： free states, free operations and resource states. The basic requirement of resource theory is that free operations cannot generate a resource state from a free one. Free states can be created and performed at no cost, and any state outside of the set of free states is called a resource state.

As an important physical resource, quantum coherence [6] has found use in a variety of physical tasks in quantum information processing, such as quantum algorithm [7], quantum thermodynamics [8, 9], metrology [10], and quantum biology [11]. Let $\{|i\rangle\}$ $(i = 1, ..., d)$ be a paticular basis in a $d-$dimensional Hilbert space $\mathcal{H}_d$. A state is called incoherent state if it is diagonal in this basis and otherwise coherent. The structure of the incoherent states is as follows

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|, \tag{1}$$

where $\sum_{i=1}^{d} \delta_i = 1$.

Depending on the different physical requirement, there exist different types of incoherent operations. The important free operations are known as incoherent operations(IO) [5] and strictly incoherent operations(SIO) [12]. We denote $\mathcal{I}$ as the set of all incoherent states. A completely positive and trace-preserving map(CPTP) Φ is said to be an IO if Φ has a Kraus operator representation $\{K_n\}$ such that $K_n\rho K_n^\dagger / Tr[K_n\rho K_n^\dagger] \in \mathcal{I}$ for all n and $\rho \in \mathcal{I}$, while SIO require further $\{K_n\}$ and $\{K_n^\dagger\}$ are incoherent.

Recently, A. Streltsov et al. in [13] have derived the upper bound of the number of incoherent Kraus operators in a general incoherent operation. For any single qubit IO, the canonical representation of the Kraus operator is given by the set

$$\left\{\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}, \begin{pmatrix} 0 & b_4 \\ a_4 & 0 \end{pmatrix}, \begin{pmatrix} a_5 & 0 \\ 0 & 0 \end{pmatrix}\right\}, \tag{2}$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{C}$. Moreover, a_i and b_j should satisfy the equalities $\sum_{i=1}^{5} a_i^2 = \sum_{j=1}^{4} |b_j|^2 = 1$ and $a_1 b_1 + a_2 b_2 = 0$. Later, some scholars have reduced the optimal number of incoherent Kraus operators on qubit systems to 4 by using the Choi-Jamiołkowski-

Sudarshan matrix [14], which is proved to be optimal. In this work, we reduce the number of qubit and qutrit incoherent Kraus operators by constructing proper unitary matrices. We show that the number of incoherent Kraus operators for a single qubit can be reduce from 5 to 4. For qutrit systems we obtain 32 incoherent Kraus operators, while the upper bound in the research of Sterltsov gives 39 Kraus operators. Besides, we reduce the number of strictly incoherent Kraus operators from more than 15 to 13. Lastly, we consider the state transformation via SIO and IO in qutrit system. And we find the achievable region about the set of final states from a given initial qutrit state by all possible qutrit IOs.

II.THE UPPER BOUND OF (STRICTLY) INCOHERENT OPERATORS FOR QUTRIT SYSTEM

Recently, the structure of incoherent and strictly incoherent operations is studied in [13, 14]. As mentioned in [13], any single qubit IO can be decomposed into 5 incoherent Kraus operators using the structure of IO. Similarly, the number of incoherent Kraus operators can be reduced to 39 for any single qutrit incoherent operation. Besides, the upper bound of the number of strictly incoherent operator is less than 15. In the following, we first introduce an isometry about the two sets of Kraus decompositions which give rise to the same quantum operation.

Lemma 1. *The two sets of Kraus operators $\{K_j\}$ and $\{L_i\}$ are Kraus decompositions of the same quantum operation if and only if there is a unitary matrix U such that [15]*

$$L_i = \sum_j U_{i,j} K_j. \tag{3}$$

Therefore, according to the above result, the number of Kraus operators of a quantum operation is finite. There must be a set with the least number of Kraus operators. Firstly, let's study the qubit case. By using the properties of Lemma 1 and the qubit incoherent Kraus operator, we find the following conclusion

Proposition 1. *Every qubit IO can be decomposed into four incoherent Kraus operators. The canonical representation of the Kraus operators is given by the set*

$$\left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}, \begin{pmatrix} 0 & b_4 \\ a_4 & 0 \end{pmatrix} \right\}, \tag{4}$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{C}$ satisfing the equalities $\sum_{i=1}^4 a_i^2 = \sum_{j=1}^4 |b_j|^2 = 1$ and $a_1 b_1 + a_2 b_2 = 0$.

Proof. Denote the incoherent Kraus operations in Eq.(2) as follows

$$K_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \end{pmatrix}, \quad K_3 = \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix} \quad K_4 = \begin{pmatrix} 0 & b_4 \\ a_4 & 0 \end{pmatrix}, \quad K_5 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{5}$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{C}$, $*$ denotes some complex number. Now, select the following 4×4 unitary matrix

$$U = \begin{pmatrix} ka_1 & 0 & ka_3 & 0 \\ -la_3^2 a_4 |b_1||b_4| & \frac{la_2 |b_1||b_4| (a_1^2 |b_4|^2 + a_3^2 |b_1|^2 + a_3^2 |b_4|^2)}{b_1^* b_4} & la_1 a_3 a_4 |b_1||b_4| & la_3^2 a_4 |b_1|^2 \\ -\frac{ma_2 a_3 b_4 |b_1|}{b_1} & -ma_3 a_4 |b_1| & \frac{ma_1 a_2 b_4 |b_1|}{b_1} & ma_2 a_3 |b_1| \\ \frac{na_3^2 b_1^* |b_4|}{b_4^*} & 0 & \frac{-na_1 a_3 b_1^* |b_4|}{b_4^*} & n(a_1^2 + a_3^2)|b_4| \end{pmatrix}, \tag{6}$$

where the parameters k, l, m and n are chosen as

$$
\begin{aligned}
k^2 &= \frac{1}{a_1^2 + a_3^2},\\
l^2 &= \frac{1}{a_3^4 a_4^2 |b_1|^2 |b_4|^2 + a_2^2 (a_1^2 |b_4|^2 + a_3^2 |b_1|^2 + a_3^2 |b_4|^2)^2 + a_1^2 a_3^2 a_4^2 |b_1|^2 |b_4|^2 + a_3^4 a_4^2 |b_1|^4},\\
m^2 &= \frac{1}{a_2^2 a_3^2 |b_4|^2 + a_3^4 a_4^2 |b_1|^2 + a_1^2 a_2^2 |b_4|^2 + a_2^2 a_3^2 |b_1|^2},\\
n^2 &= \frac{1}{a_3^4 |b_1|^2 + a_1^2 a_3^2 |b_1|^2 + (a_1^2 + a_3^2)^2 |b_4|^2}.
\end{aligned}
\tag{7}
$$

We then introduce a unitary matrix V defined by

$$V = U \oplus I_1, \tag{8}$$

where I_1 is the identity operator with dimension 1. According to Lemma 1, we have

$$L_i = \begin{cases} \sum_{j=1}^4 V_{i,j} K_j & for\ 1 \le i \le 4, \\ K_i & for\ i = 5. \end{cases} \tag{9}$$

Then one computes that

$$L_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}, \quad L_3 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad L_5 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{10}$$

where $*$ denote some complex numbers that are the combinations of a_i and b_i. It is obvious that L_5 has the same form as L_3. Thus, we can reduce the set $\{L_3, L_5\}$ to one Kraus operator. This proves that every IO in qubit system can be decomposed into at most four incoherent Kraus operators as given in Eq.(4). From the normalization property $\sum_{i=1}^4 K_i^\dagger K_i = I$, we have $\sum_{i=1}^4 a_i^2 = \sum_{j=1}^4 |b_j|^2 = 1$ and $a_1 b_1 + a_2 b_2 = 0$. □

Using the Choi-Jamiołkowski-Sudarshan matrix for a quantum operation, Rana et al. have proved that the optimal number of incoherent Kraus operators for an incoherent qubit operation is four. However, we observe that it is more convenient to draw the conclusion using the isometry of Kraus decompositions. For most incoherent operations, the above result is the optimal form of incoherent Kraus decomposition. We cannot find a general unitary matrix to reduce the number of incoherent Kraus operator. But some special quantum operations could be decomposed into least four incoherent Kraus operators, such as the phase damping channel and amplitude damping channel [16].

For qutrit system, any incoherent operation admits a decomposition with at most 39 incoherent Kraus operators. A canonical representation of the Kraus operators for a qutrit IO can be obtained from the proof of Proposition 5 in Ref. [13] as follows,

$$
\begin{aligned}
&K_1=\begin{bmatrix} a_1 & b_1 & c_1\\ 0&0&0\\ 0&0&0\end{bmatrix},\quad
K_2=\begin{bmatrix} a_2 & b_2 & 0\\ 0&0&c_2\\ 0&0&0\end{bmatrix},\quad
K_3=\begin{bmatrix} a_3 & b_3 & 0\\ 0&0&0\\ 0&0&c_3\end{bmatrix},\quad
K_4=\begin{bmatrix} a_4 & b_4 & 0\\ 0&0&0\\ 0&0&0\end{bmatrix},\quad
K_5=\begin{bmatrix} 0 & 0 & c_5\\ a_5&b_5&0\\ 0&0&0\end{bmatrix},\\
&K_6=\begin{bmatrix} 0&0&0\\ a_6 & b_6 & c_6\\ 0&0&0\end{bmatrix},\quad
K_7=\begin{bmatrix} 0&0&0\\ a_7 & b_7 & 0\\ 0&0&c_7\end{bmatrix},\quad
K_8=\begin{bmatrix} 0&0&0\\ a_8 & b_8 & 0\\ 0&0&0\end{bmatrix},\quad
K_9=\begin{bmatrix} a_9&0&c_9\\ 0 & b_9 & 0\\ 0&0&0\end{bmatrix},\quad
K_{10}=\begin{bmatrix} a_{10}&0&0\\ 0 & b_{10} & c_{10}\\ 0&0&0\end{bmatrix},\\
&K_{11}=\begin{bmatrix} a_{11}&0&0\\ 0 & b_{11} & 0\\ 0&0&c_{11}\end{bmatrix},\quad
K_{12}=\begin{bmatrix} a_{12}&0&0\\ 0 & b_{12} & 0\\ 0&0&0\end{bmatrix},\quad
K_{13}=\begin{bmatrix} 0&b_{13}&c_{13}\\ a_{13} & 0 & 0\\ 0&0&0\end{bmatrix},\quad
K_{14}=\begin{bmatrix} 0&b_{14}&0\\ a_{14} & 0 & c_{14}\\ 0&0&0\end{bmatrix},\quad
K_{15}=\begin{bmatrix} 0&b_{15}&0\\ a_{15} & 0 & 0\\ 0&0&c_{15}\end{bmatrix},\\
&K_{16}=\begin{bmatrix} 0&b_{16}&0\\ a_{16} & 0 & 0\\ 0&0&0\end{bmatrix},\quad
K_{17}=\begin{bmatrix} a_{17}&0&c_{17}\\ 0 & 0 & 0\\ 0&b_{17}&0\end{bmatrix},\quad
K_{18}=\begin{bmatrix} a_{18}&0&0\\ 0 & 0 & c_{18}\\ 0&b_{18}&0\end{bmatrix},\quad
K_{19}=\begin{bmatrix} a_{19}&0&0\\ 0 & 0 & 0\\ 0&b_{19}&c_{19}\end{bmatrix},\quad
K_{20}=\begin{bmatrix} a_{20}&0&0\\ 0 & 0 & 0\\ 0&b_{20}&0\end{bmatrix},\\
&K_{21}=\begin{bmatrix} 0&0&c_{21}\\ a_{21} & 0 & 0\\ 0&b_{21}&0\end{bmatrix},\quad
K_{22}=\begin{bmatrix} 0&0&0\\ a_{22} & 0 & c_{22}\\ 0&b_{22}&0\end{bmatrix},\quad
K_{23}=\begin{bmatrix} 0&0&0\\ a_{23} & 0 & 0\\ 0&b_{23}&c_{23}\end{bmatrix},\quad
K_{24}=\begin{bmatrix} 0&0&0\\ a_{24} & 0 & 0\\ 0&b_{24}&0\end{bmatrix},\quad
K_{25}=\begin{bmatrix} 0&0&c_{25}\\ 0 & b_{25} & 0\\ a_{25}&0&0\end{bmatrix},\\
&K_{26}=\begin{bmatrix} 0&0&0\\ 0 & b_{26} & c_{26}\\ a_{26}&0&0\end{bmatrix},\quad
K_{27}=\begin{bmatrix} 0&0&0\\ 0 & b_{27} & 0\\ a_{27}&0&c_{27}\end{bmatrix},\quad
K_{28}=\begin{bmatrix} 0&0&0\\ 0 & b_{28} & 0\\ a_{28}&0&0\end{bmatrix},\quad
K_{29}=\begin{bmatrix} 0&0&c_{29}\\ 0 & 0 & 0\\ a_{29}&b_{29}&0\end{bmatrix},\quad
K_{30}=\begin{bmatrix} 0&0&0\\ 0 & 0 & c_{30}\\ a_{30}&b_{30}&0\end{bmatrix},\\
&K_{31}=\begin{bmatrix} 0&0&0\\ 0 & 0 & 0\\ a_{31}&b_{31}&c_{31}\end{bmatrix},\quad
K_{32}=\begin{bmatrix} 0&0&0\\ 0 & 0 & 0\\ a_{32}&b_{32}&0\end{bmatrix},\quad
K_{33}=\begin{bmatrix} 0&b_{33}&c_{33}\\ 0 & 0 & 0\\ a_{33}&0&0\end{bmatrix},\quad
K_{34}=\begin{bmatrix} 0&b_{34}&0\\ 0 & 0 & c_{34}\\ a_{34}&0&0\end{bmatrix},\quad
K_{35}=\begin{bmatrix} 0&b_{35}&0\\ 0 & 0 & 0\\ a_{35}&0&c_{35}\end{bmatrix},\\
&K_{36}=\begin{bmatrix} 0&b_{36}&0\\ 0 & 0 & 0\\ a_{36}&0&0\end{bmatrix},\quad
K_{37}=\begin{bmatrix} 0&0&0\\ 0 & 0 & 0\\ a_{37}&0&0\end{bmatrix},\quad
K_{38}=\begin{bmatrix} a_{38}&0&0\\ 0 & 0 & 0\\ 0&0&0\end{bmatrix},\quad
K_{39}=\begin{bmatrix} 0&0&0\\ a_{39} & 0 & 0\\ 0&0&0\end{bmatrix}.
\end{aligned}
\tag{11}
$$

Theorem 1. *Any incoherent operation acting on a single qutrit system admits a decomposition with at most 32 incoherent Kraus operators.*

Proof. Firstly, we choose the Kraus operators K_{32}, K_{24}, K_{37} and K_{39}. Denote K_{32}, K_{24}, K_{37} and K_{39} as M_1, M_2, M_3 and M_4 respectively. Define a 3×3 unitary matrix U_1 by

$$
U_1=\begin{pmatrix} l_1 a_{32}^* & 0 & l_1 a_{37}^*\\ m_1 b_{32}^*|a_{37}|^2 & m_1(|a_{32}|^2+|a_{37}|^2)b_{24}^* & -m_1 a_{37}^* b_{32}^* a_{32}\\ n_1 a_{37} b_{24} & -n_1 a_{37} b_{32} & -n_1 a_{32} b_{24}\end{pmatrix},
\tag{12}
$$

where the parameters l_1, m_1 and n_1 are chosen as

$$
\begin{aligned}
l_1^2 &= \frac{1}{a_{32}^2+a_{37}^2},\\
m_1^2 &= \frac{1}{(a_{32}^2+a_{37}^2)(|a_{37}|^2(|b_{32}|^2+|b_{24}|^2)+|a_{37}b_{24}|^2)},\\
n_1^2 &= \frac{1}{|a_{37}|^2(|b_{32}|^2+|b_{24}|^2)+|a_{37}b_{24}|^2}.
\end{aligned}
\tag{13}
$$

According to Lemma 1 and the construction of the unitary matrix such as Eq.(8), we find that the operator L_4 has the same form as L_3. In other words, the set $\{M_1,\ M_2,\ M_3,\ M_4\}$ can be reduce to the set $\{M_1,\ M_2,\ M_4\}$, that is, the number of the incoherent Kraus operators can be reduce to 38.

Similarly, define U_2 as a unitary matrix by,

$$U_2 = \begin{pmatrix} l_2 a_8^* & 0 & l_2 a_{39}^* \\ m_2 b_8^* |a_{39}|^2 & m_2(|a_8|^2 + |a_{39}|^2) b_{12}^* & -m_2 a_{39}^* b_8^* a_8 \\ n_2 a_{39} b_{12} & -n_2 a_{39} b_8 & -n_2 a_8 b_{12} \end{pmatrix}, \tag{14}$$

where the parameters l_2, m_2 and n_2 are chosen as

$$\begin{aligned} l_2^2 &= \frac{1}{a_8^2 + a_{39}^2}, \\ m_2^2 &= \frac{1}{(a_8^2 + a_{39}^2)(|a_{39}|^2(|b_8|^2 + |b_{12}|^2) + |a_{39} b_{12}|^2)}, \\ n_2^2 &= \frac{1}{|a_{39}|^2(|b_8|^2 + |b_{12}|^2) + |a_{39} b_{12}|^2}. \end{aligned} \tag{15}$$

One finds the set $\{K_8,\ K_{12},\ K_{39},\ K_{38}\}$ can be reduce to $\{K_8,\ K_{12},\ K_{38}\}$. Thus the number of the incoherent Kraus operators can be reduce to 37.

Then, we discover that there is a unitary matrix U_3 which can reduce the set $\{K_4,\ K_8,\ K_{38},\ K_{16},\ K_{12}\}$ to $\{K_4,\ K_8,\ K_{12},\ K_{16}\}$. The specific form of U_3 is as follows,

$$U_3 = \begin{pmatrix} k_3 a_4 & 0 & k_3 a_{38} & 0 \\ -l_3 a_{38}^2 a_{16} |b_4||b_{16}| & \frac{l_3 a_8 |b_4||b_{16}|(a_4^2|b_{16}|^2 + a_{38}^2|b_4|^2 + a_{38}^2|b_{16}|^2)}{b_4^* b_{16}} & l a_4 a_{38} a_{16} |b_4||b_{16}| & l_3 a_{38}^2 a_{16} |b_4|^2 \\ -\frac{m_3 a_8 a_{38} b_{16} |b_4|}{b_4} & -m_3 a_{38} a_{16} |b_4| & \frac{m_3 a_4 a_8 b_{16} |b_4|}{b_4} & m a_8 a_{38} |b_4| \\ \frac{n_3 a_{38}^2 b_4^* |b_{16}|}{b_{16}^*} & 0 & \frac{-n_3 a_4 a_{38} b_4^* |b_{16}|}{b_{16}^*} & n_3(a_4^2 + a_{38}^2)|b_{16}| \end{pmatrix}, \tag{16}$$

where the parameters k_3, l_3, m_3 and n_3 are chosen as

$$\begin{aligned} k_3^2 &= \frac{1}{a_4^2 + a_{38}^2}, \\ l_3^2 &= \frac{1}{a_{38}^4 a_{16}^2 |b_4|^2 |b_{16}|^2 + a_8^2(a_4^2|b_{16}|^2 + a_{38}^2|b_4|^2 + a_{38}^2|b_{16}|^2)^2 + a_4^2 a_{38}^2 a_{16}^2 |b_4|^2 |b_{16}|^2 + a_{38}^4 a_{16}^2 |b_4|^4}, \\ m_3^2 &= \frac{1}{a_8^2 a_{38}^2 |b_{16}|^2 + a_{38}^4 a_{16}^2 |b_4|^2 + a_4^2 a_8^2 |b_{16}|^2 + a_8^2 a_{38}^2 |b_4|^2}, \\ n_3^2 &= \frac{1}{a_{38}^4 |b_4|^2 + a_4^2 a_{38}^2 |b_4|^2 + (a_4^2 + a_{38}^2)^2 |b_{16}|^2}. \end{aligned} \tag{17}$$

Similarly, we can find some special unitary matrices reducing the set $\{K_{11},\ K_{12},\ K_{19},\ K_{20}\}$, $\{K_{15},\ K_{16},\ K_{35},\ K_{36}\}$, $\{K_{15},\ K_{16},\ K_{23},\ K_{24}\}$ and $\{K_{11},\ K_{12},\ K_{27},\ K_{28}\}$ to the set $\{K_{11},\ K_{12},\ K_{19}\}$, $\{K_{15},\ K_{16},\ K_{35}\}$, $\{K_{15},\ K_{16},\ K_{23}\}$ and $\{K_{11},\ K_{12},\ K_{27}\}$ respectively. Therefore, the upper bound on the number of incoherent Kraus operators is 32.

□

Theorem 2. *Any strictly incoherent operation acting on a single qutrit system admits a decomposition with at most 13 strictly incoherent Kraus operators.*

Proof. In Ref. [13], the authors verify that any qutrits strictly incoherent operation admits a decomposition with at most 15 incoherent Kraus operators. The specific form is as follows,

$$K_1=\begin{bmatrix}a_1&0&0\\0&b_1&0\\0&0&c_1\end{bmatrix},\quad K_2=\begin{bmatrix}a_2&0&0\\0&0&c_2\\0&b_2&0\end{bmatrix},\quad K_3=\begin{bmatrix}0&b_3&0\\a_3&0&0\\0&0&c_3\end{bmatrix},\quad K_4=\begin{bmatrix}0&0&c_4\\a_4&0&0\\0&b_4&0\end{bmatrix},\quad K_5=\begin{bmatrix}0&0&c_5\\0&b_5&0\\a_5&0&0\end{bmatrix},$$
$$K_6=\begin{bmatrix}0&b_6&0\\0&0&c_6\\a_6&0&0\end{bmatrix},\quad K_7=\begin{bmatrix}a_7&0&0\\0&b_7&0\\0&0&0\end{bmatrix},\quad K_8=\begin{bmatrix}a_8&0&0\\0&0&0\\0&b_8&0\end{bmatrix},\quad K_9=\begin{bmatrix}0&b_9&0\\a_9&0&0\\0&0&0\end{bmatrix},\quad K_{10}=\begin{bmatrix}0&0&0\\a_{10}&0&0\\0&b_{10}&0\end{bmatrix},\tag{18}$$
$$K_{11}=\begin{bmatrix}0&0&0\\0&b_{11}&0\\a_{11}&0&0\end{bmatrix},\quad K_{12}=\begin{bmatrix}0&b_{12}&0\\0&0&0\\a_{12}&0&0\end{bmatrix},\quad K_{13}=\begin{bmatrix}a_{13}&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad K_{14}=\begin{bmatrix}0&0&0\\a_{14}&0&0\\0&0&0\end{bmatrix},\quad K_{15}=\begin{bmatrix}0&0&0\\0&0&0\\a_{15}&0&0\end{bmatrix}.$$

By defining the following 3×3 unitary matrix

$$U_1=\begin{pmatrix}-l_1(|a_{12}|^2+|a_{15}|^2)b_9 & -l_1|a_{15}|^2b_{12}^* & l_1a_{12}a_{15}^*b_{12}^*\\ 0 & -m_1a_{12}^* & -m_1a_{15}^*\\ n_1a_{15}b_{12} & -n_1a_{15}b_9^* & n_1a_{12}b_9^*\end{pmatrix},\tag{19}$$

where the parameters l_1, m_1 and n_1 are chosen as

$$\begin{aligned}l_1^2&=\frac{1}{(|a_{12}|^2+|a_{15}|^2)(|a_{12}|^2|b_9|^2+|a_{15}|^2|b_9|^2+|a_{15}|^2|b_{12}|^2)},\\ m_1^2&=\frac{1}{|a_{12}|^2+|a_{15}|^2},\\ n_1^2&=\frac{1}{|a_{12}|^2|b_9|^2+|a_{15}|^2|b_9|^2+|a_{15}|^2|b_{12}|^2}.\end{aligned}\tag{20}$$

we find the set $\{K_9,K_{12},K_{14},K_{15}\}$ can be reduced to $\{K_9,K_{12},K_{14}\}$.

Besides, we take the unitary matrix

$$U_2=\begin{pmatrix}-l_2(|a_{10}|^2+|a_{14}|^2)b_8 & -l_2|a_{14}|^2b_{10}^* & l_2a_{10}a_{14}^*b_{10}^*\\ 0 & -m_2a_{10}^* & -m_2a_{14}^*\\ n_2a_{14}b_{10} & -n_2a_{14}b_8^* & n_2a_{10}b_8^*\end{pmatrix},\tag{21}$$

where the parameters l_2, m_2 and n_2 are chosen as

$$\begin{aligned}l_2^2&=\frac{1}{(|a_{10}|^2+|a_{14}|^2)(|a_{10}|^2|b_8|^2+|a_{14}|^2|b_8|^2+|a_{14}|^2|b_{10}|^2)}\\ m_2^2&=\frac{1}{|a_{10}|^2+|a_{14}|^2},\\ n_2^2&=\frac{1}{|a_{10}|^2|b_8|^2+|a_{14}|^2|b_8|^2+|a_{14}|^2|b_{10}|^2}.\end{aligned}\tag{22}$$

The combination of K_8, K_{10} and K_{14} has the same form of K_{13}. So the set $\{K_8,K_{10},K_{13},K_{14}\}$ can be reduced to $\{K_8,K_{10},K_{13}\}$. In other words, any single-qutrit strictly incoherent operation admits a decomposition with at most 13 strictly incoherent Kraus operators. □

In 3-dimension Hilbert space, an arbitrary quantum state is expressed as

$$\rho=\frac{1}{3}I+\frac{1}{2}\sum_{i=1}^{8}t_i\lambda_i,\tag{23}$$

where $\vec{t} = \{t_1, t_2, ..., t_8\}$ is the 8 dimensional Bloch vector, λ_i is a generator of SU(3), where the length of $\vec{t}$ should be less than or equal to $\frac{2}{\sqrt{3}}$ [17]. In order to visualize the state transformation via single qutrit SIO and IO, we consider two-dimensional sections of $\sum_3(i, j)$ [18] which are constructed as $\sum_3(i, j) = \{\mathbf{t} \in B(\mathbb{R}^8) : \mathbf{t} = \{0, ..., 0, t_i, 0, ..., t_j, ..., 0\}\}$. For a given Bloch vector of $\mathbf{t} = \{0, ..., 0, t_i, 0, ..., t_j, ..., 0\}$, we can find the achievable region for the final state $\mathbf{m} = \{0, ..., 0, m_i, 0, ..., m_j, ..., 0\}$ via single qutrit SIO and IO. In the two-dimensional sections, we can find the limited conditions, the proof can be found in Appendix A.

1: In the $\{m_1, ..., m_6\} - \{m_7, m_8\}$ plane, the following inequalities should be satisfied:

$$\begin{aligned} m_i^2 &\le t_i^2, \quad \{i = 1, ..., 6\} \\ m_7 &\in [\frac{1-\sqrt{3}}{3}, \frac{2}{\sqrt{3}}], \\ m_8 &\in [-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}]. \end{aligned} \tag{24}$$

2: In the $\{m_7\} - \{m_8\}$ plane, the following equality should be satisfied:

$$-\sqrt{3}m_7 + m_8 - \frac{2\sqrt{3}}{3} = 0. \tag{25}$$

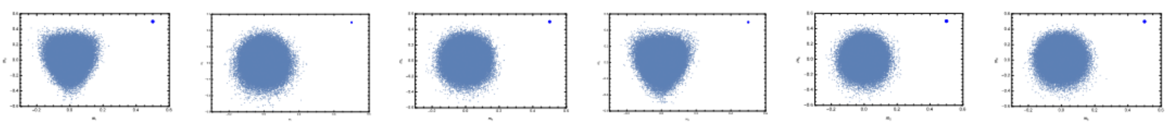
3: In the $\{m_1\} - \{m_4\}$, $\{m_2\} - \{m_5\}$ and $\{m_3\} - \{m_6\}$ planes, the following inequality should be satisfied respectively:

$$\begin{aligned} m_1^2 + m_4^2 &\le t_1^2 + t_4^2, \\ m_2^2 + m_5^2 &\le t_2^2 + t_5^2, \\ m_3^2 + m_6^2 &\le t_3^2 + t_6^2. \end{aligned} \tag{26}$$

4: In the other planes, we find the following inequality:

$$(|m_i| + |m_j|)^2 \le (|t_i| + |t_j|)^2. \tag{27}$$

In the following Fig.1–Fig.4, we show the projection of the achievable region into the $\{m_1, ..., m_6\} - \{m_7, m_8\}, \{m_7\} - \{m_8\}, \{m_1\} - \{m_4\}, \{m_2\} - \{m_5\}, \{m_3\} - \{m_6\}$ and other planes for the corresponding different initial states. These images, which are numerically simulated the set of final states, coincide with our conclusion.

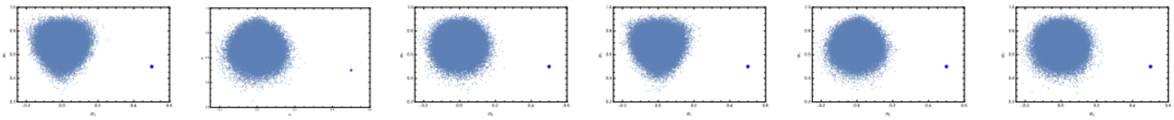


(a) $m_1 - m_8$ plane (b) $m_2 - m_8$ plane (c) $m_3 - m_8$ plane (d) $m_4 - m_8$ plane (e) $m_5 - m_8$ plane (f) $m_6 - m_8$ plane

FIG. 1: The achievable region for single qutrit SIO and IO in condition 1. The blue colored area shows the projection of the achievable region in the $m_i - m_8$ $(i = 1, ..., 6)$ plane. We have set $t_i = t_8 = 0.5$ in the initial state.

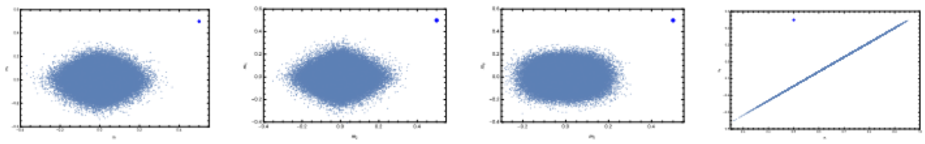
III. DISCUSSION AND CONCLUSION

In this paper, we have discussed how to reduce the number of incoherent Kraus operators. Furthermore, we have shown that the number of incoherent Kraus operators for a single qubit can be reduce from 5 to 4. For qutrit system, we have found that any incoherent operation or strictly incoherent operation admits decomposition with at most 32 or 13 Kraus operator respectively. We have also investigated the achievable region for a fixed state via single qutrit SIO and IO. An open question is that whether the upper bound can be further reduced to a much tight level. Besides, it is still yet to be solved to compute the optimal number



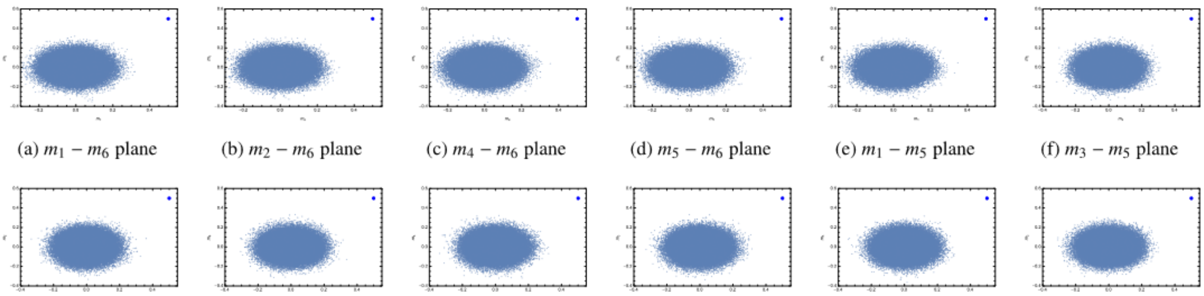
(a) $m_1 - m_7$ plane (b) $m_2 - m_7$ plane (c) $m_3 - m_7$ plane (d) $m_4 - m_7$ plane (e) $m_5 - m_7$ plane (f) $m_6 - m_7$ plane

FIG. 2: The achievable region for single qutrit SIO and IO in condition 2. The blue colored area shows the projection of the achievable region in the $m_i - m_7$ $(i = 1, ..., 6)$ plane. In the initial state we set $t_i = 0.5,\ t_7 = 0.5$ respectively.



(a) $m_1 - m_4$ plane (b) $m_2 - m_5$ plane (c) $m_3 - m_6$ plane (d) $m_7 - m_8$ plane

FIG. 3: The achievable region for single qutrit SIO and IO in condition 3. The blue colored area shows the projection of the achievable region in the $m_1 - m_4$, $m_2 - m_5$, $m_3 - m_6$ and $m_7 - m_8$ plane, where in the initial state we set $t_i = 0.5, t_j = 0.5$ respectively.



(a) $m_1 - m_6$ plane (b) $m_2 - m_6$ plane (c) $m_4 - m_6$ plane (d) $m_5 - m_6$ plane (e) $m_1 - m_5$ plane (f) $m_3 - m_5$ plane

(g) $m_4 - m_5$ plane (h) $m_2 - m_4$ plane (i) $m_3 - m_4$ plane (j) $m_1 - m_3$ plane (k) $m_2 - m_3$ plane (l) $m_1 - m_2$ plane

FIG. 4: The achievable region for single qutrit SIO and IO in condition 4. The blue colored area shows the projection of the achievable region in the planes that are not mentioned above, where in the initial state we set $t_i = 0.5, t_j = 0.5$ respectively.

of incoherent Kraus operator when $d \geq 4$. We suspect that the number of d dimensional incoherent Kraus operators is related to the number of d-1 dimensional incoherent Kraus operators. In addition, the form of incoherent Kraus operator in d dimension is also related to the d-1 dimensional incoherent Kraus operators. More importantly, according to the relationship between the superposition-free operation and incoherent operation, we can obtain the structure of the resource theory of superposition[19].

Acknowledgments This work is supported by NSFC (11775306, 11701568, 11675113), the Fundamental Research Funds for the Central Universities (18CX02035A, 18CX02023A, 19CX02050A), Beijing Municipal Commission of Education under Grant No. KZ201810028042, and Beijing Natural Science Foundation (Z190005).

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APPENDIX A: THE COMPUTATIONS OF THE CONDITIONS FOR THE STATE TRANSFORMATION

In this appendix, we will introduce the proof of the conditions for the state transformation. For the 2-dimensional sections $\sum_3(i,j)$, where $i \in \{1,2,...,6\}$ and $j \in \{7,8\}$, the coefficients m_i and m_j of the final state via a strictly incoherent channel can be derived as follows:

$$
\begin{aligned}
m_1 &= t_1(a_1Re[b_1] + a_3Re[b_3] + a_7Re[b_7] + a_9Re[b_9]),\\
m_2 &= t_2(a_1Re[c_1] + a_5Re[c_5]),\\
m_3 &= t_3(\frac{1}{2}(b_1c_1^\dagger + b_1^\dagger c_1 + b_2c_2^\dagger + b_2^\dagger c_2)),\\
m_4 &= t_4(a_1Re[b_1] - a_3Re[b_3] + a_7Re[b_7] - a_9Re[b_9]),\\
m_5 &= t_5(a_1Re[c_1] - a_5Re[c_5]),\\
m_6 &= t_6(\frac{1}{2}(b_1c_1^\dagger + b_1^\dagger c_1 - b_2c_2^\dagger + b_2^\dagger c_2)),\\
m_7 &= \frac{1}{3}(1-|c_1|^2-|c_3|^2) + (1-|a_5|^2-|a_6|^2-|a_{11}|^2-|a_{12}|^2)(\frac{1}{3}+\frac{t_7}{2}) + (1-|b_2|^2-|b_4|^2-|b_8|^2-|b_{10}|^2)(\frac{1}{3}-\frac{t_7}{2}),\\
m_8 &= \frac{1}{\sqrt{3}} - \sqrt{3}((|c_1|^2+|c_3|^2)(\frac{1}{3}-\frac{t_8}{\sqrt{3}}) + (|a_5|^2+|a_6|^2+|a_{11}|^2+|a_{12}|^2+|b_2|^2+|b_4|^2+|b_8|^2+|b_{10}|^2)(\frac{1}{3}+\frac{t_8}{2\sqrt{3}})).
\end{aligned}
\tag{28}
$$

Due to the completeness of Kraus operators, $\sum_i K_i^\dagger K_i = \mathcal{I}$, we obtain $\sum_{i=1}^{13} a_i^2 = \sum_{j=1}^{12} |b_j|^2 = \sum_{k=1}^{6} |c_k|^2 = 1$, where a_i can be chosen as real numbers, b_j and c_k as complex numbers. It is easy to obtain the conditions 1 by using the length of the Bloch vector, the normalization of the parameters and the Cauchy – Schwarz inequality.

In the $m_7 - m_8$ plane, we can obtain the explicit relation. The final form of $m_7,\ m_8$ are as follows:

$$m_7 = (|c_2|^2 + |c_4|^2 + |c_5|^2 + |c_6|^2)(\frac{1}{3} - \frac{t_8}{\sqrt{3}}) + (|b_1|^2 + |b_3|^2 + |b_5|^2 + |b_6|^2 + |b_7|^2 + |b_9|^2 + |b_{11}|^2 + |b_{12}|^2)(\frac{1}{3} + \frac{1}{2}(-t_7 + \frac{t_8}{\sqrt{3}}))$$
$$+ (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_7^2 + a_8^2 + a_9^2 + a_{10}^2 + a_{13}^2)(\frac{1}{3} + \frac{1}{2}(t_7 + \frac{t_8}{\sqrt{3}})),$$
$$m_8 = \frac{1}{\sqrt{3}} - \sqrt{3}(|c_1|^2 + |c_3|^2)(\frac{1}{3} - \frac{t_8}{\sqrt{3}}) + (|b_2|^2 + |b_4|^2 + |b_8|^2 + |b_{10}|^2)(\frac{1}{3} + \frac{1}{2}(-t_7 + \frac{t_8}{\sqrt{3}})) + (a_5^2 + a_6^2 + a_{11}^2 + a_{12}^2)(\frac{1}{3} + \frac{1}{2}(t_7 + \frac{t_8}{\sqrt{3}})). \tag{29}$$

It is obvious that $-\sqrt{3}m_7 + m_8 - \frac{2\sqrt{3}}{3} = 0$, using the conditions of $\sum_{i=1}^{13} a_i^2 = \sum_{j=1}^{12} |b_j|^2 = \sum_{k=1}^{6} |c_k|^2 = 1$.

In $m_1 - m_4$ plane, the final state form of $m_1,\ m_4$ are as follows:

$$m_1 = \frac{1}{2}((a_1 b_1^\dagger + a_3 b_3^\dagger + a_7 b_7^\dagger + a_9 b_9^\dagger)(t_1 - it_4) + ((a_1 b_1 + a_3 b_3 + a_7 b_7 + a_9 b_9)(t_1 + it_4)),$$
$$m_4 = \frac{-i}{2}((-a_1 b_1^\dagger + a_3 b_3^\dagger - a_7 b_7^\dagger + a_9 b_9^\dagger)(t_1 - it_4) + ((a_1 b_1 - a_3 b_3 + a_7 b_7 - a_9 b_9)(t_1 + it_4)). \tag{30}$$

Then

$$m_1^2 + m_4^2 = (a_1^2|b_1|^2 + a_3^2|b_3|^2 + a_7^2|b_7|^2 + a_9^2|b_9|^2 + a_1 a_7 b_1 b_7^\dagger + a_3 a_9 b_3 b_9^\dagger + a_1 a_7 b_1^\dagger b_7 + a_3 a_9 b_3^\dagger b_9)(t_1^2 + t_4^2)$$
$$+ (a_1 a_3 b_1^\dagger b_3^\dagger + a_3 a_7 b_3^\dagger b_7^\dagger + a_1 a_9 b_1^\dagger b_9^\dagger + a_7 a_9 b_7^\dagger b_9^\dagger + a_1 a_3 b_1 b_3 + a_3 a_7 b_3 b_7 + a_1 a_9 b_1 b_9 + a_7 a_9 b_7 b_9)(t_1^2 - t_4^2) \tag{31}$$

They are classified into 3 types of conditions:

(1) When $t_1 = t_4$, we get $m_1^2 + m_4^2 \le t_1^2 + t_4^2$ directly by using the Cauchy - Schwarz inequality.

(2) When $(a_1 a_3 b_1^\dagger b_3^\dagger + a_3 a_7 b_3^\dagger b_7^\dagger + a_1 a_9 b_1^\dagger b_9^\dagger + a_7 a_9 b_7^\dagger b_9^\dagger + a_1 a_3 b_1 b_3 + a_3 a_7 b_3 b_7 + a_1 a_9 b_1 b_9 + a_7 a_9 b_7 b_9)(t_1^2 - t_4^2) \le 0$, $m_1^2 + m_4^2 \le t_1^2 + t_4^2$ holds.

(3) When $(a_1 a_3 b_1^\dagger b_3^\dagger + a_3 a_7 b_3^\dagger b_7^\dagger + a_1 a_9 b_1^\dagger b_9^\dagger + a_7 a_9 b_7^\dagger b_9^\dagger + a_1 a_3 b_1 b_3 + a_3 a_7 b_3 b_7 + a_1 a_9 b_1 b_9 + a_7 a_9 b_7 b_9)(t_1^2 - t_4^2) \ge 0$, by setting $t_1 > t_4$, we find

$$m_1^2 + m_4^2 = (a_1^2|b_1|^2 + a_3^2|b_3|^2 + a_7^2|b_7|^2 + a_9^2|b_9|^2 + a_1 a_7 b_1 b_7^\dagger + a_3 a_9 b_3 b_9^\dagger + a_1 a_7 b_1^\dagger b_7 + a_3 a_9 b_3^\dagger b_9)(t_1^2 + t_4^2)$$
$$+ (a_1 a_3 b_1^\dagger b_3^\dagger + a_3 a_7 b_3^\dagger b_7^\dagger + a_1 a_9 b_1^\dagger b_9^\dagger + a_7 a_9 b_7^\dagger b_9^\dagger + a_1 a_3 b_1 b_3 + a_3 a_7 b_3 b_7 + a_1 a_9 b_1 b_9 + a_7 a_9 b_7 b_9)(t_1^2 - t_4^2)$$
$$\le (a_1^2|b_1|^2 + a_3^2|b_3|^2 + a_7^2|b_7|^2 + a_9^2|b_9|^2 + 2a_1 a_7|b_1||b_7| + a_3 a_9|b_3||b_9|)(t_1^2 + t_4^2)$$
$$+ 2(a_1 a_3|b_1||b_3| + a_3 a_7|b_3||b_7| + a_1 a_9|b_1||b_9| + a_7 a_9|b_7||b_9|)(t_1^2 - t_4^2)$$
$$\le (a_1|b_1| + a_3|b_3| + a_7|b_7| + a_9|b_9|)^2(t_1^2 + t_4^2)$$
$$\le (t_1^2 + t_4^2) \tag{32}$$

Together with the three conditions, we show that the inequality $m_1^2 + m_4^2 \le t_1^2 + t_4^2$ holds. Similar conditions can be derived for $m_2 - m_5$ plane and $m_3 - m_6$ plane.

Without loss of generality, we take $m_1 - m_2$ plane for the other planes as an example. An IO maps a density matrix $\{t_1, t_2, 0, 0, 0, 0, 0, 0\}$ to another density matrix $\{m_1, m_2, 0, 0, 0, 0, 0, 0\}$, we have

$$m_1 = (a_1 Re[b_1] + a_3 Re[b_3] + a_7 Re[b_7] + a_9 Re[b_9])t_1 + ((a_2 Re[c_2] + a_4 Re[c_4])t_1,$$
$$m_2 = (a_2 Re[b_2] + a_6 Re[b_6] + a_8 Re[b_8] + a_{12} Re[b_{12}])t_1 + ((a_1 Re[c_1] + a_5 Re[c_5])t_2. \tag{33}$$

Then, we find the relation between initial vector and the final vector as

$$(|m_1| + |m_2|)^2 = (|(a_1 Re[b_1] + a_3 Re[b_3] + a_7 Re[b_7] + a_9 Re[b_9] + a_2 Re[b_2] + a_6 Re[b_6] + a_8 Re[b_8] + a_{12} Re[b_{12}])t_1|$$
$$+ |(a_1 Re[c_1] + a_2 Re[c_2] + a_4 Re[c_4] + a_5 Re[c_5])t_2|)^2$$
$$\le (|t_1 + t_2|)^2. \tag{34}$$

Other 2-dimensional Bloch vectors have the similar relationship.