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Two-Copy Quantum Teleportation Based on GHZ Measurement

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Abstract

We investigate quantum teleportation in the two-copy setting based on GHZ measurement and propose the detailed protocol. The output state after the teleportation is analyzed and the protocol is proved to be trace preserving. The general expression of the optimal teleportation fidelity is derived. The optimal teleportation fidelity is shown to be a linear function of two-copy fully entangled fraction, which is invariant under local unitary transformations. At last, we show two-copy teleportation based on GHZ measurement can be better than one copy teleportation by an explicit example, which is amenable to demonstration in experiments. Our study is significant for improving the fidelity of teleportation for some limited resource which cannot be significantly distilled. Moreover, it can inspire us to find many other more efficient protocols for teleportation.

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I. INTRODUCTION

Quantum teleportation plays an important role in quantum information processing [1], which gives ways to transmit an unknown quantum state from a sender traditionally named "Alice" to a receiver "Bob" who are spatially separated by using classical communication and quantum resources [2–10]. In [11, 12], the authors considered the one copy optimal teleportation: Alice and Bob previously share a pair of particles in an arbitrary mixed entangled state χ. In order to teleport an unknown state to Bob, Alice performs a joint Bell measurement on her particles and tells her results to Bob by classical communication. Bob tries his best to choose a particular unitary transformation to get the maximal transmission fidelity. The transmission fidelity of such optimal teleportation is given by the fully entangled fraction [13, 14] of the quantum resource. It shows that when the resource χ is a maximally entangled pure state, the corresponding optimal fidelity is equal to 1. However, Alice and Bob usually share a mixed entangled state due to decoherence, and the fidelity is less than 1. In this sense, one goal of quantum teleportation or quantum information theory is to find the optimal ways to make use of noisy resources or establish better entanglement [15].

For quantum teleportation, to gain better entanglement by distillation first[16], which aims to converting mixed entanglement to singlets by using many copies of the entangled resources, is one available way to increase quantum teleportation fidelity. The distillation of pure states is often referred to as entanglement concentration [17]. For mixed states, since the distillation protocol presented in [16], fruitful results have been obtained [18–21]. It has been found that some entangled quantum mixed states, called bound entangled states, are not distillable [22]. The distillation procedure is complicated and may have to be repeated for infinitely many times to bring out a singlet. Moreover, in each round the desired results are usually obtained probabilistically, sometimes with an extremely low possibility to get an expected measurement outcome.

Another method to increase quantum teleportation fidelity is to improve the traditional teleportation protocol. In Ref. [23], we have proposed a two-copy quantum teleportation protocol based on Bell measurement. The corresponding optimal teleportation fidelity is proved to be better than that in traditional one copy teleportation. Different teleportation protocols give rises to different fidelities. Therefore it is interesting and meaningful to develop novel teleportation protocols.

In this paper, we design a teleportation protocol in two-copy setting. This teleportation is based on three-particle GHZ measurement. We analyze the output state in this protocol for arbitrary input state and show this protocol is trace preserving. After that, the corresponding optimal teleportation fidelity is calculated, which is shown to be a linear function of two-copy fully entangled fraction we defined in this corresponding protocol. Finally, this two-copy teleportation protocol is demonstrated to be better than one copy traditional teleportation protocol in some cases, which is amenable to demonstration in experiments. Our work could simulate further study on efficient teleportation protocols.

II. TWO-COPY TELEPORTATION PROTOCOL BASED ON GHZ MEASUREMENT

Suppose H is the Hilbert space. The picture of the protocol is like Figure 1: Alice and Bob share two pairs of particles, which are both in the same mixed entangled state χ. Alice wants to transmit an unknown state $|\phi\rangle$ to Bob. Firstly, Alice performs a joint rotation W on particle 1 and particle 3, then she conducts a joint GHZ measurement on her three particles. After that, her measurement results are delivered to Bob by classical means. Secondly, according to the measurement results, Bob chooses corresponding unitary transformations $\{T\} : H \otimes H \to H \otimes H$ to meet the maximum teleportation fidelity. Since Bob's two particles are in $H \otimes H$ space, while the unknown state $|\phi\rangle$ Alice wants to deliver is in H space, Bob also needs to take partial trace on his particles, then we can realize the two-copy optimal teleportation based on GHZ measurement.

Let $\{|j\rangle, j = 0, ..., n-1, n < \infty\}$ be an orthogonal normalized basis of an n-dimensional Hilbert space H. A is an arbitrary linear operator: $H \to H$, which can be defined by the corresponding $n \times n$ matrix as $A|j\rangle = \sum_{k=0}^{n-1} a_{jk}|k\rangle, a_{ij} \in \mathbb{C}$. Consider that W is a unitary matrix: $H \otimes H \to H \otimes H$ defined as $W|jk\rangle = \sum_{j',k'=0}^{n-1} W_{j'k'}^{jk}|j'k'\rangle$. One reason why we introduce the unitary matrix W into this problem is to ensure that the two-copy fully entangled fraction we defined in this protocol is an invariant under local unitary transformation, as it can be seen from the final result.

We also need to introduce a set of unitary matrices U_{st} in H as follows: $U_{st} = h^t g^s$, in which both h and g are $n \times n$ matrices: $h|j\rangle = |(j+1)/ \mod n\rangle$ and $g|j\rangle = w^j|j\rangle$, with $w = \exp\{-2i\pi/n\}$. Then we have the following relations [24]: $tr(U_{st}^\dagger U_{s't'}) = n\delta_{tt'}\delta_{ss'}$, $U_{st}U_{st}^\dagger =$

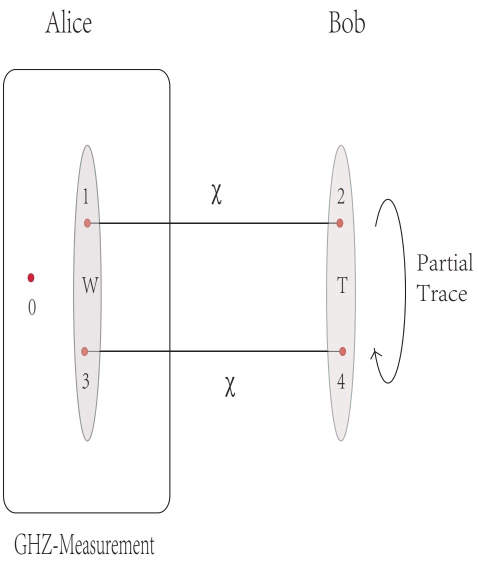

FIG. 1. Scheme of two-copy teleportation protocol based on GHZ measurement. Assume Alice and Bob share two copies of entangled resources $\chi_{12} \otimes \chi_{34}$, with particles 1 and 3 in Alice's side, while particles 2 and 4 in Bob's side. To teleport input state ρ_{in} in particle 0, firstly, Alice performs a joint local unitary operation W on particles 1 and 3 to make these particles correlated. Then she makes a joint GHZ measurement on particles 0, 1 and 3 and informs Bob the measurement results by classical communication. According to these measurement results, Bob chooses corresponding unitary transformations $\{T\}$ on particles 2 and 4 to restore the input state ρ_{in} on particle 2.

$I_{n\times n}$. The generalized Bell states [11] can be given as: $|\Phi_{st}\rangle = (1 \otimes U_{st})|\Phi\rangle$, where the usual maximal entangled pure state $|\Phi_{00}\rangle = |\Phi\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |jj\rangle$. The n^2 generalized Bell states $\{|\Phi_{st}\rangle\}$ form a complete orthogonal normalized basis of $H \otimes H$ space.

In the new protocol, we need to define a series of complete orthogonal normalized generalized GHZ-states $\{|\Phi^s_{rm}\rangle\}$ in $H \otimes H \otimes H$:

$$|\Phi^s_{rm}\rangle = (I \otimes U^s_{rm})|\Phi^0_{00}\rangle = \frac{1}{\sqrt{n}} \sum_{j,k,l=0}^{n-1} (U^s_{rm})_{jkl} |jkl\rangle, \tag{1}$$

where $|\Phi^0_{00}\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |jjj\rangle$, to realize Alice's joint GHZ measurement of her three particles. The corresponding unitary matrices $\{U^s_{rm}\}$ are given by: $U^s_{rm} = h^r g^s \otimes h^m$, thus we have the relation: $tr(U^s_{rm} U^{s'\dagger}_{r'm'}) = n^2 \delta_{r,r'} \delta_{m,m'} \delta_{s,s'}$, $\ U^s_{rm} U^{s\dagger}_{rm} = I_{n^2\times n^2}$. Then the explicit expressions of the generalized GHZ-states $\{|\Phi^s_{rm}\rangle\}$ read:

$$|\Phi^s_{rm}\rangle = \frac{1}{\sqrt{n}} \sum_j w^{js} |j, j+r, j+m\rangle. \tag{2}$$

Associated with the generalized GHZ-states $\{|\Phi^s_{rm}\rangle\}$, one can introduce a set of linear operators $\{\widetilde{U}^{s\dagger}_{rm}\}$:

$$\widetilde{U}^{s\dagger}_{rm}|j\rangle = \sum_{kl}(U^s_{rm})^*_{jkl}|kl\rangle = \sum_{j',k,l}((U^s_{rm})^*_{j'kl}|k,l\rangle\langle j'|)|j\rangle, \tag{3}$$

which map $H \to H \otimes H$. It is easy to show that the correspondence between $\{|\Phi^s_{rm}\rangle\}$ and $\{\widetilde{U}^{s\dagger}_{rm}\}$ is indeed one to one . Moreover one can give the explicit expressions of the operators :

$$\widetilde{U}^{s\dagger}_{rm} = \sum_j w^{-js}|j+r, j+m\rangle\langle j| = (h^r g^{s*} \otimes h^m)E, \tag{4}$$

where $E = \sum_j |jj\rangle\langle j|$. These unitary operators $\{\widetilde{U}^{s\dagger}_{rm}\}$ satisfy that: $\widetilde{U}^s_{rm}\widetilde{U}^{s\dagger}_{rm} = I_{n\times n}$, which maps $H \to H$; $\widetilde{U}^{s\dagger}_{rm}\widetilde{U}^s_{rm} = (I_{n\times n} \otimes h^{m-r})(\sum_j |jj\rangle\langle jj|)(I_{n\times n} \otimes h^{m-r})^\dagger$, which maps $H \otimes H \to H \otimes H$. Any operator A in H, satisfies the following relation:

$$\sum_{r,m,s} \widetilde{U}^{s\dagger}_{rm}\, A\, \widetilde{U}^s_{rm} = ntr\{A\}\, \mathrm{I_{n\times n}} \otimes \mathrm{I_{n\times n}}. \tag{5}$$

Throughout this paper we adopt the standard notations: for any matrix $A \in \mathrm{End}(H)$, A_j is an embedding operator in the tensor space $H \otimes H \otimes \cdots \otimes H$, which acts as A on the j-th space and identity on the other factor spaces; for any matrix $U \in \mathrm{End}(H \otimes H)$, U_{jk} is an embedding operator in the tensor space, which acts as U on the j-th and k-th spaces and identity on the factor spaces. The similar procedure can be generalized to any matrices by embedding them in the tensor space. After some tedious calculation, we have:

Theorem 1. *If Alice and Bob share two pairs of particles which are both in the same arbitrary mixed entangled state χ, the quantum channel of the two-copy teleportation protocol with the choices of W and $\{T^s_{rm}\}$ is:*

$$\Lambda^{\{T^s_{rm},W\}}_{(\chi^{\otimes 2})}(\rho_{in}) = \frac{1}{n^3}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s'_1,t'_1}\sum_{s'_2,t'_2}\langle\Phi_{s_1t_1}|\chi|\Phi_{s'_1t'_1}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s'_2t'_2}\rangle$$
$$\times\, \mathrm{tr}_4\{\sum_{r,m,s}(T^s_{rm})_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}^{s\dagger}_{rm})_2(\rho_{in})_2$$
$$(\widetilde{U}^s_{rm})_{24}(W^\dagger)_{24}(U_{s'_1t'_1})^\dagger_2(U_{s'_2t'_2})^\dagger_4(T^s_{rm})^\dagger_{24}\}.$$

where ρ_{in} is an arbitrary input state.

Proof. Suppose the unknown initial state that Alice wants to teleport is $|\phi\rangle = \sum_\nu \alpha_\nu|\nu\rangle$. Next we divide the proof into two parts: pure state as input state and mixed state as input state.

1). Entangled pure state as resource

Let $\chi^{(2)}$ be a two-copy of an arbitrary entangled pure state: $\chi^{(2)} = |\Psi\rangle\langle\Psi|$, and $|\Psi\rangle$ can be written as: $|\Psi\rangle = \sum_{j,k=0}^{n-1}\sum_{l,m=0}^{n-1} a_{jk}|jk\rangle \otimes a_{lm}|lm\rangle$, $where \sum_{j,k=0}^{n-1}|a_{jk}|^2 = 1$. Alice acts unitary operator W on her two resource particles: $|\Psi'\rangle = W_{13}|\Psi\rangle_{1234} = \sum_{j,k=0}^{n-1}\sum_{l,m=0}^{n-1}\sum_{j',l'=0}^{n-1} a_{jk}a_{lm}W_{jl}^{j'l'}|j'kl'm\rangle$. Now the initial states that Alice and Bob share together is: $|\phi\rangle \otimes |\Psi'\rangle = \sum_{j,k=0}^{n-1}\sum_{l,m=0}^{n-1}\sum_{j',l',\nu=0}^{n-1} a_{jk}a_{lm}W_{jl}^{j'l'}\alpha_\nu|\nu j'kl'm\rangle$.

After Alice's joint GHZ measurement based on $|\Phi_{rm}^s\rangle$, we get: $\langle\Phi_{rm}^s|(|\phi\rangle \otimes |\Psi'\rangle) = \frac{1}{\sqrt{n}}A_2A_4W_{24}(\widetilde{U}_{rm}^{s\dagger})_2|\phi\rangle_2$. Bob receives Alice's measurement results through a classical channel, and according to the results, he acts unitary operators $T_{rm}^s \in \{T\}$ on his two resource particles to meet the maximum fidelity. Then the resulting state in Bob becomes $\frac{1}{\sqrt{n}}(T_{rm}^s)_{24}A_2A_4W_{24}(\widetilde{U}_{rm}^{s\dagger})_2|\phi\rangle_2$. After Bob taking partial trace on particle 4, the resulting quantum channel and the output state can be expressed as:

$$\Lambda_{(\chi^{\otimes 2})}^{\{T_{rm}^s,W\}}(\rho_{in}) = \frac{1}{n}\,\mathrm{tr}_4\{\sum_{r,m,s}(T_{rm}^s)_{24}(A)_2(A)_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2(\rho_{in})_2(\widetilde{U}_{rm}^{s})_{24}(W^\dagger)_{24}(A)_2^\dagger(A)_4^\dagger(T_{rm}^s)_{24}^\dagger\}.$$

2). Entangled mixed state as resource

Let $\chi^{(2)}$ be a two-copy of an arbitrary entangled mixed state $\chi^{(2)} = \sum_{\alpha,\beta}P_\alpha P_\beta|\Psi_{\alpha\beta}\rangle\langle\Psi_{\alpha\beta}|$, $|\Psi_{\alpha\beta}\rangle = \sum_{j,k=0}^{n-1}\sum_{l,m=0}^{n-1} a_{jk}^{(\alpha)}|jk\rangle \otimes a_{lm}^{(\beta)}|lm\rangle$, where $0 \le P_\alpha \le 1$, and $\sum_\alpha P_\alpha = 1$. Following the same procedure as that of the pure state case, we have

$$\Lambda_{(\chi^{\otimes 2})}^{\{T_{rm}^s,W\}}(\rho_{in}) = \frac{1}{n}\sum_{\alpha,\beta}P_\alpha P_\beta\,\mathrm{tr}_4\{\sum_{r,m,s}(T_{rm}^s)_{24}(A^{(\alpha)})_2(A^{(\beta)})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2(\rho_{in})_2 (\widetilde{U}_{rm}^{s})_{24}(W^\dagger)_{24}(A^{(\alpha)})_2^\dagger(A^{(\beta)})_4^\dagger(T_{rm}^s)_{24}^\dagger\}.$$

Since each matrix $A^{(\alpha)}$ can be decomposed in the basis of U_{st}: $A^{(\alpha)} = \sum_{s,t}a_{st}^{(\alpha)}U_{st}$, by using the relation [11]: $n\sum_\alpha p_\alpha a_{st}^{(\alpha)}a_{s't'}^{(\alpha)*} = \langle\Phi_{st}|\chi|\Phi_{s't'}\rangle$, we find:

$$\Lambda_{(\chi^{\otimes 2})}^{\{T_{rm}^s,W\}}(\rho_{in}) = \frac{1}{n^3}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle \times \mathrm{tr}_4\{\sum_{r,m,s}(T_{rm}^s)_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2(\rho_{in})_2 (\widetilde{U}_{rm}^{s})_{24}(W^\dagger)_{24}(U_{s_1't_1'})_2^\dagger(U_{s_2't_2'})_4^\dagger(T_{rm}^s)_{24}^\dagger\}.$$

Despite that we have supposed the input state is pure, $\rho_{in} = |\phi\rangle\langle\phi|$, in the above derivation, the results can be generalized directly to arbitrary mixed state because the channel is linear.

□

Remark The quantum channel of two-copy teleportation protocol based on GHZ measurement is trace preserving. In fact,

$$
\begin{aligned}
& \mathrm{tr}[\Lambda_{(\chi^{\otimes 2})}^{(\{T_{rm}^{s},W\})}(\rho_{in})] \\
= & \frac{1}{n^3}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle \\
\times & \ \mathrm{tr}_2\{\mathrm{tr}_4\{\sum_{r,m,s}(T_{rm}^{s})_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2(\rho_{in})_2 \\
& (\widetilde{U}_{rm}^{s})_{24}(W^{\dagger})_{24}(U_{s_1't_1'})_2^{\dagger}(U_{s_2't_2'})_4^{\dagger}(T_{rm}^{s})_{24}^{\dagger}\}\} \\
= & \frac{1}{n^3}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle \\
& \times\{\sum_{r,m,s}\mathrm{tr}_{24}\left\{(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2(\rho_{in})_2(\widetilde{U}_{rm}^{s})_{24}(W^{\dagger})_{24}(U_{s_1't_1'})_2^{\dagger}(U_{s_2't_2'})_4^{\dagger}\right\} \\
= & \frac{1}{n^2}\,\mathrm{tr}\{\rho_{in}\}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle\,\mathrm{tr}\{U_{s_1t_1}U_{s_1't_1'}^{\dagger}\}\,\mathrm{tr}\{U_{s_2t_2}U_{s_2't_2'}^{\dagger}\} \\
= & \ \mathrm{tr}(\chi)\,\mathrm{tr}(\chi)=1.
\end{aligned} \tag{6}
$$

In the third equality, we have used the identity (5).

Based on Theorem 1, we are ready to obtain the optimal fidelity of the above teleportation protocol. At first, we introduce two-copy fully entangled fraction based on GHZ measurement as

$$
\begin{aligned}
F(\chi^{\otimes 2}) \ = \ & \tfrac{1}{n}\max_{\{W,T_{rm}^{s}\}}\textstyle\sum_{r,m,s,j}\{\langle\Phi|_{12}\langle\Phi|_{34}[W_{24}(h^r g^{s*})_2(h^m)_4(M_j)_{24}(T_{rm}^{s})_{24}](\chi_{12}\chi_{34}) \\
& [(T_{rm}^{s\dagger})_{24}(M_j^{\dagger})_{24}(h^r g^{s*})_2^{\dagger}(h^m)_4^{\dagger}W_{24}^{\dagger}]|\Phi\rangle_{12}|\Phi\rangle_{34}\}
\end{aligned} \tag{7}
$$

where $M_j=\sum_k|kk\rangle\langle kj|$.

Theorem 2. *Two-copy fully entangled fraction $F(\chi^{\otimes 2})$ is invariant under local unitary transformations.*

Proof. Suppose unitary operator $\Omega_a\in\{U_{n\times n}\otimes U_{n\times n}\}$ acts on Alice's resource particles and

$\Omega_b \in \{U_{n\times n} \otimes U_{n\times n}\}$ acts on Bob's particles, then we have

$$
\begin{aligned}
& F((\Omega_a)_{13} \otimes (\Omega_b)_{24}\chi^{\otimes 2}(\Omega_a)^\dagger_{13} \otimes (\Omega_b)^\dagger_{24}) \\
= & \frac{1}{n} \max_{\{W, T^s_{rm}\}} \sum_{r,m,s,j} \{\langle\Phi|_{12}\langle\Phi|_{34}[W_{24}(h^r g^{s*})_2(h^m)_4(M_j)_{24}(T^s_{rm})_{24}]((\Omega_a)_{13}(\Omega_b)_{24}\chi_{12}\chi_{34} \\
& (\Omega_a)^\dagger_{13}(\Omega_b)^\dagger_{24})[(T^{s\dagger}_{rm})_{24}(M^\dagger_j)_{24}(h^r g^{s*})^\dagger_2(h^m)^\dagger_4 W^\dagger_{24}]|\Phi\rangle_{12}|\Phi\rangle_{34}\} \\
= & \frac{1}{n} \max_{\{W, T^s_{rm}\}} \sum_{r,m,s,j} \{\langle\Phi|_{12}\langle\Phi|_{34}[(\Omega_a)^T_{24} W_{24}(h^r g^{s*})_2(h^m)_4(M_j)_{24}(T^s_{rm})_{24}]((\Omega_b)_{24}\chi_{12}\chi_{34}(\Omega_b)^\dagger_{24}) \\
& [(T^{s\dagger}_{rm})_{24}(M^\dagger_j)_{24}(h^r g^{s*})^\dagger_2(h^m)^\dagger_4 W^\dagger_{24}(\Omega_a)^*_{24}]|\Phi\rangle_{12}|\Phi\rangle_{34}\} \\
= & F(\chi^{\otimes 2}).
\end{aligned}
$$

□

Theorem 3. *The optimal teleportation fidelity* $f_{\max}(\chi^{\otimes 2})$ *for the two-copy teleportation protocol based on GHZ measurement only depends on the fully entangled fraction* $F(\chi^{\otimes 2})$ *in Eq. (7) as follow:*

$$f_{\max}(\chi^{\otimes 2}) = \frac{nF(\chi^{\otimes 2})}{(n+1)} + \frac{1}{n+1}. \tag{8}$$

Proof. We first introduce $U(n)$ as an irreducible n-dimensional representation of unitary group denoted by G [11]. Using the Schur's lemma, we can get the fidelity of our two-copy teleportation protocol by the method developed in [11]:

$$
\begin{aligned}
f(\chi^{\otimes 2}) &= \overline{\langle\phi_{in}|\Lambda_{(\chi)}^{\{T_{rm}^{s},W\}}(\rho_{in})|\phi_{in}\rangle} \\
&= \frac{1}{n^3}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle \\
&\times\sum_{r,m,s,j}\langle 00|\int_G[U(g)^\dagger\otimes U(g)^\dagger]\langle j|_4[(T_{rm}^{s})_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2] \\
&\otimes[(\widetilde{U}_{rm}^{s})_{24}(W^\dagger)_{24}(U_{s_1't_1'}^\dagger)_2(U_{s_2't_2'}^\dagger)_4(T_{rm}^{s\dagger})_{24}]|j\rangle_4[U(g)\otimes U(g)]dg|00\rangle \\
&= \frac{1}{n^4(n+1)}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle \\
&\times\sum_{r,m,s,j}\{\mathrm{tr}_{24}[(T_{rm}^{s})_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2\langle j|_4] \\
&\quad\mathrm{tr}_{24}[|j\rangle_4(\widetilde{U}_{rm}^{s})_{24}W_{24}^\dagger(U_{s_1't_1'}^\dagger)_2(U_{s_2't_2'}^\dagger)_4(T_{rm}^{s\dagger})_{24}] \\
&+\mathrm{tr}_2[\langle j|_4(T_{rm}^{s})_{24}(U_{s_1t_1})_2(U_{s_2t_2})_4(W)_{24}(\widetilde{U}_{rm}^{s\dagger})_2 \\
&\quad(\widetilde{U}_{rm}^{s})_{24}(W^\dagger)_{24}(U_{s_1't_1'}^\dagger)_2(U_{s_2't_2'}^\dagger)_4(T_{rm}^{s\dagger})_{24}|j\rangle_4]\} \\
&= \frac{1}{n^4(n+1)}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\sum_{r,m,s,j}\{\langle\Phi|_{12}\langle\Phi|_{34}[(U_{s_1t_1}^\dagger)_2(U_{s_2t_2}^\dagger)_4] \\
&\quad\mathrm{tr}_{24}[(U_{s_1t_1})_2(U_{s_2t_2})_4W_{24}(h^rg^{s*})_2(h^m)_4(M_j)_{24}(T_{rm}^{s})_{24}](\chi_{12}\chi_{34}) \\
&\quad\mathrm{tr}_{24}[(T_{rm}^{s\dagger})_{24}(M_j^\dagger)_{24}(h^rg^{s*})_2^\dagger(h^m)_4^\dagger W_{24}^\dagger(U_{s_1't_1'}^\dagger)_2(U_{s_2't_2'}^\dagger)_4][(U_{s_1't_1'})_2(U_{s_2't_2'})_4]|\Phi\rangle_{12}|\Phi\rangle_{34} \\
&+\frac{1}{n^2(n+1)}\sum_{s_1,t_1}\sum_{s_2,t_2}\sum_{s_1',t_1'}\sum_{s_2',t_2'}\langle\Phi_{s_1t_1}|\chi|\Phi_{s_1't_1'}\rangle\langle\Phi_{s_2t_2}|\chi|\Phi_{s_2't_2'}\rangle\ \mathrm{tr}\{U_{s_1t_1}U_{s_1't_1'}^\dagger\}\ \mathrm{tr}\{U_{s_2t_2}U_{s_2't_2'}^\dagger\} \\
&= \frac{1}{(n+1)}\sum_{r,m,s,j}\{\langle\Phi|_{12}\langle\Phi|_{34}[W_{24}(h^rg^{s*})_2(h^m)_4(M_j)_{24}(T_{rm}^{s})_{24}](\chi_{12}\chi_{34}) \\
&\quad[(T_{rm}^{s\dagger})_{24}(M_j^\dagger)_{24}(h^rg^{s*})_2^\dagger(h^m)_4^\dagger W_{24}^\dagger]|\Phi\rangle_{12}|\Phi\rangle_{34}\}+\frac{1}{n+1},
\end{aligned}
$$

where we have used Eq. (5) in the fourth equation.

Employing the definition of fully entangled fraction in Eq. (7), the optimal teleportation fidelity can be expressed as

$$
f_{\max}(\chi^{\otimes 2}) = \frac{nF(\chi^{\otimes 2})}{(n+1)}+\frac{1}{n+1}.
$$

□

To reach the optimal teleportation fidelity, we need to run over all unitary operators W and measurements $\{T_{rm}^{s}\}$ to achieve the maximum.

The relation between optimal teleportation fidelity and fully entangled fraction in Eq. (8) is consistent with the previous results[11, 12, 23]. In traditional one copy teleportation, the optimal teleportation fidelity is given by [11, 12]

$$f_{\max}^{(1)}(\chi) = \frac{nF_1(\chi)}{n+1} + \frac{1}{n+1}, \tag{9}$$

where $F_1(\chi) = \max_{U\in U(n)}\{\langle\Phi|_{12}U_2^{\dagger}\chi_{12}U_2|\Phi\rangle_{12}\}$ is the original fully entangled fraction. In two-copy teleportation based on Bell measurement [23], the optimal teleportation fidelity is

$$f_{\max}^{(2)}(\chi^{\otimes 2}) = \frac{nF_2(\chi^{\otimes 2})}{(n+1)} + \frac{1}{n+1}, \tag{10}$$

where $F_2(\chi^{\otimes 2}) = \max_{\Omega,V\in U(n^2)}\{\langle\Phi|_{12}\,\mathrm{tr}_{34}[\Omega_{13}V_{24}\chi_{12}\chi_{34}\Omega_{13}^{\dagger}V_{24}^{\dagger}]|\Phi\rangle_{12}\}$ is two-copy fully entangled fraction based on Bell measurement. From Eq. (8), Eq. (9), and Eq. (10), one can see that optimal teleportation fidelities for one copy and two-copy teleportation protocols are all linear functions of the corresponding fully entangled fractions. These fully entangled fractions characterize the usefulness of the entangled resource states in quantum teleportation.

Now let's consider an explicit example in two-copy teleportation based on GHZ measurement. For a general entangled pure state $|\psi\rangle = \sum_j \lambda_j|jj\rangle$ with at least two nonzero coefficients [25], where λ_j are non-negative real numbers satisfying $\sum_j \lambda_j^2 = 1$ known as Schmidt coefficients, its one copy fully entangled fraction has been given as [26–28]: $1 \geq F_1(|\psi\rangle) = \frac{1}{n}(\sum_j \lambda_j)^2 > \frac{1}{n}$.

From Eq.(7), we can further calculate that

$$F(\chi^{\otimes 2}) = n^2 \max_{\{U,V\}} \sum_j \left\{\langle\Phi|_{12}\langle\Phi|_{34}[U_{24}(M_j)_{24}V_{24}]\chi_{12}\chi_{34}[U_{24}(M_j)_{24}V_{24}]^{\dagger}|\Phi\rangle_{12}|\Phi\rangle_{34}\right\}, \tag{11}$$

where U, V are unitary matrices in $H\otimes H$. Suppose the extreme point of $F(\chi^{\otimes 2})$ is $M_j(\chi)$, that is, $F(\chi^{\otimes 2}) = n^2\sum_j \left\{\langle\Phi|_{12}\langle\Phi|_{34}(M_j(\chi))_{24}\chi_{12}\chi_{34}(M_j(\chi))_{24}^{\dagger}|\Phi\rangle_{12}|\Phi\rangle_{34}\right\}$. Let

$$M_j = \sum_k |kk\rangle\langle kj| = \sum_{k_1l_1k_2l_2} m^j_{k_1l_1k_2l_2}|k_1l_1\rangle\langle k_2l_2|,$$
$$M_j(\chi) = \sum_{k_1l_1k_2l_2} m^j(\chi)_{k_1l_1k_2l_2}|k_1l_1\rangle\langle k_2l_2|,$$

where $m^j(\chi)_{k_1l_1k_2l_2}$ is a parameter that is related to the channel χ. Then after some simple calculation, one can find that the two-copy fully entangled fraction of $|\psi\rangle$ is: $F((|\psi\rangle\langle\psi|)^{\otimes 2}) = \sum_j|\sum_{j_1,j_2}\lambda_{j_1}\lambda_{j_2}m^j(|\psi\rangle\langle\psi|)_{j_1j_2j_1j_2}|^2$. For the two-dimension circumstance, λ_0 and λ_1 are both positive real number. Suppose $\lambda_0 \geq \lambda_1$, and we can calculate that $F((|\psi\rangle\langle\psi|)^{\otimes 2}) = \sum_j |\sum_{j_1,j_2} \lambda_{j_1}\lambda_{j_2} m^j(\psi\rangle\langle\psi|)_{j_1 j_2 j_1 j_2}|^2 \geq (\lambda_0^2 + \lambda_0\lambda_1)^2$. The one copy fully entangled fraction is $F_1(|\psi\rangle\langle\psi|) = \frac{(\lambda_0+\lambda_1)^2}{2}$. One can easily find that

$$F(\chi^{\otimes 2}) - F_1(|\psi\rangle\langle\psi|) \geq (\lambda_0^2 + \lambda_0\lambda_1)^2 - \frac{(\lambda_0+\lambda_1)^2}{2} = \frac{(\lambda_0^2 - \lambda_1^2)(\lambda_0+\lambda_1)^2}{2} > 0, \quad (12)$$

which demonstrates that for all arbitrary 2-dimensional entangled pure state $|\psi\rangle$, the transmission fidelity of our two-copy optimal teleportation protocol based on GHZ measurement is higher than that of one copy protocol. Evidence for that one can meet $F'((|\psi\rangle\langle\psi|)^{\otimes 2})$ with certain W and r, m, s is presented as follows, where I is 2×2 identity matrix and X is Pauli X matrix.

TABLE I. Corresponding values to meet $F'((|\psi\rangle\langle\psi|)^{\otimes}2)$.

r	m	s	$(h^r g^{s^*})_2 h_4^m$	W_{24}	$(T_{rm}^s)_{24}$
0	0	0	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
0	0	1	$\begin{pmatrix} I & 0 \\ 0 & w^* I \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & \frac{1}{w^*} X \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
0	1	0	$\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$	$\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
0	1	1	$\begin{pmatrix} X & 0 \\ 0 & w^* X \end{pmatrix}$	$\begin{pmatrix} X & 0 \\ 0 & \frac{1}{w} I \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
1	0	0	$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & I \\ X & 0 \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
1	0	1	$\begin{pmatrix} 0 & w^* I \\ I & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & I \\ \frac{1}{w^*} X & 0 \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
1	1	0	$\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & X \\ I & 0 \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
1	1	1	$\begin{pmatrix} 0 & w^* X \\ X & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & X \\ \frac{1}{w^*} I & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$

III. CONCLUSIONS

We have proposed a general two-copy quantum teleportation protocol based on GHZ measurement and showed this process is trace preserving. The corresponding optimal teleportation fidelity has been explicitly derived. It turns out that the optimal teleportation fidelity only depends on the two-copy fully entangled fraction, which is invariant under local unitary transformations on the resource states. The two-copy teleportation based on GHZ measurement is illustrated to be better than one copy teleportation in some cases by an explicit example, which is amenable to demonstration in simple experiments. We hope our work could simulate further study on efficient teleportation protocols.

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