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We study the nonlocality of high dimensional quantum systems based on quantum entanglement and projection. First, a quantitative relationship between the maximal expectation value B of Bell operators and the quantum entanglement concurrence C is obtained for even dimensional pure and mixed states, with the lower bounds of B governed by C. Second, by projecting the high dimension bipartite and tripartite quantum states to "two-qubit" and "three-qubit" quantum states, respectively, the nonlocality of the high dimensional quantum states is revealed by the violations of Bell inequalities of the projected qubits states. If the projected qubits states violate Bell inequalities but the violation is less than certain values, there exist kinds of "hidden" nonlocality of the high dimensional states, we call it locally-preprocessed-allowed nonlocality. Examples of high dimensional isotropic states are presented to illustrate the relationship between nonlocality and locally-preprocessed-allowed nonlocality.

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INTRODUCTION

Quantum nonlocality, such as that revealed by the violation of Bell inequalities by quantum entangled states [1], is one of the most startling predictions of quantum mechanics. Recently, as confirmed in loophole-free experiments [2], nonlocality has been proven to be useful in many quantum tasks such as device-independent cryptography [3] and randomness certification [4, 5].

Although all entangled pure states can display nonlocal correlations [6, 7], some mixed entangled states can provably satisfy all the Bell inequalities [8–12]. Namely, there exist entangled mixed states that never lead to nonlocality by any local POVM measurements [9]. Under local filtering operations on many copies, it was shown in Ref. [13] that the set of bipartite quantum states violating the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [14] is precisely the set of distillable [15] quantum states. However, this does not imply that bound entangled states [16] must satisfy all Bell inequalities [17]. Following the earlier results of [10, 18], this phenomenon has recently been shown to be true in general for multipartite case as well: there exist genuinely multipartite entangled states which admit a fully local hidden-variable (LHV) model [19]. Quantum nonlocality is usually associated with entangled states that violate at least one of the Bell inequalities. However, separable bipartite states can also show some nonlocal properties [20, 21]. Despite all these progresses, the precise relationship between entanglement and Bell violations has remained less known, particularly for high dimensional bipartite and multipartite cases.

Different ways to quantify the quantum nonlocality have been presented [22–33],etc., for examples, the volume of the violation of Bell-type inequalities [27, 28]. By employing the probability of violation of local realism under random measurements [34], in [30] the authors investigated the nonlocality of entangled qudits with dimensions ranging from $d = 2$ to $d = 10$. In [31] the authors proposed a machine learning approach for detection and quantification of nonlocality. Quantifying Bell nonlocality by the trace distance has been studied in [29].

In the following we use concurrence C [35, 36] as the measure of quantum entanglement. Let H_i denote the Hilbert space associated with the ith subsystem. For a pure state $|\psi\rangle \in H_1 \otimes H_2$, the concurrence is defined by [37–39], $C(|\psi\rangle) = \sqrt{2(1 - Tr\rho_1^2)}$, where the reduced density matrix $\rho_1 = Tr_2|\psi\rangle\langle\psi|$ is obtained by tracing over the second subsystem. The concurrence is then extended to mixed states ρ by convex roof,

$$C(\rho) \equiv \min_{p_i,|\psi_i\rangle} \sum_i p_i C(|\psi_i\rangle),$$

where the minimization goes over all possible ensemble realizations $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$ and $\sum_i p_i = 1$. For two-qubit states C can be calculated directly [35]. For high dimensional quantum states one has no general results [40, 41].

For two-qubit states, the well-known CHSH Bell inequality [14] has been used to detect the nonlocality. The corresponding operator is given by $\mathbb{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$. The CHSH inequality is $|\langle\mathbb{B}\rangle| \le 2$, where $\langle\mathbb{B}\rangle = Tr(\mathbb{B}\rho)$, $A_i = \vec{a}_i \cdot \vec{\sigma}$, $B_j = \vec{b}_j \cdot \vec{\sigma}$, $\vec{a}_i$ and $\vec{b}_j$ are three-dimensional real unit vectors, $i, j = 1, 2$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_1, σ_2 and σ_3 the standard Pauli matrices.

The concurrence C and the maximal Bell value $B \equiv \max\langle\mathbb{B}\rangle$ satisfy the relation: $2\sqrt{2}C \le B \le 2\sqrt{1 + C^2}$ [42]. Therefore, the Bell inequality is violated if $C > 1/\sqrt{2}$. Such relations have been also investigated by using randomly generated two-qubit states [43]. The Bell inequality for three-qubit states has been also studied [44]. The relationship between tripartite entanglement and genuine tripartite nonlocality for three qubit Greenberger-Horne-Zeilinger class is also investigated [45]. The authors in [46, 47] studied the relation between the upper bound of Bell violation and a generalized concurrence for some n-qubit states. In [48] the nonlocality distributions among multiqubit systems have been studied based on the maximal violations of the CHSH inequality of reduced pairwise qubit systems. Furthermore, from the reduced three-qubit density matrices of the four-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states and W-states, a trade-off relation among the mean values of the Svetlichny operators associated with these reduced states has been presented [49].

The so called hidden nonlocality has been illustrated in [50, 51]. It is found that locally correlated states can violate a Bell inequality under local filtering operations, demonstrated in photonic experiments later [52]. Ref. [53] shows how local filtering can increase the entanglement. It has been also shown that the local filtering can "activate" the CHSH-violation [51, 54], and a necessary and sufficient condition is derived [55]. In [56] it has been shown that all entangled states violate a Bell inequality when combined with another state which on its own cannot violate the same Bell inequality. Ref. [57] studies whether all entangled states can violate a Bell inequality after well-chosen local filtering. The answer is shown to be negative due to that there exist entangled states without hidden nonlocality. For a review on the activation of quantum nonlocality see Ref. [58].

For high dimensional quantum systems less is known about the relationship between concurrence and Bell violations. The main difficulty lies in finding the maximin mean value of suitable Bell operators. In this paper, we explore the quantitative relationship between concurrence C and the maximin Bell value B for high dimensional quantum systems. This part of research corresponds to E and e in FIG. 1. We also study the "hidden" nonlocality of the original quantum state by means of projection (f of FIG. 1).

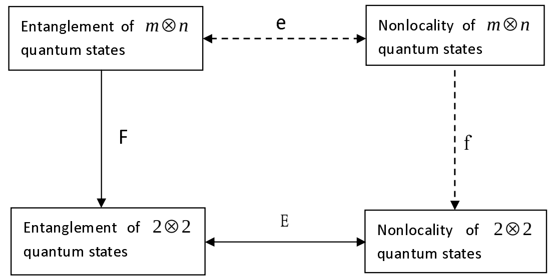

FIG. 1: The E corresponds to the research in Ref. [42], F is about the research in Ref. [40, 41]. e and f are the main researches in this article.

BELL NON-LOCALITY AND CONCURRENCE OF BIPARTITE QUANTUM STATES

Based on Bell's idea [1], for any given $n \times n$ real matrix N with entries N_{ij}, one can define a classical quantity,

$$J(N) = \sup |\sum_{i,j=1}^{n} N_{ij} a_i b_j|,$$

where the supremum is taken over all possible assignment $a_i, b_j \in \{-1, 1\}$, $1 \le i, j \le n$. For any bipartite state ρ, the corresponding Bell operator is defined by

$$\mathbb{B}(N) = \sum_{i,j=1}^{n} N_{ij} A_i \otimes B_j,$$

where A_i and B_j are arbitrary observables whose absolute values of all eigenvalues are less or equal to one.

A state ρ is said to be nonlocal if it violates the following Bell inequality,

$$B(N) \le J(N),$$

where $B(N) = tr(\mathbb{B}(N)\rho)$ is the mean value of the Bell operator. If one takes $N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, one gets the CHSH inequality with $J(N) = 2$. For given N and state, it is still an open problem to find the maximum Bell value $B \equiv \max \mathbb{B}(N)$ over all possible A_i and B_j.

A pure $m \otimes n (m \le n)$ quantum state has the standard Schmidt form,

$$|\psi\rangle = \sum_{i=1}^{m} c_i |a_i b_i\rangle, \quad (1)$$

where c_i $(i = 1, \cdots, m)$ are the Schmidt coefficients, $|a_i\rangle$ and $|b_i\rangle$ are the orthonormal bases in H_1 and H_2, respectively. The concurrence of $|\psi\rangle$ is given by

$$C = 2\sqrt{\sum_{i<j} c_i^2 c_j^2}, \quad (2)$$

which varies from 0 for pure product states to $\sqrt{2(m-1)/m}$ for maximally entangled pure states [39].

By selecting the matrix N as that from the CHSH inequality (with $J(N) = 2$), for any bipartite pure state $|\psi\rangle$ as given by (1), it has been shown that [59],

$$B_{sub}(|\psi\rangle) = 2\sqrt{(1-\gamma)^2 + K^2} + 2\gamma, \tag{3}$$

where $K = 2(c_1c_2 + c_3c_4 + \cdots)$, $\gamma = c_m^2$ for odd m, and $\gamma = 0$ for even m, B_{sub} is not definitely the maximum value B but obtained by choosing some specific local observables A_i and B_j.

The result of (3) has been further optimized in [60, 61]. However, one has no maximin Bell value B yet. In the following we use (3) to obtain the following facts, which does not depend on the optimality of (3).

Theorem 1. *For any pure $m\otimes n (m \leq n)$ quantum state $|\psi\rangle$, with the standard Schmidt form (1), then for even m, we have*

$$B \geq \sqrt{2[1 + C^2]}. \tag{4}$$

Proof. According to the definition of B, (2) and (3), in order to prove the inequality (4), we only need to prove $B_{sub} \geq \sqrt{2[1 + C^2]}$ since $B \geq B_{sub}$. To prove $B_{sub} \geq \sqrt{2[1 + C^2]}$ is equivalent to prove $1 + 2K^2 \geq C^2$, namely, we need to prove

$$1 + 2[4(c_1c_2 + c_3c_4 + \cdots + c_{m-1}c_m)^2] \geq 4\sum_{i<j} c_i^2c_j^2.$$

That is, for odd k, l, we have

$$1 + 2[4\sum_{k=1,l=1}^{m-1}(c_kc_{k+1}c_lc_{l+1})] \geq 2\sum_{i\neq j} c_i^2c_j^2. \tag{5}$$

Without loss of generality, we assume that the Schmidt coefficients in (1) satisfy $c_i \geq c_{i+1}, i = 1, 2, \cdots, m$. Then we have the following facts, $c_kc_{k+1}c_lc_{l+1} \geq c_{k+1}^2c_{l+1}^2$, $c_kc_{k+1}c_lc_{l+1} \geq c_{k+1}^2c_{l+2}^2$, $c_kc_{k+1}c_lc_{l+1} \geq c_{k+2}^2c_{l+1}^2$ and $c_kc_{k+1}c_lc_{l+1} \geq c_{k+2}^2c_{l+2}^2$. Hence, we have

$$\begin{aligned} & 4\sum_{k=1,l=1,odd}^{m-1}(c_kc_{k+1}c_lc_{l+1}) \\ \geq & \sum_{k=1,l=1,odd}^{m-1}[(c_{k+1}^2c_{l+1}^2) + (c_{k+1}^2c_{l+2}^2) \\ & +(c_{k+2}^2c_{l+1}^2) + (c_{k+2}^2c_{l+2}^2)] \end{aligned}$$

and

$$1 = (c_1^2 + \sum_{i\neq 1}^{m} c_i^2)^2 \geq 4c_1^2(\sum_{i\neq 1}^{m} c_i^2).$$

Combining above relations we obtain

$$\begin{aligned} & 1 + 2[4\sum_{k=1,l=1}^{m-1}(c_kc_{k+1}c_lc_{l+1})] \\ \geq\ & 2\sum_{k=1,l=1,odd}^{m-1}[(c_{k+1}^2c_{l+1}^2) + (c_{k+1}^2c_{l+2}^2) \\ & +(c_{k+2}^2c_{l+1}^2) + (c_{k+2}^2c_{l+2}^2)] + 4\sum_{i\neq 1}^{m} c_1^2c_i^2 \\ \geq\ & 2\sum_{i\neq j} c_i^2c_j^2, \end{aligned}$$

which gives rise to the inequality (5), and proves the inequality (4). $\square$

From Theorem 1, it is obvious that if $C > 1$, the state $|\psi\rangle$ shows non-locality. In fact, this result of pure states can be extended to the case of mixed states.

Theorem 2. *For any mixed $m \otimes n$ $(m \leq n)$ quantum state $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$, we have*

$$B(\rho) \geq \sqrt{2[1 + C^2(|\psi_{\min}\rangle)]} \tag{6}$$

for even m, where $|\psi_{\min}\rangle$ minimizes the values $B_{sub}(|\psi_i\rangle)$ of all $|\psi_i\rangle$.

Proof.

$$\begin{aligned} B(\rho) &= \max Tr(\mathbb{B}(N)\rho) \\ &= \max\sum_i p_iTr(\mathbb{B}(N)|\psi_i\rangle\langle\psi_i|) \\ &\geq \sum_i p_iB_{sub}(|\psi_i\rangle) \\ &\geq \min_i B_{sub}(|\psi_i\rangle) \\ &\equiv B_{sub}(|\psi_{\min}\rangle) \\ &\geq \sqrt{2[1 + C^2(|\psi_{\min}\rangle)]}, \end{aligned}$$

where $\mathbb{B}(N)$ is the Bell operator admitted in [59], which gives rise to the formula (3), the last inequality is due to the proof of Theorem 1. $\square$

Theorem 2 says that if a mixed state ρ has a pure state decomposition $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, then ρ is nonlocal if $B_{sub}(|\psi_i\rangle) > 2$ for any pure state $|\psi_i\rangle$. Let us consider the following example.

Example 1: Consider the mixed state $\rho = \frac{1}{d^2}\sum_{m,n}|\Psi_{m,n}\rangle\langle\Psi_{m,n}|$, where$|\Psi_{m,n}\rangle$ are generalized Bell states [62, 63] of the form

$$|\Psi_{m,n}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\zeta^{nj}|j, j+m\rangle,$$

where $\zeta = e^{\frac{2\pi i}{d}}$ and $m, n = 0, ..., d-1$. In this case, we have $C(|\Psi_{m,n}\rangle) = \sqrt{2(1-\frac{1}{d})}$. Therefore, from Theorem 2 we have $B(\rho) \geq \sqrt{2(3-\frac{2}{d})} \geq 2$, namely, ρ has nonlocality as $d > 2$.

NONLOCALITY AND LOCALLY-PREPROCESSED-ALLOWED NONLOCALITY OF BIPARTITE QUANTUM STATES

It is formidably difficult to determine the nonlocality of a mixed state in general. For two-qubit systems, a necessary and sufficient condition for the violation of the CHSH inequality has been derived [64]. However, for the two-qubit Werner states [8], $\rho(p) = (1-p)I_4/4 + p|\psi^-\rangle\langle\psi^-|$, where I_4 is 4×4 identity matrix, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, its nonlocality problem is not completely solved yet. From the violation of the CHSH inequality, one assures that $\rho(p)$ is nonlocal if $0.7071 < p \leq 1$. The Bell inequality given in [65] improves this parameter region to be $0.7056 < p \leq 1$, which is further improved to be $0.7054 < p \leq 1$ [66].

A two-qubit mixed state ρ can be expressed in general,

$$\rho = \frac{1}{4}(I_2\otimes I_2 + \vec{a}\cdot\vec{\sigma}\otimes I_2 + I_2\otimes\vec{b}\cdot\vec{\sigma} + \sum_{n,m=1}^{3} t_{m,n}\sigma_m\otimes\sigma_n)$$

where I_2 is the 2×2 identity operator. Denote T the 3×3 matrix with entries given by $t_{nm} = Tr(\rho\sigma_n\otimes\sigma_m)$. Let μ_1 and μ_2 be the two greater eigenvalues of $U \equiv T^tT$, where t stands for transpose. It has been proven that for CHSH inequality, $B = 2\sqrt{\mu_1+\mu_2}$ [64].

We consider now $d\otimes d$ $(d\geq 3)$ bipartite mixed state $\rho \in H_1\otimes H_2$. Let $|e_\alpha\rangle$, $\alpha = 0,1,2,\cdots,d-1$, be the basis of a d-dimensional space. Set $P = P_1\otimes P_2$, where $P_1 = (|e_\alpha\rangle, |e_\beta\rangle)^t$ for some $\alpha\neq\beta$ and $P_2 = (|e_\gamma\rangle, |e_\lambda\rangle)^t$ for some $\gamma\neq\lambda$ are $2\times d$ matrices, which project d-dimensional vectors to two-dimensional ones. The following filtering projects ρ to a "two-qubit" one, $\widetilde{\rho} = \frac{P\rho P^\dagger}{k}$ with $k = tr[P\rho P^\dagger]\neq 0$. Analogously, induced from the CHSH operator $\mathbb{B}$, one has $d^2\times d^2$ operator $\widetilde{\mathbb{B}} = P^\dagger\mathbb{B}P$. Correspondingly we have the induced CHSH inequalities for high dimensional bipartite states.

We first give a theorem to judge the usual nonlocality of high dimensional bipartite states from the "two-qubit" projected state $\widetilde{\rho}$.

Theorem 3. *For any $d\otimes d$ quantum state ρ, if there exists one "two-qubit" projected state $\widetilde{\rho}$ which violates CHSH inequality such that $Tr(\mathbb{B}\widetilde{\rho}) \geq \frac{2}{k}$, then ρ has nonlocality, where $k = tr[P\rho P^\dagger]$.*

Proof. Since $P_1^\dagger\sigma_i P_1$ and $P_2^\dagger\sigma_i P_2$ have the same eigenvalues as σ_i, for high dimensional state ρ, suppose there is one $2\otimes 2$ CHSH operator $\mathbb{B}$, such that

$$\langle\mathbb{B}\tilde{\rho}\rangle = \langle\widetilde{\mathbb{B}}\rho\rangle/k = s > 2,$$

where $\widetilde{\mathbb{B}} = P^\dagger\mathbb{B}P$ induced from $\mathbb{B}$, $k = tr[P\rho P^\dagger]$. Then $\langle\widetilde{\mathbb{B}}\rho\rangle = ks$. If $ks > 2$, ρ violates a Bell inequality, then ρ has nonlocality. □

If there exists one "two-qubit" projected state $\widetilde{\rho}$ which violates CHSH inequality such that $2 < Tr(\mathbb{B}\widetilde{\rho}) < \frac{2}{k}$, it implies that the state ρ still has some kind of "hidden" nonlocalities. In the following we say that if a "two-qubit" projected state $\widetilde{\rho}$ violates the CHSH inequality, $2 < Tr(\mathbb{B}\widetilde{\rho}) < \frac{2}{k}$, then ρ has locally-preprocessed-allowed nonlocality, even if ρ itself does not violate any Bell inequalities. Let us consider the $d\otimes d$ isotropic states [67],

$$\rho(p) = (1-p)\frac{I_{d^2}}{d^2} + pP_+,$$

where $P_+ = |\psi_+\rangle\langle\psi_+|$, with $|\psi_+\rangle = \sum_{i=1}^d |i\rangle\langle i|/\sqrt{d}$, $0\leq p\leq 1$. For $d = 2$, the isotropy states are just Werner states. Isotropy states are entangled iff $p > P_{ent} = \frac{1}{d+1}$. Denote P_{Bell} the critical value such that $\rho(p)$ is Bell-nonlocal iff $p > P_{Bell}$. A non-trivial upper bound of P_{Bell} has been derived in [68]. For $d = 3$ one has $p_{Bell} = 0.69615$. Denote P_{steer} the value of p such that $\rho(p)$ is steerable iff $p > P_{steer}$. In [69] the authors presented a lower bound on $P_{steer} = (H_d - 1)/(d-1)$, where $H_d = \sum_{n=1}^d (1/n)$ is the harmonic series.

Denote P_{hb} the critical point such that $\rho(p)$ has locally-preprocessed-allowed nonlocality if $p > P_{hb}$. In the following, we will give a specific example to show that how to calculate the value P_{hb} of high dimensional isotropic states with locally-preprocessed-allowed nonlocality. Moreover, P_{hb} will also give a lower bound of P_{Bell}.

Consider the case of $d = 3$ first. Let $|e_\alpha\rangle$, $\alpha = 0,1,2$, be the basis of H_1 and H_2. Set $P = P_1\otimes P_2$, where $P_1 = P_2 = (|e_0\rangle, |e_1\rangle)^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}$. We obtain the projected state of $\rho(p)$,

$$\widetilde{\rho_2} = \frac{P\rho P^\dagger}{k} = \frac{9}{2p+4} \cdot \begin{pmatrix} \frac{p}{3}+\frac{1-p}{9} & 0 & 0 & \frac{p}{3} \\ 0 & \frac{1-p}{9} & 0 & 0 \\ 0 & 0 & \frac{1-p}{9} & 0 \\ \frac{p}{3} & 0 & 0 & \frac{p}{3}+\frac{1-p}{9} \end{pmatrix},$$

which can be equivalently expressed as $\frac{1}{4}(I_2\otimes I_2 + \frac{3p}{2+p}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z))$. Therefore, $\widetilde{\rho_2}$ has nonlocality if $p > \frac{2}{3\sqrt{2}-1} \approx 0.61678$. Then, $\rho(p)$ has locally-preprocessed-allowed nonlocality if $p > \frac{2}{3\sqrt{2}-1}$, and $P_{hb}\approx 0.61678$ is a new lower bound of P_{Bell} for $3\otimes 3$ isotropic states. One has the relation $P_{steer} = 5/12 \leq P_{hb} \leq P_{Bell} < 0.69615$.

By means of the projection $P_1 = P_2 = (|e_0\rangle, |e_1\rangle)^t$, we can get the values of P_{hb} for different dimensions d of isotropic states $\rho(p)$, see Fig. 2. They are less than the best known values of P_{Bell}, obtained from the Collins-Gisin-Linden-Massar-Popescu inequalities [68].

d	2	3	4	5	6
p_{ent}	0.3333	0.2500	0.2000	0.1667	0.1429
p_{steer}	0.5000	0.4167	0.3611	0.3208	0.2900
p_{hb}	0.7054	0.6168	0.5469	0.4913	0.4871
p_{Bell}	0.7054	0.6961	0.6905	0.6872	0.6849

FIG. 2: P_{ent}, P_{hb}, P_{steer} and P_{Bell} for isotropic states with different dimension d.

NONLOCALITY AND LOCALLY-PREPROCESSED-ALLOWED NONLOCALITY OF TRIPARTITE QUANTUM STATES

The two-qubit Bell operator $\mathbb{B}_{CHSH}$ can be extended to n-qubit one [70],

$$\mathbb{B}_n = \mathbb{B}_{n-1} \otimes \frac{1}{2}(A_n + A'_n) + \mathbb{B}'_{n-1} \otimes \frac{1}{2}(A_n - A'_n), \quad (7)$$

where A_n and A'_n are the observablesOn the n-th qubit. The operators $\frac{1}{2}\mathbb{B}_{n-1}$ and $\frac{1}{2}\mathbb{B}'_{n-1}$ act on the rest $n-1$ qubits. For an n-qubit system, one has $B = Tr(\rho\mathbb{B}_n) \leq 2^{\frac{n+1}{2}}$ [70].

The nonlocality of three-qubit systems can be investigated by Mermin's inequality [71] and Svetlichny's inequality [72]. One of the Mermin inequalities for tripartite systems is of the following form,

$$\langle M\rangle = \langle A_0B_0C_1 + A_0B_1C_0 + A_1B_0C_0 - A_1B_1C_1\rangle \leq 2,$$

where A_0, A_1, B_0, B_1 and C_0, C_1 are observables on systems H_1, H_2 and H_3, respectively.

One of the Svetlichny inequalities reads

$$\begin{aligned}\langle S\rangle &= \langle A_0B_0C_1 + A_0B_1C_0 + A_1B_0C_0 - A_1B_1C_1\rangle \\ &+\langle A_0B_1C_1 + A_1B_0C_1 + A_1B_1C_0 - A_0B_0C_0\rangle \leq 4.\end{aligned}$$

This inequality can detect the genuine tripartite nonlocality [72].

In [73], the authors derived an analytical formula for the maximum expectation value of the Mermin operator M for three-qubit pure and mixed states. A tight upper bound is also obtained for the maximal quantum value of the Svetlichny operators for three-qubit systems [74]. More recently, the violations of the Mermins and Svetlichnys inequalities for three-qubit W-state and GHZ states have been studied by the IBMs cloud computing platform [75].

Let $|e_i\rangle$, $i = 0, 1, 2, \cdots, d-1$, be a local basis of H_1, H_2 and H_3. Set $P = P_1 \otimes P_2 \otimes P_3$, where P_i, $i = 1, 2, 3$, are $3 \times d$ matrices of the form $(|e_\alpha\rangle, |e_\beta\rangle, |e_\gamma\rangle)^t$ for some $\alpha \neq \beta \neq \gamma$. We can project the tripartite state ρ to be a "three-qubit" state $\widehat{\rho} = \frac{P\rho P^\dagger}{l}$ with $l = tr[P\rho P^\dagger] \neq 0$. Correspondingly, induced from the Mermin (Svetlichny) operator for three-qubit states, we have the operators $\widehat{M} = P^\dagger MP$ ($\widehat{S} = P^\dagger SP$) for high dimensional tripartite states. We have

Theorem 4. *For any $d\otimes d\otimes d$ quantum state ρ, if there exists one "three-qubit" projected state $\widehat{\rho}$ which violates the Mermin (Svetlichny) inequality such that $\langle M\rangle > \frac{2}{l}$ ($\langle S\rangle > \frac{4}{l}$), then ρ has tripartite nonlocality (genuine tripartite nonlocality).*

Proof. Since $P_1^\dagger\sigma_iP_1$ and $P_2^\dagger\sigma_iP_2$ and $P_3^\dagger\sigma_iP_3$ have the same eigenvalues as σ_i. If one "three-qubit" projected state $\widehat{\rho}$ violates an induced Mermin inequality, we have $Tr(\widehat{M}\rho) = Tr(P^\dagger MP\rho) = Tr(MP\rho P^\dagger) = l \cdot Tr(M\widehat{\rho}) = ls$ with $s > 2$. If $s > \frac{2}{l}$ that is $Tr(\widehat{M}\rho) > 2$, then ρ violates the Mermin inequality. The result for the Svetlichny's inequality can be obtained similarly. □

The nonlocality of tripartite quantum states is more complicated than that of bipartite ones. For any $d\otimes d\otimes d$ quantum state ρ, if there exists a "three-qubit" projected state $\widehat{\rho}$ which violates the Mermin inequality such that $2 < \langle M\rangle \leq \frac{2}{l}$, then ρ has locally-preprocessed-allowed tripartite nonlocality. If there exists an "three-qubit" projected state $\widehat{\rho}$ which violates the Svetlichny inequality such that $4 < \langle S\rangle \leq \frac{4}{l}$, then ρ has locally-preprocessed-allowed genuine tripartite nonlocality.

CONCLUSIONS AND DISCUSSIONS

Quantum nonlocality is a fundamental feature in quantum mechanics. We have investigated the relation between the maximal expectation value of Bell operators B and the entanglement concurrence C. The lower bounds of B have been derived based on C. Such relations between C and B play important roles in judging nonlocality from entanglement. Moreover, determining the nonlocality of high dimensional quantum states has been a difficult problem in the theory of quantum information. We have provided approaches to judge the nonlocality of high dimensional quantum states by projecting the states to qubits ones. Moreover, the violation of Bell inequalities of the projected qubits states reveals some kind of "hidden" nonlocalities of the original high dimensional state, we call them locally-preprocessed-allowed nonlocalities. The implications and applications of such nonlocalities need to be explored further. Our results may also highlight researches on the quantum nonlocality and related the quantum correlations such as quantum steerability.

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