Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

Sharwin Rezagholi

Preprint no.: 73 2020

Factors maps of collective dynamics and hyperspace entropy

Sharwin Rezagholi*[1]

[1]Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103, Leipzig, Germany

June 2020

Abstract

Given a topological dynamical system, we consider its induced collective dynamics on the space of probability measures with the weak topology and on the hyperspace of closed subsets with the lower Vietoris topology. We show that the support of measures is a factor map between these collective dynamics, and that the topological entropy of the induced hyperspace system equals the entropy of the base system.

Keywords: Hyperspace, topological dynamics, topological entropy.
Mathematics subject classification 2010: 37B40, 54B20, 54H20, 60B05.

1 Introduction

A topological dynamical system (X, f) consists of a topological space X and a continuous map $f : X \to X$. By a collective dynamic, we mean a dynamical system on a hyperspace of X given by the induced hyperspace map. Traditionally, such analyses have considered systems in the category of compact metric spaces and used the hyperspace of closed nonempty subsets with the Hausdorff metric, which metrizes the Vietoris topology. Another collective dynamic is induced onto the space of Borel probability measures with the topology of weak convergence of measures. These traditional approaches do not lead to a morphism from the probabilistic to the

*Correspondence: sharwin.rezagholi [at] mis.mpg.de

possibilistic representation. This note considers a possibilistic representation which admits a morphism of dynamical systems from the probabilistic representation. The requirement that this morphism exists necessitates a hyperspace topology that is not T_2. For the traditional setting of topological dynamics, compact metrizable spaces, we will see that our possibilistic representation is entropy-equivalent to the original system.
(The spaces of subsets and of measures which appear in this note are special cases of constructions whose category-theoretical structure is discussed in [FPR19].)

2 The exponential space and the space of measures

Definition 1 (Exponential space [Vie22]). *Let X be a topological space. Its exponential space is*

$$\exp X := \{ C \subseteq X \big| C \text{ is closed and nonempty} \}$$

with the Vietoris topology, which is generated by the subbasis of sets of the following forms,

$$\mathrm{Hit}(U) := \{ C \in \exp X \big| C \cap U \neq \varnothing \}$$

where $U \subseteq X$ is an open subset, and

$$\mathrm{Miss}(K) := \{ C \in \exp X \big| C \cap K = \varnothing \}$$

where $K \subseteq X$ is a closed subset.

If the underlying space is compact and metrizable, the Vietoris topology is metrized by the Hausdorff metric [IN99, Chapters I.2-I.3].

Example 2. Let X be a set equipped with the discrete topology. Then $\exp X$ is the set 2^X equipped with the discrete topology.

Example 3. Consider the closed interval $[0, 1]$. Its exponential space is homeomorphic to the Hilbert cube whose exponential space is homeomorphic to itself [SW72; CS74].

Example 4. A Cantor space is homeomorphic to its exponential space [Cho48].

Proposition 5 ([Mic51]). *To a continuous map $f : X \to Y$, we may assign the continuous map $\exp f : \exp X \to \exp Y$ given by $\exp f(C) = \mathrm{cl}\big(f(C)\big)$. In particular, to a dynamical system (X, f), we may assign the system $(\exp X, \exp f)$.*

Recall that the weak convergence of measures $\mu_i \to \mu$ means that $\int f d\mu_i \to \int f d\mu$ for any continuous function $f : X \to [0,1]$. The following is well-known.

Proposition 6. *To a continuous map $f : X \to Y$, we may assign a continuous map $Pf : PX \to PY$ whose domain is the set of Borel probability measures on X with the topology of weak convergence and which maps into the analogous space of probability measures over Y. This map acts by pushing forward, $Pf(\mu)(A) = \mu\big(f^{-1}(A)\big)$. In particular, to a dynamical system (X, f), we may assign the system (PX, Pf).*

Recall that the support of $\mu \in PX$, denoted by $\operatorname{supp}(\mu) \in HX$, is defined as the smallest closed subset of X with full measure.

The following proposition shows that $\operatorname{supp} : PX \to \exp X$ is not continuous, hence it may also not be a morphism between the dynamical systems (PX, Pf) and $(\exp X, \exp f)$. In fact, assuming a compact metrizable space, the proposition shows that the continuity of the support requires a topology where the closure of a closed set includes all its closed subsets.

Proposition 7. *Let X be a metric space and consider two compact subsets $C \subsetneq K \subseteq X$. Then there exists a sequence $\{\mu_i\}$ in PX such that*

$$C = \operatorname{supp}\Big(\lim_{i\to\infty} \mu_i\Big) \subsetneq \lim_{i\to\infty} \operatorname{supp}(\mu_i) = K$$

with respect to the Vietoris topology.

Proof. We denote $Y := \operatorname{cl}(K \setminus C)$. Let $\{C_i\}$ be a sequence of finite $\frac{1}{i}$-nets in C such that $C_i \subseteq C_j$ whenever $i \leq j$. The existence of such a sequence is guaranteed by compactness. Let $\{Y_i\}$ be such a sequence in Y. The measures

$$c_i := \frac{1}{|C_i|} \sum_{x \in C_i} \delta_x$$

and

$$y_i := \frac{1}{|Y_i|} \sum_{x \in Y_i} \delta_x$$

are Borel probability measures, being linear combinations of finitely many Dirac measures. Define $\mu_i := \big(1 - \frac{1}{2i}\big)c_i + \frac{1}{2i} y_i$.
By construction, $\operatorname{supp}(\mu_i) = Y_i \cup C_i$ and $\operatorname{supp}(\mu_i) \subseteq \operatorname{supp}(\mu_j)$ whenever $i \leq j$. Hence

$$\lim_{i\to\infty} \operatorname{supp}(\mu_i) = \operatorname{cl}\left(\bigcup_{i \in \mathbb{N}} (Y_i \cup C_i)\right) = K.$$

Since, for all $i \in \mathbb{N}$, we have $\mu_i \in PK$ and since PK is compact and metrizable, there exists an accumulation point μ of $\{\mu_i\}$ which is necessarily an accumulation point of $\{c_i\}$ too. Suppose, aiming for a contradiction, that there exists $x \in C_i$ such that $x \notin \operatorname{supp}(\mu)$. Then there exists an open neighborhood $U \ni x$ such that $U \cap \operatorname{supp}(\mu) = \varnothing$. We pick a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(X \setminus U) \equiv 0$. Then $V := \left\{\nu \in PX \middle| \int f d\nu > 0\right\}$ is open. Since μ is an accumulation point of $\{\mu_i\}$ there exists a subsequence $\{\mu_{i_l}\}$ such that $\mu_{i_l} \to \mu$. Since $\mu \notin V$ there exists l_0 such that $\mu_{i_l} \notin V$ whenever $l \geq l_0$. But since $C_i \subseteq C_j$ whenever $i \leq j$, this contradicts $\mu_{i_j} \to \mu$. We have

$$\operatorname{supp}(\mu) \supseteq \operatorname{cl}\left(\bigcup_{i \in \mathbb{N}} C_i\right) = C.$$

To show the reverse inclusion, suppose, aiming for a contradiction, that there exists $x \in \operatorname{supp}(\mu) \setminus C$. We must have $\operatorname{dist}(x, C) = \delta > 0$. The sequence of open balls $\operatorname{ball}(x, \delta/j)$ is a local filter of x disjoint from C. For every j we may pick a continuous function $f_j : X \to [0,1]$ such that $f_j(x) = 1$ and $f_j(X \setminus \operatorname{ball}(x, \delta/j)) \equiv 0$. Consider the open sets $U_j := \left\{\nu \in PX \middle| \int f_j d\nu > 0\right\}$. Since $x \in \operatorname{supp}(\mu)$ by hypothesis, there exists a subsequence $\{\mu_{i_j}\}$ such that $\mu_{i_j} \in U_j$. In turn there must exist a sequence $\{x_{i_j}\}$ in X such that $x_{i_j} \in \operatorname{supp}(\mu_{i_j})$ and, since $\operatorname{dist}(x, x_{i_j}) < \delta/j$, we have $x_{i_j} \to x \in \operatorname{supp}(\mu) \cap C$, the desired contradiction. We have $\operatorname{supp}(\mu) \subseteq C$. □

3 Topological (collective) dynamics

Definition 8. *Let X be a topological space. We define*

$$HX := \left\{C \subseteq X \middle| C \text{ is nonempty and closed}\right\}$$

and equip it with the lower Vietoris topology generated by the subbasis of subsets of the form

$$\operatorname{Hit}(U) := \left\{C \in HX \middle| C \cap U \neq \varnothing\right\}$$

where U runs through the open subsets of X.

The hyperspaces HX are the topological analoga of the Hoare powerdomains in domain theory. They have already be defined by Schalk [Sch93], who calls them Hoare powerspaces.

Proposition 9. *Let X be a topological space. The specialization order on HX is the order of set inclusion: we have $C \in \operatorname{cl}(\{D\})$ if and only if $C \subseteq D$.*

Proof. Suppose that $C \subseteq D$. If $C \in \operatorname{Hit}(U)$, then $D \in \operatorname{Hit}(U)$. Suppose that $C \not\subseteq D$. Then $C \in \operatorname{Hit}(X \setminus D)$, but $D \notin \operatorname{Hit}(X \setminus D)$. □

Corollary 10. *For any topological space X, its hyperspace HX is T_0.*

Proof. The T_0 property is equivalent to the antisymmetry of the specialization order. The order of set inclusion is antisymmetric. $\square$

Example 11. Consider a finite set $X = \{x_1, ..., x_n\}$ with its discrete topology. Then $HX = 2^X$ with the topology where a subset is open if and only if it is $\subseteq$-upper.

Example 12. Let I be the set $\mathbb{R}$ with the topology whose nontrivial open sets are of the form (l, ∞) where $l \in \mathbb{R}$. Then $HI \simeq I$. The homeomorphism $HI \to I$ assigns $(-\infty, r] \mapsto r$.

Example 13. Let C be a Cantor space. Since $C \simeq \exp C$ and since HC is a coarsening of $\exp C$, considering the hyperspace of C effectively means to coarsen the topology of C.

Proposition 14. *Let X and Y be topological spaces. To a continuous map $f : X \to Y$, we may assign the continuous map $Hf : HX \to HY$ given by $Hf(C) = \mathrm{cl}\big(f(C)\big)$. In particular, we may assign to a dynamical system (X, f) the hyperspace system (HX, Hf).*

Proof. It suffices to note that $(Hf)^{-1}(\mathrm{Hit}(U)) = \mathrm{Hit}(f^{-1}(U))$. $\square$

Lemma 15. *Let X be a $T_{3\frac{1}{2}}$-space. Then* $\mathrm{supp} : PX \to HX$ *is continuous.*

Proof. Let $\mu \in \mathrm{supp}^{-1}\big(\mathrm{Hit}(U)\big)$, or rather $\mathrm{supp}(\mu) \in \mathrm{Hit}(U)$. We will show that μ is an interior point of $\mathrm{supp}^{-1}\big(\mathrm{Hit}(U)\big)$. Let $x \in \mathrm{supp}(\mu) \cap U$. By the $T_{3\frac{1}{2}}$-property, there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) \equiv 0$. Then $\big\{\nu \in PX \big| \int f d\nu > 0\big\} \subseteq \mathrm{supp}^{-1}\big(\mathrm{Hit}(U)\big)$ is an open neighborhood of μ. $\square$

We want to remark that the lemma above is a special case of a more general theory discussed elsewhere [FPR19].

Lemma 16. *Let X be a compact metrizable space and let $C \subseteq X$ be closed. Then there exists a Borel probability measure whose support is C. In particular, the maps* $\mathrm{supp} : PX \to HX$, *and* $\mathrm{supp} : PX \to \exp X$ *are surjective.*

Proof. Fix a metric for X and let $C \subseteq X$ be closed, and therefore compact. We want to show that $\mathrm{supp}^{-1}(C) \neq \varnothing$. Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of finite subsets of C such that N_i is a $\frac{1}{i}$-net in C and $N_i \subseteq N_j$ for all $i \leq j$. Define the measures

$$\mu_i := \frac{1}{|N_i|} \sum_{x \in N_i} \delta_x.$$

The sequence $\{\mu_i\}$ consists of Borel probability measures. Since PX is compact, there exists an accumulation point μ of this sequence. By an argument analogous to the proof of Proposition 7, we conclude that $\text{supp}(\mu) = C$. $\square$

Recall that a morphism between the dynamical systems (X, f) and (Y, g) is a continuous equivariant surjection $m : X \to Y$, hence $g \circ m = f \circ m$.

Proposition 17. *Let X be a compact metrizable space and let $f : X \to X$ be continuous. Then* $\text{supp} : PX \to HX$ *is a morphism of dynamical systems between* (PX, Pf) *and* (HX, Hf).

Proof. Continuity is proven in Lemma 15, and surjectivity in Lemma 16. It remains to show equivariance. Let $\mu \in PX$. We need to show that $x \in \text{supp} \circ Pf(\mu) = \text{supp}(f_*\mu)$ if and only if $x \in Hf \circ \text{supp}(\mu) = f(\text{supp}(\mu))$. (In the last equality we used the closedness of f.)

Let $x \in \text{supp}(f_*\mu)$ and suppose, aiming for a contradiction, that $x \notin f(\text{supp}(\mu))$, or rather $f^{-1}(x) \not\subseteq \text{supp}(\mu)$. There exists $\epsilon > 0$ such that $\mu((f^{-1}(x))_\epsilon) = 0$. Since f must be an open map, $f((f^{-1}(x))_\epsilon)$ is an open neighborhood of x. Since $x \in \text{supp}(f_*\mu)$ by hypothesis, we must have

$$0 < (f_*\mu)\Big(f\Big(\big(f^{-1}(x)\big)_\epsilon\Big)\Big) = \mu\Big(f^{-1} \circ f\Big(\big(f^{-1}(x)\big)_\epsilon\Big)\Big) = \mu\Big(\big(f^{-1}(x)\big)_\epsilon\Big),$$

the desired contradiction. We conclude that $\text{supp}(f_*\mu) \subseteq f(\text{supp}(\mu))$.

Let $x \in f(\text{supp}(\mu))$ and suppose, aiming for a contradiction, that $x \notin \text{supp}(f_*\mu)$. Then there exists an open ball around x such that

$$(f_*\mu)\big(\,\text{ball}(x, \epsilon)\big) = \mu\Big(f^{-1}\big(\,\text{ball}(x, \epsilon)\big)\Big) = 0.$$

Since $f^{-1}(\text{ball}(x, \epsilon))$ is an open neighborhood of $f^{-1}(x) \subseteq X$, there exists some $\delta > 0$ such that $\mu((f^{-1}(x))_\delta) = 0$. This implies that $f^{-1}(x) \not\subseteq \text{supp}(\mu)$ and $x \notin f(\text{supp}(\mu))$: the desired contradiction. We conclude that $f(\text{supp}(\mu)) \subseteq \text{supp}(f_*\mu)$. $\square$

Recall that a dynamical system (X, f) is called *(topologically) transitive*, if, for any open and nonempty $U, V \subseteq X$, there exists $n \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \varnothing$. It is called *(strongly) mixing* if, for all open and nonempty $U, V \subseteq X$, there exists $n_0 \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \varnothing$ holds for all $n \geq n_0$. Clearly, mixing implies transitivity.

Proposition 18 (Bauer and Sigmund [BS75])**.** *Let (X, f) be a dynamical system where X is a compact metrizable space. Then the following three statements hold.*

(a) *If* (X, f) *is mixing, then* (PX, Pf) *and* $(\exp X, \exp f)$ *are mixing.*

(b) *If* (PX, Pf) *(equivalently* $(\exp X, \exp f)$*) is transitive, then* (X, f) *is transitive.*

(c) *If* (PX, Pf) *(equivalently* $(\exp X, \exp f)$*) is mixing, then* (X, f) *is mixing.*

Example 19. Consider the irrational rotation $f(x) = x + c \mod 1$ where $c \in \mathbb{R} \setminus \mathbb{Q}$. The system (S^1, f) is transitive (every orbit is dense). Pick $x_1, x_2 \in S^1$ with $x_1 \neq x_2$. Then a sufficiently small open neighborhood of $\frac{1}{2}(\delta_{x_1} + \delta_{x_2})$ in PX is never hit by the iterates of a sufficiently small neighborhood of δ_{x_1} in the system (PS^1, Pf). The same holds for small neighborhoods of $\{x_1\}$ and $\{x_1, x_2\}$ in $(\exp S^1, \exp f)$. This example is due to Bauer and Sigmund [BS75]. But for any open neighborhood $U \ni \{x_1\}$ and $V \ni \{x_1, x_2\}$ in HS^1 we have $(Hf)^n(U) \cap V \neq \varnothing$ for all $n \in \mathbb{N}$, since $\{x_1\} \in \mathrm{cl}(\{x_1, x_2\})$.

The notion of mixing is trivial for H-systems, as the following property of the H-topology shows.

Proposition 20 (Hyperconnectivity)**.** *For any topological space* X*, the space* HX *is hyperconnected: Any two nonempty open subsets intersect.*

Proof. Since closed sets are $\subseteq$-lower, open sets are $\subseteq$-upper. In particular every subbasic open set $\mathrm{Hit}(U) \subseteq HX$ contains X, and hence does every open set. □

Corollary 21. *Let* X *be a topological space and let* Y *be a* T_2*-space. Then* $f : HX \to Y$ *is continuous if and only if it is constant.*

Corollary 22. *Let* $f : HX \to HX$ *be a continuous map. Then* (HX, f) *is mixing. In particular, the endofunctor* $(X, g) \mapsto (HX, Hg)$ *takes image in the mixing systems.*

4 Topological entropy

Given a positive real sequence $\{x_t\}_{t \in \mathbb{N}}$ we denote its exponential growth rate by

$$\mathsf{GR}_t(x_t) \coloneqq \limsup_{t \to \infty} \frac{1}{t} \ln(x_t).$$

Given a topological dynamical system (X, f) on a compact space its topological entropy [AKM65] is

$$h(X, f) \coloneqq \sup \left\{ \mathsf{GR}_t \left(\# \bigvee_{i=0}^{t} f^{-i}(\mathcal{U}) \right) \middle| \, \mathcal{U} \text{ is an open cover of } X \right\}$$

where $\mathcal{A} \vee \mathcal{B} := \{A \cap B\}_{A \in \mathcal{A}, B \in \mathcal{B}}$, and $\#\mathcal{C}$ denotes the minimal cardinality of a finite subcover of $\mathcal{C}$.

We recall two well-known properties of entropy which we will use in the following. (i) Whenever the system (X, f) is such that $f|_K : K \to K$ for some closed subset $K \subseteq X$, we have $h(X, f) \geq h(K, f|_K)$. (ii) Consider systems (X, f) and (Y, g) and a continuous surjection $m : X \to Y$ such that $m \circ f = g \circ m$, a morphism of dynamical systems, then $h(X, f) \geq h(Y, g)$.

Theorem 23 (Glasner and Weiss [GW95])**.** *Let (X, f) be a dynamical system where X is a compact metrizable space and $h(X, f) = 0$. Then $h(PX, Pf) = 0$.*

Combining the theorem of Glasner and Weiss with Proposition 17, we obtain the following corollary.

Corollary 24. *Let (X, f) be a dynamical system where X is a compact metrizable space and $h(X, f) = 0$. Then $h(HX, Hf) = 0$.*

Proof. We have $h(HX, Hf) \leq h(PX, Pf) = 0$. □

Example 25. The shift $\big(\{0,1\}^{\mathbb{Z}}, \sigma\big)$ contains the subshift

$$I := \big\{ w \in \{0,1\}^{\mathbb{Z}} \,\big|\, \text{the letter 1 appears at most once} \big\}.$$

We have $h(I, \sigma) = 0$. Consider the continuous surjection $m : \exp I \to \{0,1\}^{\mathbb{Z}}$ that assigns $m(C) = \{v_i(C)\}_{i \in \mathbb{Z}}$ where

$$v_i(C) := \begin{cases} 1 \text{ if } C \text{ contains the sequence whose } i\text{'th letter is } 1 \\ 0 \text{ otherwise.} \end{cases}$$

It is a morphism from $(\exp I, \exp \sigma)$ to the full shift $\big(\{0,1\}^{\mathbb{Z}}, \sigma\big)$, since $\sigma \circ m(C) = \phi \circ \exp \sigma(C) = \{v_{i+1}(C)\}$. It is uniformly finite-to-one and therefore, see Lemma 27, we have $h(\exp I, \exp \sigma) = h\big(\{0,1\}^{\mathbb{Z}}, \sigma\big) = \ln(2)$. This example is due to Kwietnak and Oprocha [KO07]. Note that $h(HI, H\sigma) = h(I, \sigma) = 0$, by Corollary 24.

Theorem 26 (Bauer and Sigmund [BS75])**.** *Let (X, f) be a dynamical system where X is a compact T_2-space and $h(X, f) > 0$. Then $h(PX, Pf) = h(\exp X, \exp f) = \infty$.*

The theorem of Bauer and Sigmund shows that the P- and exp-induced systems must either have vanishing or infinite entropy. As the proof below indicates, positive entropy in the base system leads to infinite entropy in the induced system because there is a countable sequence of invariant subsystems of strictly increasing entropy.

Lemma 27 (Rufus Bowen, for example Theorem 1.8 in [Rob95])**.** *Let* (X, f) *and* (Y, g) *be dynamical systems on compact metrizable spaces. Suppose that these systems are related by a continuous surjection* $m : X \to Y$ *with* $|m^{-1}(y)| \leq c < \infty$ *for all* $y \in Y$*. Then* $h(X, f) = h(Y, g)$*.*

Sketch of proof of Theorem 26. It is well-known that the subspaces

$$\exp_n X := \{C \in \exp X \big| |C| \leq n\}$$

and

$$P_n X := \left\{ p \in PX \middle| p = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \right\}$$

are closed subspaces of PX and $\exp X$, they are homeomorphic. These spaces admit $n!$-to-one continuous equivariant surjections $X^n \to \exp_n X$ and $X^n \to P_n X$. By Lemma 27, one concludes that

$$h(\exp X, \exp f) \geq h(\exp_n X, \exp f) = h(X^n, f^{\otimes n}) = n \cdot h(X, f) \xrightarrow{n \to \infty} \infty.$$

The proof for P is analogous. □

The above proof crucially relies on Lemma 27. The next example shows that this lemma has no analog if the morphism of systems maps a compact metric space to a compact T_0-space.

Example 28. Let $f : [0, 1] \to [0, 1]$ be a continuous map with positive entropy. Denote by L the set $[0, 1]$ with the topology whose nontrivial open sets are intervals of the form $(l, 1]$ for $l \in [0, 1)$. The identity map $[0, 1] \to L$ is a continuous bijection from a compact metric space to a compact T_0-space. While f has positive entropy its image $f : L \to L$ has zero entropy, since every open cover of L admits a subcover of cardinality one.

We will now prove that the H-lifting is entropy-preserving under the separation axiom T_2.

Proposition 29. *Let* X *be a* T_2*-space. Then the space*

$$H_n X := \{C \in HX \big| |C| \leq n\}$$

of subsets of X *with at most* n *elements is closed in* HX*. Furthermore, the natural map* $\pi : X^n \to H_n X$ *is continuous.*

Proof. Since X is T_2, all its finite subsets are closed, hence $H_nX \subseteq HX$. We show that every point $K \in HX \setminus H_nX$ is interior. Since $|K| > n$, we may pick $n+1$ distinct points $\{x_1, ..., x_{n+1}\} \subseteq K$. Since X is T_2, there exist disjoint open neighborhoods $\{U_1, ..., U_{n+1}\}$ that separate these points. We have $K \in \mathrm{Hit}(U_1) \cap ... \cap \mathrm{Hit}(U_{n+1})$. This open set contains only sets of cardinality at least $n+1$. Hence H_nX is closed since its complement is open.
Consider the subbasic open set $\mathrm{Hit}(U) \subseteq H_nX$. Then

$$\pi^{-1}(\mathrm{Hit}(U)) = \big\{\{x_1, ..., x_n\} \big| x_i \in U \text{ for some } i\big\} = \bigcup_{i=1}^{n} \big\{\{x_1, ..., x_n\} \big| x_i \in U\big\}$$

where the latter is a finite union of subbasic open sets for the product topology. □

Noting that, for any continuous map $f : X \to X$, we have $Hf(H_nX) \subseteq H_nX$, we obtain the following corollary.

Corollary 30. *Let X be a T_2-space. Then the map $X \hookrightarrow HX$ given by $x \mapsto \{x\}$ is a closed embedding whose image is H_1X. In particular, $(X, f) \hookrightarrow (H_1X, Hf)$ is an isomorphism onto a closed subsystem.*

Lemma 31. *Let X be a T_1-space and let $\mathcal{V}$ be a collection of open subsets of HX. Then $\mathcal{V}$ is an open cover of HX if and only if it covers H_1X. Furthermore, $\mathcal{V}$ is a minimal open cover of HX if and only if it is a minimal cover of H_1X.*

Proof. Suppose that $\mathcal{V}$ covers H_1X. Since open subsets of HX are upper in the lattice of closed subsets of X ordered by inclusion, an open subset of HX that contains the singleton subset $\{x\}$ does also contain every element of the principal filter $\{C \in HX | x \in C\}$. We conclude that a cover of the singleton subsets is a cover of HX. The other direction is clear.
Suppose that $\mathcal{V}$ is a minimal cover of HX while $\mathcal{U} \subseteq \mathcal{V}$ is a minimal cover of H_1X. By the previous reasoning $\mathcal{U}$ is a cover of HX, hence $\mathcal{U} = \mathcal{V}$. Similarly, a minimal cover of H_1X must be minimal for HX as well. □

Proposition 32. *Let X be a T_2-space and let $f : X \to X$ be a continuous map. Then $h(HX, Hf) = h(X, f)$.*

Proof. By Corollary 30, the systems (X, f) and (H_1X, Hf) are isomorphic, in particular $h(X, f) = h(H_1X, Hf)$. From Lemma 31, we conclude that

$$\begin{aligned} h(H_1X, Hf) &:= \sup\big\{h_{\mathcal{V}}(H_1X, Hf) \big| \mathcal{V} \text{ is an open cover of } H_1X\big\} \\ &= \sup\big\{h_{\mathcal{V}}(HX, Hf) \big| \mathcal{V} \text{ is an open cover of } H_1X\big\} \end{aligned}$$

$$= \sup\{h_{\mathcal{V}}(HX,Hf) \big| \mathcal{V} \text{ is an open cover of } HX\}$$
$$= h(HX,Hf).$$

□

This result illustrates the degree to which the H-topology is coarser than the exp-topology. Whenever (X,f) is a system with positive entropy on a T_2-space, we have $h(\exp X, \exp f) = \infty$ while $h(HX,Hf) = h(X,f) < \infty$. The proof, via Lemma 31, uses that the topology of HX, being an order-topology, allows, at least in the T_2-case, to extent the bijection between X and principal filters of singletons in HX to open covers. We do not know whether weaker separation axioms may yield an entropy-increasing or entropy-decreasing H-representation.

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